# Post-Newtonian gravitational radiation and equations of motion via direct integration of the relaxed Einstein equations. IV. Radiation reaction for binary systems with spin-spin coupling

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Using post-Newtonian equations of motion for fluid bodies that include radiation-reaction terms at 2.5 and 3.5 post-Newtonian (PN) order ($O[(v/c)^5]$ and $O[(v/c)^7]$ beyond Newtonian order), we derive the equations of motion for binary systems with spinning bodies, including spin-spin effects. In particular we determine the effects of radiation-reaction coupled to spin-spin effects on the two-body equations of motion, and on the evolution of the spins. We find that radiation damping causes a 3.5PN order, spin-spin induced precession of the individual spins. This contrasts with the case of spin-*orbit* coupling, where we earlier found no effect on the spins at 3.5PN order. Employing the equations of motion and of spin precession, we verify that the loss of total energy and total angular momentum induced by spin-spin effects precisely balances the radiative flux of those quantities calculated by Kidder *et al.*

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## I. INTRODUCTION AND SUMMARY

The relativistic effects of spin may play an important role in the inspiral of compact binary systems, particularly involving black holes, and may have observable effects on the gravitational-wave signal emitted. Spin-orbit and spin-spin coupling leads to precessions of the spins of the bodies and of the orbital plane, the latter effect resulting in modulations of the amplitude of the gravitational waveform received at a detector. Furthermore, spin effects contribute directly to the gravitational waveform, and to the overall emission of energy and angular momentum from the system.

In the post-Newtonian (PN) approximation to general relativity, the effects of spin have been derived at various levels of the PN approximation. Formally, spin-orbit and spin-spin couplings begin to affect the equations of motion at the first post-Newtonian order, since they behave as $\mathbf{S}\cdot\mathbf{L}/mr^4 \sim \mathbf{S}_1\cdot\mathbf{S}_2/mr^4 \sim (mRV)(mrv)/mr^4 \sim (mRV)^2/mr^4 \sim (m/r^2)\epsilon$, where $m$, $v$, $r$ and $L$ denote mass, orbital velocity, separation and orbital anglar momentum, respectively, $R$ and $V$ denote the body's size and rotational velocity, $\mathbf{S}$ denotes spin or rotational angular momentum, and $\epsilon \sim v^2 \sim V^2 \sim m/r \sim m/R$ denotes the standard small "bookeeping parameter" of post-Newtonian theory (we use units in which $G = c = 1$) [1]. Spin evolution effects can also be seen to be 1PN-order effects. Indeed, the 1PN effects of spin have been derived by numerous authors from a variety of points of view, ranging from formal developments of the GR equations of motion in multipole expansions [2,3], to post-Newtonian calculations [4], to treatments of linearized GR as a spin-two quantum theory [5,6]. For a review of these various approaches, see [7]. The effects of spin on the gravitational waveform and on the energy and angular momentum flux were worked out by Kidder *et al.* [8,9]. Post-Newtonian corrections of the leading spin terms have also been analyzed [10,11].

In Paper III of this series [12], we derived, from first principles, the leading effects of spin-orbit coupling in the equations of motion at radiation-reaction order, specifically at 3.5PN order, or $O(\epsilon^{7/2})$ beyond Newtonian gravity. We also showed explicitly that radiation reaction had no effect, via spin-orbit coupling, on the individual spins themselves. In this paper, we extend this analysis to spin-spin coupling. As before, the leading contributions occur at 3.5PN order.

We use the hydrodynamic equations of motion derived through 3.5PN order in Papers I and II [13,14], and calculate the equations of motion and spin evolution for two spinning, finite-sized bodies. We restrict our attention to contributions that involve the products of the two spins. To this end, for each body $A$, we decompose velocities into a center-of-mass part and an internal (rotational) part according to $\mathbf{v} = \mathbf{v}_A + \bar{\mathbf{v}}$, and expand all gravitational potentials about the center of mass of each body using a similar decomposition, $\mathbf{x} = \mathbf{x}_A + \bar{\mathbf{x}}$, and retain only terms that contain the product $(m\bar{x}\,\bar{v})_1(m\bar{x}\,\bar{v})_2 \sim S_1S_2$. We do not keep terms proportional to the squares of individual spins; these represent another class of spin effects that will be studied elsewhere.

Adopting a specific definition of "proper spin" $\mathcal{S}_A$, as defined in Paper III [see Eq. (2.15)], we find the two-body equations of motion

$$\mathbf{a} = -\frac{m}{r^2}\mathbf{n} + \mathbf{a}_{\rm PN} + \mathbf{a}_{\rm PN-SO} + \mathbf{a}_{\rm PN-SS} + \ldots + \mathbf{a}_{\rm 2.5PN} + \mathbf{a}_{\rm 3.5PN} + \mathbf{a}_{\rm 3.5PN-SO} + \mathbf{a}_{\rm 3.5PN-SS} + \ldots, \tag{1.1}$$

where $\mathbf{a} = \mathbf{a}_1 - \mathbf{a}_2$ is the relative acceleration. The 1PN spin-spin terms are standard, and are given by

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$$\mathbf{a}_{\rm PN-SS} = -\frac{3}{\mu r^4}[\mathbf{n}(\mathcal{S}_1 \cdot \mathcal{S}_2) + \mathcal{S}_1(\mathbf{n} \cdot \mathcal{S}_2) + \mathcal{S}_2(\mathbf{n} \cdot \mathcal{S}_1) - 5\mathbf{n}(\mathbf{n} \cdot \mathcal{S}_1)(\mathbf{n} \cdot \mathcal{S}_2)], \tag{1.2}$$

and the 3.5PN spin-spin contributions, derived in this paper, are given by

$$\begin{aligned}\mathbf{a}_{3.5\rm PN-SS} = \tfrac{1}{r^5}\{&\mathbf{n}[(287\dot r^2 - 99v^2 + \tfrac{541}{5}\tfrac{m}{r})\dot r(\mathcal{S}_1 \cdot \mathcal{S}_2) - (2646\dot r^2 - 714v^2 + \tfrac{1961}{5}\tfrac{m}{r})\dot r(\mathbf{n} \cdot \mathcal{S}_1)(\mathbf{n} \cdot \mathcal{S}_2)\\ &+ (1029\dot r^2 - 123v^2 + \tfrac{629}{10}\tfrac{m}{r})((\mathbf{n} \cdot \mathcal{S}_1)(\mathbf{v} \cdot \mathcal{S}_2) + (\mathbf{n} \cdot \mathcal{S}_2)(\mathbf{v} \cdot \mathcal{S}_1)) - 336\dot r(\mathbf{v} \cdot \mathcal{S}_1)(\mathbf{v} \cdot \mathcal{S}_2)]\\ &+ \mathbf{v}[(\tfrac{171}{5}v^2 - 195\dot r^2 - 67\tfrac{m}{r})(\mathcal{S}_1 \cdot \mathcal{S}_2) - (174v^2 - 1386\dot r^2 - \tfrac{1038}{5}\tfrac{m}{r})(\mathbf{n} \cdot \mathcal{S}_1)(\mathbf{n} \cdot \mathcal{S}_2)\\ &- 438\dot r((\mathbf{n} \cdot \mathcal{S}_1)(\mathbf{v} \cdot \mathcal{S}_2) + (\mathbf{n} \cdot \mathcal{S}_2)(\mathbf{v} \cdot \mathcal{S}_1)) + 96(\mathbf{v} \cdot \mathcal{S}_1)(\mathbf{v} \cdot \mathcal{S}_2)]\\ &+ (\tfrac{27}{10}v^2 - \tfrac{75}{2}\dot r^2 - \tfrac{509}{30}\tfrac{m}{r})((\mathbf{v} \cdot \mathcal{S}_2)\mathcal{S}_1 + (\mathbf{v} \cdot \mathcal{S}_1)\mathcal{S}_2) + (\tfrac{15}{2}v^2 + \tfrac{77}{2}\dot r^2 + \tfrac{199}{10}\tfrac{m}{r})\dot r((\mathbf{n} \cdot \mathcal{S}_2)\mathcal{S}_1 + (\mathbf{n} \cdot \mathcal{S}_1)\mathcal{S}_2)\},\end{aligned} \tag{1.3}$$

where $\mathbf{x} \equiv \mathbf{x}_1 - \mathbf{x}_2$, $r \equiv |\mathbf{x}|$, $\mathbf{n} \equiv \mathbf{x}/r$, $m \equiv m_1 + m_2$, $\mu \equiv m_1 m_2/m$, $\mathbf{v} \equiv \mathbf{v}_1 - \mathbf{v}_2$, $\dot r = dr/dt = \mathbf{n} \cdot \mathbf{v}$, and $\tilde{\mathbf{L}}_N = \mathbf{x} \times \mathbf{v}$. The PN, PN spin-orbit, and 2.5PN contributions are standard (see Paper III Eqs. (1.2) and (1.5) for the formulae), the 3.5PN terms were derived in Paper II, Eq. (1.3), and the 3.5PN spin-orbit terms were derived in Paper III, Eq. (1.6).

The equations of spin evolution are given by

$$\dot{\mathcal{S}}_1 = (\dot{\mathcal{S}}_1)_{\rm PN-SO} + (\dot{\mathcal{S}}_1)_{\rm PN-SS} + (\dot{\mathcal{S}}_1)_{3.5\rm PN-SS}, \tag{1.4}$$

where the PN spin-orbit terms are standard (see Paper III, Eq. (1.3)). The PN spin-spin terms are also standard, given by

$$(\dot{\mathcal{S}}_1)_{\rm PN-SS} = -\tfrac{1}{r^3}(\mathcal{S}_2 - 3(\mathbf{n} \cdot \mathcal{S}_2)\mathbf{n}) \times \mathcal{S}_1. \tag{1.5}$$

There is no 3.5PN spin-orbit contribution (Paper III), but we find a 3.5PN spin-spin contribution, given by

$$(\dot{\mathcal{S}}_1)_{3.5\rm PN-SS} = \frac{\mu m}{r^5}\left(\frac{2}{3}(\mathbf{v} \cdot \mathcal{S}_2) + 30\dot r(\mathbf{n} \cdot \mathcal{S}_2)\right)(\mathbf{n} \times \mathcal{S}_1), \tag{1.6}$$

with the equations for $\mathcal{S}_2$ obtained by interchanging the spins.

As a check of these results, we verify explicitly that the loss of total energy and total angular momentum (including both orbital and spin) implied by these equations matches the energy and angular momentum radiated in gravitational waves, as calculated by Kidder *et al.* [8,9].

In Paper III, we found that spin-orbit contributions to radiation reaction had no effect on the proper spin of each body, i.e. $(\dot{\mathcal{S}}_1)_{3.5\rm PN-SO} = 0$, and we argued that this made sense, given that a spinning, axisymmetric body should not couple to gravitational radiation. Here, however, when the coupling between the two spins is taken into account, there *is* a nontrivial radiation-reaction effect on the spins. Nevertheless, the effect is a pure precession; the magnitude of the spins is unaffected. Furthermore, if either of the spins is aligned with the orbital angular momentum (i.e. perpendicular to $\mathbf{v}$ and $\mathbf{n}$), the other spin is not affected by radiation reaction.

These equations of motion do not impose any limitations on the orbits. In particular, they can be used to evolve the quasicircular inspiral orbits that are typical of those considered as sources of gravitational radiation detectable by ground-based later-interferometric detectors of the LIGO-VIRGO type, as well as highly eccentric orbits of extreme mass ratio systems that are relevant for the proposed space-based detector, LISA.

The remainder of the paper presents details. In Sec. II we derive the equations of motion to PN order, including spin-spin terms, and show that no spin-spin effects occur at leading radiation-reaction, or 2.5PN order. This section illustrates some basic features of the technique of obtaining the spin effects from the hydrodynamical equations. In Sec. III we move to 3.5PN order, where the spin-spin radiation-reaction effects first appear. Section IV presents concluding remarks.

## II. POST-NEWTONIAN AND 2.5PN EQUATIONS OF MOTION AND SPIN EVOLUTION

### A. Foundations

As in Paper III [12], we analyze a binary system consisting of balls of perfect fluid that are sufficiently small compared to their separation that tidal interactions (and their relativistic generalizations) can be ignored, but that are sufficiently extended that they can support a finite rotational angular momentum, or spin. At Newtonian order, the result is essentially trivial: the equation of motion for body 1 is $d^2\mathbf{x}_1/dt^2 = -m_2\mathbf{x}/r^3 + O(mR^2/r^4)$, where $R$ is the characteristic size of the bodies. Spin plays no role whatsoever, because the Newtonian interaction does not depend on velocity. But at post-Newtonian order, there are velocity-dependent accelerations of the schematic form $mv^2/r^2$, and thus, taking into account the finite size of the body and expanding about its center of mass, we expect to find acceleration terms of the form $(mVR) \times (mVR)'/mr^4 \sim S_1S_2/mr^4$. However, the combination of finite size and spin introduces an ambiguity in the definition of the center of mass of each body. This has given rise to the concept of "spin supplementary condition" (SSC), a statement about which center-of-mass definition is being used; this concept is discussed in Paper III, Appendix A. It turns out that this is an issue only for spin-orbit effects; the choice of SSC or of center of mass has no effect on spin-spin effects at PN or at 3.5PN order.

We will define centers of mass and spins provisionally using the "conserved", or baryonic density, given by

$$\rho^* \equiv \rho\sqrt{-g}u^0, \tag{2.1}$$

where $\rho$ is the mass energy density as measured by an observer in a local inertial frame momentarily at rest with respect to the fluid, $g$ is the determinant of the metric, and $u^0$ is the time component of the fluid four-velocity. Assuming that $\rho$ is proportional to the baryon number density, then conservation of baryon number leads to the useful *exact* continuity equation

$$\partial\rho^*/\partial t + \nabla\cdot(\rho^*\mathbf{v}) = 0, \tag{2.2}$$

where $v^i = u^i/u^0$ is the ordinary (coordinate) velocity of the fluid. (Greek indices range over spacetime values 0, 1, 2, 3, while Latin indices range over spatial values 1, 2, 3. Henceforth, spatial vectorial quantities will be handled using a Cartesian metric.) The baryonic mass, center of mass and baryonic spin of each body in our system are defined to be

$$m_A \equiv \int_A \rho^* d^3x, \tag{2.3a}$$

$$\mathbf{x}_A \equiv m_A^{-1}\int_A \rho^*\mathbf{x}d^3x, \tag{2.3b}$$

$$\mathbf{S}_A \equiv \int_A \rho^*\bar{\mathbf{x}}\times\bar{\mathbf{v}}d^3x, \tag{2.3c}$$

where $\bar{\mathbf{x}} = \mathbf{x} - \mathbf{x}_A$ and $\bar{\mathbf{v}} = \mathbf{v} - \mathbf{v}_A$. We also define a two-index spin quantity

$$S_A^{ij} \equiv 2\int_A \rho^*\bar{x}^{[i}\bar{v}^{j]}d^3x, \qquad S_A^{ij} = \epsilon^{ijk}S_A^k, \qquad S_A^i = \frac{1}{2}\epsilon^{ijk}S_A^{jk}, \tag{2.4}$$

where [] around indices denotes antisymmetrization. With these definitions, the baryonic mass $m_A$ is constant, and the velocity, acceleration and rate of change of spin of body $A$ are given by

$$\mathbf{v}_A = m_A^{-1}\int_A \rho^*\mathbf{v}d^3x, \tag{2.5a}$$

$$\mathbf{a}_A = m_A^{-1}\int_A \rho^*\mathbf{a}d^3x, \tag{2.5b}$$

$$d\mathbf{S}_A/dt = \int_A \rho^*\bar{\mathbf{x}}\times\mathbf{a}d^3x. \tag{2.5c}$$

Notice that, by virtue of the definition of center of mass, the "bar" can be dropped from the acceleration in Eq. (2.5c).

### B. Baryonic equations of motion and spin evolution

We begin by working to 1PN and 2.5PN order, reproducing the standard 1PN formulae for spin-spin interactions, and establishing some results that will be useful when we go on to 3.5PN order. Since we are only interested in radiation-reaction aspects of spin, we can ignore the 2PN terms in the equations of motion; these produce only conservative PN corrections to the spin equations of motion [10].

We use the hydrodynamic equations of motion derived in Paper II, Eqs. 2.23, 2.24a, 2.24c, with all quantities expressed in terms of the conserved density $\rho^*$. They are given by

$$d^2x^i/dt^2 = U^{,i} + a^i_{\rm PN} + a^i_{\rm 2.5PN}, \tag{2.6}$$

where $U$ is the Newtonian potential, and where

$$a^i_{\rm PN} = v^2U^{,i} - 4v^iv^jU^{,j} - 4UU^{,i} - 3v^i\dot{U} + 4\dot{V}^i + 8v^jV^{[i,j]} + \tfrac{3}{2}\Phi_1^{,i} - \Phi_2^{,i} + \tfrac{1}{2}\ddot{X}^{,i}, \tag{2.7a}$$

$$a^i_{\rm 2.5PN} = \frac{3}{5}x^j\big(\overset{(5)}{I}{}^{ij} - \tfrac{1}{3}\delta^{ij}\overset{(5)}{I}{}^{kk}\big) + 2v^j\overset{(4)}{I}{}^{ij} + 2U^{,j}\overset{(3)}{I}{}^{ij} + \tfrac{4}{3}U^{,i}\overset{(3)}{I}{}^{kk} - X^{,ijk}\overset{(3)}{I}{}^{jk}, \tag{2.7b}$$

where commas denote partial derivatives, overdots denote partial time derivatives, and $(n)$ above quantities denotes the number of total time derivatives. The potentials used here and elsewhere in the paper are given by the general definitions

$$\Sigma(f) \equiv \int\frac{\rho^*(t,\mathbf{x}')f(t,\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x', \qquad X(f) \equiv \int\rho^*(t,\mathbf{x}')f(t,\mathbf{x}')|\mathbf{x}-\mathbf{x}'|d^3x', \tag{2.8}$$

with specific potentials given by

$$U = \Sigma(1), \quad V^i = \Sigma(v^i), \quad \Phi_1 = \Sigma(v^2), \quad \Phi_2 = \Sigma(U), \quad X = X(1), \quad X^i = X(v^i). \tag{2.9}$$

The multipole moment of the system $I^{ij}$, as well as additional moments, $\mathcal{J}^{ij}$, $\mathcal{J}^{ijk}$, and $\mathcal{M}^{ijkl}$, that will be relevant at 3.5PN order, are defined in Paper III, Appendix C (see also Eq. (3.2)); note that there are no explicit spin-spin terms in $I^{ij}$, to the PN order considered.

We now multiply the equation of motion (2.6) by $\rho^*$ and integrate over body 1, expressing the variables $\mathbf{x}$ and $\mathbf{v}$ as $\mathbf{x} = \mathbf{x}_A + \bar{\mathbf{x}}$ and $\mathbf{v} = \mathbf{v}_A + \bar{\mathbf{v}}$, where $A = 1, 2$, depending on the body in which the point lies. To get the acceleration of body 1, we divide the result by $m_1$. We use Eqs. (2.3) and (2.5) to simplify where possible. We expand the various potentials in powers of $\bar{x}/r$, and keep only terms proportional to the product of $\bar{v}\times\bar{x}$ for one body with $\bar{v}\times\bar{x}$ for the other body.

In Paper III, we also kept internal terms proportional to $\bar{v}^2$ and used virial relations derived in Paper III, Appendix E to simplify expressions dependent on the internal structure of each body. While the use of those virial theorems generated spin-orbit terms at PN order, it turns out that they generate no spin-spin terms at this order.

We will deal with the effect of virial theorems on 3.5PN spin-spin terms in Sec. III.

The Newtonian term gives $a^i_{\rm N} = -m_2 x^i_{12}/r^3$, where, in this paragraph, to avoid confusion, we denote $x^i_{12} \equiv x^i_1 - x^i_2$, $r \equiv |\mathbf{x}_{12}|$, and $\mathbf{n} = \mathbf{x}_{12}/r$. The only PN terms in Eq. (2.7a) that can have a $v$ in one body and a $v$ in the other body and that could therefore lead to a spin-spin effect are the terms $-3v^i\dot{U}$ and $8v^jV^{[i,j]}$. Keeping only the relevant terms, the first of these gives, for example,

$$-3\frac{1}{m_1}\int_1 \rho^* v^i \dot{U} d^3x = -3\frac{1}{m_1}\int_1 \rho^*(v^i_1 + \bar{v}^i)d^3x \times \int_2 \rho^{*\prime}(v^j_2 + \bar{v}^{\prime j})\left[\frac{n^j}{r^2} + (\bar{x} - \bar{x}^\prime)^k\nabla^k\left(\frac{x^j_{12}}{r^3}\right) + \frac{1}{2}(\bar{x} - \bar{x}^\prime)^k(\bar{x} - \bar{x}^\prime)^l\nabla^k\nabla^l\left(\frac{x^j_{12}}{r^3}\right) + \dots\right]d^3x^\prime = \frac{3}{4m_1}S^{ki}_1 S^{lj}_2\nabla^k\nabla^l\left(\frac{x^j_{12}}{r^3}\right), \tag{2.10}$$

where unprimed (primed) barred variables are in body 1(2). We also assume that each body is in stationary equilibrium, with $\dot{I}^{ij}_A = (d/dt)\int_A \rho^*\bar{x}^i\bar{x}^j d^3x = 0$, so that $\int_A \rho^*\bar{x}^i\bar{v}^j d^3x = S^{ij}_A/2$.

In the combination of PN terms $4\dot{V}^i + \frac{1}{2}\ddot{X}^{,i}$ in Eq. (2.7a), the time derivatives generate accelerations inside the potentials. To the order needed for our purposes, we must therefore substitute the Newtonian and 2.5PN continuum terms for those accelerations and carry out the same procedures for the integrals as described above. However, while this produces spin-orbit effects (Paper III), it produces *no* spin-spin terms.

The resulting 1PN spin-spin contribution to the equation of motion for body 1 is

$$(a^i_1)_{\rm PN-SS} = \frac{3}{m_1 r^4}(S^{ij}_1 S^{kj}_2 n^k + S^{ij}_2 S^{kj}_1 n^k + S^{kj}_1 S^{kj}_2 n^i - 5S^{kj}_1 S^{lj}_2 n^i n^k n^l). \tag{2.11}$$

Substituting Eq. (2.4) and calculating the relative acceleration gives

$$\mathbf{a}_{\rm PN-SS} = -\frac{3}{\mu r^4}[\mathbf{n}(\mathbf{S}_1\cdot\mathbf{S}_2) + \mathbf{S}_1(\mathbf{n}\cdot\mathbf{S}_2) + \mathbf{S}_2(\mathbf{n}\cdot\mathbf{S}_1) - 5\mathbf{n}(\mathbf{n}\cdot\mathbf{S}_1)(\mathbf{n}\cdot\mathbf{S}_2)], \tag{2.12}$$

where the spins here are baryonic spins.

Turning to the 2.5PN terms, Eq. (2.7b), the integrations lead to no explicit spin-spin terms, so that the 2.5PN relative acceleration terms are given by

$$(a^i)_{\rm 2.5PN} = \tfrac{3}{5}x^j(\overset{(5)}{I}{}^{ij} - \tfrac{1}{3}\delta^{ij}\overset{(5)}{I}{}^{kk}) + 2v^j\overset{(4)}{I}{}^{ij} - \tfrac{1}{3}\tfrac{m}{r^2}n^i\overset{(3)}{I}{}^{kk} - 3\tfrac{m}{r^2}n^in^jn^k\overset{(3)}{I}{}^{jk}. \tag{2.13}$$

However, when we work at 3.5PN order, even though the multipole moments themselves contain no explicit spin-spin terms, time derivatives acting on them *will* produce spin-spin contributions via the PN spin-spin terms in the equations of motion.

We calculate the evolution of the spin in a similar manner. Starting with $dS^i_1/dt = \epsilon^{ijk}\int_1 \rho^*\bar{x}^j a^k d^3x$, we expand about the baryonic centers of mass, keeping only terms that depend on a product of $\bar{x}\times\bar{v}$ for each body. At 1PN order, the only term in Eq. (2.7a) that contributes is $8v^jV^{[i,j]}$. The result, at 1PN order is

$$(\dot{\mathbf{S}}_1)_{\rm PN-SS} = -\tfrac{1}{r^3}(\mathbf{S}_2 - 3(\mathbf{n}\cdot\mathbf{S}_2)\mathbf{n})\times\mathbf{S}_1, \tag{2.14}$$

where again these are baryonic spins. At 2.5PN order, there is no spin-spin contribution.

### C. The proper spin

In Paper III, we defined the proper spin of each body to be

$$\mathcal{S}^i_1 \equiv S^i_1(1 + \tfrac{1}{2}v^2_1 + 3\tfrac{m_2}{r}) - \tfrac{1}{2}[\mathbf{v}_1\times(\mathbf{v}_1\times\mathbf{S}_1)]^i - S^i_1\overset{(3)}{I}{}^{jj} + S^j_1\overset{(3)}{I}{}^{ij},$$
$$\mathcal{S}_2 \equiv (1 \rightleftharpoons 2). \tag{2.15}$$

The post-Newtonian corrections in Eq. (2.15) arise from transforming the baryonic spin from our coordinate frame to a suitable inertial frame comoving with the spinning body. The 2.5PN terms involving time derivatives of $I^{ij}$ arise from the fact that the equations of motion at 2.5PN order may be written in various gauges, such as Burke-Thorne gauge [15] (in which the 2.5PN radiation-reaction terms in the acceleration are given by $a^i_{\rm 2.5PN} = \frac{2}{5}x^j d^5 I^{<ij>}/dt^5$), or Damour-Deruelle gauge [16,17] (the gauge used in this paper). Including the 2.5PN terms as in (2.15) is equivalent to defining our spins in the Burke-Thorne gauge. In any case, such quantities as angular momentum and energy are well-defined only up to the order at which they conserved, and one is free to add 2.5PN and 3.5PN order terms to them without affecting their fundamental conserved properties; including the 2.5PN terms in (2.15) has the property that radiation-reaction effects in the proper spin do not appear (if they appear at all) until 3.5PN order. With this definition, the proper spins $\mathcal{S}_A$ also satisfy the standard spin-orbit precession equations, Paper III, Eq. (1.3). Transforming from our baryonic spin to the proper spin will generate some spin-spin terms at 3.5PN order.

### D. Conserved total energy and angular momentum

The Newtonian and PN spin-spin terms in the equation of motion (2.12), and the PN spin-spin terms in the spin precession Eq. (2.14), together imply conservation of the total energy and angular momentum of the system, given to Newtonian and PN spin-spin order by

$$E = \mu(\tfrac{1}{2}v^2 - \tfrac{m}{r}) - \tfrac{1}{r^3}[(\mathcal{S}_1 \cdot \mathcal{S}_2) - 3(\mathbf{n} \cdot \mathcal{S}_1)(\mathbf{n} \cdot \mathcal{S}_2)], \tag{2.16a}$$

$$\mathbf{J} = \mu \tilde{\mathbf{L}}_{\mathrm{N}} + \mathcal{S}, \tag{2.16b}$$

where $\mathcal{S} = \mathcal{S}_1 + \mathcal{S}_2$. Notice that there is no PN spin-spin contribution to the total angular momentum. In Eq. (2.16) we have converted the baryonic spins to the proper spins; the PN corrections in Eq. (2.15) do not introduce new spin-spin contributions to $E$ or $J$ to the required order, and the 2.5PN corrections can always be dropped as meaningless terms that have no effect on the conserved quantities.

In Sec. III C we will use these expressions together with the 3.5PN equations of motion to compare $\dot{E}$ and $\dot{\mathbf{J}}$ with the corresponding fluxes of radiation to infinity.

## III. 3.5PN EQUATIONS OF MOTION AND SPIN EVOLUTION

### A. Equation of motion

To obtain the 3.5PN contributions to the equations of motion including spin-spin coupling terms, we take the 3.5PN fluid expressions shown in Eq. (D4) of Paper III, multiply by $\rho^*$, and integrate over body 1. We follow the same procedure as in Sec. II B, expanding potentials about the baryonic centers of mass of the bodies, keeping only spin-spin terms (terms involving products of $\bar{x} \times \bar{v}$ for one body with that for the other body). Most terms in Eq. (D4) of Paper III make only point-mass or spin-orbit contributions; the only terms that can possibly produce nontrivial spin-spin terms in either the acceleration or the spin evolution are:

$$\begin{aligned}\delta a^i_{3.5\mathrm{PN}} = {}& -8v^k V^{k,j} \overset{(3)}{I}{}^{ij} + \tfrac{16}{3} v^j V^{[i,j]} \overset{(3)}{I}{}^{kk} + 8(v^j V^{k,i} - v^l X^{[i,l]jk}) \overset{(3)}{I}{}^{jk} - \tfrac{16}{45} x^j v^k \epsilon^{qjk} \overset{(5)}{\mathcal{J}}{}^{qi} + \tfrac{2}{45}(2(\mathbf{v}\cdot\mathbf{x})\epsilon^{qik} - 2x^i v^j \epsilon^{qjk} \\ & + 5x^j v^i \epsilon^{qjk} + 12 x^j v^k \epsilon^{qij} + 4x^k v^j \epsilon^{qij}) \overset{(5)}{\mathcal{J}}{}^{qk} + \tfrac{2}{9}(4v^j v^k \epsilon^{qij} - v^2 \epsilon^{qik}) \overset{(4)}{\mathcal{J}}{}^{qk} \\ & - \tfrac{2}{9}(v^i U^{,j} + 2v^j U^{,i} + V^{j,i} + \dot{X}^{,ij}) \epsilon^{qjk} \overset{(3)}{\mathcal{J}}{}^{qk} + \tfrac{1}{15} v^j (\epsilon^{qjk} \overset{(5)}{\mathcal{J}}{}^{qik} - \epsilon^{qik} \overset{(5)}{\mathcal{J}}{}^{qjk} - \epsilon^{qij} \overset{(5)}{\mathcal{J}}{}^{qkk}) - \tfrac{1}{6} v^i \overset{(4)}{\mathcal{M}}{}^{kkjj} + \tfrac{2}{3} v^j \overset{(4)}{\mathcal{M}}{}^{ijkk},\end{aligned} \tag{3.1}$$

where, to the required order,

$$\begin{aligned} I^{ij} &= \mu x^i_{12} x^j_{12}, \\ \mathcal{J}^{ij} &= -\eta \delta m \tilde{L}^i_{\mathrm{N}} x^j_{12} - \tfrac{1}{2}\eta(3\Delta^i x^j_{12} - \delta^{ij} \Delta \cdot \mathbf{x_{12}}), \\ \mathcal{J}^{ijk} &= \eta m(1-3\eta) \tilde{L}^i_{\mathrm{N}} x^j_{12} x^k_{12} \\ &\quad + \eta(2\xi^i x^j_{12} x^k_{12} - \xi \cdot \mathbf{x_{12}} \delta^{i(j} x^{k)}_{12}), \\ \mathcal{M}^{ijkk} &= \eta m (1-3\eta) r^2 (v^i_{12} v^j_{12} - \tfrac{m}{3r} n^i_{12} n^j_{12}) \\ &\quad - \tfrac{1}{6}\eta m^2 r (n^i_{12} n^j_{12} - 3\delta^{ij}) - 2\eta(\mathbf{x}_{12} \times \xi)^{(i} v^{j)}_{12}, \end{aligned} \tag{3.2}$$

where $\delta m = m_1 - m_2$, $\eta = \mu/m$, $\xi^i = (m_2/m_1)S^i_1 + (m_1/m_2)S^i_2$, and $\Delta^i = m(S^i_2/m_2 - S^i_1/m_1)$. Spin-spin contributions come from terms such as $v^j V^{k,i} d^3 I^{jk}/dt^3$, with a $\bar{v}$ in body 1 and a $\bar{v}$ in body 2 together with suitable $\bar{x}$'s. They also come from terms involving the current moments $\mathcal{J}^{qk}$, where a single spin generated by the prefactor (eg. $x^i v^j \epsilon^{qjk}$, or $v^i U^{,j} \epsilon^{qjk}$) is multiplied by the spin of the other body that appears in $\mathcal{J}^{qk}$. The terms involving $d^4\mathcal{J}^{qk}/dt^4$ $d^5\mathcal{J}^{qpk}/dt^5$ and $d^4\mathcal{M}^{pqkk}/dt^4$ do not generate spin-spin terms in the equation of motion (no free $\bar{x}$ to go with a velocity), but *do* generate terms in the spin evolution.

In addition, when the prefactor of $d^3\mathcal{J}^{qk}/dt^3$ is integrated over body 1, it yields a prefactor given by $-(2/9) \times (4\mathcal{H}^{(ij)}_1 - 3\mathcal{K}^{ij}_1)$, where

$$\begin{aligned} \mathcal{H}^{\,ij}_A &\equiv \int_A \int_A \rho^* \rho^{*\prime} \frac{\bar{v}'^i (x - x')^j}{|\mathbf{x} - \mathbf{x}'|^3} d^3x d^3x', \\ \mathcal{K}^{ij}_A &\equiv \int_A \int_A \rho^* \rho^{*\prime} \frac{\bar{\mathbf{v}}' \cdot (\mathbf{x} - \mathbf{x}')(x - x')^i (x - x')^j}{|\mathbf{x} - \mathbf{x}'|^5} d^3x d^3x'. \end{aligned} \tag{3.3}$$

However, a virial theorem derived from the requirement that $\dddot{I}^{ij}_A = 0$ gives, to the required PN order,

$$2\mathcal{H}^{(ij)}_1 - \tfrac{3}{2}\mathcal{K}^{ij}_1 = -\tfrac{3m_2}{2r^3} S^{k(i}_1 n^{j)k}. \tag{3.4}$$

(See Paper III, Appendix E, for a discussion of virial relations.) This spin term, combined with those generated directly by the potentials, and multiplied by the appropriate spin term in $d^3\mathcal{J}^{qk}/dt^3$, gives a 3.5PN spin-spin term. This is the only place where the virial theorems play a role in spin-spin effects.

In addition, the combination of 1PN terms $4\dot{V}^i + \frac{1}{2}\ddot{X}^{,i}$ in Eq. (2.7a), will generate accelerations whose 2.5PN terms will produce 3.5PN contributions; however these produce no spin-spin terms. We must also re-express the 1PN spin-spin terms of Eq. (2.11) in terms of the proper spin of Eq. (2.15); the 2.5PN contributions there will generate 3.5PN spin-spin terms in the equation of motion. Finally, in the 2.5PN accelerations of Eq. (2.13), we must include the 1PN spin-spin terms in the equations of motion that are generated by the many time derivatives; the explicit 1PN corrections to those moments do not contain spin-spin terms.

The result for the 3.5PN acceleration of body 1 is an expression too lengthy to reproduce here. Calculating $a_1^i - a_2^i$ and converting all variables to relative coordinates using $\mathbf{x}_1 = (m_2/m)\mathbf{x}$ and $\mathbf{x}_2 = -(m_1/m)\mathbf{x}$, we obtain Eq. (1.3).

### B. Spin evolution

We now want to calculate the evolution of the proper spin $\mathcal{S}_1$ to 3.5PN order. A time derivative of Eq. (2.15) gives

$$\dot{\mathcal{S}}_1^i = \dot{S}_1^i\left(1 + v_1^2 + 3\frac{m_2}{r}\right) - \frac{1}{2}v_1^i(\mathbf{v}_1\cdot\dot{\mathbf{S}}_1) + S_1^i\left(2\mathbf{v}_1\cdot\mathbf{a}_1 - 3\frac{m_2\dot{r}}{r^2}\right) - \frac{1}{2}a_1^i(\mathbf{v}_1\cdot\mathbf{S}_1) - \frac{1}{2}v_1^i(\mathbf{a}_1\cdot\mathbf{S}_1) - S_1^i\overset{(4)}{I}{}^{jj} + S_1^j\overset{(4)}{I}{}^{ij} - \dot{S}_1^i\overset{(3)}{I}{}^{jj} + \dot{S}_1^j\overset{(3)}{I}{}^{ij}. \tag{3.5}$$

We repeat the method of Sec. II B to determine the contributions of 3.5PN fluid terms to the time derivative of the baryonic spin $\mathbf{S}_1$, by calculating $\epsilon^{ijk}\int_1 \rho^*\bar{x}^j a^k_{3.5\mathrm{PN}} d^3x$. Only the terms displayed in Eq. (3.1) will contribute spin-spin terms. Notice that, as we have discussed, the 2.5PN contribution to $\dot{S}_1^i$ (actually a spin-orbit term) cancels the first two terms in the last line of Eq. (3.5). For $\mathbf{a}_1$, which appears in the 1PN terms in Eq. (3.5), we must substitute the 2.5PN equations of motion; however these make no spin-spin contribution. For $\dot{S}_1^i$ in the final 2.5PN terms in Eq. (3.5) we must substitute the 1PN spin-spin precession equations; finally we must use Eq. (2.15) to convert from $\mathbf{S}_1$ in the 1PN spin-spin terms back to the proper spin $\mathcal{S}_1$; the 2.5PN terms there will generate 3.5PN contributions to the spin evolution.

The result is the 1PN spin precession of Eq. (1.5), plus a lengthy 3.5PN expression. However, using the fact that, to lowest order $\dot{\mathcal{S}}_1 = 0$, together with the identities listed in Appendix A, it is straightforward to show that our lengthy 3.5PN expression is almost, but not quite, a pure total time derivative, given by

$$\begin{aligned}(\dot{\mathcal{S}}_1)_{3.5\mathrm{PN-SS}} = {}& \frac{\mu m}{r^5}\left[\frac{2}{3}(\mathbf{v}\cdot\mathcal{S}_2) + 30\dot{r}(\mathbf{n}\cdot\mathcal{S}_2)\right](\mathbf{n}\times\mathcal{S}_1) + \frac{d}{dt}\left[\frac{\mu}{15r^3}\left\{(\mathcal{S}_1\times\mathcal{S}_2)\left[(14-\alpha)v^2 - (42-3\alpha)\dot{r}^2 + (21+\alpha)\frac{m}{r}\right]\right.\right.\\ &- 10\mathbf{n}[\mathbf{n}\cdot(\mathcal{S}_1\times\mathcal{S}_2)]\left[3(1+\alpha)v^2 - 15(1+\alpha)\dot{r}^2 - (2-\alpha)\frac{m}{r}\right] - 90\alpha\dot{r}\mathbf{n}[\mathbf{v}\cdot(\mathcal{S}_1\times\mathcal{S}_2)]\\ &- 45(1+2\alpha)\dot{r}\mathbf{v}[\mathbf{n}\cdot(\mathcal{S}_1\times\mathcal{S}_2)] - 5(1-8\alpha)\mathbf{v}[\mathbf{v}\cdot(\mathcal{S}_1\times\mathcal{S}_2)]\\ &+ 3(\mathbf{n}\cdot\mathcal{S}_2)(\mathbf{n}\times\mathcal{S}_1)\left[3(1+\alpha)v^2 - 15(1+\alpha)\dot{r}^2 + (56+\alpha)\frac{m}{r}\right] + 27(1+\alpha)\dot{r}(\mathbf{v}\cdot\mathcal{S}_2)(\mathbf{n}\times\mathcal{S}_1)\\ &\left.\left.- 9(2-3\alpha)\dot{r}(\mathbf{n}\cdot\mathcal{S}_2)(\mathbf{v}\times\mathcal{S}_1) + 3(1-4\alpha)(\mathbf{v}\cdot\mathcal{S}_2)(\mathbf{v}\times\mathcal{S}_1)\right\}\right],\end{aligned} \tag{3.6}$$

where $\alpha = m_1/m_2$. The total time derivative can be eliminated by moving it to the left-hand side and absorbing it into a redefined proper spin $\mathcal{S}_1$, which differs from the original by meaningless 3.5PN correction terms. This is the same philosophy by which we absorbed the 2.5PN terms into the initial definition of proper spin in Eq. (2.15). In the spin-orbit case of Paper III, *all* the 3.5PN terms could be so absorbed. However, in the spin-spin case, we find that this is not the case, and there is a residual contribution to the spin evolution, given by the first line in Eq. (3.6). Because there is no unique way to absorb total time derivatives into $\dot{\mathcal{S}}_1$, the expression for this residual is not unique, although its time average over an orbit is. Notice that the residual term is orthogonal to the spin. In other words, the magnitude of the proper spin of body 1 is not affected by radiation reaction to 3.5PN order, again not surprising for a spinning axisymmetric body.

Redefining the proper spin, we obtain finally Eq. (1.6). This represents a pure precession of the spin of body 1 about the radial direction $\mathbf{n}$; however, if the companion spin is perpendicular to the orbital plane, there is no effect.

### C. Comparison with fluxes of energy and angular momentum

The fluxes of energy and angular momentum in gravitational waves from a binary with spin-spin interactions were derived by Kidder *et al.* [8,9], and are given by

$$\frac{dE}{dt} = \dot{E}_{\rm N} + \dot{E}_{\rm SS}, \qquad \frac{d\mathbf{J}}{dt} = \dot{\mathbf{J}}_{\rm N} + \dot{\mathbf{J}}_{\rm SS}, \tag{3.7}$$

where we include only the lowest-order "Newtonian" and 1PN spin-spin contributions, given by

$$\dot{E}_N = -\frac{8}{15}\frac{\eta^2 m^4}{r^4}(12v^2 - 11\dot{r}^2), \tag{3.8a}$$

$$\dot{E}_{SS} = -\frac{4}{15}\frac{\eta m^2}{r^6}[(\mathcal{S}_1\cdot\mathcal{S}_2)(141v^2 - 165\dot{r}^2) + (\mathbf{n}\cdot\mathcal{S}_1)(\mathbf{n}\cdot\mathcal{S}_2)(807\dot{r}^2 - 504v^2) + 71(\mathbf{v}\cdot\mathcal{S}_1)(\mathbf{v}\cdot\mathcal{S}_2) - 171\dot{r}(\mathbf{v}\cdot\mathcal{S}_1)(\mathbf{n}\cdot\mathcal{S}_2) - 171\dot{r}(\mathbf{v}\cdot\mathcal{S}_2)(\mathbf{n}\cdot\mathcal{S}_1)], \tag{3.8b}$$

$$\dot{\mathbf{J}}_N = -\frac{8}{5}\frac{\eta^2 m^3}{r^3}\tilde{\mathbf{L}}_{\rm N}\left(2v^2 - 3\dot{r}^2 + 2\frac{m}{r}\right), \tag{3.8c}$$

$$\begin{aligned}\dot{\mathbf{J}}_{SS} = \tfrac{2\mu}{5r^4}\{&(\mathbf{n}\times\mathcal{S}_1)\tfrac{m}{r}[6(\mathbf{n}\cdot\mathcal{S}_2)\dot{r} - 5(\mathbf{v}\cdot\mathcal{S}_2)] + (\mathbf{n}\times\mathcal{S}_2)\tfrac{m}{r}[6(\mathbf{n}\cdot\mathcal{S}_1)\dot{r} - 5(\mathbf{v}\cdot\mathcal{S}_1)]\\ &+ (\mathbf{v}\times\mathcal{S}_1)[(\mathbf{n}\cdot\mathcal{S}_2)(18v^2 - 30\dot{r}^2 + 11\tfrac{m}{r}) + 6(\mathbf{v}\cdot\mathcal{S}_2)\dot{r}] + (\mathbf{v}\times\mathcal{S}_2)[(\mathbf{n}\cdot\mathcal{S}_1)(18v^2 - 30\dot{r}^2 + 11\tfrac{m}{r}) + 6(\mathbf{v}\cdot\mathcal{S}_1)\dot{r}]\\ &+ \tfrac{1}{r}\tilde{\mathbf{L}}_{\rm N}[(\mathcal{S}_1\cdot\mathcal{S}_2)(60\dot{r}^2 - 24v^2 - 46\tfrac{m}{r}) + 5(\mathbf{n}\cdot\mathcal{S}_1)(\mathbf{n}\cdot\mathcal{S}_2)(24v^2 - 84\dot{r}^2 + 36\tfrac{m}{r})\\ &+ 90\dot{r}((\mathbf{n}\cdot\mathcal{S}_1)(\mathbf{v}\cdot\mathcal{S}_2) + (\mathbf{v}\cdot\mathcal{S}_1)(\mathbf{n}\cdot\mathcal{S}_2)) - 12(\mathbf{v}\cdot\mathcal{S}_1)(\mathbf{v}\cdot\mathcal{S}_2)]\}.\end{aligned} \tag{3.8d}$$

Kidder [9] did not determine $\dot{\mathbf{J}}_{SS}$ explicitly, but rather left it in the form of

$$\begin{aligned}\dot{J}^{\,i}_{SS} = -\frac{2}{5}\mu\epsilon^{ijk}\Big\{&\overset{(2)}{I}{}^{\langle jl\rangle}(\mathbf{a}_{\rm N})\overset{(3)}{I}{}^{\langle kl\rangle}(\mathbf{a}_{\rm PN-SS})\\ &+ \overset{(2)}{I}{}^{\langle jl\rangle}(\mathbf{a}_{\rm PN-SS})\overset{(3)}{I}{}^{\langle kl\rangle}(\mathbf{a}_{\rm N})\\ &+ \frac{16}{9}\overset{(2)}{\mathcal{J}}{}^{\langle jl\rangle}_{\rm S}(\mathbf{a}_{\rm N})\overset{(3)}{\mathcal{J}}{}^{\langle kl\rangle}_{\rm S}(\mathbf{a}_{\rm N})\Big\},\end{aligned} \tag{3.9}$$

where the mass and current multipole moments are given by Eqs. (3.2), where the angular brackets around indices denote the symmetric, trace-free part. The notation $(\mathbf{a}_{\rm N})$ or $(\mathbf{a}_{\rm PN-SS})$ denotes which acceleration, Newtonian or spin-spin, is to be used for the acceleration generated by the time derivatives, and the subscript S denotes the spin part of the current moment $\mathcal{J}$. Following this procedure and keeping only terms involving the product of $S_1$ with $S_2$, we obtain Eq. (3.8d).

We now calculate the time derivative of the energy and angular momentum expressions (2.16), and substitute the equations of motion, PN spin-spin, 2.5PN point-mass and 3.5PN spin-spin terms, along with the 1PN and 3.5PN spin precession equations. After recovering the fact that all 1PN spin-spin contributions cancel, leaving $E$ and $\mathbf{J}$ conserved to that order, we find that the changes in $E$ and $\mathbf{J}$ due to 2.5PN and 3.5PN spin-spin radiation reaction are obtained from the following expressions,

$$\begin{aligned}\dot{E} &= \mu\mathbf{v}\cdot(\mathbf{a}_{2.5\rm PN} + \mathbf{a}_{3.5\rm PN-SS}),\\ \dot{\mathbf{J}} &= \mu\mathbf{x}\times(\mathbf{a}_{2.5\rm PN} + \mathbf{a}_{3.5\rm PN-SS}) + \dot{\mathcal{S}}.\end{aligned} \tag{3.10}$$

Initially, the results do not match the flux expressions above. However, by making use of the identities listed in Appendix A, we can show that the difference between the expressions in all cases is a total time derivative. These can thus be absorbed into meaningless 2.5PN and 3.5PN corrections to the definition of total energy and angular momentum. Notice that the residual 3.5PN precession term from $\dot{\mathcal{S}}$ given by the sum of $(\dot{\mathcal{S}}_1)_{3.5\rm PN}$ and $(\dot{\mathcal{S}}_2)_{3.5\rm PN}$ from Eq. (1.6) exactly balances a corresponding effect in the orbital part, so that the net $\dot{\mathbf{J}}$ matches the flux modulo a total time derivative. Thus we have established a proper energy and angular momentum balance between the radiation flux and the evolution of the orbit, including spin-spin effects.

## IV. CONCLUSIONS

We have derived the equations of motion for binary systems of spinning bodies from first principles, including the effects of gravitational radiation reaction, and incorporating the contributions of spin-spin coupling at 3.5PN order. We found that the spin-spin coupling combined with radiation reaction leads to a small 3.5PN-order precession of the individual spins. The resulting equations of motion are instantaneous, dynamical equations, and do not rely on assumptions of energy balance, or orbital averaging. They may be used to study the effects of spin-spin interactions on the inspiral of compact binaries numerically. We have focussed attention on effects involving products $S_1 S_2$ of the spins; effects depending quadratically on the individual spins can in principle also be calculated with our approach. This will be the subject of future work.

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## APPENDIX: EXTRACTING TOTAL TIME DERIVATIVES

Using the Newtonian equations of motion plus the 1PN spin-spin terms, it is straightforward to establish a number of identities, which may be used to extract time derivatives from 2.5PN and 3.5PN terms in the expressions (3.6) and (3.10). For any non-negative integers $s$, $p$ and $q$, we obtain

$$\begin{aligned}
\frac{d}{dt}\left(\frac{v^{2s}\dot{r}^p}{r^q}\right) &= \frac{v^{2s-2}\dot{r}^{p-1}}{r^{q+1}}\bigg\{pv^4-(p+q)v^2\dot{r}^2-2s\dot{r}^2\frac{m}{r}-pv^2\frac{m}{r}-3p\frac{v^2}{\mu r^3}((\mathcal{S}_1\cdot\mathcal{S}_2)-3(\mathbf{n}\cdot\mathcal{S}_1)(\mathbf{n}\cdot\mathcal{S}_2))\\
&\quad -6s\frac{\dot{r}}{\mu r^3}(\dot{r}(\mathcal{S}_1\cdot\mathcal{S}_2)+(\mathbf{v}\cdot\mathcal{S}_1)(\mathbf{n}\cdot\mathcal{S}_2)+(\mathbf{v}\cdot\mathcal{S}_2)(\mathbf{n}\cdot\mathcal{S}_1)-5\dot{r}(\mathbf{n}\cdot\mathcal{S}_1)(\mathbf{n}\cdot\mathcal{S}_2))\bigg\},\\
\frac{d}{dt}\left(\frac{v^{2s}\dot{r}^p}{r^q}\tilde{\mathbf{L}}_{\rm N}\right) &= \frac{v^{2s-2}\dot{r}^{p-1}}{r^{q+1}}\bigg\{\bigg[pv^4-(p+q)v^2\dot{r}^2-2s\dot{r}^2\frac{m}{r}-pv^2\frac{m}{r}-3p\frac{v^2}{\mu r^3}((\mathcal{S}_1\cdot\mathcal{S}_2)-3(\mathbf{n}\cdot\mathcal{S}_1)(\mathbf{n}\cdot\mathcal{S}_2))\\
&\quad -6s\frac{\dot{r}}{\mu r^3}(\dot{r}(\mathcal{S}_1\cdot\mathcal{S}_2)+(\mathbf{v}\cdot\mathcal{S}_1)(\mathbf{n}\cdot\mathcal{S}_2)+(\mathbf{v}\cdot\mathcal{S}_2)(\mathbf{n}\cdot\mathcal{S}_1)-5\dot{r}(\mathbf{n}\cdot\mathcal{S}_1)(\mathbf{n}\cdot\mathcal{S}_2))\bigg]\tilde{\mathbf{L}}_{\rm N}\\
&\quad -\frac{v^2\dot{r}}{\mu r^2}((\mathbf{n}\times\mathcal{S}_1)(\mathbf{n}\cdot\mathcal{S}_2)+(\mathbf{n}\times\mathcal{S}_2)(\mathbf{n}\cdot\mathcal{S}_1))\bigg\}
\end{aligned} \tag{A1}$$

Another set of identities, to be used only in 3.5PN terms, requires only the Newtonian equations of motion:

$$\begin{aligned}
\frac{d}{dt}\left(\frac{v^{2s}\dot{r}^p}{r^q}x^ix^j\right) &= \frac{v^{2s-2}\dot{r}^{p-1}}{r^{q+1}}\bigg\{\bigg[pv^4-(p+q)v^2\dot{r}^2-2s\dot{r}^2\frac{m}{r}-pv^2\frac{m}{r}\bigg]x^ix^j+2v^2\dot{r}rx^{(i}v^{j)}\bigg\},\\
\frac{d}{dt}\left(\frac{v^{2s}\dot{r}^p}{r^q}v^iv^j\right) &= \frac{v^{2s-2}\dot{r}^{p-1}}{r^{q+1}}\bigg\{\bigg[pv^4-(p+q)v^2\dot{r}^2-2s\dot{r}^2\frac{m}{r}-pv^2\frac{m}{r}\bigg]v^iv^j-2m\frac{v^2\dot{r}}{r^2}x^{(i}v^{j)}\bigg\},\\
\frac{d}{dt}\left(\frac{v^{2s}\dot{r}^p}{r^q}x^iv^j\right) &= \frac{v^{2s-2}\dot{r}^{p-1}}{r^{q+1}}\bigg\{\bigg[pv^4-(p+q)v^2\dot{r}^2-2s\dot{r}^2\frac{m}{r}-pv^2\frac{m}{r}\bigg]x^iv^j+v^2\dot{r}r\left(v^iv^j-\frac{m}{r}n^in^j\right)\bigg\}.
\end{aligned} \tag{A2}$$


[1] In discussions of inspiralling compact binaries, one often sees the statement that spin-spin terms are 2PN order. This is because, for such systems, one treats the radius of each body as being of order $m$, and the rotational velocity of each body $V$ as being of order unity (especially for rapidly rotating black holes); consequently, the spin-spin term in this case is *effectively* of order $(m^2)^2/(mr^4) \sim (m/r^2)\epsilon^2$. Because the equations derived in this paper apply to arbitrary systems treatable with PN methods, we will stick with the formal PN ordering of spin terms.

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