Unified first law and the thermodynamics of the apparent horizon in the FRW universe

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In this paper we revisit the relation between the Friedmann equations and the first law of thermodynamics. We find that the unified first law first proposed by Hayward to treat the outertrapping horizon of a dynamical black hole can be used to the apparent horizon (a kind of inner trapping horizon in the context of the FRW cosmology) of the FRW universe. We discuss three kinds of gravity theorties: Einstein theory, Lovelock thoery, and scalar-tensor theory. In Einstein theory, the first law of thermodynamics is always satisfied on the apparent horizon. In Lovelock theory, treating the higher derivative terms as an effective energy-momentum tensor, we find that this method can give the same entropy formula for the apparent horizon as that of black hole horizon. This implies that the Clausius relation holds for the Lovelock theory. In scalar-tensor gravity, we find, by using the same procedure, the Clausius relation no longer holds. This indicates that the apparent horizon of the FRW universe in the scalar-tensor gravity corresponds to a system of nonequilibrium thermodynamics. We show this point by using the method developed recently by Eling *et al.* for dealing with the $f(R)$ gravity.

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I. INTRODUCTION

Quantum mechanics together with general relativity predicts that a black hole behaves like a black body, emitting thermal radiations, with a temperature proportional to its surface gravity at the black hole horizon and with an entropy proportional to its horizon area [1,2]. The Hawking temperature and horizon entropy together with the black hole mass obey the first law of black hole thermodynamics ($dM = TdS$) [3]. The formulas of black hole entropy and temperature have a certain universality in the sense that the horizon area and surface gravity are purely geometric quantities determined by the space-time geometry.

Since the discovery of black hole thermodynamics in the 1970's, physicists have been speculating that there should be some relation between the thermodynamic laws and Einstein equations. Otherwise, how does general relativity know that the horizon area of a black hole is related to its entropy and the surface gravity to its temperature [4]? Indeed, Jacobson [4] was able to derive Einstein equations from the proportionality of entropy to the horizon area, A, together with the fundamental relation (Clausius relation) $\delta Q = TdS$, assuming the relation holds for all local Rindler causal horizons through each space-time point. Here δQ and T are the variation of heat flow and Unruh temperature seen by an accelerated observer just inside the horizon. More recently, Eling *et al.* found that one cannot get the right equations of motion for $f(R)$ gravity if one simply uses the Clausius relation and the entropy assumption $S = \alpha f'(R)A$. In order to get the equations of motion, an entropy production term has to be added to the Clausius relation. They have argued that this corresponds to the nonequilibrium thermodynamics of space-time [5]. It is interesting to see whether the nonequilibrium thermodynamics is needed in other gravity theories. In this paper, we will discuss the scalar-tensor gravity by following [5]. By using entropy assumption $S = \alpha F(\phi)A$ in the scalar-tensor gravity and Clausius relation with an appropriate entropy production term, we can obtain correct equations of motion for the scalar-tensor gravity. This suggests that for the scalar-tensor gravity the nonequilibrium thermodynamics also has to be employed to derive the dynamic equations of motion of space-times.

On the other hand, most discussions of black hole thermodynamics have been focused on the stationary black holes. For dynamical (i.e., nonstationary) black holes, Hayward has proposed a method to deal with thermodynamics associated with a trapping horizon of a dynamic black hole in four-dimensional Einstein theory [6–9]. In this method, for spherical symmetric space-times, Einstein equations can be rewritten in a form called "unified first law." Projecting this unified first law along a trapping horizon, one gets the first law of thermodynamics for a dynamical black hole. A definition of energy-supply, Ψ, is introduced in the unified first law. It is an energy flux defined by the energy-momentum tensor of matter. After projecting along a vector ξ tangent to the trapping horizon, one finds $\langle A\Psi, \xi\rangle = \frac{\kappa}{8\pi G}\langle dA, \xi\rangle$. This equation can be regarded as the Clausius relation of the dynamical black hole. The Friedmann-Robertson-Walker (FRW) universe is one kind of nonstationary spherically symmetric space-times. Certainly, we can discuss its thermodynamics on

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the trapping horizon. However, in the FRW universe, the trapping horizon (''outer trapping horizon'') is absent; instead there exists a kind of cosmological horizon like trapping horizon (''inner trapping horizon''). This horizon coincides with the apparent horizon in the context of the FRW cosmology. In this paper, we therefore do not distinguish them. We will focus on this apparent horizon and discuss associated thermodynamics. An interesting question is whether the field equations for non-Einstein gravity can be written to a form as the unified first law in Einstein gravity. We do not try to resolve this issue, instead we rewrite the field equations for non-Einstein gravity to a form of the Einstein gravity by introducing an effective energy-momentum tensor. Thus, we can use unified first law to non-Einstein gravity theories and get the thermodynamics of apparent horizon in the FRW universe in those theories. As a result, we have the equation $\langle A\Psi, \xi\rangle = \frac{\kappa}{8\pi G}\langle dA, \xi\rangle$ even for the non-Einstein gravity theories. But, the energy-supply Ψ here includes the contribution of the effective energy-momentum tensor.

In order to get the heat δQ (defined by integration of pure matter energy flux) in the Clausius relation, we have to extract the contribution of pure matter fields from the $\langle A\Psi, \xi\rangle$ in the left-hand side of the equation $\langle A\Psi, \xi\rangle = \frac{\kappa}{8\pi G}\langle dA, \xi\rangle$ and put the others into the right-hand side of the equation. Now, does the right-hand side of the equation have the correct form of TdS in Clausius relation? If the answer is yes, the Clausius relation holds and we are treating equilibrium thermodynamics. On the contrary, if the answer is no, the Clausius relation in equilibrium thermodynamics no longer holds, and we have to treat the system with nonequilibrium thermodynamics. In this paper, we will use this idea to treat the Lovelock gravity and scalar-tensor gravity. We find that the Clausius relation holds in Lovelock gravity, while it breaks for the scalar-tensor gravity. We will also show this point by using the method of Eling *et al.* developed for the $f(R)$ gravity [5].

There exist some works dealing with the relation between the Einstein equations and the first law of thermodynamics. In a setup of a special kind of spherically symmetric black hole space-times, Padmanabhan *et al.* [10] showed that the Einstein equations on the black hole horizon can be written into the first law of thermodynamics, $dE = TdS - PdV$. This also holds in Lovelock gravity [10]. In the setting of the FRW universe, some authors investigated the relation between the first law and the Friedmann equations describing the dynamic evolution of the Universe [11]. In particular, Cai and Kim in [12] derived the Friedmann equations by applying the fundamental relation $\delta Q = T\delta S$ to the apparent horizon of the FRW universe with any spatial curvature and assuming that the apparent horizon has temperature and entropy

$$T = \frac{1}{2\pi R_A}, \qquad S = \frac{\pi R_A^2}{G}, \tag{1.1}$$

where R_A is the apparent horizon radius. Further they showed that using the same procedure, the Friedmann equations can be derived also in the Gauss-Bonnet gravity and more general Lovelock gravity. For the scalar-tensor gravity and $f(R)$ gravity, the possibility to derive the corresponding Friedmann equations in those theories was investigated in [13]. More recently, Akbar and Cai [14] have shown that at the apparent horizon, the Friedmann equation can be written into a form of the first law of thermodynamics with a volume change term, not only in Einstein gravity, but also in Lovelock gravity.

This paper is organized as follows. In Sec. II, we give a brief review on the unified first law by generalizing it to $(n+1)$-dimensional Einstein gravity. In Sec. III, we consider a FRW universe and give the projection vector which will be used in the following sections. In Sec. IV, we give the rigorous first law of thermodynamics on the apparent horizon for the FRW universe in Einstein theory. In Sec. V, we treat the thermodynamics of apparent horizon in the Lovelock gravity and find that the Clausius relation holds for the Lovelock gravity. In Sec. VI, we discuss the scalar-tensor gravity and show that the Clausius relation no longer holds. In Sec. VII, we derive the equations of motion for the scalar-tensor gravity by using the method of Eling *et al.* developed for dealing with the $f(R)$ gravity. We end this paper with a conclusion in Sec. VIII.

II. A BRIEF REVIEW ON THE UNIFIED FIRST LAW

Hayward has proposed a general definition of black hole dynamics on a trapping horizon in four-dimensional Einstein theory [6–9]. In this section, we will make a brief review and generalize his discussions to the $(n+1)$-dimensional Einstein gravity.

For an arbitrary $(n+1)$-dimensional spherical symmetric space-time, locally we can put its metric in the double-null form

$$ds^2 = -2e^{-f}d\xi^+d\xi^- + r^2d\Omega_{n-1}^2, \tag{2.1}$$

where $d\Omega_{n-1}^2$ is the line element of an $(n-1)$-sphere with unit radius, r and f are functions of (ξ^+, ξ^-). Certainly, there are some remainder freedoms to choose the double-null coordinates. Assume that the space-time is time orientable and $\partial_\pm = \partial/\partial\xi^\pm$ are future pointing. Considering radial null geodesic congruence, from $ds^2 = 0$, one can find that there are two kinds of null geodesics corresponding to $\xi^+ =$ constant and $\xi^- =$ constant, respectively. It is easy to get the expansions of these two congruences

$$\theta_\pm = (n-1)\frac{\partial_\pm r}{r}. \tag{2.2}$$

The expansion measures whether the light rays normal to the sphere are diverging ($\theta_\pm > 0$) or converging ($\theta_\pm < 0$) or equivalently, whether the area of the sphere is increasing or decreasing in the null directions.

A sphere is said to be *trapped*, *untrapped*, or *marginal* if (on this sphere)

$$\theta_+\theta_- > 0, \qquad \theta_+\theta_- < 0, \qquad \text{or} \quad \theta_+\theta_- = 0. \tag{2.3}$$

Note that

$$g^{ab}\partial_a r \partial_b r = -\frac{2}{(n-1)^2} e^f r^2 \theta_+\theta_-. \tag{2.4}$$

We can write this definition to be

$$g^{ab}\partial_a r\partial_b r < 0, \qquad g^{ab}\partial_a r\partial_b r > 0, \qquad g^{ab}\partial_a r\partial_b r = 0. \tag{2.5}$$

If $e^f\theta_+\theta_-$ is a function with nonvanishing derivatives, the space-time is divided into trapped and untrapped regions, separated by marginal surface. Some subdivisions may be made as follows.

(i) A trapped sphere is said to be future, if $\theta_\pm < 0$ ($\partial_\pm r < 0$) and past if $\theta_\pm > 0$ ($\partial_\pm r > 0$).

(ii) On an untrapped sphere, a spatial or null normal vector z is outgoing if $\langle dr, z\rangle > 0$ and ingoing if $\langle dr, z\rangle < 0$. Equivalently, fixing the orientation by locally $\theta_+ > 0$ and $\theta_- < 0$, z is outgoing if $g(z, \partial_+) > 0$ or $g(z, \partial_-) < 0$ and ingoing if $g(z, \partial_+) < 0$ or $g(z, \partial_-) > 0$. In particular, ∂_+ and ∂_- are, respectively, the outgoing and ingoing null normal vectors.

(iii) A marginal sphere with $\theta_+ = 0$ is future if $\theta_- < 0$, past if $\theta_- > 0$, bifurcating if $\theta_- = 0$, outer if $\partial_-\theta_+ < 0$, inner if $\partial_-\theta_+ > 0$, and degenerate if $\partial_-\theta_+ = 0$.

The closure of a hypersurface foliated by future or past, outer or inner marginal sphere is called a (nondegenerate) *trapping horizon*.

In the works of Hayward, the future (past) outer trapping horizon is taken as the definition of black (white) holes, i.e., on the marginal sphere of the trapping horizon, we have

$$\theta_+ = 0, \qquad \theta_- < 0, \qquad \partial_-\theta_+ < 0, \tag{2.6}$$

or equivalently

$$\partial_+ r = 0, \qquad \partial_- r < 0, \qquad \partial_-\partial_+ r < 0. \tag{2.7}$$

However, in the FRW universe, we will treat a horizon which is similar to the cosmological horizon in the de Sitter space-time. In this case, the surface gravity is negative. So, we need not the requirement of "outer." In fact, we will take the future inner trapping horizon as a system on which the thermodynamics will be established.

The Misner-Sharp energy [7,15,16] is defined to be

$$E = \frac{1}{16\pi G}(n-1)\Omega_{n-1} r^{n-2}(1 - g^{ab}\partial_a r\partial_b r). \tag{2.8}$$

This energy is the total energy (not only the passive energy) inside the sphere with radius r. It is a pure geometric quantity. From the definition, the ratio E/r^{n-2} controls the formation of black and white holes and trapped sphere generally [7,16]. There are a lot of definitions for energy in general relativity, such as, ADM mass for asymptotically flat space-time, Bondi-Sachs energy defined at null infinity of the asymptotically flat space-time, Brown-York energy and Liu-Yau energy, etc. [17–21]. These definitions can be found in a recent review [22]. The physical meanings of Misner-Sharp energy, and the comparison of Misner-Sharp energy to ADM mass and Bondi-Sachs energy have been given in [7,15]. For spherical space-time, Brown-York energy agrees with the Liu-Yau energy, but they both differ from the Misner-Sharp energy. For example, for the four-dimensional Reissner-Nordström black hole, the Misner-Sharp energy differs from the Brown-York or Liu-Yau mass by a term which is the energy of the electromagnetic field inside the sphere. The Misner-Sharp energy has the relation to the structure of the space-time and one can relate it to Einstein equations [see Eq. (2.11) below]. This is an important advantage of Misner-Sharp energy.

From the energy-momentum tensor T_{ab}, we can give two useful invariants—*work* and *energy-supply*:

$$W = -\tfrac{1}{2}\mathrm{trace}T = -g_{+-}T^{+-}, \tag{2.9}$$

$$\Psi_a = T_a{}^b\partial_b r + W\partial_a r. \tag{2.10}$$

By using the Misner-Sharp energy and these two quantities, one can find that the (0,0) component of Einstein equations can be written as

$$dE = A\Psi + WdV, \tag{2.11}$$

where $A = \Omega_{n-1}r^{n-1}$ and $V = \frac{1}{n}\Omega_{n-1}r^n$ with $\Omega_{n-1} = 2\pi^{n/2}/\Gamma(n/2)$ are the area and volume of a sphere with radius r. Equation (2.11) is called *unified first law*. It is the natural result of Einstein equations for spherical symmetric space-times. The unified first law (2.11) contains rich information. After projecting it along different directions, this equation gives different meanings. For instance, (i) projecting the unified first law along the future null infinity, one has the Bondi energy loss equation; (ii) projecting the unified first law along the flow of a thermodynamic material yields the first law of relativistic thermodynamics; (iii) projecting the unified first law along the trapping horizon, one obtains the first law of black hole thermodynamics. Here, we will concentrate on the first law of black hole thermodynamics, which has the form

$$\langle dE, z\rangle = \frac{\kappa}{8\pi G}\langle dA, z\rangle + W\langle dV, z\rangle, \tag{2.12}$$

where κ is the surface gravity of the trapping horizon and is defined by

$$\kappa = \tfrac{1}{2}\nabla^a\nabla_a r, \tag{2.13}$$

where ∇ corresponds to the covariant derivative of two-dimension space normal to the sphere, and z is a vector which is tangent to the trapping horizon. (Certainly, z is not arbitrary and must satisfy some conditions on the trapping horizon.) In the double-null coordinates, z can be expressed as

$$z = z^+\partial_+ + z^-\partial_-. \tag{2.14}$$

The Eq. (2.12) is obtained through projecting the unified first law along the vector z. The nontrivial part is to show

$$\langle A\Psi, z\rangle = \frac{\kappa}{8\pi G}\langle dA, z\rangle. \tag{2.15}$$

As mentioned above, we take the horizon to be

$$\partial_+ r = 0. \tag{2.16}$$

Then, one has on the marginal sphere,

$$z^a\partial_a(\partial_+ r) = z^+\partial_+\partial_+ r + z^-\partial_-\partial_+ r = 0. \tag{2.17}$$

Using Einstein equations and the definition of surface gravity, one can arrive at (2.15). The most important thing is to note that z is not arbitrary, on the trapping horizon it must satisfy the equation above, then

$$\frac{z^-}{z^+} = -\frac{\partial_+\partial_+ r}{\partial_-\partial_+ r}. \tag{2.18}$$

Therefore z belongs to the one-dimensional subspace of the tangent space. Through the above projection along z, the extra differential on the space-time is replaced by the differential on the state space of a dynamical black hole, and one can arrive at the exact first law of the trapping horizon for dynamical black holes [8]. This Eq. (2.18) plays a curial role in getting the projection vector. We will give this ratio for the FRW universe in the next section.

III. TRAPPING HORIZON AND APPARENT HORIZON OF THE FRW UNIVERSE

Now, we consider an $(n+1)$-dimensional FRW universe. We put the FRW metric in the form

$$ds^2 = h_{ab}dx^a dx^b + \tilde{r}^2 d\Omega_{n-1}^2, \tag{3.1}$$

where $x^0 = t$, $x^1 = r$, $\tilde{r} = ar$ is the radius of the sphere and a is the scale factor. It should be noted here that $\tilde{r} = \tilde{r}(t, r)$ plays the role of sphere radius r defined in the previous section. Defining

$$\begin{aligned} d\xi^+ &= -\frac{1}{\sqrt{2}}\left(dt - \frac{a}{\sqrt{1-kr^2}}dr\right),\\ d\xi^- &= -\frac{1}{\sqrt{2}}\left(dt + \frac{a}{\sqrt{1-kr^2}}dr\right), \end{aligned} \tag{3.2}$$

where k is the spatial curvature parameter of the FRW universe, we can put the FRW metric into a double-null form

$$ds^2 = -2d\xi^+ d\xi^- + \tilde{r}^2 d\Omega_{n-1}^2. \tag{3.3}$$

It is easy to find

$$\begin{aligned} \partial_+ &= \frac{\partial}{\partial\xi^+} = -\sqrt{2}\left(\partial_t - \frac{\sqrt{1-kr^2}}{a}\partial_r\right),\\ \partial_- &= \frac{\partial}{\partial\xi^-} = -\sqrt{2}\left(\partial_t + \frac{\sqrt{1-kr^2}}{a}\partial_r\right), \end{aligned} \tag{3.4}$$

where the minus signs ensure that $\partial_\pm$ are future pointing. The trapping horizon, we denote it by $\tilde{r}_A$, is defined to be

$$\partial_+\tilde{r}|_{\tilde{r}=\tilde{r}_A} = 0. \tag{3.5}$$

Solving this equation, one finds

$$\tilde{r}_A^2 = \frac{1}{H^2 + \frac{k}{a^2}}. \tag{3.6}$$

This radius has the same form as apparent horizon [16]. It is not surprising because the trapping horizon and apparent horizon coincide with each other in the FRW universe. On the other hand, we have

$$\partial_-\tilde{r}|_{\tilde{r}=\tilde{r}_A} = -2\tilde{r}_A H < 0, \tag{3.7}$$

that is, this trapping horizon is future. A similar calculation on the trapping horizon gives

$$\begin{aligned} \partial_-\partial_+\tilde{r}|_{\tilde{r}_A} &= 2\tilde{r}_A\left(\dot{H} + 2H^2 + \frac{k}{a^2}\right),\\ \partial_+\partial_+\tilde{r}|_{\tilde{r}_A} &= 2\tilde{r}_A\left(\dot{H} - \frac{k}{a^2}\right). \end{aligned} \tag{3.8}$$

By definition, one can find the surface gravity

$$\kappa = -\frac{\tilde{r}_A}{2}\left(\dot{H} + 2H^2 + \frac{k}{a^2}\right) = -\frac{1}{\tilde{r}_A}\left(1 - \frac{\dot{\tilde{r}}_A}{2H\tilde{r}_A}\right). \tag{3.9}$$

Further, we define

$$\epsilon = \frac{\dot{\tilde{r}}_A}{2H\tilde{r}_A}. \tag{3.10}$$

Here, we assume $\epsilon < 1$ such that $\kappa < 0$. In another words, we are treating an "inner" trapping horizon, rather than "outer " trapping horizon (with positive surface gravity) discussed by Hayward. Note that in Refs. [11,12], in fact, an approximation $\epsilon \ll 1$ has been used in calculating the energy flow crossing the apparent horizon. In the present paper, no approximation will be used. In terms of the horizon radius $\tilde{r}_A$, we have

$$\dot{H} - \frac{k}{a^2} = -\frac{2\epsilon}{\tilde{r}_A^2}. \tag{3.11}$$

Substituting this into (3.8), we get

$$\frac{z^-}{z^+} = -\frac{\partial_+\partial_+\tilde{r}}{\partial_-\partial_+\tilde{r}}\bigg|_{\tilde{r}_A} = \frac{\epsilon}{1-\epsilon}. \tag{3.12}$$

Let $z^+ = 1$, then $z^- = \frac{\epsilon}{1-\epsilon}$, in the coordinates (t, r), we then can express z as

$$z = -\frac{\sqrt{2}}{1-\epsilon}\left(\frac{\partial}{\partial t} - (1-2\epsilon)Hr\frac{\partial}{\partial r}\right), \tag{3.13}$$

where we have used the relation

$$\left.\frac{\sqrt{1-kr^2}}{a}\right|_{\tilde{r}_A} = Hr|_{\tilde{r}_A}, \tag{3.14}$$

on the trapping horizon. Thus, if we use the (t, r) coordinates instead of double-null coordinates, any project vector ξ must have the form

$$\xi = \xi^t\left(\frac{\partial}{\partial t} - (1-2\epsilon)Hr\frac{\partial}{\partial r}\right). \tag{3.15}$$

Certainly, we can choose $\xi^t = 1$. All calculations in this section are pure geometrical. Therefore, the results in this section are applicable for the FRW universe in any gravity theory.

IV. THERMODYNAMICS OF APPARENT HORIZON IN EINSTEIN GRAVITY

Consider a Lagrangian for an $(n+1)$-dimensional Einstein gravity with perfect fluid

$$\mathcal{L} = \frac{1}{16\pi G}R + \overset{m}{\mathcal{L}}, \tag{4.1}$$

where $\overset{m}{\mathcal{L}}$ denotes the Lagrangian for the perfect fluid. In the FRW universe, the energy-momentum tensor of the perfect fluid has the form

$$T_{ab} = \overset{m}{T}_{ab} = (\rho_m + p_m)U_aU_b + p_m g_{ab}, \tag{4.2}$$

where ρ_m and p_m are the energy density and pressure of the perfect fluid, respectively. It is easy to find that the energy-momentum tensor projecting onto two-dimensional spacetime normal to the sphere has the form

$$T_{ab} = \mathrm{diag}\left(\rho_m, \frac{p_m a^2}{1-kr^2}\right), \qquad T_a{}^b = \mathrm{diag}(-\rho_m, p_m), \tag{4.3}$$

and then the work term and energy supply are

$$W = \tfrac{1}{2}(\rho_m - p_m), \tag{4.4}$$

$$\Psi_t = -\tfrac{1}{2}(\rho_m + p_m)H\tilde{r}, \tag{4.5}$$

$$\Psi_r = \tfrac{1}{2}(\rho_m + p_m)a, \tag{4.6}$$

respectively. One then has

$$\begin{aligned}\Psi &= \Psi_t dt + \Psi_r dr \\ &= -\tfrac{1}{2}(\rho_m + p_m)H\tilde{r}dt + \tfrac{1}{2}(\rho_m + p_m)adr.\end{aligned} \tag{4.7}$$

Thus, on the trapping horizon/apparent horizon we have

$$\begin{aligned}dE &= A\Psi + WdV = A\Psi + AWd\tilde{r}_A \\ &= -A(\rho_m + p_m)H\tilde{r}_A dt + A\rho_m d\tilde{r}_A \\ &= Vd\rho_m + \rho_m dV = d(\rho_m V).\end{aligned} \tag{4.8}$$

Substituting the first Friedmann equation $[H^2 + k/a^2 = 16\pi G\rho_m/(n(n-1))]$ into the last line in the above equation, one can get nothing but the differential of the Misner-Sharp energy. On the other hand, if we use the Misner-Sharp energy (2.8) inside the apparent horizon with the radius (3.6), the above Eq. (4.8) gives us the first Friedmann equation [14].

Let ξ be a vector tangent to the apparent horizon, which can be expressed as (3.15). From now on we choose $\xi^t = 1$, thus we have

$$\langle dE, \xi\rangle = -AH\tilde{r}_A[(1-2\epsilon)\rho_m + p_m)]. \tag{4.9}$$

On the other hand,

$$\frac{\kappa}{8\pi G}dA = -(n-1)(1-\epsilon)\frac{A}{8\pi G\tilde{r}_A^2}(H\tilde{r}_A dt + adr). \tag{4.10}$$

By using the Friedmann equation, we have

$$\begin{aligned}\frac{\kappa}{8\pi G}\langle dA, \xi\rangle &= -(n-1)2\epsilon(1-\epsilon)\frac{A}{8\pi G\tilde{r}_A^2}H\tilde{r}_A \\ &= -(1-\epsilon)AH\tilde{r}_A(\rho_m + p_m).\end{aligned} \tag{4.11}$$

Similarly, one can show

$$\langle WdV, \xi\rangle = \epsilon AH\tilde{r}_A(\rho_m - p_m). \tag{4.12}$$

Combining them yields

$$\frac{\kappa}{8\pi G}\langle dA, \xi\rangle + \langle WdV, \xi\rangle = -AH\tilde{r}_A[(1-2\epsilon)\rho_m + p_m)]. \tag{4.13}$$

Thus we have shown that the unified first law on the inner trapping horizon has the form

$$\langle dE, \xi\rangle = \frac{\kappa}{8\pi G}\langle dA, \xi\rangle + \langle WdV, \xi\rangle. \tag{4.14}$$

Here some remarks are in order. (i) The unified first law (2.11) is not a real first law of thermodynamics, but just an identity concerning the (0,0) component of Einstein equations. However, the projection of the unified first law along a trapping horizon (or apparent horizon in FRW cosmology context) gives a real first law of thermodynamics. (ii) The Misner-Sharp energy plays an important role in the unified first law and the definition of work and energy supply is very useful. The separation of work and energy supply gives a very similar form as the first law of thermodynamics before projecting it along the horizon. (iii) On the horizon, the energy supply has the form

$$\langle A\Psi, \xi\rangle = \frac{\kappa}{8\pi G}\langle dA, \xi\rangle. \tag{4.15}$$

This is the Clausius relation in the version of black hole thermodynamics. The left-hand side of the above equation is nothing but heat flow δQ defined by the matter energy-momentum tensor. The right-hand side has the form TdS if one identities that temperature $T = \kappa/2\pi$ and $S = A/4G$. Thus we conclude that in Einstein theory, the unified first law also implies the Clausius relation $\delta Q = TdS$. (iv) For the FRW cosmology in Einstein theory, the Misner-Sharp energy inside the trapping horizon is $E = \rho_m V$ [see (4.8)]—the energy density times the volume. This energy form for the FRW cosmology in Einstein theory is very special. In general, the Misner-Sharp energy has no such a form in other gravity theories, which will be seen in the next sections.

V. THERMODYNAMICS OF APPARENT HORIZON IN LOVELOCK GRAVITY

The Lagrangian of the Lovelock gravity [23] consists of the dimensionally extended Euler densities

$$\mathcal{L} = \sum_{i=0}^{m} c_i \mathcal{L}_i, \tag{5.1}$$

where c_i is an arbitrary constant, $m \le [n/2]$, and $\mathcal{L}_i$ is the Euler density of a $2i$-dimensional manifold

$$\mathcal{L}_i = \frac{1}{2^i} \delta^{a_1 b_1 \cdots a_i b_i}_{c_1 d_1 \cdots c_i d_i} R^{c_1 d_1}{}_{a_1 b_1} \cdots R^{c_i d_i}{}_{a_i b_i}. \tag{5.2}$$

$\mathcal{L}_0$ corresponds to the cosmological term. $\mathcal{L}_1$ is just the Einstein-Hilbert term, and $\mathcal{L}_2$ corresponds to the so called "Gauss-Bonnet" term. Although the Lagrangian of the Lovelock theory contains higher curvature terms, there are no terms with more than second order derivatives of metric in equations of motion. This point can be directly found from the equations of motion

$$\mathcal{G}^a_b = \sum_{i=0}^{m} \frac{c_i}{2^{i+1}} \delta^{a a_1 b_1 \cdots a_i b_i}_{b c_1 d_1 \cdots c_i d_i} R^{c_1 d_1}{}_{a_1 b_1} \cdots R^{c_i d_i}{}_{a_i b_i} = 0. \tag{5.3}$$

If we introduce matter fields, the equations of motion become

$$\mathcal{G}^a_b = \sum_{i=0}^{m} \frac{c_i}{2^{i+1}} \delta^{a a_1 b_1 \cdots a_i b_i}_{b c_1 d_1 \cdots c_i d_i} R^{c_1 d_1}{}_{a_1 b_1} \cdots R^{c_i d_i}{}_{a_i b_i} = 8\pi G \overset{m}{T}{}^a_b. \tag{5.4}$$

In the FRW cosmology, the energy-momentum tensor is still taken to be that of perfect fluid. We can put this equation of motion into the standard form in Einstein gravity by moving those terms except the Einstein tensor into the right-hand side of the equation

$$G^a_b = 8\pi G(\overset{m}{T}{}^a_b + \overset{e}{T}{}^a_b), \tag{5.5}$$

where the effective energy-momentum tensor $\overset{e}{T}{}^a_b$ has the expression

$$\overset{e}{T}{}^a_b = -\frac{1}{8\pi G} \sum_{i=0, i\neq 1}^{m} \frac{c_i}{2^{i+1}} \delta^{a a_1 b_1 \cdots a_i b_i}_{b c_1 d_1 \cdots c_i d_i} R^{c_1 d_1}{}_{a_1 b_1} \cdots R^{c_i d_i}{}_{a_i b_i}. \tag{5.6}$$

In the FRW metric, some nonvanishing components of Riemann tensor have the forms

$$\begin{aligned} R^{tr}{}_{tr} &= \dot{H} + H^2, \qquad & R^{ti}{}_{tj} &= (\dot{H} + H^2)\delta^i_j, \\ R^{ri}{}_{rj} &= \left(H^2 + \frac{k}{a^2}\right)\delta^i_j, \qquad & R^{ij}{}_{kl} &= \left(H^2 + \frac{k}{a^2}\right)\delta^{ij}_{kl}. \end{aligned} \tag{5.7}$$

Substituting these into (5.6), we have

$$\overset{e}{T}{}^t_t = -\frac{1}{8\pi G} \sum_{i=0, i\neq 1}^{m} \frac{\hat{c}_i}{2} \left(H^2 + \frac{k}{a^2}\right)^i, \tag{5.8}$$

$$\begin{aligned} \overset{e}{T}{}^r_r = -\frac{1}{8\pi G} \sum_{i=0, i\neq 1}^{m} \frac{\hat{c}_i}{2} \bigg[\left(H^2 + \frac{k}{a^2}\right)^i \\ + \frac{2i}{n}\left(H^2 + \frac{k}{a^2}\right)^{i-1}\left(\dot{H} - \frac{k}{a^2}\right)\bigg], \end{aligned} \tag{5.9}$$

where

$$\hat{c}_i = \frac{n!}{(n-2i)!} c_i. \tag{5.10}$$

The work term can be decomposed as

$$W = \overset{m}{W} + \overset{e}{W}, \tag{5.11}$$

where

$$\overset{m}{W} = -\tfrac{1}{2} h^{ab} \overset{m}{T}_{ab} = \tfrac{1}{2}(\rho_m - p_m), \qquad \overset{e}{W} = -\tfrac{1}{2} h^{ab} \overset{e}{T}_{ab}. \tag{5.12}$$

The higher curvature terms produce the effective work term

$$\begin{aligned} \overset{e}{W} = \frac{1}{8\pi G} \sum_{i=0, i\neq 1}^{m} \frac{\hat{c}_i}{2} \bigg[\left(H^2 + \frac{k}{a^2}\right)^i \\ + \frac{i}{n}\left(H^2 + \frac{k}{a^2}\right)^{i-1}\left(\dot{H} - \frac{k}{a^2}\right)\bigg]. \end{aligned} \tag{5.13}$$

Similarly, the energy-supply Ψ can also be divided into

$$\Psi = \overset{m}{\Psi} + \overset{e}{\Psi}, \tag{5.14}$$

where

$$\overset{m}{\Psi}_a = \overset{m}{T}{}^b_a \partial_b \tilde{r} + \overset{m}{W} \partial_a \tilde{r}, \qquad \overset{e}{\Psi}_a = \overset{e}{T}{}^b_a \partial_b \tilde{r} + \overset{e}{W} \partial_a \tilde{r}. \tag{5.15}$$

After some calculations, we arrive at

$$\overset{m}{\Psi} = -\tfrac{1}{2}(\rho_m + p_m) H \tilde{r} dt + \tfrac{1}{2}(\rho_m + p_m) a dr, \tag{5.16}$$

$$\overset{e}{\Psi} = \frac{1}{16\pi G}\sum_{i=0,i\neq 1}^{m}\left[\hat{c}_i\frac{i}{n}\left(H^2+\frac{k}{a^2}\right)^{i-1}\left(\dot{H}-\frac{k}{a^2}\right)\right]H\tilde{r}dt - \frac{1}{16\pi G}\sum_{i=0,i\neq 1}^{m}\left[\hat{c}_i\frac{i}{n}\left(H^2+\frac{k}{a^2}\right)^{i-1}\left(\dot{H}-\frac{k}{a^2}\right)\right]adr. \tag{5.17}$$

Thus, we have put the Lovelock theory into the form of Einstein theory with an effective energy-momentum tensor, and the effective energy-momentum tensor has been used in the work and energy-supply terms. These imply that the unified first law discussed previously is applicable as well here and the energy has the form of the Misner-Sharp energy (2.8).

The Friedmann equations of Lovelock gravity read [12]

$$\sum_{i=0}^{m}\hat{c}_i\left(H^2+\frac{k}{a^2}\right)^{i} = 16\pi G\rho_m, \tag{5.18}$$

$$\sum_{i=0}^{m} i\hat{c}_i\left(H^2+\frac{k}{a^2}\right)^{i-1}\left(\dot{H}-\frac{k}{a^2}\right) = -n8\pi G(\rho_m+p_m). \tag{5.19}$$

Using the Friedmann equations and the work and energy-supply terms given above, one can obtain the differential of the Misner-Sharp energy from the unified first law. Turn around, if we use the Misner-Sharp energy and work and energy-supply terms, we can obtain the Friedmann equations in the Lovelock gravity from the unified first law. An interesting point is here that for the Lovelock gravity, we cannot put the Misner-Sharp energy inside the trapping horizon into the form $E=\rho_m V$ since the effective energy-momentum tensor will make a contribution to the energy inside the horizon [14]. Projecting the unified first law along the trapping horizon, we have

$$\langle dE,\xi\rangle = \frac{\kappa}{8\pi G}\langle dA,\xi\rangle + \langle WdV,\xi\rangle, \tag{5.20}$$

since we have written the equation of motion for the Lovelock gravity to the form of Einstein gravity. This projection implies that we have

$$\langle A\Psi,\xi\rangle = \frac{\kappa}{8\pi G}\langle dA,\xi\rangle. \tag{5.21}$$

Namely, we have

$$\langle A\overset{m}{\Psi},\xi\rangle = \frac{\kappa}{8\pi G}\langle dA,\xi\rangle - \langle A\overset{e}{\Psi},\xi\rangle. \tag{5.22}$$

Clearly, the left-hand side is just the "energy supply" projecting along ξ, which is nothing but the heat flow δQ defined by pure matter energy-momentum tensor. Thus, an interesting question is whether the right-hand side of the equation can be of the form TdS in the Clausius relation as is the case in Einstein gravity? The answer is yes: the right-hand side of the equation can indeed be written to a form with the surface gravity times the differential of the entropy in Lovelock theory projecting along ξ. Now we show this. From the definition of $\overset{e}{\Psi}$, we have

$$\frac{\kappa}{8\pi G}\langle dA,\xi\rangle - \langle A\overset{e}{\Psi},\xi\rangle = -\frac{A}{4\pi G\tilde{r}_A^2}\epsilon(1-\epsilon)(n-1)H\tilde{r}_A - \frac{A}{4\pi G}\epsilon(1-\epsilon)\sum_{i=0,i\neq 1}^{m}\left[c_i\frac{i(n-1)!}{(n-2i)!}\left(\frac{1}{\tilde{r}_A^2}\right)^{i}\right]H\tilde{r}_A. \tag{5.23}$$

Having considered $c_1=1$, we can rewrite the above equation into

$$\begin{aligned}\frac{\kappa}{8\pi G}\langle dA,\xi\rangle - \langle A\overset{e}{\Psi},\xi\rangle &= -\frac{A}{4\pi G}\epsilon(1-\epsilon)\sum_{i=0}^{m}\left[c_i\frac{i(n-1)!}{(n-2i)!}\left(\frac{1}{\tilde{r}_A^2}\right)^{i}\right]H\tilde{r}_A\\ &= -\frac{1}{2\pi\tilde{r}_A}(1-\epsilon)\Omega_{n-1}\left\langle\frac{1}{4G}\sum_{i=0}^{m}\left[c_i\frac{i(n-1)!}{(n-2i)!}\tilde{r}_A^{n-2i}\right]d\tilde{r}_A,\xi\right\rangle\\ &= \frac{\kappa}{2\pi}\Omega_{n-1}\left\langle\frac{1}{4G}\sum_{i=0}^{m}\left[c_i\frac{i(n-1)!}{(n-2i+1)!}d\tilde{r}_A^{n-2i+1}\right],\xi\right\rangle\\ &= \frac{\kappa}{2\pi}\left\langle d\left\{\frac{A}{4G}\sum_{i=0}^{m}\left[c_i\frac{i(n-1)!}{(n-2i+1)!}\tilde{r}_A^{2-2i}\right]\right\},\xi\right\rangle = T\langle dS,\xi\rangle,\end{aligned} \tag{5.24}$$

where $T=\kappa/2\pi$ and

$$S = \frac{A}{4G}\sum_{i=0}^{m}\left[c_i\frac{i(n-1)!}{(n-2i+1)!}\tilde{r}_A^{2-2i}\right]. \tag{5.25}$$

Thus we have shown that $\frac{\kappa}{8\pi G}\langle dA,\xi\rangle - \langle A\overset{e}{\Psi},\xi\rangle$ is exactly the surface gravity times a total differential projecting along the tangent direction of the trapping horizon. This total differential is nothing but the differential of the horizon entropy defined in Lovelock gravity [12,14,24].

The above discussions tell us: In Lovelock gravity, if one uses the pure matter energy momentum to define δQ, i.e., $\delta Q = \langle\overset{m}{\Psi},\xi\rangle$, then $\frac{\kappa}{8\pi G}\langle dA,\xi\rangle - \langle A\overset{e}{\Psi},\xi\rangle$ is of the form TdS. That is, *the Clausius relation $\delta Q = TdS$ still holds in Lovelock gravity.*

An interesting question arises: Does the Clausius relation always hold for any gravity theory? In the next section, using the method developed in this section, we will show that the Clausius relation does not hold in the scalar-tensor theory.

VI. THERMODYNAMICS OF APPARENT HORIZON IN SCALAR-TENSOR GRAVITY

In the Jordan frame, the Lagrangian of the scalar-tensor gravity in $(n+1)$-dimensional space-times can be written as

$$\mathcal{L} = \frac{1}{16\pi G}F(\phi)R - \frac{1}{2}g^{ab}\partial_a\phi\partial_b\phi - V(\phi) + \mathcal{L}_m, \tag{6.1}$$

where $F(\phi)$ is a positive continuous function of the scalar field ϕ and $V(\phi)$ is its potential. Varying the action, we have the equations of motion

$$FG_{ab} + g_{ab}\nabla^2 F - \nabla_a\nabla_b F = 8\pi G(\overset{\phi}{T}_{ab} + \overset{m}{T}_{ab}), \tag{6.2}$$

$$\nabla^2\phi - V'(\phi) + \frac{1}{16\pi G}F'(\phi)R = 0, \tag{6.3}$$

where $\overset{m}{T}_{ab}$ is the energy-momentum tensor of matter. We denote $\overset{\phi}{T}_{ab}$ by

$$\overset{\phi}{T}_{ab} = \partial_a\phi\partial_b\phi - g_{ab}(\tfrac{1}{2}g^{cd}\partial_c\phi\partial_d\phi + V(\phi)). \tag{6.4}$$

Note that here $\overset{\phi}{T}_{ab}$ is not the energy-momentum tensor of the scalar field. As in the case of Lovelock gravity, in order to use the unified first law in Einstein gravity, we rewrite the equations of motion into the following form:

$$G_{ab} = 8\pi G T_{ab} = 8\pi G\frac{1}{F}(\overset{\phi}{T}_{ab} + \overset{m}{T}_{ab} + \overset{e}{T}_{ab}), \tag{6.5}$$

where

$$\overset{e}{T}_{ab} = \frac{1}{8\pi G}(-g_{ab}\nabla^2 F + \nabla_a\nabla_b F). \tag{6.6}$$

In the FRW metric, it is easy to find that $\nabla^2 F$ and the nonvanishing components of $\nabla_a\nabla_b F$ are

$$\nabla^2 F = -\ddot{F} - nH\dot{F}, \qquad \nabla_t\nabla_t F = \ddot{F},$$
$$\nabla_r\nabla_r F = -\frac{a^2}{1-kr^2}H\dot{F}, \tag{6.7}$$

respectively. We then have

$$\overset{e}{T}{}^t_t = \frac{1}{8\pi G}nH\dot{F}, \qquad \overset{e}{T}{}^r_r = \frac{1}{8\pi G}(\ddot{F} + (n-1)H\dot{F}). \tag{6.8}$$

The work term can be decomposed as

$$W = \overset{\phi}{W} + \overset{m}{W} + \overset{e}{W}, \tag{6.9}$$

with

$$\overset{\phi}{W} + \overset{m}{W} = \frac{1}{2F}(\rho_\phi + \rho_m - p_\phi - p_m), \tag{6.10}$$

$$\overset{e}{W} = -\frac{1}{16\pi GF}(\ddot{F} + (2n-1)H\dot{F}), \tag{6.11}$$

where

$$\rho_\phi = \tfrac{1}{2}\dot{\phi}^2 + V(\phi), \qquad p_\phi = \tfrac{1}{2}\dot{\phi}^2 - V(\phi). \tag{6.12}$$

Similarly, the energy supply has the form

$$\Psi = \overset{\phi}{\Psi} + \overset{m}{\Psi} + \overset{e}{\Psi}, \tag{6.13}$$

with

$$\overset{\phi}{\Psi} + \overset{m}{\Psi} = -\frac{1}{2F}(\rho_\phi + \rho_m + p_\phi + p_m)H\tilde{r}dt + \frac{1}{2F}(\rho_\phi + \rho_m + p_\phi + p_m)adr, \tag{6.14}$$

$$\overset{e}{\Psi} = -\frac{1}{16\pi GF}(\ddot{F} - H\dot{F})H\tilde{r}dt + \frac{1}{16\pi GF}(\ddot{F} - H\dot{F})adr. \tag{6.15}$$

On the trapping horizon/apparent horizon, the unified first law tells us

$$\begin{aligned} dE &= A\Psi + WdV = A\Psi + AWd\tilde{r}_A \\ &= \frac{A}{F}\bigg[-(\rho_\phi + \rho_m + p_\phi + p_m)H\tilde{r}_A dt + (\rho_\phi + \rho_m)d\tilde{r}_A \\ &\quad - \frac{1}{8\pi G}(\ddot{F} - H\dot{F})H\tilde{r}_A dt - \frac{1}{8\pi G}nH\dot{F}d\tilde{r}_A\bigg]. \end{aligned} \tag{6.16}$$

By using Friedmann equations, one can find that dE is nothing but the exterior differential of the Misner-Sharp energy

$$\begin{aligned} E &= \frac{1}{16\pi G}(n-1)\Omega_{n-1}\tilde{r}_A^{n-2} \\ &= \frac{V}{F}\bigg(\rho_\phi + \rho_m - \frac{1}{8\pi G}nH\dot{F}\bigg). \end{aligned} \tag{6.17}$$

On the other hand, if substituting the Misner-Sharp energy, work and energy-supply terms defined above into the unified first law (6.16), we can obtain the Friedmann equations in the scalar-tensor gravity [13]:

$$\frac{1}{2}n(n-1)F\bigg(H^2 + \frac{k}{a^2}\bigg) + nH\dot{F} = 8\pi G(\rho_\phi + \rho_m), \tag{6.18}$$

$$\begin{aligned} &-(n-1)F\bigg(\dot{H} - \frac{k}{a^2}\bigg) - (\ddot{F} - H\dot{F}) \\ &= 8\pi G(\rho_\phi + p_\phi + \rho_m + p_m), \end{aligned} \tag{6.19}$$

where ρ_ϕ and p_ϕ have the same form as (6.12). Since we have rewritten the equations of motion of the scalar-tensor gravity into a form as in Einstein theory, the following equation should hold:

$$\langle dE, \xi\rangle = \frac{\kappa}{8\pi G}\langle dA, \xi\rangle + \langle WdV, \xi\rangle, \tag{6.20}$$

where W is given by (6.9). Further, we have

$$\langle A\Psi, \xi\rangle = \frac{\kappa}{8\pi G}\langle dA, \xi\rangle, \tag{6.21}$$

where Ψ is given by (6.13). Substituting Ψ into the above equation, we have

$$F\langle A\overset{\phi}{\Psi}, \xi\rangle + F\langle A\overset{m}{\Psi}, \xi\rangle = \frac{\kappa F}{8\pi G}\langle dA, \xi\rangle - F\langle A\overset{e}{\Psi}, \xi\rangle. \tag{6.22}$$

The left-hand side of this equation can be explicitly expressed as

$$\begin{aligned} F\langle A\overset{\phi}{\Psi}, \xi\rangle + F\langle A\overset{m}{\Psi}, \xi\rangle = \langle -\tfrac{1}{2}(\rho_\phi + \rho_m + p_\phi + p_m)H\tilde{r}dt \\ + \tfrac{1}{2}(\rho_\phi + \rho_m + p_\phi + p_m)adr, \xi\rangle. \end{aligned} \tag{6.23}$$

This is just the energy-supply term provided by matter and the scalar field. We denote it by δQ as is the case in Einstein gravity. Can we put the right-hand side of Eq. (6.22) into a form with the surface gravity times a total differential projecting along the vector ξ as the case of Lovelock gravity? The answer is no in the present case. To see this, let us note that the right-hand side of the equation can be expressed as

$$\begin{aligned} \frac{\kappa F}{8\pi G}\langle dA, \xi\rangle - F\langle A\overset{e}{\Psi}, \xi\rangle &= \frac{\kappa F}{8\pi G\tilde{r}_A}A(n-1)2\epsilon H\tilde{r}_A \\ &\quad - \frac{\kappa}{8\pi G}A\tilde{r}_A(\ddot{F} - H\dot{F})H\tilde{r}_A \\ &= T\langle dS, \xi\rangle \\ &\quad + T\frac{A}{4G}\tilde{r}_A^2\left(\frac{\dot{F}}{\tilde{r}_A^2} - H\ddot{F} + H^2\dot{F}\right), \end{aligned} \tag{6.24}$$

where

$$T = \frac{\kappa}{2\pi}, \qquad S = \frac{F(\phi)A}{4G}. \tag{6.25}$$

Here S has the form of the entropy of black holes in the scalar-tensor gravity [25]. Thus Eq. (6.22) can be reexpressed as

$$\delta Q = TdS + Td_iS, \tag{6.26}$$

where

$$d_iS = \frac{A}{4G}\tilde{r}_A^2\left(\frac{\dot{F}}{\tilde{r}_A^2} - H\ddot{F} + H^2\dot{F}\right). \tag{6.27}$$

The Eq. (6.26) implies that the Clausius relation $\delta Q = TdS$ does not hold for the scalar-tensor gravity. The term d_iS in (6.27) can be interpreted as the entropy production term in the nonequilibrium thermodynamics associated with the apparent horizon. Indeed, in Einstein gravity, Jacobson [4] used the Clausius relation $\delta Q = TdS$ and derived the Einstein field equations. However, recently Eling *et al.* [5] have found that the Clausius relation does not hold for the $f(R)$ gravity, and that in order to obtain the equations of motion for the $f(R)$ gravity, an entropy production term has to be added to the Clausius relation like (6.26). In the next section, following [4,5], we will show that indeed for the scalar-tensor gravity, an additional entropy production term is needed for deriving the equations of motion.

VII. SCALAR-TENSOR GRAVITY AND NONEQUILIBRIUM THERMODYNAMICS

In [4], Jacobson derived Einstein equations from the proportionality of entropy to the horizon area, A, together with the fundamental Clausius relation $\delta Q = TdS$, assuming that the relation holds for all local Rindler causal horizons through each space-time point. Here δQ and T are the variation of heat and Unruh temperature seen by an accelerated observer just inside the horizon. Recently, Eling *et al.* [5] have shown that the Clausius relation plus the entropy assumption $S = \alpha A f'(R)$ cannot give the correct equations of motion for the $f(R)$ gravity. In order to get correct equations of motion, one has to modify the equilibrium Clausius relation to a nonequilibrium one; an entropy production term needs to be added to the Clausius relation of equilibrium thermodynamics. Namely, the $f(R)$ gravity corresponds to a nonequilibrium thermodynamics of space-time. In this section, we will deal with the scalar-tensor gravity by using their method.

For a space-time point p in $(n+1)$ dimensions, locally, one can define a causal horizon as in [5]: Choose a spacelike $(n-1)$-surface patch B including p and then choose one side of the boundary of the past of B. Near the point p, this boundary is a congruence of the null geodesics orthogonal to B. These comprise the horizon. To define the heat flux, we can employ an approximate boost killing vector χ which is future pointing on the causal horizon and vanishes at p. χ has a relation with the tangent vector of causal horizon k: $\chi = -\lambda k$, where λ is the affine parameter of the corresponding null geodesic line. The heat is defined to be the boost energy current of matter (including the scalar field in the scalar-tensor gravity) across the horizon

$$\delta Q = \int T_{ab}\chi^a d\Sigma^b. \tag{7.1}$$

Using relation $\chi = -\lambda k$, and temperature $T = \hbar/2\pi$, we have

$$\frac{\delta Q}{T} = \frac{2\pi}{\hbar}\int T_{ab}k^a k^b(-\lambda)d\lambda d^{n-1}A, \tag{7.2}$$

where

$$T_{ab} = \overset{\phi}{T}_{ab} + \overset{m}{T}_{ab}. \tag{7.3}$$

Assuming that the entropy is of the form

$$S = \alpha F(\phi)A, \tag{7.4}$$

where α is a constant depending on the number and nature of quantum fields, we have

$$\delta S = \alpha\int(\theta F + \dot{F})d\lambda d^{n-1}A, \tag{7.5}$$

where θ is the expansion of null geodesic congruence and the overdot means the derivative with respect to the affine parameter λ (do not confuse this with the derivative with respect to t in the previous sections). To extract the $O(\lambda)$ term from the integrand, we differentiate it with respect to λ and use the requirement [5]: $(\theta F + \dot{F})(p) = 0$, at the point p, we have

$$\frac{d}{d\lambda}(\theta F + \dot{F})\bigg|_{\lambda=0} = \dot{\theta}F - F^{-1}\dot{F}^2 + \ddot{F}. \tag{7.6}$$

By using the Raychaudhuri equation for the null geodesic congruence in $(n+1)$ dimensions

$$\frac{d}{d\lambda}\theta = -\frac{1}{n-1}\theta^2 - \sigma_{ab}\sigma^{ab} - R_{ab}k^a k^b, \tag{7.7}$$

and the geodesic equation, we can rewrite Eq. (7.6) as

$$-k^a k^b(R_{ab} - \nabla_a\nabla_b F + F^{-1}\nabla_a F\nabla_b F) - \frac{1}{n-1}F\theta^2 - F\sigma_{ab}\sigma^{ab}. \tag{7.8}$$

We assume that the shear term vanishes in the whole space-time. With the Clausius relation, we have

$$FR_{ab} - \nabla_a\nabla_b F + \frac{n}{n-1}F^{-1}\nabla_a F\nabla_b F + \Phi g_{ab} = \frac{2\pi}{\alpha\hbar}(\overset{\phi}{T}_{ab} + \overset{m}{T}_{ab}), \tag{7.9}$$

where Φ is an undetermined function and $\overset{\phi}{T}_{ab} = \partial_a\phi\partial_b\phi - g_{ab}(\frac{1}{2}g^{cd}\partial_c\phi\partial_d\phi + V(\phi))$. We assume that the matter stress tensor is divergence free, i.e., $\nabla^a\overset{m}{T}_{ab} = 0$, but due to the coupling function $F(\phi)$ between ϕ and scalar curvature R, the $\overset{\phi}{T}_{ab}$ is not conserved, namely, $\nabla^a\overset{\phi}{T}_{ab} \neq 0$. Instead the divergence of $\overset{\phi}{T}_{ab}$ is

$$\nabla^a\overset{\phi}{T}_{ab} = \nabla^2\phi\nabla_b\phi - V'(\phi)\nabla_b\phi. \tag{7.10}$$

Considering the identity

$$\nabla^a(FR_{ab} - \nabla_a\nabla_b F) = \tfrac{1}{2}\nabla_b(FR) - \nabla_b\nabla^2 F - \tfrac{1}{2}RF'\nabla_b\phi \tag{7.11}$$

and taking divergence for both sides of the Eq. (7.9), we find

$$\nabla_b\Phi = \nabla_b\Big(\nabla^2 F - \frac{1}{2}FR\Big) - \nabla^a\Big(\frac{n}{n-1}F^{-1}\nabla_a F\nabla_b F\Big) + \Big(\frac{1}{2}F'R + \frac{2\pi}{\alpha\hbar}(\nabla^2\phi - V')\Big)\nabla_b\phi. \tag{7.12}$$

The last term in the right-hand side of the above equation is nothing but the equation of motion for the scalar field. Therefore, this term always vanishes. While the left-hand side is a pure gradient of a scalar, the second term in the right-hand side of the equation $\nabla^a(\frac{n}{n-1}F^{-1}\nabla_a F\nabla_b F)$ cannot always be written as the gradient of a scalar. As a result, as the case of $f(R)$ gravity [5], there is a contradiction here. To resolve this, we add an entropy production term $d_i S$ to the Clausius relation. It is easy to find if we choose

$$d_i S = \int\sigma d\lambda d^{n-1}A \tag{7.13}$$

with entropy production density

$$\sigma = -\frac{n}{n-1}\alpha F^{-1}\dot{F}^2\lambda \tag{7.14}$$

and use the equation of motion for the scalar field

$$\tfrac{1}{2}F'R + \frac{2\pi}{\alpha\hbar}(\nabla^2\phi - V') = 0. \tag{7.15}$$

The field equations (7.9) for the scalar-tensor theory become

$$FG_{ab} - \nabla_a\nabla_b F + g_{ab}\nabla^2 F = \frac{2\pi}{\alpha\hbar}(\overset{\phi}{T}_{ab} + \overset{m}{T}_{ab}), \tag{7.16}$$

where G_{ab} is the Einstein tensor. Taking $\alpha = \frac{1}{4G\hbar}$, Eqs. (7.15) and (7.16) are just the equations of motion for g_{ab} and ϕ in the scalar-tensor gravity theory, whose Lagrangian is given by (6.1).

Thus we conclude that in order to get the equations of motion in the scalar-tensor gravity, the Clausius relation has to be modified; we have to add an entropy production term $d_i S$ (7.13) to the Clausius relation. This suggests that as in the case of $f(R)$ gravity, the scalar-tensor gravity corresponds to nonequilibrium thermodynamics of space-time. We note that this entropy production term (7.13) does not coincide with the entropy production term (6.27) in the setup of the FRW universe although they look similar to each other. This is not surprising because the analysis of Eling *et al.* is locally at each space-time point, while our previous analysis is focused on the trapping horizon/apparent horizon of the FRW universe. The common point between them is that both of them imply that the scalar-tensor gravity requires a nonequilibrium thermodynamic treatment as the $f(R)$ gravity.

VIII. CONCLUSIONS

In this paper we have revisited the relation between the Friedmann equations and the thermodynamics on the trapping horizon/apparent horizon in the FRW universe. We have generalized the unified first law to the case of $(n+1)$-dimensional Einstein theory. After projecting the unified first law along inner trapping horizon/apparent horizon, we have obtained the first law of thermodynamics of the FRW universe, which is very similar to the thermodynamics of dynamic black holes on their outer trapping horizon. The form of the first law of thermodynamics is rigorous without any approximation. For non-Einstein gravity theories, we have rewritten the field equations to a form of Einstein gravity by introducing an effective energy-momentum tensor and treated them as Einstein gravity theory. In these theories the first law of thermodynamics for the apparent horizon in the FRW universe has the same form

$$\langle dE, \xi\rangle = \frac{\kappa}{8\pi G}\langle dA, \xi\rangle + \langle WdV, \xi\rangle,$$

where E is the Misner-Sharp energy, as the case of Einstein gravity theory. But here W is an effective work term, the Misner-Sharp energy is fixed to be the form of Eq. (2.8) because it is defined through space-time geometry. In the Lovelock gravity, if we define the heat δQ by pure matter energy supply projecting along the horizon, we find that the Clausius relation

$$\delta Q = TdS$$

still holds, where $T = \kappa/2\pi$ and S is of the exact form of the entropy of black hole horizon in the Lovelock gravity. However, the same treatment tells us that the Clausius relation cannot be fulfilled for the scalar-tensor gravity. We have to introduce an entropy production term to the Clausius relation

$$\delta Q = TdS + Td_iS.$$

This implies that the thermodynamics of apparent horizon is nonequilibrium thermodynamics for the scalar-tensor theory.

Following Eling *et al.* [5], we have treated the scalar-tensor gravity and shown that the Clausius relation plus the entropy form $S = \alpha F(\phi)A$ cannot give the correct equations of motion for the theory. In order to resolve this issue, we have to modify the Clausius relation by introducing an entropy production term. This also indicates that the scalar-tensor gravity is the nonequilibrium thermodynamics of space-time as the case of $f(R)$ gravity.

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