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Black diring and infinite nonuniqueness

Hideo Iguchi and Takashi Mishima
Laboratory of Physics, College of Science and Technology, Nihon University, Narashinodai, Funabashi, Chiba 274-8501, Japan
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We show that the S^1-rotating black rings can be superposed by the solution-generating technique. We analyze the black diring solution for the simplest case of multiple rings. There exists an equilibrium black diring where the conical singularities are cured by the suitable choice of physical parameters. Also there are infinite numbers of black dirings with the same mass and angular momentum. These dirings can have two different continuous limits of single black rings. Therefore, we can transform the fat black ring to the thin ring with the same mass and angular momentum by way of the diring solutions.

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I. INTRODUCTION

One of the most important recent findings of the higher-dimensional general relativity is a single-rotational black ring solution by Emparan and Reall [1]. (See also [2].) This solution is an axially symmetric and asymptotically flat solution of the five-dimensional vacuum Einstein equations. The topology of the event horizon is $S^1 \times S^2$. The black ring rotates along the direction of S^1. The balanced black ring, which has no conical singularity, has a minimum of angular momentum for a fixed mass parameter. When the angular momentum is near this minimum, there are two different black rings with the same angular momentum. They are called fat and thin black rings according to their shapes. In addition, we have a single-rotational spherical black hole [3] with the same asymptotic parameters. This finding entails the discrete nonuniqueness of the five-dimensional vacuum solutions. It has been shown that the black rings can have dipole charges which are independent of all conserved charges [4]. Therefore, the dipole rings imply the infinite violation of uniqueness by a continuous parameter.

We recently found a black ring solution with S^2 rotation by using a solitonic solution-generating technique [5,6]. The seed solution of this ring is a simple Minkowski spacetime. Because the effect of rotation cannot compensate for the gravitational attractive force, the ring has a kind of strut structure. We also have generated the black ring with S^1 rotation by the same solitonic solution-generating technique [7]. The seed solution is not a Minkowski spacetime but an Euclidean C-metric solution. It has been shown that these two solutions can be obtained by the inverse scattering method [8–11]. Rotating dipole black ring solutions have been systematically generated in five-dimensional Einstein-Maxwell-dilaton gravity [12,13]. The relations between the seed and the solitonic solutions can be understood easily through the analysis of their rod structures [14,15]. Furthermore, the seed of the S^1-rotating black ring has been constructed by the help of the rod structure analysis. Thus, the rod structure analysis is expected to be a useful guide to construct seed solutions for new solutions.

In this paper we consider the multiplexed S^1-rotating black rings arranged in a concentric pattern. The seed solution can be constructed by the help of rod structure analysis as in the case of the S^1-rotating black ring [7]. The exact expressions of metric functions can be written down by the solitonic transformation, but in rather complicated forms. Here we analyze diring solutions as the first step in the black ring multiplication. In the supersymmetric system, the solution of multiple black rings do indeed exist [16,17]. Also, the solution of concentric static extremal black rings has been considered [18].

The simplest multiple black rings solution is a black diring. This solution has two ringlike event horizons of different radii with the same topology of $S^1 \times S^2$. Both horizons can rotate along the direction of S^1. As similar as the single black ring solutions, this solution has conical singularities for general values of parameters. However, these conical singularities can be cured by an appropriate choice of the parameters as in the case of the S^1-rotating black ring. The black dirings can have the same mass and angular momentum for infinite numbers of sets of parameters. Therefore, the black dirings realize an infinite nonuniqueness without dipole charges. There can exist one Myers-Perry black hole, two S^1-rotating black rings, and infinite numbers of black dirings for the same mass and angular momentum. Also, these single black rings are continuous limits in the black dirings. Therefore, these two different black rings are connected by the black dirings with the same mass and angular momentum. When we shrink the inner ring down to zero radius, we obtain a solution describing a black hole sitting at the common center of the outer ring.

The paper is organized as follows. In Sec. II we briefly review the solution-generating technique used in the analysis. The rod structure analysis is explained in Sec. III. In Sec. IV, we give the seed solutions of black multiring and diring solutions and analyze some features of diring solutions, and we give a summary in Sec. V.

II. BRIEF REVIEW OF SOLUTION-GENERATING TECHNIQUE

First, we briefly explain the procedure to generate axisymmetric solutions in the five-dimensional general relativity. The spacetimes which we considered satisfy the following conditions: (c1) five dimensions, (c2) asymptotically flat spacetimes, (c3) the solutions of vacuum Einstein equations, (c4) having three commuting Killing vectors including time translational invariance, and (c5) having a single nonzero angular momentum component. Under the conditions (c1)–(c5), we can employ the following Weyl-Papapetrou metric form

$$ds^2 = -e^{2U_0}(dx^0 - \omega d\phi)^2 + e^{2U_1}\rho^2(d\phi)^2 + e^{2U_2}(d\psi)^2 + e^{2(\gamma+U_1)}(d\rho^2 + dz^2), \tag{1}$$

where U_0, U_1, U_2, ω, and γ are functions of ρ and z. Then we introduce new functions $S := 2U_0 + U_2$ and $T := U_2$ so that the metric form (1) is rewritten into

$$ds^2 = e^{-T}[-e^S(dx^0 - \omega d\phi)^2 + e^{T+2U_1}\rho^2(d\phi)^2 + e^{2(\gamma+U_1)+T}(d\rho^2 + dz^2)] + e^{2T}(d\psi)^2. \tag{2}$$

Using this metric form, the Einstein equations are reduced to the following set of equations:

$$\text{(i)}\ \nabla^2 T = 0, \qquad \text{(ii)} \begin{cases} \partial_\rho \gamma_T = \frac{3}{4}\rho[(\partial_\rho T)^2 - (\partial_z T)^2] \\ \partial_z \gamma_T = \frac{3}{2}\rho[\partial_\rho T \partial_z T], \end{cases}$$

$$\text{(iii)}\ \nabla^2 \mathcal{E}_S = \frac{2}{\mathcal{E}_S + \bar{\mathcal{E}}_S}\nabla\mathcal{E}_S \cdot \nabla\mathcal{E}_S,$$

$$\text{(iv)} \begin{cases} \partial_\rho \gamma_S = \frac{\rho}{2(\mathcal{E}_S + \bar{\mathcal{E}}_S)}(\partial_\rho\mathcal{E}_S\partial_\rho\bar{\mathcal{E}}_S - \partial_z\mathcal{E}_S\partial_z\bar{\mathcal{E}}_S) \\ \partial_z \gamma_S = \frac{\rho}{2(\mathcal{E}_S + \bar{\mathcal{E}}_S)}(\partial_\rho\mathcal{E}_S\partial_z\bar{\mathcal{E}}_S + \partial_\rho\mathcal{E}_S\partial_z\bar{\mathcal{E}}_S), \end{cases}$$

$$\text{(v)}\ (\partial_\rho\Phi, \partial_z\Phi) = \rho^{-1}e^{2S}(-\partial_z\omega, \partial_\rho\omega),$$

$$\text{(vi)}\ \gamma = \gamma_S + \gamma_T, \qquad \text{(vii)}\ U_1 = -\frac{S+T}{2},$$

where Φ is defined through the equation (v) and the function $\mathcal{E}_S$ is defined by $\mathcal{E}_S := e^S + i\Phi$. The most nontrivial task to obtain new metrics is to solve the equation (iii) because of its nonlinearity. To overcome this difficulty here we use the method similar to the Neugebauer's Bäcklund transformation [19] or the Hoenselaers-Kinnersley-Xanthopoulos transformation [20].

To write down the exact form of the metric functions, we follow the procedure given by Castejon-Amenedo and Manko [21]. In the five-dimensional spacetime we start from the following form of a seed static metric:

$$ds^2 = e^{-T^{(0)}}[-e^{S^{(0)}}(dx^0)^2 + e^{-S^{(0)}}\rho^2(d\phi)^2 + e^{2\gamma^{(0)}-S^{(0)}}(d\rho^2 + dz^2)] + e^{2T^{(0)}}(d\psi)^2.$$

For this static seed solution, $e^{S^{(0)}}$, of the Ernst equation (iii), a new Ernst potential can be written in the form

$$\mathcal{E}_S = e^{S^{(0)}}\frac{x(1+ab) + iy(b-a) - (1-ia)(1-ib)}{x(1+ab) + iy(b-a) + (1-ia)(1-ib)},$$

where x and y are the prolate spheroidal coordinates: $\rho = \sigma\sqrt{x^2-1}\sqrt{1-y^2}$, $z = \sigma xy$, with $\sigma > 0$. The ranges of these coordinates are $1 \le x$ and $-1 \le y \le 1$. The functions a and b satisfy the following simple first-order differential equations:

$$\begin{aligned} (x-y)\partial_x a &= a[(xy-1)\partial_x S^{(0)} + (1-y^2)\partial_y S^{(0)}], \\ (x-y)\partial_y a &= a[-(x^2-1)\partial_x S^{(0)} + (xy-1)\partial_y S^{(0)}], \\ (x+y)\partial_x b &= -b[(xy+1)\partial_x S^{(0)} + (1-y^2)\partial_y S^{(0)}], \\ (x+y)\partial_y b &= -b[-(x^2-1)\partial_x S^{(0)} + (xy+1)\partial_y S^{(0)}]. \end{aligned} \tag{3}$$

For the typical seed

$$S^{(0)} = \tfrac{1}{2}\ln[R_d + (z-d)], \tag{4}$$

the following a and b satisfy the differential Eqs. (3),

$$a = l_\sigma^{-1}e^{2\phi_{d,\sigma}}, \qquad b = -l_{-\sigma}e^{-2\phi_{d,-\sigma}}, \tag{5}$$

where $R_d = \sqrt{\rho^2 + (z-d)^2}$ and

$$\phi_{d,c} = \tfrac{1}{2}\ln[e^{-\tilde{U}_d}(e^{2U_c} + e^{2\tilde{U}_d})]. \tag{6}$$

Here the functions $\tilde{U}_d$ and U_c are defined as $\tilde{U}_d := \frac{1}{2} \times \ln[R_d + (z-d)]$ and $U_c := \frac{1}{2}\ln[R_c - (z-c)]$. Because of the linearity of the differential Eqs. (3) for $S^{(0)}$, we can easily obtain a and b which correspond to a general seed function.

The metric functions for the five-dimensional metric (2) are obtained by using the formulas shown by [21],

$$e^S = e^{S^{(0)}}\frac{A}{B}, \tag{7}$$

$$\omega = 2\sigma e^{-S^{(0)}}\frac{C}{A} + C_1, \tag{8}$$

$$e^{2\gamma} = C_2(x^2-1)^{-1}Ae^{2\gamma'}, \tag{9}$$

where C_1 and C_2 are constants and A, B, and C are given by

$$\begin{aligned} A &:= (x^2-1)(1+ab)^2 - (1-y^2)(b-a)^2, \\ B &:= [(x+1) + (x-1)ab]^2 + [(1+y)a + (1-y)b]^2, \\ C &:= (x^2-1)(1+ab)[(1-y)b - (1+y)a] \\ &\quad + (1-y^2)(b-a)[x+1-(x-1)ab]. \end{aligned}$$

In addition, the γ' in Eq. (9) is a γ function corresponding to the static metric,

$$ds^2 = e^{-T^{(0)}}[-e^{2U_0^{(\mathrm{BH})}+S^{(0)}}(dx^0)^2 + e^{-2U_0^{(\mathrm{BH})}-S^{(0)}}\rho^2(d\phi)^2 + e^{2(\gamma'-U_0^{(\mathrm{BH})})-S^{(0)}}(d\rho^2 + dz^2)] + e^{2T^{(0)}}(d\psi)^2, \tag{10}$$

where $U_0^{(\mathrm{BH})} = \frac{1}{2}\ln(\frac{x-1}{x+1})$. Therefore, the function γ' obeys the following equations:

$$\partial_\rho \gamma' = \tfrac{1}{4}\rho[(\partial_\rho S')^2 - (\partial_z S')^2] + \tfrac{3}{4}\rho[(\partial_\rho T')^2 - (\partial_z T')^2], \tag{11}$$

$$\partial_z \gamma' = \tfrac{1}{2}\rho[\partial_\rho S' \partial_z S'] + \tfrac{3}{2}\rho[\partial_\rho T' \partial_z T'], \tag{12}$$

where the first terms are contributions from equation (iv) and the second terms come from equation (ii). Here the functions S' and T' can be read out from Eq. (10) as

$$S' = 2U_0^{(BH)} + S^{(0)}, \tag{13}$$

$$T' = T^{(0)}. \tag{14}$$

To integrate these equations we can use the following fact that the partial differential equations

$$\partial_\rho \gamma'_{cd} = \rho[\partial_\rho \tilde{U}_c \partial_\rho \tilde{U}_d - \partial_z \tilde{U}_c \partial_z \tilde{U}_d], \tag{15}$$

$$\partial_z \gamma'_{cd} = \rho[\partial_\rho \tilde{U}_c \partial_z \tilde{U}_d + \partial_\rho \tilde{U}_d \partial_z \tilde{U}_c], \tag{16}$$

have the following solution:

$$\gamma'_{cd} = \tfrac{1}{2}\tilde{U}_c + \tfrac{1}{2}\tilde{U}_d - \tfrac{1}{4}\ln Y_{cd}, \tag{17}$$

where $Y_{cd} := R_c R_d + (z-c)(z-d) + \rho^2$. The general solution of γ' is given by the linear combination of the functions γ'_{cd}. And then the function T is equal to $T^{(0)}$, and U_1 is given by the Einstein equation (vii).

III. ROD STRUCTURE ANALYSIS

We give a brief explanation of the rod structure analysis elaborated by Harmark [15]. See [15] for complete explanations.

Here we denote the D-dimensional axially symmetric stationary metric as

$$ds^2 = G_{ij} dx^i dy^j + e^\nu (d\rho^2 + dz^2), \tag{18}$$

where G_{ij} and ν are functions only of ρ and z and $i, j = 0, 1, \ldots, D-3$. The $D-2$ by $D-2$ matrix field G satisfies the following constraint:

$$\rho = \sqrt{|\det G|}. \tag{19}$$

The equations for the matrix field G can be derived from the Einstein equation $R_{ij} = 0$ as

$$G^{-1}\nabla G = (G^{-1}\nabla G)^2, \tag{20}$$

where the differential operator ∇ is the gradient in three-dimensional unphysical flat space with metric

$$d\rho^2 + \rho^2 d\omega^2 + dz^2. \tag{21}$$

Because of the constraint $\rho = \sqrt{|\det G|}$, at least one eigenvalue of $G(\rho, z)$ goes to zero for $\rho \to 0$. However, it was shown that if more than one eigenvalue goes to zero as $\rho \to 0$, we have a curvature singularity there. Therefore, we consider solutions which have only one eigenvalue go to zero for $\rho \to 0$, except at isolated values of z. Denoting these isolated values of z as $a_1, a_2, \ldots, a_N$, we can divide the z axis into the $N+1$ intervals $[-\infty, a_1]$,$[a_1, a_2]$, $\ldots$, $[a_N, \infty]$, which are called rods. These rods correspond to the source added to Eq. (20) at $\rho = 0$ to prevent the breakdown of the equation there.

The eigenvector for the zero eigenvalue of $G(0, z)$

$$\mathbf{v} = v^i \frac{\partial}{\partial x^i}, \tag{22}$$

which satisfies

$$G_{ij}(0, z) v^i = 0, \tag{23}$$

determines the direction of the rod. If the value of $\frac{G_{ij} v^i v^j}{\rho^2}$ is negative (positive) for $\rho \to 0$, the rod is called timelike (spacelike). Each rod corresponds to the region of the translational or rotational invariance of its direction. The timelike rod corresponds to a horizon. The spacelike rod corresponds to a compact direction.

IV. S^1-ROTATING BLACK MULTIRING

The n-multiplexed S^1-rotating black ring can be obtained in the following manner. First we prepare the seed solution of multirings as in Fig. 1. To assure the asymptotical flatness, we need two semi-infinite spacelike rods in the different directions. There is a finite spacelike rod with the direction vector $\partial/\partial\phi$ around the $z = 0$. Between this finite rod and the semi-infinite spacelike rod of the ϕ direction, we alternately arrange n spacelike rods in a ψ direction and $(n-1)$ static finite timelike rods. The finite spacelike rod of the ϕ direction is changed to a finite timelike rod with ϕ rotation by the solitonic transformation [6,7]. In addition, the finite timelike rods of seed solution can get the ϕ components in their direction vectors through the transformation.

In the following we investigate the simplest multiple black rings, i.e., the black diring solution. The rod structure of the seed and the diring solution are given in Fig. 2. The rod structure of the diring is determined by 4 lengths of finite rods and 2 angular velocities of timelike rods. We have 5 physical parameters: η_1, η_2, δ_1, δ_2, and λ, except for the freedom of scaling. These parameters should satisfy the condition $-1 < \eta_1 < \eta_2 < 1 < \delta_1 < \delta_2 < \lambda$ for the diring solution. Note that when we set $\delta_2 = \lambda$, the inner ring shrinks to an S^3 sphere. When $\delta_1 = \delta_2$, these structures are exactly the same as the case of a single S^1-rotating black ring.

The seed functions of black diring are given by the following functions:

$$T^{(0)} = \tilde{U}_{\lambda\sigma} + \tilde{U}_{\delta_1\sigma} - \tilde{U}_{\delta_2\sigma} + \tilde{U}_{\eta_1\sigma} - \tilde{U}_{\eta_2\sigma}, \tag{24}$$

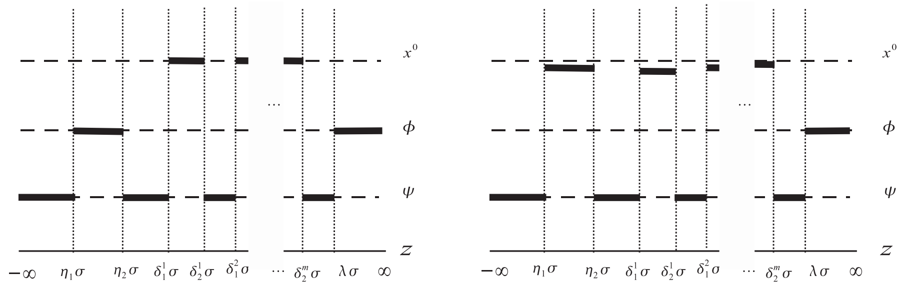

FIG. 1. Schematic pictures of rod structures of multiring and its seed. The left panel shows the rod structure of seed metric of S^1-rotating black multiring. The right panel shows the rod structure of S^1-rotating black multiring. The finite spacelike rod $[\eta_1\sigma, \eta_2\sigma]$ in the left panel is altered to the finite timelike rod by the solution-generating transformation. All static timelike rods may be transformed to stationary ones by the solitonic transformation. To denote the rotation of the event horizons, we put the finite timelike rods between the lines of x^0 and ϕ.

$$S^{(0)} = \tilde{U}_{\lambda\sigma} - (\tilde{U}_{\delta_1\sigma} - \tilde{U}_{\delta_2\sigma}) + \tilde{U}_{\eta_1\sigma} - \tilde{U}_{\eta_2\sigma}. \tag{25}$$

The corresponding auxiliary potentials of solitonic solutions are obtained as

$$a = \frac{\alpha}{2\sigma^{1/2}} \frac{e^{2U_\sigma} + e^{2\tilde{U}_{\lambda\sigma}}}{e^{\tilde{U}_{\lambda\sigma}}} \frac{e^{\tilde{U}_{\delta_1\sigma}}}{e^{2U_\sigma} + e^{2\tilde{U}_{\delta_1\sigma}}} \frac{e^{2U_\sigma} + e^{2\tilde{U}_{\delta_2\sigma}}}{e^{\tilde{U}_{\delta_2\sigma}}} \times \frac{e^{2U_\sigma} + e^{2\tilde{U}_{\eta_1\sigma}}}{e^{\tilde{U}_{\eta_1\sigma}}} \frac{e^{\tilde{U}_{\eta_2\sigma}}}{e^{2U_\sigma} + e^{2\tilde{U}_{\eta_2\sigma}}}, \tag{26}$$

$$b = 2\sigma^{1/2}\beta \frac{e^{\tilde{U}_{\lambda\sigma}}}{e^{2U_{-\sigma}} + e^{2\tilde{U}_{\lambda\sigma}}} \frac{e^{2U_{-\sigma}} + e^{2\tilde{U}_{\delta_1\sigma}}}{e^{\tilde{U}_{\delta_1\sigma}}} \frac{e^{\tilde{U}_{\delta_2\sigma}}}{e^{2U_{-\sigma}} + e^{2\tilde{U}_{\delta_2\sigma}}} \times \frac{e^{\tilde{U}_{\eta_1\sigma}}}{e^{2U_{-\sigma}} + e^{2\tilde{U}_{\eta_1\sigma}}} \frac{e^{2U_{-\sigma}} + e^{2\tilde{U}_{\eta_2\sigma}}}{e^{\tilde{U}_{\eta_2\sigma}}}, \tag{27}$$

where α and β are integration constants. The functions S' and T' in Eqs. (11) and (12) are obtained as

$$\begin{aligned} S' &= 2U_0^{(BH)} + S^{(0)} \\ &= 2(\tilde{U}_\sigma - \tilde{U}_{-\sigma}) + \tilde{U}_{\lambda\sigma} - (\tilde{U}_{\delta_1\sigma} - \tilde{U}_{\delta_2\sigma}) \\ &\quad + \tilde{U}_{\eta_1\sigma} - \tilde{U}_{\eta_2\sigma}, \end{aligned} \tag{28}$$

$$T' = T^{(0)} = \tilde{U}_{\lambda\sigma} + \tilde{U}_{\delta_1\sigma} - \tilde{U}_{\delta_2\sigma} + \tilde{U}_{\eta_1\sigma} - \tilde{U}_{\eta_2\sigma}, \tag{29}$$

therefore, the function γ' becomes the following sum of the functions γ'_{cd},

$$\begin{aligned} \gamma' &= \gamma'_{\sigma,\sigma} + \gamma'_{-\sigma,-\sigma} + \gamma'_{\lambda\sigma,\lambda\sigma} + \gamma'_{\delta_1\sigma,\delta_1\sigma} + \gamma'_{\delta_2\sigma,\delta_2\sigma} \\ &\quad + \gamma'_{\eta_1\sigma,\eta_1\sigma} + \gamma'_{\eta_2\sigma,\eta_2\sigma} - 2\gamma'_{\sigma,-\sigma} + \gamma'_{\sigma,\lambda\sigma} - \gamma'_{\sigma,\delta_1\sigma} \\ &\quad + \gamma'_{\sigma,\delta_2\sigma} + \gamma'_{\sigma,\eta_1\sigma} - \gamma'_{\sigma,\eta_2\sigma} - \gamma'_{-\sigma,\lambda\sigma} + \gamma'_{-\sigma,\delta_1\sigma} \\ &\quad - \gamma'_{-\sigma,\delta_2\sigma} - \gamma'_{-\sigma,\eta_1\sigma} + \gamma'_{-\sigma,\eta_2\sigma} + \gamma'_{\lambda\sigma,\delta_1\sigma} \\ &\quad - \gamma'_{\lambda\sigma,\delta_2\sigma} + 2\gamma'_{\lambda\sigma,\eta_1\sigma} - 2\gamma'_{\lambda\sigma,\eta_2\sigma} - 2\gamma'_{\delta_1\sigma,\delta_2\sigma} \\ &\quad + \gamma'_{\delta_1\sigma,\eta_1\sigma} - \gamma'_{\delta_1\sigma,\eta_2\sigma} - \gamma'_{\delta_2\sigma,-\eta_1\sigma} + \gamma'_{\delta_2\sigma,\eta_2\sigma} \\ &\quad - 2\gamma'_{\eta_1\sigma,\eta_2\sigma}. \end{aligned}$$

Using these functions we can write down the metric functions of a black diring. The constants C_1 and C_2 of Eq. (8) and (9) are fixed as

$$C_1 = \frac{2\sigma^{1/2}\alpha}{1 + \alpha\beta}, \qquad C_2 = \frac{1}{\sqrt{2}(1 + \alpha\beta)^2},$$

to assure that the spacetime does not have global rotation and that the periods of ϕ and ψ become 2π at infinity, respectively.

To make the metric component $g_{\phi\phi}$ be regular, we have to set the integration constants α and β as

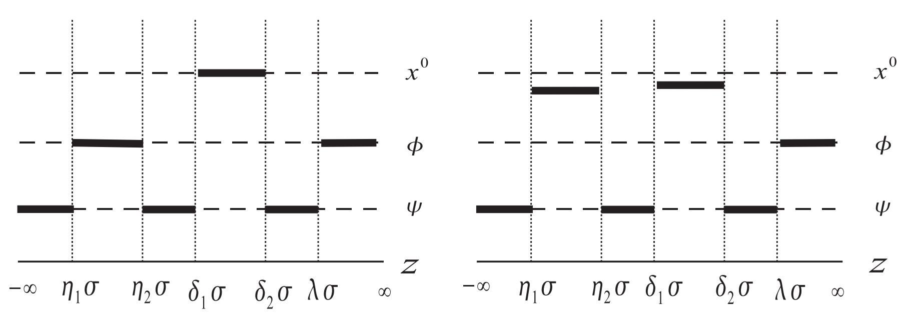

FIG. 2. Schematic pictures of rod structures of black diring and its seed. The left panel shows the rod structure of seed metric of S^1-rotating black diring. The right panel shows the rod structure of S^1-rotating black diring. The finite spacelike rod $[\eta_1\sigma, \eta_2\sigma]$ in the left panel is altered to the finite timelike rod by the solution-generating transformation.

$$\alpha = \pm\sqrt{\frac{2(\delta_1 - 1)(1 - \eta_2)}{(\lambda - 1)(\delta_2 - 1)(1 - \eta_1)}},\qquad \beta = \pm\sqrt{\frac{(\lambda + 1)(\delta_2 + 1)(1 + \eta_1)}{2(\delta_1 + 1)(1 + \eta_2)}}. \tag{30}$$

These conditions also assure the nonexistence of closed timelike curves around the event horizons.

To cure the conical singularities, we have to set the periods of angular coordinates appropriately. The periods of the coordinates ϕ and ψ are defined as

$$\Delta\phi = 2\pi \lim_{\rho\to 0}\sqrt{\frac{\rho^2 g_{\rho\rho}}{g_{\phi\phi}}} \quad \text{and} \quad \Delta\psi = 2\pi \lim_{\rho\to 0}\sqrt{\frac{\rho^2 g_{\rho\rho}}{g_{\psi\psi}}}. \tag{31}$$

We already set the periods of ϕ and the one of ψ outside the ring to be 2π. In addition, the periods of ψ can be obtained from

$$\Delta\psi = \frac{2\pi}{1+\alpha\beta}\sqrt{\frac{(\lambda+1)(\lambda-\delta_2)}{(\lambda-1)(\lambda-\delta_1)}}\left(\frac{\lambda-\eta_2}{\lambda-\eta_1}\right)\times\left(1 + \frac{\lambda-1}{\lambda+1}\alpha\beta\right), \tag{32}$$

for $\delta_2\sigma < z < \lambda\sigma$ and

$$\Delta\psi = \frac{2\pi}{1+\alpha\beta}\sqrt{\frac{(\lambda+1)(\delta_1-1)(\delta_2+1)(\delta_1-\eta_2)(\delta_2-\eta_1)}{(\lambda-1)(\delta_1+1)(\delta_2-1)(\delta_1-\eta_1)(\delta_2-\eta_2)}}\times\left(\frac{\lambda-\eta_2}{\lambda-\eta_1}\right)\left(1+\frac{(\lambda-1)(\delta_1+1))(\delta_2-1)}{(\lambda+1)(\delta_1-1))(\delta_2+1)}\alpha\beta\right), \tag{33}$$

for $\eta_2\sigma < z < \delta_1\sigma$. The parameters can be adjusted to make both values of $\Delta\psi$ equal to 2π.

The asymptotic form of $\mathcal{E}_S$ near the infinity $\tilde{r} = \infty$ becomes

$$\mathcal{E}_S = \tilde{r}\cos\theta\left[1 - \frac{\sigma}{\tilde{r}^2}\frac{P(\alpha,\beta,\lambda)}{(1+\alpha\beta)^2} + \cdots\right] + 2i\sigma^{1/2}\left[\frac{\alpha}{1+\alpha\beta} - \frac{2\sigma\cos^2\theta}{\tilde{r}^2}\frac{Q(\alpha,\beta,\lambda)}{(1+\alpha\beta)^3} + \cdots\right],$$

where we introduced the new coordinates $\tilde{r}$ and θ through the relations

$$x = \frac{\tilde{r}^2}{2\sigma} + \lambda + (\eta_1 - \eta_2) - (\delta_1 - \delta_2), \qquad y = \cos 2\theta, \tag{34}$$

and

$$P = 4(1 + \alpha^2 - \alpha^2\beta^2),$$
$$Q = \alpha(2\alpha^2 - \delta_1 + \delta_2 + \eta_1 - \eta_2 + \lambda + 3) - 2\alpha^2\beta^3 - \beta[2(2\alpha\beta + 1)(\alpha^2 + 1) + (\delta_1 - \delta_2 - \eta_1 + \eta_2 - \lambda - 1)\alpha^2(\alpha\beta + 2)].$$

From the asymptotic behavior of the Ernst potential, we can compute the mass parameter m^2 and rotational parameter $m^2 a_0$ as

$$m^2 = \sigma\frac{P}{(1+\alpha\beta)^2}, \qquad m^2 a_0 = 4\sigma^{3/2}\frac{Q}{(1+\alpha\beta)^3}.$$

The angular velocities of event horizons are obtained from the direction vectors of finite timelike rods. For the finite timelike rod of inner ring $[\delta_1\sigma < z < \delta_2\sigma]$, the direction vector is calculated as

$$\mathbf{v} = (1, \Omega_1, 0),\qquad \Omega_1 = -\frac{2\beta(1+\alpha\beta)}{\sqrt{\sigma}((\lambda-1)\alpha\beta + \lambda + 1)((\delta_2 - 1)\alpha\beta + \delta_2 + 1)}. \tag{35}$$

The outer ring $[\eta_1\sigma < z < \eta_2\sigma]$ has a direction vector

$$\mathbf{v} = (1, \Omega_2, 0), \qquad \Omega_2 = \frac{(1+\alpha\beta)((2\beta(\delta_1+1)(1+\eta_2) - \alpha(\lambda+1)(\delta_2+1)(1-\eta_1))}{2\sqrt{\sigma}(2\alpha\beta(\delta_1+1)(1+\eta_2) - (\lambda+1)(\delta_2+1)(\alpha^2(1-\eta_1) + 2\alpha\beta + 2))}. \tag{36}$$

When $\eta_1 = -1$, the inner ring becomes static because of $\Omega_1 = 0$. In this case the rotation of the outer ring only can cause the absence of the conical singularity. When $\eta_2 = 1$, both rings rotate along the same direction.

Analyzing the mass and angular momentum parameters, we can show the infinite nonuniqueness of a black diring which means that the diring solution has a continuous parameter region to have the same mass and angular momentum. In addition, there can be two different single ring limits, thin and fat black rings, of the black diring with the same mass and angular momentum. Therefore, these two single rings can be transformed into each other through the black diring of the same mass and angular momentum.

To show this fact, we consider the black diring of $\eta_2 = 1$. In Fig. 3, we plot the variable

$$\frac{a_0^2}{m^2} = \frac{16Q^2}{P^3} \tag{37}$$

as a function of δ_1 and η_1. At first we numerically decide the values of λ and δ_2 for the balanced black diring for which the right-hand sides of Eqs. (32) and (33) become 2π with respect to given δ_1 and η_1. Next we obtain the value of $\frac{a_0^2}{m^2}$ by substituting the parameters which satisfy the condition $\delta_1 < \delta_2 < \lambda$. The bold line of Fig. 3 is the single ring limit where $\delta_1 = \delta_2$. When $\delta_1 = \delta_2$, we can show

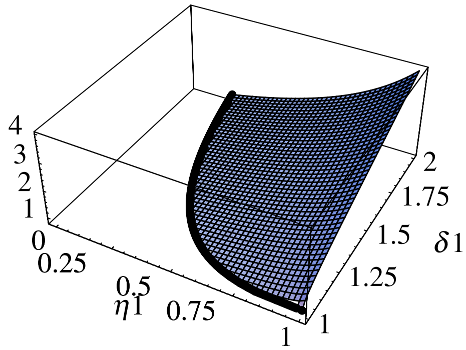

FIG. 3 (color online). Plot of $\frac{a_0^2}{m^2}$ as a function of η_1 and δ_1 where λ and η_2 are determined by the equilibrium conditions and $\eta_2 = 1$. The bold line corresponds to the single black ring of $\delta_1 = \delta_2$.

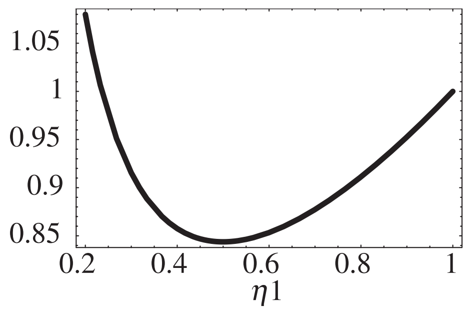

FIG. 4. Plot of $\frac{a_0^2}{m^2}$ of single black ring as a function of η_1. The region $0.5 < \eta_1 < 1$ corresponds to fat rings and $\eta_1 < 0.5$ to thin rings.

that Eq. (37) of the equilibrium ring is reduced to the following form:

$$\frac{a_0^2}{m^2} = \frac{(1 + \eta_1)^3}{8\eta_1}, \tag{38}$$

which corresponds with the single S^1-rotating black ring. The Fig. 4 is a plot of Eq. (38), where $0.5 < \eta_1 < 1$ corresponds to the fat ring and $\eta_1 < 0.5$ the thin ring. Along the bold line in Fig. 3, the value of $\frac{a_0^2}{m^2}$ has the same η_1 dependence of Eq. (38). Apparently there is a continuous path of diring between fat and thin black rings which have the same mass and angular momentum.

V. SUMMARY AND DISCUSSION

In this paper we have shown that the S^1-rotating black ring can be superposed concentrically. The solution of these multiple rings can be written down by the solitonic transformation for the appropriately arranged seed solution. We have obtained the functions needed to write down the metric of a black diring which is the simplest multiple S^1-rotating black ring. To regularize the metric function, the integration constants α and β should be set appropriately. For the equilibrium black diring we need the two additional conditions of parameters. We have analyzed the mass and angular momentum of black diring from the asymptotic form of Ernst potential.

The most important feature of black dirings is that they entail the infinite nonuniqueness of the vacuum neutral solutions of five-dimensional general relativity. To show this, we have numerically plotted the spin parameter of the equilibrium black dirings as a function of the two independent parameters. This plot shows that there are infinite numbers of black dirings with the same mass and angular momentum. In addition we have shown that the black diring can be a pathway between the fat and thin S^1-rotating black rings.

The nonuniqueness we have shown is derived from the existence of a one-parameter family of black dirings with the same conserved parameters because we have fixed one parameter in the analysis. The parameters set for which the general black dirings have the same conserved parameters would be a two-dimensional surface in the three-dimensional parameter space. The physical features of a black diring will be analyzed in detail.

The generalization of the solution to have two angular momenta would be important. Recently, the generalization of the single black ring solution to this direction has been considered by the inverse scattering method [22] and by a numerical study [23]. After this work was completed we noticed Ref. [24], which considers a black saturn: a spherical black hole surrounded by a black ring. It would be important to consider the relation between the black saturn and the black diring solutions.

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