$f(R)$ gravity theories in Palatini formalism: Cosmological dynamics and observational constraints

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We make a systematic study of the cosmological dynamics for a number of $f(R)$ gravity theories in Palatini formalism, using phase space analysis as well as numerical simulations. Considering homogeneous and isotropic models, we find a number of interesting results: (i) models based on theories of the type (a) $f(R) = R - \beta/R^n$ and (b) $f(R) = R + \alpha \ln R - \beta$, unlike the metric formalism, are capable of producing the sequence of radiation-dominated, matter-dominated, and de Sitter periods, and (ii) models based on theories of the type (c) $f(R) = R + \alpha R^m - \beta/R^n$ can produce early as well as late accelerating phases but an early inflationary epoch does not seem to be compatible with the presence of a subsequent radiation-dominated era. Thus, for the classes of models considered here, we have been unable to find the sequence of all four dynamical epochs required to account for the complete cosmological dynamics, even though three out of four phases are possible. We also place observational constraints on these models using the recently released supernovae data by the Supernova Legacy Survey as well as the baryon acoustic oscillation peak in the Sloan Digital Sky Survey luminous red galaxy sample and the cosmic microwave background shift parameter. The best-fit values are found to be $n = 0.027$, $\beta = 4.63$ for the models based on (a) and $\alpha = 0.11$, $\beta = 4.62$ for the models based on (b), neither of which are significantly preferred over the ΛCDM model. Moreover, the logarithmic term alone is not capable of explaining the late acceleration. The models based on (c) are also consistent with the data with suitable choices of their parameters. We also find that some of the models for which the radiation-dominated epoch is absent prior to the matter-dominated era also fit the data. The reason for this apparent contradiction is that the combination of the data considered here does not place stringent enough constraints on the cosmological evolution prior to the decoupling epoch, which highlights the importance of our combined theoretical-observational approach to constrain models.

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I. INTRODUCTION

Recent high-precision observations by the Wilkinson microwave anisotropy probe (WMAP), together with other cosmic microwave background (CMB) and high redshift surveys have produced a wealth of information regarding the early Universe. The analysis of the resulting data has provided strong evidence for the core predictions of the inflationary cosmology, including the almost spatial flatness of the Universe [1,2]. Furthermore, these observations coupled with the low redshift supernovae surveys [3–7] and observations of large scale structure [8,9] and baryon acoustic oscillations [10] suggest that the Universe is at present undergoing a phase of accelerated expansion (see Refs. [11,12] for reviews). Thus a "standard" model of cosmology has emerged which is characterized by four distinct phases: accelerated expansions at both early and late times, mediated by radiation-dominated and matter-dominated eras. The central question in cosmology at present is, therefore, how to successfully account for these distinct dynamical phases in the history of the Universe, and, in particular, whether they can all be realized simultaneously within a single theoretical framework, motivated by a fundamental theory of quantum gravity.

A number of scenarios have been proposed to account for these dynamical modes of behavior. These fall into two categories: (i) those involving the introduction of exotic matter sources, and (ii) those introducing changes to the gravitational sector of general relativity. Among the latter are $f(R)$ gravity theories, which involve nonlinear generalizations to the (linear) Hilbert action. Nonlinear modifications are expected to be present in the effective action of the gravitational field when string/M-theory corrections are considered [13–16].

A great deal of effort has recently gone into the study of such theories. An important reason for this interest has been the demonstration that generalized Lagrangians of this type—which include negative as well as positive powers of the curvature scalar—can lead to accelerating phases both at early [17] and late [18,19] times in the history of the Universe (see also Ref. [20]).

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In deriving the Einstein field equations from the Hilbert action, the variations are taken with respect to the metric coefficients, while the connections are assumed to be the Christoffel symbols defined in terms of the metric. An alternative procedure—the so-called Palatini approach, originally considered by Einstein himself—is to treat both the metric and the (affine) connections as independent variables and perform the variations with respect to both. In the case of linear Hilbert action both approaches produce identical results, as long as the energy-momentum tensor does not depend on the connection [21]. This, however, is not the case once the gravitational Lagrangian is allowed to be nonlinear. In that case, the two methods of variation produce different field equations with nontrivial differences in the resulting dynamics.

Performing variations of nonlinear actions using the metric approach results in field equations that are fourth order; which makes them difficult to deal with in practice. Furthermore, within this framework, models based on theories of the type $f(R) = R - \beta/R^n$ have difficulties passing the solar system tests [22] and having the correct Newtonian limit [23]. In addition, such theories suffer gravitational instabilities as discussed in Ref. [24]. Also recent studies have found that these theories are not able to produce a standard matter-dominated era followed by an accelerated expansion [25,26]. Finally, models based on theories of the type $f(R) = R + \alpha R^m - \beta/R^n$ have been shown to have difficulties in satisfying the set of constraints coming from early and late-time acceleration, big bang nucleosynthesis, and fifth-force experiments [27]. (See Refs. [28] for other works concerning the metric formalism.)

Variations using the Palatini approach [29,30], however, result in second order field equations which are different from the ones derived in the metric approach. While it is still an open problem to clarify whether the $f(R)$ theories in Palatini formalism are free from gravitational instabilities [31–34] and whether they satisfy the solar system tests and have the correct Newtonian limit [30,35–41], the Palatini formalism can allow a possibility to realize a successful cosmological evolution of radiation, matter, and accelerated epochs. Here we shall concentrate on the Palatini approach and consider a number of families of $f(R)$ theories recently put forward in the literature. These theories have been the focus of a great deal of interest recently with a number of studies attempting to determine their viability as cosmological theories, both theoretically [23,42–45] and observationally [46]. Despite these efforts, it is still fair to say that cosmological dynamics of models based on such theories is not fully understood. Of particular interest is the number of dynamical phases such models are capable of admitting, among the four phases required for strict cosmological viability, namely, early and late accelerating phases mediated by radiation-dominated and matter-dominated epochs. Especially it is important to determine whether they are capable of allowing all four phases required for cosmological evolution. We should note that the idea that such theories should be capable of successfully accounting for all four phases is clearly a maximal demand. It would still be of great interest if such theories could successfully account for a sequence of such phases, such as the first or the last three phases. For example, the presence of the last 3 phases could distinguish these from the corresponding theories based on the metric formalism [25]. Also observational constraints have so far only been obtained for the models based on the theories of the type $f(R) = R - \beta/R^n$. Such constraints need to be also studied for other theories considered in this context in the literature.

Here in order to determine the cosmological viability of models based on $f(R)$ theories in Palatini formalism, we shall employ a two pronged approach. Considering homogeneous and isotropic settings, we shall first provide autonomous equations applicable to any $f(R)$ gravity theory. We shall then make a systematic and detailed phase space analysis of a number of families of $f(R)$ theories. This provides a clear understanding of the dynamical modes of behavior which are admitted by these theories, and, in particular, whether these theories possess early and late accelerating phases which are mediated by the radiation-dominated and matter-dominated eras, respectively. The existence of these phases is a necessary but not sufficient condition for cosmological viability of such theories. To be cosmologically viable, it is also necessary that the subset of parameters in these theories that allows these phases to exist are also compatible with observations [47]. We constrain these parameters for a number of families of $f(R)$ theories using the data from recent observations, including recently released supernovae data by the Supernova Legacy Survey [7] as well as the baryon acoustic oscillation peak in the Sloan Digital Sky Survey (SDSS) luminous red galaxy sample [10] and the CMB shift parameter [1,2].

The plan of the paper is as follows. In Sec. II we give a brief account of the Palatini formalism and provide the basic equations for general $f(R)$ theories. We also introduce new variables which allow these equations to be written as autonomous dynamical systems. In Sec. III we proceed to study the cosmological dynamics for a number of classes of $f(R)$ theories. Particular emphasis will be placed on finding cases which can allow the largest number of dynamical phases required in the cosmological evolution. In Sec. IV we obtain observational constraints on these theories. This allows further constraints to be placed on the parameters of these theories. Finally we conclude in Sec. V.

II. $f(R)$ THEORIES IN PALATINI FORMALISM AND AUTONOMOUS EQUATIONS

We consider the classes of theories given by the generalized action

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{2\kappa}f(R) + \mathcal{L}_m + \mathcal{L}_r\right], \tag{1}$$

where f is a differentiable function of the Ricci scalar R, $\mathcal{L}_m$ and $\mathcal{L}_r$ are the Lagrangians of the pressureless dust and radiation, respectively, $\kappa = 8\pi G$, and G is the gravitational constant. Motivated by recent observations, we shall study the cosmological dynamics in these theories for a flat Friedmann-Lemaître-Robertson-Walker (FLRW) background,

$$ds^2 = -dt^2 + a^2(t)d\mathbf{x}^2, \tag{2}$$

where $a(t)$ is the scale factor and t is the cosmic time.

As was mentioned above, in Palatini formalism the metric and the affine connections are treated as independent variables with respect to which the action is varied. A generalized Ricci scalar is defined by $R = g^{\mu\nu}R_{\mu\nu}(\hat{\Gamma})$, where a generalized Ricci tensor $R_{\mu\nu}(\hat{\Gamma})$ is written in terms of the connection:

$$R_{\mu\nu}(\hat{\Gamma}) = \hat{\Gamma}^{\alpha}_{\mu\nu,\alpha} - \hat{\Gamma}^{\alpha}_{\mu\alpha,\nu} + \hat{\Gamma}^{\alpha}_{\alpha\lambda}\hat{\Gamma}^{\lambda}_{\mu\nu} - \hat{\Gamma}^{\alpha}_{\mu\lambda}\hat{\Gamma}^{\lambda}_{\alpha\nu}. \tag{3}$$

Varying the action (1) with respect to the metric $g_{\mu\nu}$ gives (see e.g. Ref. [23])

$$FR_{\mu\nu} - \tfrac{1}{2}fg_{\mu\nu} = \kappa T_{\mu\nu}, \tag{4}$$

where $F \equiv \partial f/\partial R$ and $T_{\mu\nu}$ is given by

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta[\mathcal{L}_m + \mathcal{L}_r]}{\delta g^{\mu\nu}}. \tag{5}$$

By expressing the generalized Ricci tensor $R_{\mu\nu}(\hat{\Gamma})$ in terms of the Ricci tensor $R_{\mu\nu}(g)$ associated with the metric $g_{\mu\nu}$ and covariant derivatives ∇_μ of the function f associated with the Levi-Civita connection of the metric, we obtain the generalized Friedmann equation [29]

$$6F\left(H + \frac{\dot{F}}{2F}\right)^2 - f = \kappa(\rho_m + 2\rho_r), \tag{6}$$

where a dot denotes differentiation with respect to t and $H \equiv \dot{a}/a$ is the Hubble parameter. Here we take into account the contribution of pressureless dust and radiation whose energy densities are given by ρ_m and ρ_r, respectively. These satisfy the conservation equation [48]

$$\dot{\rho}_m + 3H\rho_m = 0, \tag{7}$$

$$\dot{\rho}_r + 4H\rho_r = 0. \tag{8}$$

Contracting Eq. (4), and recalling that the trace of the radiative fluid vanishes, gives

$$FR - 2f = -\kappa\rho_m. \tag{9}$$

Using Eqs. (6)–(9) we obtain

$$\dot{R} = \frac{3\kappa H\rho_m}{F'R - F} = -3H\frac{FR - 2f}{F'R - F}, \tag{10}$$

where a prime denotes a derivative with respect to R. Combining Eqs. (6) and (10) we find

$$H^2 = \frac{2\kappa(\rho_m + \rho_r) + FR - f}{6F\xi}, \tag{11}$$

where

$$\xi \equiv \left[1 - \frac{3}{2}\frac{F'(FR - 2f)}{F(F'R - F)}\right]^2. \tag{12}$$

In the case of the Hilbert action with $f = R$, Eq. (11) reduces to the standard Friedmann equation: $H^2 = \kappa(\rho_m + \rho_r)/3$.

To obtain a clear picture of possible dynamical regimes admitted by these theories, we shall make a detailed study of a number of families of $f(R)$ theories of the type (1), using phase space analysis. It is convenient to express these systems as autonomous systems by introducing the following dimensionless variables:

$$y_1 \equiv \frac{FR - f}{6F\xi H^2}, \qquad y_2 \equiv \frac{\kappa\rho_r}{3F\xi H^2}. \tag{13}$$

In terms of these variables the constraint equation (11) becomes

$$\frac{\kappa\rho_m}{3F\xi H^2} = 1 - y_1 - y_2. \tag{14}$$

Differentiating Eq. (11) and using Eqs. (6)–(11), we obtain

$$2\frac{\dot{H}}{H^2} = -3 + 3y_1 - y_2 - \frac{\dot{F}}{HF} - \frac{\dot{\xi}}{H\xi} + \frac{\dot{F}R}{6F\xi H^3}. \tag{15}$$

Now using variables (13) together with Eqs. (14) and (15), we can derive the corresponding evolution equations for the variables y_1 and y_2 thus:

$$\frac{dy_1}{dN} = y_1[3 - 3y_1 + y_2 + C(R)(1 - y_1)], \tag{16}$$

$$\frac{dy_2}{dN} = y_2[-1 - 3y_1 + y_2 - C(R)y_1], \tag{17}$$

where $N \equiv \ln a$ and

$$C(R) \equiv \frac{R\dot{F}}{H(FR - f)} = -3\frac{(FR - 2f)F'R}{(FR - f)(F'R - F)}. \tag{18}$$

We note that the following constraint equation also holds:

$$\frac{FR - 2f}{FR - f} = -\frac{1 - y_1 - y_2}{2y_1}, \tag{19}$$

which shows that R and thus $C(R)$ can in principle be expressed in terms of variables y_1 and y_2.

The behaviors of the variables y_1 and y_2 depend on the behavior of the function $C(R)$. In particular, the divergence of $C(R)$ can prevent y_1 and y_2 from reaching equilibrium, as we shall see in the next section. To proceed here we shall assume that $C(R)$ is well behaved. In that case an important step in understanding the dynamics of such systems is to look at their equilibrium points/invariant sets and their stabilities. The fixed points (y_1, y_2) satisfy $dy_1/dN = 0 = dy_2/dN$. In this case [even when $C(R)$ depends on R, but excluding the cases $C(R) = -3, -4$], we obtain the following fixed points:

(i) P_r: $(y_1, y_2) = (0, 1)$, [We shall discuss this case further below.]
(ii) P_m: $(y_1, y_2) = (0, 0)$,
(iii) P_d: $(y_1, y_2) = (1, 0)$.

If $C(R) = -3$, as is the case with the model $f(R) \propto R^n$ $(n \neq 1, 2)$, we obtain (a) $(y_1, y_2) = (0, 1)$ and (b) $(y_1, y_2) = (y_1^{(c)}, 0)$, where $y_1^{(c)}$ is a constant. When $C(R) = -4$, the fixed points are given by (a) $(y_1, y_2) = (0, 0)$, (b) $(y_1, y_2) = (1, 0)$, and (c) $(y_1, y_2) = (y_1^{(c)}, 1 - y_1^{(c)})$. Note that in both of these cases the latter points in fact correspond to a line of (rather than isolated) fixed points.

The stability of the fixed points in the 2-dimensional phase space (y_1, y_2) can then be studied by linearizing the equations and obtaining the eigenvalues of the corresponding Jacobian matrices calculated at each equilibrium point [49]. Assuming that dC/dy_1 and dC/dy_2 remain bounded, we find the following eigenvalues for the above fixed points:

(i) P_r: $(\lambda_1, \lambda_2) = (4 + C(R), 1)$,
(ii) P_m: $(\lambda_1, \lambda_2) = (3 + C(R), -1)$,
(iii) P_d: $(\lambda_1, \lambda_2) = (-3 - C(R), -4 - C(R))$.

In the following we shall also find it useful to define an effective equation of state (EOS), $w_{\rm eff}$, which is related to the Hubble parameter via $\dot{H}/H^2 = -(3/2)(1 + w_{\rm eff})$. Using Eq. (15) we find

$$w_{\rm eff} = -y_1 + \frac{1}{3}y_2 + \frac{\dot{F}}{3HF} + \frac{\dot{\xi}}{3H\xi} - \frac{\dot{F}R}{18F\xi H^3}. \qquad (20)$$

Once the fixed points of the system are obtained, one can evaluate the corresponding effective equation of state by using this relation.

III. COSMOLOGICAL DYNAMICS FOR MODELS BASED ON $f(R)$ THEORIES

In this section we shall use the above formalism to study a number of families of $f(R)$ theories, recently considered in the literature.

A. $f(R) = R - \Lambda$

The simplest model not ruled out by observations is the ΛCDM model. To start with, therefore, it is instructive to consider this model in this context, as it represents the asymptotic state in a number of cases considered below. The Lagrangian in this case is given by

$$f(R) = R - \Lambda, \qquad (21)$$

which gives $C(R) = 0$. Equations (16) and (17) then reduce to

$$\frac{dy_1}{dN} = y_1(3 - 3y_1 + y_2), \qquad (22)$$

$$\frac{dy_2}{dN} = y_2(-1 - 3y_1 + y_2), \qquad (23)$$

with fixed points given by

$$P_r\colon (y_1, y_2) = (0, 1), \qquad P_m\colon (y_1, y_2) = (0, 0), \qquad P_d\colon (y_1, y_2) = (1, 0). \qquad (24)$$

The corresponding eigenvalues can be evaluated to be

$$P_r\colon (\lambda_1, \lambda_2) = (1, 4), \qquad P_m\colon (\lambda_1, \lambda_2) = (3, -1), \qquad P_d\colon (\lambda_1, \lambda_2) = (-3, -4). \qquad (25)$$

Thus, the fixed points P_r, P_m, and P_d correspond to an unstable node, a saddle point, and a stable node, respectively. From Eq. (20) the effective equation of state is in this case given by $w_{\rm eff} = -y_1 + y_2/3$. This then leads to $w_{\rm eff} = 1/3$, 0, and -1 for the three points given in Eq. (24), indicating radiation-dominated, matter-dominated, and de Sitter phases, respectively. We have confirmed that this sequence of behaviors does indeed occur by directly solving the autonomous equations (22) and (23).

The ΛCDM model corresponds to $f(R) = R - \Lambda$, in which case $C(R)$ vanishes because $F' = 0$. Equation (18) shows that $C(R)$ also vanishes when

$$FR - 2f = 0, \qquad (26)$$

provided that other terms in the expression of $C(R)$ do not exhibit divergent behavior. When $C(R) \to 0$, the system possesses the fixed points P_r, P_m, and P_d. In principle, the nature of the point P_d depends on the nature of the theory under study, i.e. the system (16) and (17). If the last terms in these equations vanish sufficiently fast as $C(R) \to 0$, then P_d corresponds to a de Sitter solution which is a stable node since its eigenvalues are given by $(\lambda_1, \lambda_2) = (-3, -4)$. If, on the other hand, these terms fall slower than $1/N$, then they can contribute to the evolution of y_1 and y_2 and the point P_d may be different from a de Sitter point. We note, however, that for all the theories considered in this paper P_d corresponds to a de Sitter point.

B. $f(R)$ theories with the sum of power-law terms

The families of models we shall consider in this section belong to the classes of theories given by

$$f(R) = R + \alpha R^m - \beta/R^n, \qquad (27)$$

where m and n are real constants with the same sign and α and β have dimensions $[\text{mass}]^{2(1-m)}$ and $[\text{mass}]^{2(n+1)}$, respectively. Such theories have been considered with the hope of explaining both the early and the late accelerating phases in the Universe [23,32,35,36]. In this subsection, we shall make a detailed study of models based on such theories in order to determine whether they admit viable cosmological models, with both early and late time acceleration phases. In this case Eq. (19) becomes

$$\frac{1-(m-2)\alpha R^{m-1}-(n+2)\beta R^{-n-1}}{(m-1)\alpha R^{m-1}+(n+1)\beta R^{-n-1}}=\frac{1-y_1-y_2}{2y_1}, \tag{28}$$

which in principle allows R to be expressed in terms of y_1 and y_2, at least implicitly. The function $C(R)$ is given by

$$C(R)=-3\frac{[1-(m-2)\alpha R^{m-1}-(n+2)\beta R^{-n-1}][m(m-1)\alpha R^m-n(n+1)\beta R^{-n}]}{[1-m(m-2)\alpha R^{m-1}+n(n+2)\beta R^{-n-1}][(m-1)\alpha R^m+(n+1)\beta R^{-n}]}. \tag{29}$$

Before proceeding, some comments are in order concerning the variables y_1 and y_2. From the expression for Hubble function (11), one can observe that if $FR-f>0$ then one requires $6F\xi>0$, as otherwise H^2 would be negative. Assuming (see below) that at late times Eq. (27) tends to $R-\beta/R^n$, then $FR-f\rightarrow(1+n)\beta R^{-n}$ which is thus positive since observations (see Sec. IV A) require $n>-1$ and $\beta>0$. Hence, at late times y_1 and y_2 are positive and, from Eq. (14), their sum must be smaller than 1, as otherwise ρ_m would be negative. Similarly assuming (see below) that at early times Eq. (27) tends to $R+\alpha R^m$, then we have $FR-f\rightarrow(m-1)\alpha R^m$. To obtain an early time inflation, we need $m>1$ (as we shall see below). Moreover, for the action to remain positive at early times, we require $\alpha>0$ which implies $f>0$. Therefore $FR-f$ is again positive. Thus, y_1 and y_2 are positive also at early times, with a sum which is smaller than 1 for physical reasons. This indicates that, at both early and late times, variables y_1 and y_2 can be treated as normalized.

1. Limiting behaviors

To study the full behavior of the theories of the type (27) it is useful to first consider the limiting cases where one of the nonlinear terms in the action will dominate. We shall consider these cases separately.

(I) The $\alpha=0$ case.

In this case (27) reduces to

$$f(R)=R-\beta/R^n, \tag{30}$$

and $C(R)$ becomes

$$C(R)=3n\frac{R^{1+n}-(2+n)\beta}{R^{1+n}+n(2+n)\beta}, \tag{31}$$

which allows R to be expressed in terms of y_1 and y_2 thus:

$$R^{1+n}=\frac{\beta[3y_1+n(y_1-y_2+1)-y_2+1]}{2y_1}. \tag{32}$$

The critical points in this case are given by

(i) P_r: $(y_1,y_2)=(0,1)$.
Since the numerator and the denominator of Eq. (32) tend to zero as one approaches P_r, care must be taken in this case. We therefore split the analysis into three parts:

1. P_{r1}: This point corresponds to $\beta/R^{1+n}\ll 1$ and $C\rightarrow 3n$. The eigenvalues in this case are given by $(\lambda_1,\lambda_2)=(3n+4,1)$ with $w_{\rm eff}=1/3$.
2. P_{r2}: This point corresponds to $\beta/R^{1+n}\gg 1$ and $C\rightarrow -3$. The eigenvalues in this case are given by $(\lambda_1,\lambda_2)=(1,1)$ with $w_{\rm eff}=-2/3-1/n$.
3. P_{r3}: $R^{1+n}\rightarrow$ constant. This case, however, does not occur since then $y_1\gg y_2-1$ (the reasoning would still remain the same if y_1 is of the same order of magnitude as $1-y_2$) which would imply that R^{n+1} should tend to $\beta(3+n)/2$. But from Eq. (14), one can deduce that $\kappa\rho_m\simeq(-FR+f)/2$ and from Eq. (9) that $\kappa\rho_m=-FR+2f$. Hence, this would imply $(-FR+f)/2\simeq -FR+2f$ which is not possible.

(ii) P_m: $(y_1,y_2)=(0,0)$.
In this case $C\rightarrow 3n$, $(\lambda_1,\lambda_2)=(3(n+1),-1)$ and $w_{\rm eff}=0$.

(iii) P_d: $(y_1,y_2)=(1,0)$.
In this case R is a nonzero constant satisfying $R^{1+n}=(2+n)\beta$ with $C=0$, $(\lambda_1,\lambda_2)=(-3,-4)$, and $w_{\rm eff}=-1$. This de Sitter point exists for $n>-2$ provided that $R>0$ and $\beta>0$.

(iv) P: $(y_1,y_2)=(-\frac{n+1}{n+3},0)$.
In this case $C=3n$, $(\lambda_1,\lambda_2)=(1+1/n,-1)$, and $w_{\rm eff}=-1-1/n$.
This fixed point is, however, not relevant in the asymptotic regimes that we shall consider. This is because with observationally motivated parameters (see below) $\frac{n+1}{n+3}>0$ which thus implies $y_1<0$. However, we have shown that for late and early times $0<y_1<1$.

A summary of fixed points in this case together with their stability properties is given in Table I. An inspection of this table shows that the sequence of radiation (P_{r1}), matter (P_m), and de Sitter acceleration (P_d) can be realized for $n>-1$.

TABLE I. Fixed points and their natures for models based on theories of the type $f(R) = R - \beta/R^n$. The existence of the P_d point here assumes that $\beta > 0$. For negative β the symbol "$\cdots$" in the Table should be replaced by "Stable" and vice versa.

n	P_{r1}	P_{r2}	P_m	P_d	P
$n < -2$	Saddle	Unstable	Stable	$\cdots$	Saddle
$-2 < n < -4/3$	Saddle	Unstable	Stable	Stable	Saddle
$-4/3 < n < -1$	Unstable	Unstable	Stable	Stable	Saddle
$-1 < n < 0$	Unstable	Unstable	Saddle	Stable	Stable
$n > 0$	Unstable	Unstable	Saddle	Stable	Saddle

(II) The $\beta = 0$ case.
In this case (27) reduces to

$$f(R) = R + \alpha R^m. \tag{33}$$

The fixed points in this model can be obtained from the previous case by simply taking $n \to -m$ and $\beta \to -\alpha$. Hence, depending on the energy region the fixed points can be summarized as follows.

(1) High energy points ($\alpha R^{m-1} \gg 1$)
 (1a) P_{r2}: $(y_1, y_2) = (0, 1)$, $w_{\rm eff} = -2/3 + 1/m$ and $(\lambda_1, \lambda_2) = (1, 1)$.
 (1b) P: $(y_1, y_2) = (-\frac{1-m}{3-m}, 0)$, $w_{\rm eff} = -1 + 1/m$ and $(\lambda_1, \lambda_2) = (1 - 1/m, -1)$.

(2) Low energy points ($\alpha R^{m-1} \ll 1$)
 (2a) P_{r1}: $(y_1, y_2) = (0, 1)$, $w_{\rm eff} = 1/3$ and $(\lambda_1, \lambda_2) = (4 - 3m, 1)$.
 (2b) P_m: $(y_1, y_2) = (0, 0)$, $w_{\rm eff} = 0$ and $(\lambda_1, \lambda_2) = (3(1 - m), -1)$.

(3) de Sitter point [$\alpha R^{m-1} = 1/(m - 2)$]
 (3) P_d: $(y_1, y_2) = (1, 0)$, $w_{\rm eff} = -1$ and $(\lambda_1, \lambda_2) = (-3, -4)$.

Note that the case $m = 2$ requires a separate analysis since this corresponds to $C = -6$ independent of R [see Eq. (31)]. We shall return to this case below. The fixed points and their properties for this case are summarized in Table II.

One can in principle consider the fixed points in Tables I and II as, respectively, being relevant at the future and the past, respectively. Using the above asymptotic information,

TABLE II. Fixed points and their natures for models based on theories of the type $f(R) = R + \alpha R^m$. The existence of the P_d point here assumes that $\alpha > 0$. For negative α the symbol "$\cdots$" in the Table should be replaced by Stable and vice versa.

m	P_{r1}	P_{r2}	P_m	P_d	P
$2 < m$	Saddle	Unstable	Stable	Stable	Saddle
$4/3 < m < 2$	Saddle	Unstable	Stable	$\cdots$	Saddle
$1 < m < 4/3$	Unstable	Unstable	Stable	$\cdots$	Saddle
$0 < m < 1$	Unstable	Unstable	Saddle	$\cdots$	Stable
$m < 0$	Unstable	Unstable	Saddle	$\cdots$	Saddle

we shall now consider families of theories of this type separately.

2. Theories of type $f(R) = R - \beta/R^n$

In the metric approach, models based on theories of this type were considered in Refs. [18,19] as ways of producing a late-time acceleration of the Universe. In addition to the difficulties such theories face (as discussed in the Introduction), they have recently been shown to be unable to produce a standard matter-dominated era followed by an accelerated expanding phase for $n > 0$ [25]. Using the Palatini approach, however, such theories can be shown to be able to give rise to such a standard matter-dominated phase, as we shall see below.

To proceed we shall, for the sake of compatibility with the observational constraints obtained in the following sections, confine ourselves to the cases with $n > -1$ and $\beta > 0$. In such cases, Table I shows that P_{r1} is an unstable node (source), P_m is a saddle, and P_d is a stable node. Numerical simulations confirm that such theories indeed admit the 3 postinflationary phases, namely, radiation-dominated, matter-dominated, and de Sitter phases (see Fig. 1). The point P can never be reached because, for the considered range of n, one can show that the positive variable y_1 is an increasing function of time, i.e. $dy_1/dN > 0$ [50]. The point P, however, occurs for $y_1 = 0$ and, hence,

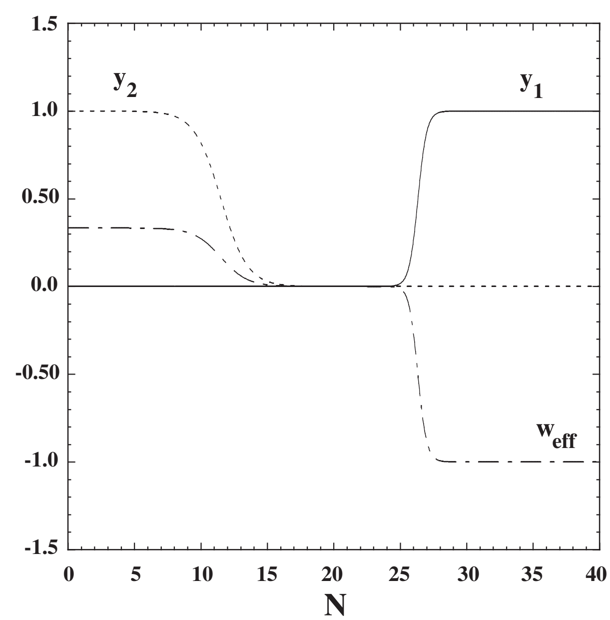

FIG. 1. Evolutions of the variables y_1 and y_2 for the model $f(R) = R - \beta/R^n$ with $n = 0.02$, together with the effective equation of state $w_{\rm eff}$. Initial conditions were chosen to be $y_1 = 10^{-40}$ and $y_2 = 1$–10^{-5}. This demonstrates that the sequence of (P_{r1}) radiation-dominated ($w_{\rm eff} = 1/3$), (P_m) matter-dominated ($w_{\rm eff} = 0$), and (P_d) de Sitter acceleration ($w_{\rm eff} = -1$) eras occur for these models.

can never be reached except by taking $y_1 < 0$ which is unphysical since it implies a negative H^2.

The initial value of the ratio $r = y_1(0)/y_2(0)$ plays an important role in determining the duration of the matter-dominated phase. This ratio is related to cosmological parameters such as the present value of Ω_m that will be constrained by observations in the next section. For the existence of a prolonged matter-dominated epoch, we require the condition $r \ll 1$. In this case the smaller the value of r the longer will be the matter-dominated phase. For large enough values of r, the solutions directly approach the stable de Sitter point P_d after the end of the radiation-dominated era.

3. Theories of type $f(R) = R + \alpha R^m - \beta/R^n$

In this section, we shall employ the above asymptotic analysis to study the dynamics of the generalized theories of this type. In particular, we shall give a detailed argument which indicates that, under some general assumptions, an initial inflationary phase cannot be followed by a radiation phase.

We shall proceed by looking at the early and late dynamics separately. In this connection it is important to decide which phases can be considered to be produced by the early and late time approximations $f(R) = R + \alpha R^m$ and $f(R) = R - \beta/R^n$, respectively. We note that in the discussion below we shall at times divide the "early" phase into high-energy ($\alpha R^{m-1} \gg 1$) and low-energy ($\alpha R^{m-1} \ll 1$) regimes. To proceed, we start by assuming that the early time approximation covers inflation and radiation-dominated phases and the late time approximation covers the matter-dominated and dark energy phases. Thus, our analysis does not cover the epochs where the two nonlinear terms are comparable. Such epochs can occur between early and late times or at late times. Nevertheless our assumption regarding early times should still be valid since R should be large in such regimes. Also, observations still require that $n > -1$ and $\beta > 0$.

Table II shows that to have an inflationary stage followed by a radiation-dominated phase we must have $m > 4/3$, since this is the only way to obtain a saddle radiation phase P_{r1} (which would be required to obtain an exit from this phase). The de Sitter point P_d exists for $m > 2$, but one cannot use this fixed point for inflation followed by the radiation era because P_d is a stable attractor which does not admit the exit from inflation. The inflationary epoch can only come from the point P_{r2} with $m > 3$ or point P with $m > 3/2$, in order to ensure that $w_{\rm eff} < -1/3$.

Let us consider the case $m > 2$. Taking into account the fact that P_{r2} is in a high energy region ($\alpha R^{m-1} \gg 1$), the possible trajectory in this case is $P_{r2} \to P \to P_d$, since P_{r1} and P_m are in the low energy region. All points P_{r2}, P, and P_d correspond to inflationary solutions for $m > 3$, whereas P_{r2} is not an inflationary solution for $2 < m < 3$. In neither case do we have a successful exit from the de Sitter epoch P_d to the radiation phase.

When $4/3 < m < 2$, it is possible to have the sequence: (i) $P_{r2} \to P \to P_m$, or (ii) $P_{r2} \to P_{r1} \to P_m$. Note that P_{r2} corresponds to a nonstandard evolution with $-1/6 < w_{\rm eff} < 1/12$. When $m > 3/2$ the point P leads to an accelerated expansion, but the case (i) is not viable because of the absence of the radiation era after inflation. Similarly, the case (ii) can be ruled out since it does not possess an inflationary phase. Now since both P and P_{r1} are saddle points, we may ask whether the sequence $P_{r2} \to P \to P_{r1} \to P_m$ can occur. To study this possibility, we proceeded numerically and solved the autonomous equations for a range of initial conditions close to $y_1 = 0$ and $y_2 = 1$ and a range of model parameters. Despite extensive numerical searches, we were unable to find a radiation-dominated phase between the de Sitter and matter epochs. Figure 2 gives a typical example of such simulations showing the evolution of the variables y_1 and y_2 together with $w_{\rm eff}$. Clearly, the radiation epoch is absent between the de Sitter and matter epochs. The sequence (i) $P_{r2} \to P \to P_m$ is, however, realized as expected, which is finally followed by a de Sitter universe as the β/R^n term becomes important. Thus, the models with $3/2 < m < 2$

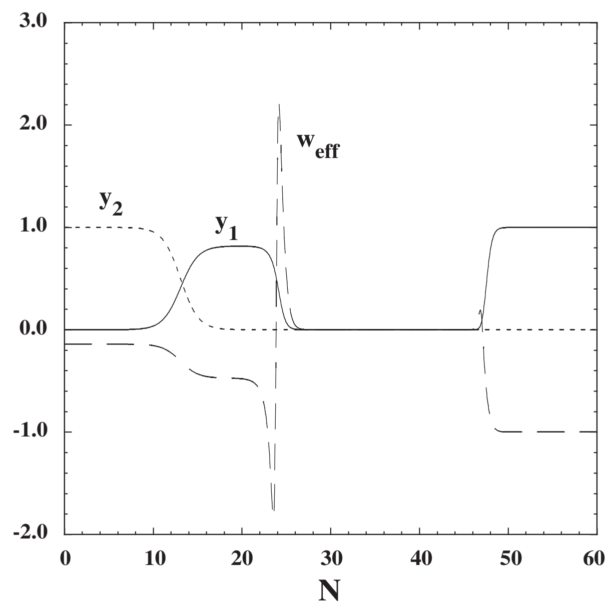

FIG. 2. Evolution of y_1 and y_2 together with the effective equation of state $w_{\rm eff}$ for the model $f(R) = R + \alpha R^m - \beta/R^n$ with $m = 1.9$, $n = 1$, $\alpha = 1$, $\beta = 10^{-60}$. The initial conditions are $y_1 = 1.71 \times 10^{-6}$ and $y_2 = 1\text{–}2.09 \times 10^{-6}$. The system starts from a nonstandard radiation-type phase (corresponding to the P_{r2} point with $w_{\rm eff} = -2/3 + 1/m = -0.14$) followed by an inflationary phase (corresponding to the P point with $w_{\rm eff} = -1 + 1/m = -0.47$) and then followed by a matter-dominated phase (corresponding to the P_m point with $w_{\rm eff} = 0$). The system finally approaches a de Sitter point P_d because of the presence of the β/R^n term.

seem to be unable to produce an initial inflationary phase leading to a radiation-dominated epoch.

The above discussion indicates that the assumption $m > 3/2$, which is necessary for an early-time acceleration phase to occur, is incompatible with the subsequent radiation-dominated phase.

To complete this analysis, we also consider the range where $1 < m < 4/3$. In this case there are two possible sequences: (i) $P_{r2} \to P \to P_m$ and (ii) $P_{r1} \to P_m$. In the case (i) the system starts from a nonstandard high energy phase P_{r2} ($\alpha R^{m-1} \gg 1$), which is followed by a noninflationary phase P. Since neither an inflationary phase nor a radiation-dominated epoch exist, the sequence (i) is therefore not cosmologically viable. In the case (ii) the system starts from a low-energy radiation-dominated phase ($\alpha R^{m-1} \ll 1$) and is followed by a matter-dominated epoch. As long as the term β/R^n becomes important at late times, the sequence (ii) is followed by a de Sitter point. Thus the case (ii) can at least account for the last three phases required in cosmology. When $0 < m < 1$ it is also possible to have the sequence $P_{r1} \to P_m \to P_d$.

Finally, we shall consider the case where the term αR^m is comparable to R around the present epoch. This corresponds to a large coupling α and is suggested by the likelihood analysis considerations of the model parameters in the next section. When $0 < m < 1$ this case is similar to the one for the model $f(R) = R - \beta/R^n$ with $-1 < n < 0$. If $m > 1$ this corresponds to a nonstandard cosmological evolution with $\alpha R^m \gg R$ in the past, which can be realized for a very large coupling α. To complement the discussion above, we shall also briefly consider this case. To start with, let us consider the case $m \neq 2$. Since $\alpha R^{m-1} \gg 1$ prior to the dark energy epoch, the fixed points correspond to either (i) P_{r2} with an EOS $w_{\rm eff} = -2/3 + 1/m$, eigenvalues $(\lambda_1, \lambda_2) = (1, 1)$ or (ii) P with an EOS $w_{\rm eff} = -1 + 1/m$, eigenvalues $(\lambda_1, \lambda_2) = (1 - 1/m, -1)$. Since we are considering the case $m > 1$, we have (i) $w_{\rm eff} < 1/3$ and (ii) $w_{\rm eff} < 0$. This shows that the radiation-dominated phase is not realized unless m is unity (i.e. Einstein gravity)

We shall consider the case $m = 2$ in the next subsection.

4. The special case $m = 2$

To conclude our consideration of theories of the type (27), we shall consider the special case of these theories with $m = 2$, which have recently attracted some attention in the literature [51].

(A) Theories of the type $f(R) = R + \alpha R^2$

In this case we have $C = -6$. The critical points, eigenvalues, and the effective EOS are

(i) P_r: $(y_1, y_2) = (0, 1)$, $(\lambda_1, \lambda_2) = (-2, 1)$.
We note that in this case $w_{\rm eff}$ is undefined, as can be seen from Eq. (A7) in the Appendix. This, however, does not play an important role in the reasoning below, as we shall see.

(ii) $P_m = (0, 0)$, $(\lambda_1, \lambda_2) = (-3, -1)$, $w_{\rm eff} = 0$.
This matter point exists only in the region $\alpha R \ll 1$ because of the relation $\alpha R = 2y_1/(1 - y_1 - y_2)$.

(iii) $P_d = (1, 0)$, $(\lambda_1, \lambda_2) = (2, 3)$, $w_{\rm eff} = 0$.
This point exists only in the region $\alpha R \gg 1$.

An important feature of this model is that the fixed point P_d in this case is an unstable node mimicking a dust universe. Depending upon whether the radiation ($y_1 \ll y_2$) or the dark energy ($y_2 \ll y_1$) dominates initially, the Universe starts with the EOS of matter (P_d) or radiation (P_r) and ends in a matter era (P_m) which is a stable node. Before reaching the final attractor P_m, there can exist a time $N = N_s$ such that $y_1 = (1 - y_2)/3$. In particular, this always occurs when $y_1 > 1/3$ at early times, as would be expected in a universe dominated by dark energy. In this case, the EOS diverges at N_s as we see from Eq. (A7) in the Appendix. Note that when $y_1 = (1 - y_2)/3$, the scalar curvature is regular ($\alpha R = 1$) with finite y_1 and y_2, indicating that there is no physical singularity, despite the singular behavior of the effective EOS. This can be understood by recalling that the $w_{\rm eff}$ so defined is just a mathematical construction for $f(R)$ theories.

(B) Theories of the type $f(R) = R + \alpha R^2 - \beta/R$

Let us consider the case in which the terms αR^2 and β/R are important for early and late times, respectively. Then as long as $\alpha > 0$ and $\beta > 0$, it is possible to have the sequence $P_d \to P_r \to P_m$ followed by a de Sitter point P_d which appears because of the presence of the term β/R. However, since P_r corresponds to a divergent EOS, we do not have a standard radiation-dominated epoch in this model.

Let us next consider the case in which the signs of α and β are different, say $\alpha > 0$ and $\beta < 0$. We will encounter this situation when we carry out a likelihood analysis of model parameters in the next section. Then it can happen that $C(R)$ diverges as R approaches either $(-2\beta/\alpha)^{1/3}$ or $\sqrt{-3\beta}$. In this case, the effective EOS diverges and dy_1/dN approaches $-\infty$. For the model parameters constrained by observations the divergence occurs at $R = (-2\beta/\alpha)^{1/3}$ before reaching $R = \sqrt{-3\beta}$, since in that case $(-2\beta/\alpha)^{1/3} > \sqrt{-3\beta}$. We also note that the radiation-dominated phase is absent in this case as well if we go back to the past.

This provides an example of what can happen when $C(R)$ diverges, which may also occur for other values of (m, n).

C. Theories of type $f(R) = R + \alpha \ln R - \beta$

Finally we shall consider theories of the type [42,52]

$$f(R) = R + \alpha \ln R - \beta. \tag{34}$$

For these theories Eqs. (18) and (19) give

$$C(R) = 3\alpha \frac{R - \alpha + 2\alpha \ln R - 2\beta}{(\alpha - \alpha \ln R + \beta)(R + 2\alpha)}, \tag{35}$$

and

$$\frac{R - \alpha + 2\alpha \ln R - 2\beta}{\alpha - \alpha \ln R + \beta} = \frac{1 - y_1 - y_2}{2y_1}. \tag{36}$$

These imply $C \to 0$ in the limit $R \to \infty$ and $C \to -3$ in the limit $R \to 0$. We also note that assuming the denominator in (35) is nonzero, then $C = 0$ for a constant R satisfying $R - \alpha + 2\alpha \ln R - 2\beta = 0$. Using the constraint equation (36), the critical points can be found to be

(1) P_r: $(y_1, y_2) = (0, 1)$.
  (a) P_{r1}: $R \to \infty$ and $C \to 0$. The eigenvalues are given by $(\lambda_1, \lambda_2) = (4, 1)$ with $w_{\rm eff} = 1/3$.
  (b) P_{r2}: $R \to 0$ with $C \to -3$. The eigenvalues are given by $(\lambda_1, \lambda_2) = (1, 1)$ with $w_{\rm eff} \to -\infty$.

(2) P_m: $(y_1, y_2) = (0, 0)$.
$R \to \infty$, $C \to 0$, $(\lambda_1, \lambda_2) = (3, -1)$ and $w_{\rm eff} = 0$.

(3) P_d: $(y_1, y_2) = (1, 0)$.
$C = 0$, $(\lambda_1, \lambda_2) = (-3, -4)$ and $w_{\rm eff} = -1$, with R behaving as a constant near the critical point.

(4) P: $(y_1, y_2) = (-1/3, 0)$.
$R \to 0$, $C = -3$, $(\lambda_1, \lambda_2) = (0, -1)$ and $w_{\rm eff} \to -\infty$.

From the above discussion it is clear that one can obtain the sequence of radiation-dominated (P_{r1}), matter-dominated (P_m), and de Sitter (P_d) eras. We have checked numerically that this sequence indeed occurs. Note that the points P_{r2} and P are irrelevant to realistic cosmology, since the system approaches the stable de Sitter point P_d with a constant R before reaching $R \to 0$.

IV. CONFRONTING $f(R)$ GRAVITY MODELS WITH OBSERVATIONAL DATA

The detailed phase space analysis given in the previous section provides a clear picture of the possible dynamical modes of behavior that can occur for models based on these theories. These analyses demonstrate that none of the theories considered here can produce all four phases required for cosmological evolution. They, however, show that a subset of these theories is capable of producing the last three phases, i.e. radiation-dominated, matter-dominated, and late time accelerating phases. As was mentioned above, the occurrence of these phases is a necessary but not a sufficient condition for cosmological viability of such theories. To be cosmologically viable, it is also necessary that the subset of parameters in these theories that allow these phases to exist are also compatible with observations.

In this section we shall study this observational compatibility by confronting models based on these theories with observational data. To proceed, we rewrite Eqs. (9)–(11), using the redshift parameter $z = a_0/a - 1$ and employing the expressions $\rho_m = \rho_{m0}(1+z)^3$ and $\rho_r = \rho_{r0}(1+z)^4$ to obtain

$$FR - 2f = -3H_0^2 \Omega_{m0}(1+z)^3, \tag{37}$$

$$\frac{dR}{dz} = -\frac{9H_0^2 \Omega_{m0}(1+z)^2}{F'R - F}, \tag{38}$$

$$\frac{H^2}{H_0^2} = \frac{3\Omega_{m0}(1+z)^3 + 6\Omega_{r0}(1+z)^4 + f/H_0^2}{6F[1 + \frac{9H_0^2 F' \Omega_{m0}(1+z)^3}{2F(F'R-F)}]^2}, \tag{39}$$

where $\Omega_{m0} \equiv \kappa \rho_{m0}/3H_0^2$ and $\Omega_{r0} \equiv \kappa \rho_{r0}/3H_0^2$ and a subscript 0 denotes evaluation at the present time. From Eqs. (37) and (38), one can express R in terms of z for a given $f(R)$ model. Equation (39) can then be used to obtain the Hubble parameter in terms of z.

We note that for a given $f(R)$ theory sourced by cold dark matter, Ω_{m0} is uniquely determined once units are chosen such that $H_0 = 1$ [46]. If radiation is also present, then both H_0 and Ω_{r0} need to be specified in order to determine Ω_{m0} uniquely. In the following, we choose $\Omega_{r0} = 5 \times 10^{-5}$ and $H_0 = 1$ (see below).

To constrain $f(R)$ models, we shall use three sets of data:

(I) The first year data set from the Supernova Legacy Survey (SNLS) [7] with 71 new supernovae below $z = 1.01$, together with another 44 low z supernovae already available, i.e. a total of 115 SNe.

(II) The baryon acoustic oscillation peak (BAO) recently detected in the correlation function of luminous red galaxies (LRG) in the Sloan Digital Sky Survey [10]. This peak corresponds to the first acoustic peak at recombination and is determined by the sound horizon. The observed scale of the peak effectively constrains the quantity

$$A_{0.35} = D_V(0.35)\frac{\sqrt{\Omega_{m0} H_0^2}}{0.35} = 0.469 \pm 0.017, \tag{40}$$

where $z = 0.35$ is the typical LRG redshift and $D_V(z)$ is the comoving angular diameter distance defined as

$$D_V(z) = \left[D_M(z)^2 \frac{z}{H(z)}\right]^{1/3}, \quad \text{with } D_M = \int_0^z \frac{dz}{H}. \tag{41}$$

(III) The CMB shift parameter. This constraint is defined by

$$R_{1089} = \sqrt{\Omega_{m0} H_0^2} \int_0^{1089} \frac{dz}{H} = 1.716 \pm 0.062, \tag{42}$$

which is a measure of the distance between $z = 0$ and $z = 1089$. It relates the angular diameter distance to the last scattering surface with the angular scale of the first acoustic peak. An important feature of this parameter is that it is model independent and insensitive to perturbations [53]. To be able to use it one has to have a standard matter-dominated era at decoupling.

For the goodness of fit we employ the standard χ^2 minimization, defined by

$$\chi^2 = \sum_{i=1}^{n} \frac{(m_i^{\rm obs} - m_i^{\rm th})^2}{\sigma_i^2},$$

where n is the number of data points and $m_i^{\rm obs}$ and $m_i^{\rm th}$ are, respectively, the observed and the theoretical magnitudes calculated from the model. We shall marginalize the Hubble constant by defining as usual a new χ^2:

$$\bar{\chi}^2 = -2\ln \int_{-\infty}^{+\infty} e^{-\chi^2/2} {\rm d}\bar{M},$$

where $\bar{M} = M - 5\ln(H_0) + 25$ and M is the magnitude zero point offset. After some algebraic manipulations and defining

$$A = \sum_{p=1}^{115} \frac{(m_i^{\rm obs} - 5\ln(D_l))^2}{\sigma_i^2},$$
$$B = \sum_{p=1}^{115} \frac{m_i^{\rm obs} - 5\ln(D_l)}{\sigma_i^2}, \qquad C = \sum_{p=1}^{115} \frac{1}{\sigma_i^2}, \tag{43}$$

we obtain

$$\bar{\chi}^2 = A - \frac{B^2}{C} + \ln\frac{C}{2\pi}. \tag{44}$$

Finally, we note that the BAO and the CMB shift parameter do not depend on the Hubble constant. Having only one measure point in each case, these tests can be used in the same way as priors.

In what follows, we shall proceed to place observational constraints on the models discussed in the previous section. Our choice of units such that $H_0 = 1$ is consistent with marginalization over H_0, similar to the choice made in the literature [46].

A. Theories of the type $f(R) = R - \beta/R^n$

We shall first consider theories of the type

$$f(R) = R - \beta/R^n. \tag{45}$$

As we already demonstrated in the previous section, one can obtain in this case a sequence of radiation-dominated, matter-dominated, and acceleration phases provided that $n > -1$.

Recently Amarzguioui *et al.* [46] have found that models of this kind are compatible with the supernova "Gold" data set from Riess *et al.* [6] together with the BAO constraint (40) and the CMB constraint (42), subject to constraints on the values of the parameters β and n. In their combined analysis the best-fit model was found to be $\beta = 3.6$ and $n = -0.09$. Here we extend this work by employing the more recent first year SNLS data as well as taking into account the presence of radiation. In Fig. 3 we show observational contour plots at the 68% and 95% confidence levels obtained from the SNLS data only.

In this case the best-fit value (smallest χ^2) is found to be $\chi^2 = 116.55$ with $\beta = 12.5$ and $n = 0.6$. This is consistent with the results obtained in Ref. [46], namely $\beta = 10.0$ and $n = 0.51$ and with the results obtained in the previous section according to which the transition from the matter-dominated era to the de Sitter era occurs for $n > -1$. As n gets closer to -1, it is difficult to realize a late-time acceleration phase since the model (45) approaches Einstein gravity.

If we further include constraints from BAO and CMB, the likelihood parameter space is reduced significantly as seen from Fig. 4, with narrower ranges of the allowed values of n and β lying in the intervals $n \in [-0.23, 0.42]$ and $\beta \in [2.73, 10.6]$ at the 95% confidence level. This difference is mainly due to the fact that the BAO data better constrains the cold dark matter parameter than the SN data alone. The best-fit values are found to be $\beta = 4.63$ and $n = 0.027$ with $\chi^2 = 116.69$.

Note that the ΛCDM model corresponding to $\beta = 4.38$ and $n = 0$ lies in the 68% confidence level. In Fig. 5, we have plotted the evolution of $w_{\rm eff}$ for the best-fit case. This shows that the models in agreement with the data undergo a transition from a radiation-dominated to a matter-dominated epoch followed by an acceleration phase which in the future asymptotically approaches a de Sitter solution. This is a nice realization of the trajectories that follow the sequence $P_{r1} \rightarrow P_m \rightarrow P_d$ described in the previous section.

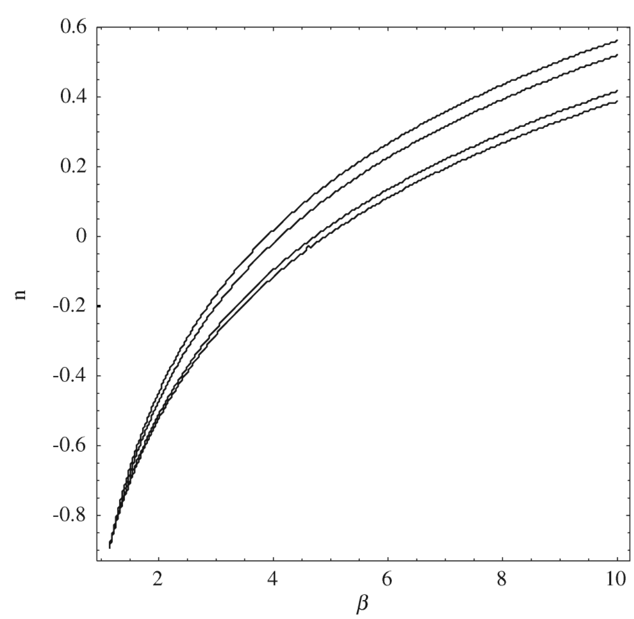

FIG. 3. The 68.3% and 95.4% confidence levels for the model based on the theory $f(R) = R - \beta/R^n$, constrained by SNLS data only.

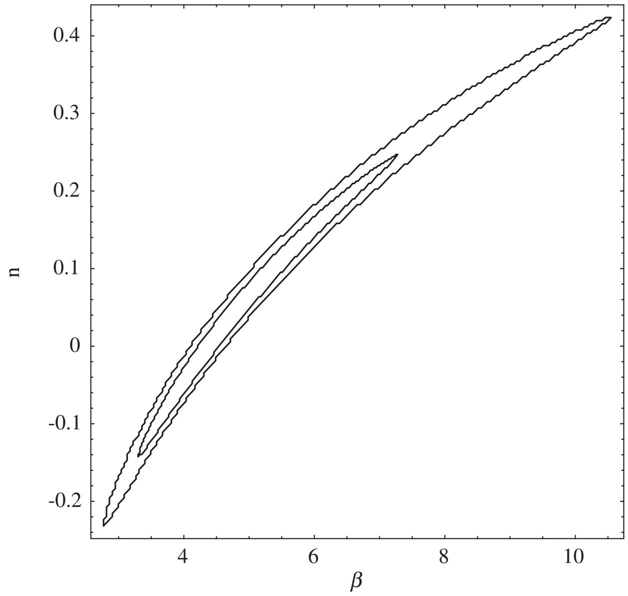

FIG. 4. The 68.3% and 95.4% confidence levels for the model based on the theory $f(R) = R - \beta/R^n$, constrained by SNLS, BAO, and CMB data.

B. Theories of the type $f(R) = R + \alpha R^m - \beta/R^n$

We next consider theories of the type

$$f(R) = R + \alpha R^m - \beta/R^n. \tag{46}$$

In the previous section we showed that an initial inflationary phase (requiring $m > 3/2$) does not seem to be compatible with a subsequent radiation-dominated epoch. However, dropping the requirement of an inflationary phase, such theories can produce the last 3 phases, namely, radiation-dominated, matter-dominated, and late acceleration epochs, provided the radiation-dominated era starts in the $\alpha R^{m-1} \ll 1$ regime. If the term αR^m becomes smaller than β/R^n prior to the decoupling epoch, observational constraints on the models based on (46) are similar to those on the models based on (45), i.e., $n \in [-0.23, 0.42]$ and $\beta \in [2.73, 10.6]$ at the 2σ level.

When $m = 2$ we also showed separately in Sec. III B 4 that the radiation-dominated epoch does not exist. To study the observational constraints in such cases, we have carried out a likelihood analysis for the model $f(R) = R + \alpha R^2 - \beta/R$ considered in Ref. [51]. Note that the four parameters α, β, m, and n cannot be constrained at the same time with the observations used here, so we have chosen special values for m and n. With these choices, we have found that the model is compatible with the observational data subject to the parameter constraints $\alpha \in [2.30, 3.29]$ and $\beta \in [-0.012, -0.004]$ (see Fig. 6). However, there is no radiation-dominated stage prior to the matter-dominated era, as is seen in Fig. 7 which gives a plot of the evolution of $w_{\rm eff}$ in the best-fit case ($\alpha = 2.69$ and $\beta = -0.008$).

When $\alpha > 0$ and $\beta < 0$ we have already shown in Sec. III B 4 that a singularity appears for $w_{\rm eff}$. In fact Fig. 7 shows that the de Sitter solution is not the late-time attractor since there is a singularity in the future (around $z \sim -0.27$) where $w_{\rm eff}$ becomes (positive) infinite (and the Hubble parameter $H \to 0$) after a short transient period during which it is smaller than $-1/3$.

The above discussion demonstrates the necessity of the theoretical considerations in the previous section. Using the observational data alone, we would obtain the misleading result that the above model is consistent with observations despite the absence of the radiation-dominated era.

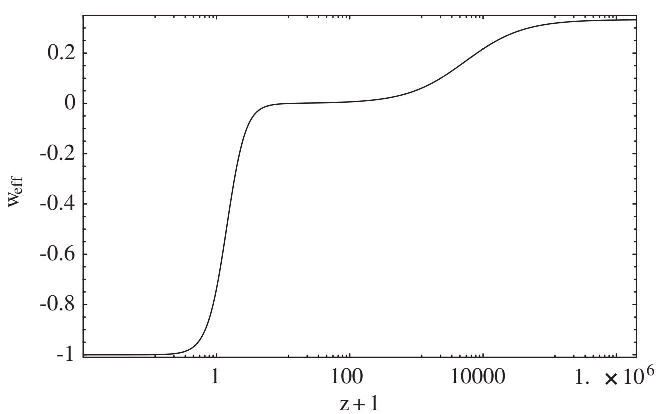

FIG. 5. Evolution of the effective equation of state $w_{\rm eff}$, for the model based on the theory $f(R) = R - \beta/R^n$, with the best-fit value $\beta = 4.63$ and $n = 0.027$. For very large z, $w_{\rm eff} = 1/3$, which then decreases and approaches 0 at redshift of a few tens. The accelerated expansion occurs around $z = 1$ after which $w_{\rm eff}$ becomes smaller than $-1/3$. In the future, $w_{\rm eff}$ asymptotically approaches -1.

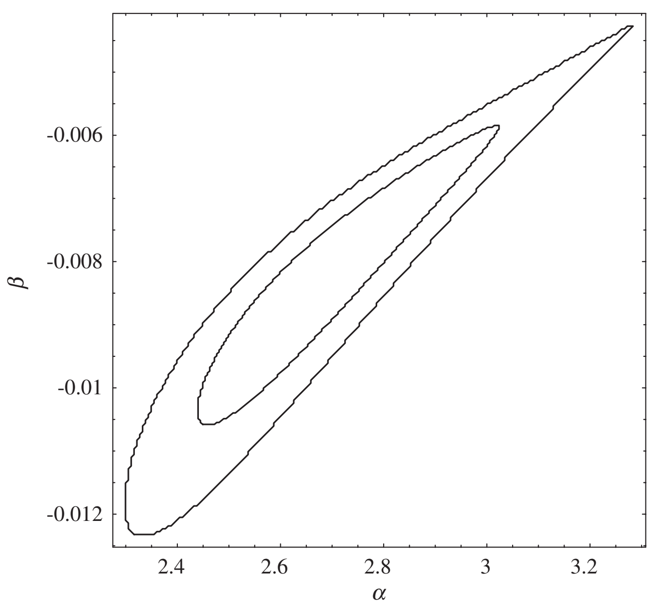

FIG. 6. The 68.3% and 95.4% confidence levels for the model $f(R) = R + \alpha R^2 - \beta/R$ constrained by SNLS, BAO, and CMB data.

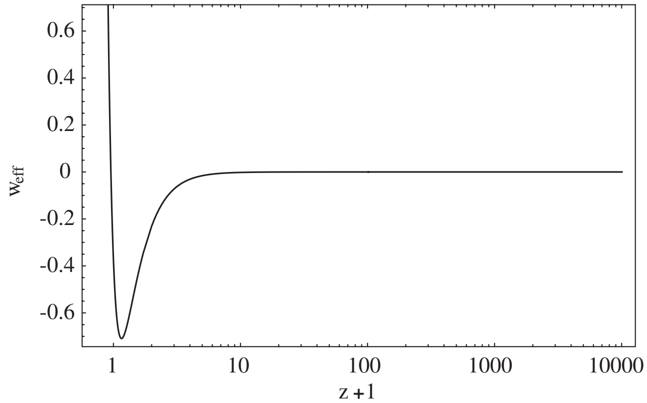

FIG. 7. Evolution of $w_{\rm eff}$ for the model $R + \alpha R^2 - \beta/R$ with the best-fit value $\alpha = 2.69$ and $\beta = -0.008$. There is no radiation-dominated epoch prior to the matter-dominated era ($w_{\rm eff} = 0$). Although $w_{\rm eff}$ can be smaller than $-1/3$ around the present epoch, the late-time evolution does not correspond to a de Sitter universe.

This is due to the fact that the observations we have considered do not probe the radiation-dominated phase.

So far we have considered the cases with $m > 1$. We have also checked that the sequence of radiation-dominated, matter-dominated, and de Sitter eras can be realized for $0 < m < 1$, although in this case the initial inflationary epoch is absent. One can perform a likelihood analysis by taking several values of m and n which exist in the regions $0 < m < 1$ and $0 < n < 1$, while allowing α and β to vary. Interestingly, there are values of (m, n) in these ranges which fit the observational data. Table III shows the best-fitting model parameters for some values of the parameters m and n. For this range of parameters, the best-fitting values of α and β are always found to be negative and positive, respectively. As n is increased we find that the magnitudes of α and β both increase, while the former still remaining negative. The increase in the amplitudes of α and β tends to compensate for the decrease in the $1/R^n$ term. On the other hand, as m increases, the amplitude of α decreases (while still remaining negative) while that of β increases. These effects have the consequence of minimizing the contribution of R^m relative to $1/R^n$. We also did not find the divergence of $w_{\rm eff}$ in such cases. Note that from the Table III the case with $n = 0.8$ and $m = 0.8$ can produce late-time acceleration. This provides an example of a case where the two nonlinear terms are comparable and necessary to produce the acceleration at late times. There would be no late-time acceleration if a single term is chosen with such an exponent.

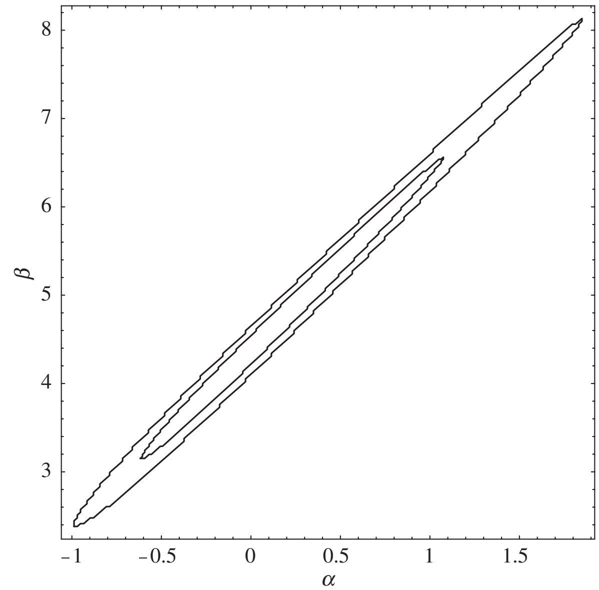

FIG. 8. 68.3% and 95.4% confidence level for the theory $f(R) = R + \alpha \ln R - \beta$ constrained by SNLS, BAO, and CMB data. The models without a cosmological constant ($\beta = 0$) are excluded by the data.

C. Theories of the type $f(R) = R + \alpha \ln R - \beta$

Finally we consider theories of the type

$$f(R) = R + \alpha \ln R - \beta, \tag{47}$$

where α and β are dimensionless constants.

The combined constraints from the SNLS, BAO, and CMB data confine the allowed values of α and β to the ranges $\alpha \in [-1, 1.86]$ and $\beta \in [-0.96, 8.16]$, respectively (see Fig. 8). The best-fit model corresponds to $\alpha = 0.11$ and $\beta = 4.62$ with $\chi^2 = 116.69$. These values are not too different from the ΛCDM model with $\alpha = 0$ and $\beta = 4.38$, indicating that the model is not significantly preferred over the ΛCDM model.

Note that the case $\beta = 0$ is excluded by the data. This therefore shows that the $\ln R$ term alone cannot drive the late-time acceleration; one still requires a cosmological constant. Thus based on the observational constraints con-

TABLE III. The best-fitting values of the parameters (χ^2, α, β, Ω_{m0}) for the models based on theories $f(R) = R + \alpha R^m - \beta/R^n$ with some particular values of (m, n).

	$m = 0.1$	$m = 0.5$	$m = 0.8$
$n = 0.1$	(116.68, −1.33, 3.37, 0.27)	(116.64, −0.22, 4.66, 0.27)	(116.61, −0.08, 4.90, 0.27)
$n = 0.3$	(116.64, −2.47, 2.54, 0.27)	(116.84, −0.45, 5.95, 0.27)	(116.76, −0.24, 6.15, 0.27)
$n = 0.5$	(116.64, −2.85, 2.56, 0.27)	(116.63, −0.82, 6.36, 0.27)	···
$n = 0.8$	(116.99, −2.85, 4.71, 0.28)	(116.76, −1.03, 9.16, 0.28)	(118.29, −0.47, 12.4, 0.28)

sidered here, only strong theoretical reasons would motivate the consideration of theories of this type.

V. CONCLUSIONS

We have made a detailed and systematic study of a number of families of $f(R)$ theories in Palatini formalism, using phase space analysis, extensive numerical simulations as well as observational constraints from recent data. These theories have been recently put forward in the literature in order to explain the different dynamical phases in the evolution of the Universe, in particular the early and/or late acceleration epochs deduced from recent observations.

Considering the FLRW setting, we have expressed the evolution equations as an autonomous dynamical system in the form of Eqs. (16) and (17). The ΛCDM model corresponds to a case where the expression $C(R)$ given by Eq. (18) vanishes. Models, based on $f(R)$ theories, in which the variable $C(R)$ asymptotically approaches 0, are able to give rise to a de Sitter expansion with constant R at late times. This includes the models based on theories of the type (a) $f(R) = R - \alpha/R^n$, (b) $f(R) = R + \alpha R^m - \beta/R^n$, and (c) $f(R) = R + \alpha \ln R - \beta$. We have carried out a detailed analysis of the cosmological evolution for such models by studying their fixed points and their stabilities against perturbations.

For models based on theories of the type $f(R) = R - \beta/R^n$, we have shown that the sequence of radiation-dominated, matter-dominated, and de Sitter eras is in fact realized for $n > -1$. This is a stark contrast to the corresponding theories of metric formalism for which it is difficult to realize such a sequence [25].

Considering models based on theories of the type $f(R) = R + \alpha R^m - \beta/R^n$, we have found that while they are capable of producing early as well as late accelerating phases an early inflationary epoch does not seem to be followed by a standard radiation-dominated era. When $m > 2$, the de Sitter point is stable, which prohibits an exit from the inflationary phase. Inflationary solutions can also be obtained for $3/2 < m < 2$ (for $\alpha > 0$), but in that case they directly exit into a matter-dominated phase which in turn leads to a late-time accelerating phase. For $0 < m < 4/3$, the radiation fixed point is unstable, which makes it possible to have the sequence of radiation-dominated, matter-dominated, and de Sitter phases without a preceding inflationary epoch. For $4/3 < m < 3/2$, inflation is not possible and radiation, which is a saddle point, can only be preceded by a nonaccelerated phase such as P_{r2} or an unknown phase not satisfying our late and early time assumptions.

We have also placed observational constraints on the parameters of these models, employing the recently released supernovae data by the Supernova Legacy Survey as well as the baryon acoustic oscillation peak in the SDSS luminous red galaxy sample and the CMB shift parameter. We have found that both classes of models are in agreement with the observations with appropriate choices of their parameters. For models based on theories of the type $f(R) = R - \alpha/R^n$, the best-fit values were found to be $\beta = 4.63$ and $n = 0.027$. This is consistent with the results obtained by Amarzguioui *et al.* [46], who used the supernova "Gold" data rather than the SNLS without taking into account the radiation.

For the models based on theories of the type $f(R) = R + \alpha R^m - \beta/R^n$, it is not possible to constrain all four parameters simultaneously. Concentrating on the special class of theories $f(R) = R + \alpha R^2 - \beta/R$ studied in literature, we have found that they are compatible with the data subject to parameter constraints $\alpha \in [2.30, 3.29]$ and $\beta \in [-0.012, -0.004]$. Despite this agreement we have found, using a phase space analysis, that such cases do not seem to produce a radiation-dominated era prior to the matter-dominated epoch, thus making them cosmologically unacceptable. The reason for this apparent contradiction is that the combination of the SN, SDSS, and the CMB shift parameter considered here does not provide constraints on early phases of the Universe prior to the decoupling epoch. We have also checked that a sequence of radiation-dominated, matter-dominated, and de Sitter phases becomes possible for models based on theories of the type $f(R) = R + \alpha R^m - \beta/R^n$ when $0 < m < 1$, in agreement with the data.

Finally, we have studied the compatibility of theories of the type $f(R) = R + \alpha \ln R - \beta$ with observations. Again we have found that such models are compatible with observations with appropriate choices of their parameters. However, the logarithmic term on its own is unable to explain the late-time acceleration consistent with observations.

In summary, we have found that using Palatini formalism it is possible to produce models based on the classes of theories considered here, which possess the sequence of radiation-dominated, matter-dominated, and late-time acceleration phases consistent with observations. However, we have been unable to find the sequence of all four phases required for a complete explanation of the cosmic dynamics. It will also be interesting to investigate whether these models can pass the local gravity test and are free from gravitational instabilities.

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APPENDIX

In this Appendix we shall give the expression of $w_{\rm eff}$ for the model $f(R) = R + \alpha R^m - \beta/R^n$. We shall write the various terms in the expression for $w_{\rm eff}$ in Eq. (20).

Using the relation (10), we obtain

$$\frac{\dot{R}}{HR} = -3\frac{1-(m-2)\alpha R^{m-1}-(n+2)\beta R^{-n-1}}{1-m(m-2)\alpha R^{m-1}+n(n+2)\beta R^{-n-1}}, \tag{A1}$$

which allows the third term in Eq. (20) to be written as

$$\frac{\dot{F}}{3HF} = \frac{\dot{R}}{HR}\frac{m(m-1)\alpha R^{m-1}-n(n+1)\beta R^{-n-1}}{3[1+m\alpha R^{m-1}+n\beta R^{-n-1}]}. \tag{A2}$$

Defining the variable ζ as

$$\zeta \equiv \frac{3F'(FR-2f)}{2F(F'R-F)} = \frac{3[m(m-1)\alpha R^{m-1}-n(n+1)\beta R^{-n-1}][-1+(m-2)\alpha R^{m-1}+(n+2)\beta R^{-n-1}]}{2(1+m\alpha R^{m-1}+n\beta R^{-n-1})[-1+m(m-2)\alpha R^{m-1}-n(n+2)\beta R^{-n-1}]}, \tag{A3}$$

allows the fourth term in Eq. (20) to be expressed as

$$\frac{\dot{\xi}}{3H\xi} = -\frac{2}{3}\frac{\zeta}{1-\zeta}\frac{\dot{R}}{HR}\left[\frac{m(m-1)^2\alpha R^{m-1}+n(n+1)^2\beta R^{-n-1}}{m(m-1)\alpha R^{m-1}-n(n+1)\beta R^{-n-1}} - \frac{(m-2)(m-1)\alpha R^{m-1}-(n+1)(n+2)\beta R^{-n-1}}{1-(m-2)\alpha R^{m-1}-(n+2)\beta R^{-n-1}} - \frac{m(m-1)\alpha R^{m-1}-n(n+1)\beta R^{-n-1}}{1+m\alpha R^{m-1}+n\beta R^{-n-1}} + \frac{m(m-1)(m-2)\alpha R^{m-1}+n(n+1)(n+2)\beta R^{-n-1}}{1-m(m-2)\alpha R^{m-1}+n(n+2)\beta R^{-n-1}}\right]. \tag{A4}$$

Finally, the last term in Eq. (20) is given by

$$\frac{\dot{F}R}{18F\xi H^3} = \frac{m(m-1)\alpha R^{m-1}-n(n+1)\beta R^{-n-1}}{3[(m-1)\alpha R^{m-1}+(n+1)\beta R^{-n-1}]}y_1\frac{\dot{R}}{HR}. \tag{A5}$$

Note that from Eq. (28) R is a function of y_1 and y_2, which in turn implies that $w_{\rm eff}$ is also a function of y_1 and y_2, albeit of a very complicated form.

Despite this complexity, however, the expression for $w_{\rm eff}$ reduces to a fairly simple form for specific cases. For example, when $m = 2$ and $\beta = 0$, we find

$$w_{\rm eff} = y_1 + \frac{1}{3}y_2 + \frac{2\alpha R(2+\alpha R)}{(1+2\alpha R)(1-\alpha R)}. \tag{A6}$$

Now since from Eq. (28) $\alpha R = 2y_1/(1-y_1-y_2)$ in this case, $w_{\rm eff}$ is expressed in terms of y_1 and y_2 thus

$$w_{\rm eff} = y_1 + \frac{1}{3}y_2 + \frac{8y_1(1-y_2)}{(1+3y_1-y_2)(1-3y_1-y_2)}. \tag{A7}$$

[1] D. N. Spergel *et al.* (WMAP Collaboration), Astrophys. J. Suppl. Ser. **148**, 175 (2003).
[2] D. N. Spergel *et al.*, astro-ph/0603449.
[3] A. G. Riess *et al.* (Supernova Search Team Collaboration), Astron. J. **116**, 1009 (1998).
[4] S. Perlmutter *et al.* (Supernova Cosmology Project Collaboration), Astrophys. J. **517**, 565 (1999).
[5] R. A. Daly and S. G. Djorgovski, Astrophys. J. **597**, 9 (2003).
[6] A. G. Riess *et al.*, Astrophys. J. **607**, 665 (2004).
[7] P. Astier *et al.*, Astron. Astrophys. **447**, 31 (2006).
[8] M. Tegmark *et al.* (SDSS Collaboration), Phys. Rev. D **69**, 103501 (2004).
[9] U. Seljak *et al.* (SDSS Collaboration), Phys. Rev. D **71**, 103515 (2005).
[10] D. J. Eisenstein *et al.*, Astrophys. J. **633**, 560 (2005).
[11] V. Sahni and A. A. Starobinsky, Int. J. Mod. Phys. D **9**, 373 (2000); V. Sahni, Lect. Notes Phys. **653**, 141 (2004); S. M. Carroll, Living Rev. Relativity **4**, 1 (2001); T. Padmanabhan, Phys. Rep. **380**, 235 (2003); P. J. E. Peebles and B. Ratra, Rev. Mod. Phys. **75**, 559 (2003).
[12] E. J. Copeland, M. Sami, and S. Tsujikawa, Int. J. Mod. Phys. D **15**, 1753 (2006).
[13] I. L. Buchbinder, S. D. Odintsov, and I. L. Shapiro, *Effective Action in Quantum Gravity* (IOP Publishing, Bristol, 1992).
[14] M. Gasperini and G. Veneziano, Phys. Lett. B **277**, 256 (1992).
[15] S. Nojiri and S. D. Odintsov, Phys. Lett. B **576**, 5 (2003).
[16] D. V. Vassilevich, Phys. Rep. **388**, 279 (2003).
[17] A. A. Starobinsky, Phys. Lett. **91B**, 99 (1980).
[18] S. Capozziello, S. Carloni, and A. Troisi, astro-ph/0303041; S. Capozziello, V. F. Cardone, S. Carloni, and A. Troisi, Int. J. Mod. Phys. D **12**, 1969 (2003).
[19] S. M. Carroll, V. Duvvuri, M. Trodden, and M. S. Turner, Phys. Rev. D **70**, 043528 (2004).

[20] S. Capozziello, F. Occhionero, and L. Amendola, Int. J. Mod. Phys. D **1**, 615 (1992).
[21] R. M. Wald, *General Relativity* (University of Chicago Press, Chicago, 1984).
[22] T. Chiba, Phys. Lett. B **575**, 1 (2003).
[23] T. P. Sotiriou, Phys. Rev. D **73**, 063515 (2006).
[24] A. D. Dolgov and M. Kawasaki, Phys. Lett. B **573**, 1 (2003).
[25] L. Amendola, D. Polarski, and S. Tsujikawa, astro-ph/0603703 [PRLC newsletter (to be published)]; see also astro-ph/0605384.
[26] L. Amendola, R. Gannouji, D. Polarski, and S. Tsujikawa, gr-qc/0612180.
[27] A. W. Brookfield, C. v. de Bruck, and L. M. H. Hall, Phys. Rev. D **74**, 064028 (2006).
[28] M. E. Soussa and R. P. Woodard, Gen. Relativ. Gravit. **36**, 855 (2004); G. Allemandi, A. Borowiec, and M. Francaviglia, Phys. Rev. D **70**, 103503 (2004); D. A. Easson, Int. J. Mod. Phys. A **19**, 5343 (2004); S. M. Carroll, A. De Felice, V. Duvvuri, D. A. Easson, M. Trodden, and M. S. Turner, Phys. Rev. D **71**, 063513 (2005); S. Carloni, P. K. S. Dunsby, S. Capozziello, and A. Troisi, Classical Quantum Gravity **22**, 4839 (2005); S. Capozziello, V. F. Cardone, and A. Troisi, Phys. Rev. D **71**, 043503 (2005); G. Cognola, E. Elizalde, S. Nojiri, S. D. Odintsov, and S. Zerbini, J. Cosmol. Astropart. Phys. 02 (2005) 010; S. Nojiri, S. D. Odintsov, and S. Tsujikawa, Phys. Rev. D **71**, 063004 (2005); S. Nojiri and S. D. Odintsov, hep-th/0601213; R. P. Woodard, astro-ph/0601672; T. Clifton and J. D. Barrow, Phys. Rev. D **72**, 103005 (2005); S. Das, N. Banerjee, and N. Dadhich, Classical Quantum Gravity **23**, 4159 (2006); S. Capozziello, V. F. Cardone, E. Elizalde, S. Nojiri, and S. D. Odintsov, Phys. Rev. D **73**, 043512 (2006); S. K. Srivastava, astro-ph/0602116; T. P. Sotiriou, Classical Quantum Gravity **23**, 5117 (2006); Phys. Lett. B **645**, 389 (2007); gr-qc/0611158; T. P. Sotiriou and S. Liberati, gr-qc/0604006; A. De Felice, M. Hindmarsh, and M. Trodden, J. Cosmol. Astropart. Phys. 08 (2006) 005; S. Nojiri and S. D. Odintsov, Phys. Rev. D **74**, 086005 (2006); A. F. Zakharov, A. A. Nucita, F. De Paolis, and G. Ingrosso, Phys. Rev. D **74**, 107101 (2006); P. Zhang, Phys. Rev. D **73**, 123504 (2006); A. Borowiec, W. Godlowski, and M. Szydlowski, Phys. Rev. D **74**, 043502 (2006); K. Atazadeh and H. R. Sepangi, gr-qc/0602028; S. Bludman, astro-ph/0605198; S. M. Carroll, I. Sawicki, A. Silvestri, and M. Trodden, New J. Phys. **8**, 323 (2006); A. Borowiec, W. Godlowski, and M. Szydlowski, astro-ph/0607639; D. Huterer and E. V. Linder, Phys. Rev. D **75**, 023519 (2007); X. h. Jin, D. j. Liu, and X. z. Li, astro-ph/0610854; N. J. Poplawski, gr-qc/0610133; V. Faraoni, Phys. Rev. D **74**, 104017 (2006); Y. S. Song, W. Hu, and I. Sawicki, Phys. Rev. D **75**, 044004 (2007); I. Navarro and K. Van Acoleyen, gr-qc/061112; R. Bean, D. Bernat, L. Pogosian, A. Silvestri, and M. Trodden, astro-ph/0611321 [Phys. Rev. D (to be published)]; T. Chiba, T. L. Smith, and A. L. Erickcek, astro-ph/0611867; V. Faraoni and S. Nadeau, Phys. Rev. D **75**, 023501 (2007); E. O. Kahya and V. K. Onemli, gr-qc/0612026; T. Faulkner, M. Tegmark, E. F. Bunn, and Y. Mao, astro-ph/0612569.
[29] D. N. Vollick, Phys. Rev. D **68**, 063510 (2003); Classical Quantum Gravity **21**, 3813 (2004).
[30] E. E. Flanagan, Phys. Rev. Lett. **92**, 071101 (2004); Classical Quantum Gravity **21**, 417 (2004).
[31] X. Meng and P. Wang, Classical Quantum Gravity **20**, 4949 (2003); **21**, 951 (2004).
[32] X. H. Meng and P. Wang, Classical Quantum Gravity **21**, 2029 (2004).
[33] J. A. R. Cembranos, Phys. Rev. D **73**, 064029 (2006).
[34] V. Faraoni, Phys. Rev. D **74**, 104017 (2006).
[35] S. Nojiri and S. D. Odintsov, Phys. Rev. D **68**, 123512 (2003).
[36] R. Dick, Gen. Relativ. Gravit. **36**, 217 (2004).
[37] E. E. Flanagan, Classical Quantum Gravity **21**, 3817 (2004).
[38] G. J. Olmo, Phys. Rev. Lett. **95**, 261102 (2005).
[39] G. J. Olmo, Phys. Rev. Lett. **98**, 061101 (2007).
[40] V. Faraoni, Phys. Rev. D **74**, 023529 (2006).
[41] T. P. Sotiriou, Gen. Relativ. Gravit. **38**, 1407 (2006).
[42] S. Nojiri and S. D. Odintsov, Gen. Relativ. Gravit. **36**, 1765 (2004).
[43] X. H. Meng and P. Wang, Phys. Lett. B **584**, 1 (2004).
[44] T. P. Sotiriou, Classical Quantum Gravity **23**, 1253 (2006).
[45] A. Dominguez and D. Barraco, Phys. Rev. D **70**, 043505 (2004); G. M. Kremer and D. S. M. Alves, Phys. Rev. D **70**, 023503 (2004); G. J. Olmo, Phys. Rev. D **72**, 083505 (2005); G. Olmo and W. Komp, gr-qc/0403092; M. Han, Y. Ma, Y. Ding, and L. Qin, Mod. Phys. Lett. A **20**, 725 (2005); S. Capozziello, V. F. Cardone, and M. Francaviglia, Gen. Relativ. Gravit. **38**, 711 (2006); G. Allemandi, M. Capone, S. Capozziello, and M. Francaviglia, Gen. Relativ. Gravit. **38**, 33 (2006); P. Wang, G. M. Kremer, D. S. M. Alves, and X. H. M. Meng, Gen. Relativ. Gravit. **38**, 517 (2006); T. Koivisto and H. Kurki-Suonio, Classical Quantum Gravity **23**, 2355 (2006); T. Koivisto, Phys. Rev. D **73**, 083517 (2006); B. Li and M. C. Chu, Phys. Rev. D **74**, 104010 (2006); N. J. Poplawski, Phys. Rev. D **74**, 084032 (2006); Phys. Lett. B **640**, 135 (2006); B. Li, K. C. Chan, and M. C. Chu, astro-ph/0610794; K. Kainulainen, V. Reijonen, and D. Sunhede, gr-qc/0611132; S. Baghram, M. Farhang, and S. Rahvar, astro-ph/0701013.
[46] M. Amarzguioui, O. Elgaroy, D. F. Mota, and T. Multamaki, Astron. Astrophys. **454**, 707 (2006).
[47] The idea that such theories should be able to successfully account for all four phases in the cosmological evolution is clearly a maximal demand. It would still be of great interest if such theories could successfully account for a sequence of such phases including an accelerating phase, such as the first or the last three phases.
[48] T. Koivisto, Classical Quantum Gravity **23**, 4289 (2006).
[49] R. Tavakol, in *Dynamical Systems in Cosmology*, edited by J. Wainwright and G. Ellis (Cambridge University Press, Cambridge, England, 1997); E. J. Copeland, A. R. Liddle, and D. Wands, Phys. Rev. D **57**, 4686 (1998); B. Gumjudpai, T. Naskar, M. Sami, and S. Tsujikawa, J. Cosmol. Astropart. Phys. 06 (2005) 007; S. Tsujikawa, Phys. Rev. D **73**, 103504 (2006).
[50] To show that $dy_1/dN > 0$, we solve $dy_1/dN = 0$ with respect to n to obtain $n(y_1, y_2)$. Then plotting the function

$n(y_1, y_2)$ in the physical range of variables $(y_1, y_2) \in [0, 1]$, we find that the set of triplets (n, y_1, y_2), such that $\mathrm{d}y_1/\mathrm{d}N = 0$, all satisfy $n < -1$ apart from the special (and therefore the zero measure) set of fixed points which are also solutions of $\mathrm{d}y_1/\mathrm{d}N = 0$ for any n.
[51] X. H. J. Meng and P. Wang, Classical Quantum Gravity **21**, 2029 (2004); X. H. Meng and P. Wang, Classical Quantum Gravity **22**, 23 (2005).
[52] X. H. Meng and P. Wang, Phys. Lett. B **584**, 1 (2004).
[53] R. Maartens and E. Majerotto, Phys. Rev. D **74**, 023004 (2006).