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Quartification on an orbifold

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We investigate quartification models in five dimensions, with the fifth dimension forming an $S^1/Z_2 \times Z_2'$ orbifold. The orbifold construction is combined with a boundary Higgs sector to break the quartified gauge group directly to a group $H \subset SU(3)^4$ which is operative at the electroweak scale. We consider $H = G_{\rm SM} \otimes SU(2)_\ell$ and $H = G_{\rm SM}$, where $G_{\rm SM}$ is the standard model gauge group, and find that unification occurs only when the remnant leptonic color symmetry $SU(2)_\ell$ remains unbroken. Furthermore, the demands of a realistic low-energy fermion spectrum specify a unique symmetry breaking route for the unifying case of $H = G_{\rm SM} \otimes SU(2)_\ell$. We contrast this with four-dimensional quartification models where unification may be achieved via a number of different symmetry breaking routes both with and without the remnant $SU(2)_\ell$ symmetry. The boundary Higgs sector of our model may be decoupled to achieve a Higgsless limit, and we show that the electroweak Higgs doublet may be identified as the fifth component of a higher-dimensional gauge field.

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I. INTRODUCTION

The notion of quartification is predicated upon the hypothesis of quark-lepton universality at high energies. Quark-lepton universality is typically implemented by the imposition of a discrete interchange symmetry between quarks and leptons, a demand which further requires the introduction of a leptonic color gauge group $SU(3)_\ell$. This leads one to the so-called quark-lepton symmetric model [1–8] wherein the known leptons are identified as one member of a generalized $SU(3)_\ell$ lepton triplet.

One of the primary goals of modern theoretical particle physics is to identify extensions to the standard model (SM), with some of the most appealing extensions being grand unified theories (GUTs). Adding a leptonic color group to the SM clearly renders the traditional approaches to unification, namely, $SU(5)$ and $SO(10)$, inapplicable. It was the desire to uncover a unified theory capable of accommodating the notion of quark-lepton universality which motivated the construction of the quartification model [9,10]. Borrowing from the notion of trinification [11,12], which postulates the gauge group $G_T = SU(3)_c \otimes SU(3)_L \otimes SU(3)_R$ augmented by a cyclic Z_3 symmetry, the quartification model posits the gauge group $SU(3)^4 \equiv G_T \otimes SU(3)_\ell$ with a cyclic Z_4 symmetry permuting the group factors. The discrete groups ensure a single coupling constant at the unification scale and thus, a crucial feature of GUTs, namely, the unification of the gauge coupling parameters, can ensue. While in its original implementation only partial unification was achieved [9], it was subsequently demonstrated that the modification of the exotic lepton mass spectrum permitted full unification [10]. This motivated a recent study of quartification models with intermediate symmetry breaking scales [13,14] where it was shown that unification could be achieved via a number of different symmetry breaking routes.

In [13,14] a subset of schemes which allowed for unification and maintained phenomenological consistency were uncovered. Although representing an improvement on the original quartified theories of Refs. [9,10], these models exhibited some of the unsatisfactory features common to conventional GUTs, resulting mostly from the Higgs sector. A large number of scalar multiplets were required to achieve a realistic model and any potential describing their self-interactions would contain a plethora of unknown parameters. Furthermore, the vacuum expectation values (VEVs) and mass spectra of the fields was not predicated by the theory, forcing a variety of assumptions on this sector.

Orbifold compactifications, however, uncover a new arena by which to explore GUTs. The compactification process provides a geometrical origin for the symmetry breaking, allowing for the removal of the scalar sectors which complicate conventional four-dimensional unified theories. The ability to obtain a consistent effective field theory on these constructions has motivated the use of orbifolds to probe grand unified theories [15,16]. Even the SM Higgs field has a natural origin in orbifold models, with gauge-Higgs unified theories identifying the Higgs doublet as higher-dimensional components of the gauge fields [17].

This type of hybrid model was recently explored in a supersymmetric trinification theory by Carone and Conroy [18,19]. They placed the trinification gauge supermultiplets on a five-dimensional orbifold, and localized the full trinified matter content onto a brane. The orbifold action reduced the symmetry down to a subgroup on this subspace. This symmetry was then broken to the SM gauge group by a boundary scalar sector which decoupled from

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the low-energy theory. Additionally, the SM Higgs doublets were recovered as remnant zero modes of the gauge fields.

The realization of a GUT that is not reliant upon a scalar sector and its associated problems provides the motivation to explore quartification in this orbifold context. We follow a similar approach to Carone and Conroy [18], considering a five-dimensional quartification model and studying the prospects for unification within this framework. Interestingly we find that unification is only achieved when the remnant leptonic color group $SU(2)_\ell \subset SU(3)_\ell$ remains unbroken. Furthermore, the demands of a phenomenologically acceptable fermion spectrum specifies the choice of orbifold symmetry reduction on the SM matter brane and thus we arrive at a unique five-dimensional quartification model which achieves unification.

The layout of the paper is as follows. Section II provides the reader with a brief summary of the quartification framework. In Sec. III we consider the higher-dimensional symmetry breaking necessary to reduce $SU(3)^4$ to an acceptable low-energy subgroup. The symmetry breaking occurs in two distinct ways, via both orbifolding and the introduction of a boundary Higgs sector (which we ultimately decouple from the theory), and we detail each of these mechanisms. In Sec. IV we consider fermion mass in the model and demonstrate that the exotic fermions all obtain GUT scale masses while the SM fermions remain massless until electroweak symmetry breaking occurs. Section V covers the issue of gauge coupling unification within the model and we contrast our five-dimensional model with conventional approaches in Sec. VII. We conclude in Sec. VIII.

II. QUARTIFICATION PRELIMINARIES

In this section we briefly surmise the conventional quartification framework. For more details refer to [10,13,14]. The quartification gauge group is

$$G_4 = SU(3)_q \otimes SU(3)_L \otimes SU(3)_\ell \otimes SU(3)_R. \qquad (2.1)$$

A Z_4 symmetry which cyclicly permutes the gauge groups as per $q \to L \to \ell \to R \to q$ is also imposed to ensure a single gauge coupling constant. The fermion assignment is anomaly-free, with each family contained within a left-handed **36** of Eq. (2.1)

$$\begin{aligned} \mathbf{36} &= (\mathbf{1},\mathbf{1},\mathbf{3},\bar{\mathbf{3}}) \oplus (\bar{\mathbf{3}},\mathbf{1},\mathbf{1},\mathbf{3}) \oplus (\mathbf{3},\bar{\mathbf{3}},\mathbf{1},\mathbf{1}) \oplus (\mathbf{1},\mathbf{3},\bar{\mathbf{3}},\mathbf{1}) \\ &= \ell^c \oplus q^c \oplus q \oplus \ell. \end{aligned} \qquad (2.2)$$

q denotes the left-handed quarks, ℓ the left-handed leptons, and q^c and ℓ^c the left-handed antiquarks and antileptons, respectively. These are represented by 3×3 matrices, with the first family having the form

$$\begin{aligned} &q \sim \begin{pmatrix} d & u & h \\ d & u & h \\ d & u & h \end{pmatrix}, \qquad q^c \sim \begin{pmatrix} d^c & d^c & d^c \\ u^c & u^c & u^c \\ h^c & h^c & h^c \end{pmatrix}, \\ &\ell \sim \begin{pmatrix} x_1 & x_2 & \nu \\ y_1 & y_2 & e \\ z_1 & z_2 & N \end{pmatrix}, \qquad \ell^c \sim \begin{pmatrix} x_1^c & y_1^c & z_1^c \\ x_2^c & y_2^c & z_2^c \\ \nu^c & e^c & N^c \end{pmatrix}. \end{aligned} \qquad (2.3)$$

The quark multiplets contain exotic quark color triplets h and h^c, and exotic fermions are also required to fill the lepton representations. The SM leptons have partners x_i, x_i^c, y_i, y_i^c, z_i, and z_i^c, $i = 1, 2$ which transform as $SU(2)_\ell$ doublets, and there are two exotic singlets N and N^c. Under $H = G_{\rm SM} \otimes SU(2)_\ell$, the generator of hypercharge is

$$Y = \frac{1}{\sqrt{3}}(T_{8L} + T_{8\ell} + T_{8R}) + T_{3R}, \qquad (2.4)$$

where the T's refer to the Gell-Mann generators. In this framework, the exotic particles have the electric charges $Q(x_i, y_i, z_i) = (1/2, -1/2, 1/2)$, $Q(N) = 0$, and $Q(h) = -2/3$. In the case where the leptonic color symmetry is entirely broken $H = G_{\rm SM}$, the x_1, x_1^c, y_2, y_2^c, z_1, and z_1^c leptons become neutral.

The Higgs sector required to reproduce acceptable low-energy phenomenology in four-dimensional quartification models [10,13,14] consists of two different **36**'s of G_4 which are labeled as per

$$\begin{aligned} &\Phi_a \sim (\mathbf{1},\mathbf{3},\mathbf{1},\bar{\mathbf{3}}), \qquad \Phi_b \sim (\bar{\mathbf{3}},\mathbf{1},\mathbf{3},\mathbf{1}), \\ &\Phi_c \sim (\mathbf{1},\bar{\mathbf{3}},\mathbf{1},\mathbf{3}), \qquad \Phi_d \sim (\mathbf{3},\mathbf{1},\bar{\mathbf{3}},\mathbf{1}), \\ &\Phi_\ell \sim (\mathbf{1},\mathbf{3},\bar{\mathbf{3}},1), \qquad \Phi_{\ell^c} \sim (\mathbf{1},\mathbf{1},\mathbf{3},\bar{\mathbf{3}}), \\ &\Phi_{q^c} \sim (\bar{\mathbf{3}},\mathbf{1},\mathbf{1},\mathbf{3}), \qquad \Phi_q \sim (\mathbf{3},\bar{\mathbf{3}},\mathbf{1},\mathbf{1}). \end{aligned} \qquad (2.5)$$

These fields are closed under the Z_4 symmetry and generate realistic fermion masses and mixings. We shall not require the two **36**'s of scalars to accomplish the necessary symmetry breaking in the five-dimensional model, relying instead on the orbifold geometry. The subsequent absence of scalars with nontrivial $SU(3)_q$ quantum numbers removes issues of proton stability in this higher-dimensional context.

III. GAUGE SYMMETRY BREAKING FRAMEWORK

A. Orbifold symmetry breaking

We consider a pure quartification gauge theory defined in a five-dimensional spacetime, with the extra-dimensional coordinate labeled y. This extra dimension is compactified on an $S^1/Z_2 \times Z_2'$ orbifold. This procedure constrains the spacetime geometry as per

$$y \to y + 2\pi R, \qquad y \to -y, \qquad y' \to -y', \qquad (3.1)$$

where $y' \equiv y + \pi R/2$. These identifications have the effect of reducing the physical region to the interval $y \in [0, \pi R/2]$ with two fixed points located at $y = 0$ and $y = \pi R/2$. These points are geometrical singularities and are thus chosen as the location of four-dimensional branes. For simplicity, we shall consider all matter fields to be localized on the $y = \pi R/2$ brane, which we shall refer to as the matter brane, while the brane at $y = 0$ remains "hidden."

The orbifold action also has a definition on the space of gauge fields which freely propagate in the bulk. The quartification gauge fields are denoted by

$$A^M(x^\mu, y) = A_q^M(x^\mu, y) \oplus A_L^M(x^\mu, y) \oplus A_\ell^M(x^\mu, y) \oplus A_R^M(x^\mu, y) \tag{3.2}$$

$$= A_q^{Ma}T^a \oplus A_L^{Ma}T^a \oplus A_\ell^{Ma}T^a \oplus A_R^{Ma}T^a, \tag{3.3}$$

where $a = 1, 2, \ldots, 8$ is the gauge index and M is the Lorentz index $M = \mu, 5$. We define P and P' to be 3×3 matrix representations of the orbifold actions Z_2 and Z_2', respectively. To maintain gauge invariance under these projections, the gauge fields are required to have the transformations

$$\begin{aligned} A_\mu(x^\mu, y) &\to A_\mu(x^\mu, -y) = PA_\mu(x^\mu, y)P^{-1}, \\ A_5(x^\mu, y) &\to A_5(x^\mu, -y) = -PA_5(x^\mu, y)P^{-1}, \\ A_\mu(x^\mu, y') &\to A_\mu(x^\mu, -y') = P'A_\mu(x^\mu, y')P'^{-1}, \\ A_5(x^\mu, y') &\to A_5(x^\mu, -y') = -P'A_5(x^\mu, y')P'^{-1}. \end{aligned} \tag{3.4}$$

Given that P and P' define a representation of reflection symmetries, their eigenvalues are ± 1, and thus we can express these matrices in diagonal form, with a freedom in the parity choice of the entries. The exact nature of these actions then completely determines the gauge symmetry which remains unbroken at the fixed points. Unless P is the identity matrix, not all the gauge fields will commute with the orbifold action. These fields are projected off the brane, and thus only a subset of the five-dimensional gauge theory is manifest at the fixed points. Ideally, one would desire the matter brane to respect only the SM gauge group; however, this is not directly possible here via orbifolding. The $Z_2 \times Z_2'$ actions are Abelian and commute with the diagonal quartification generators. Subsequently, the rank of the quartification group must be preserved on the matter brane. This means that breaking unwanted $SU(3)$ factors has the trade-off of retaining the spurious $U(1)$ subgroups. Thus one must invoke a mechanism in tandem to orbifolding in order to accomplish the breaking to $G_{\rm SM}$.

We shall choose to have Higgs fields χ_i localized on $y = \pi R/2$ to instigate the rank-reducing breaking, giving the general symmetry breaking framework

$$F \equiv SU(3)^4 \overset{\rm orbifold}{\to} G \overset{\rm Higgs}{\to} H \equiv G_{\rm SM} \otimes SU(2)_\ell. \tag{3.5}$$

This type of hybrid model has been explored recently in the context of $SO(10)$ [20] and the aforementioned $SU(3)^3$ [18,21] orbifold GUTs. We shall consider the case where there is a residual $SU(2)_\ell$ symmetry in what follows. It is phenomenologically acceptable to retain $SU(2)_\ell$ as an exact low-energy symmetry. This symmetry acts only on exotic leptons which fill out the $SU(3)_\ell$ multiplets (known as liptons in the literature). These are constrained to be heavier than a TeV and will be much heavier than this in our construct. We shall comment on the case when $SU(2)_\ell$ is broken (namely, $H = G_{\rm SM}$) in Sec. VI.

The group G is determined by the desire to reproduce the correct SM matter content on the brane at the $y = \pi R/2$ fixed point. It turns out that the only feasible choice is to consider the orbifold returning the symmetry $G_{\pi R/2} \equiv SU(3)_q \otimes SU(2)_L \otimes SU(3)_\ell \otimes SU(3)_R \otimes U(1)_L$. All other options do not return a favorable low-energy theory. For example, we require the $SU(3)_R$ symmetry to generate realistic quark masses, and the $SU(2)_L$ symmetry must be realized so as to return a SM Higgs. The full leptonic color symmetry $SU(3)_\ell$ also needs to be respected on the matter brane. If this symmetry was broken, then indistinguishability issues surface between the SM leptons and their liptonic partners.

The orbifold action can be decomposed as $(P, P') = (P_q \oplus P_L \oplus P_\ell \oplus P_R, P_q' \oplus P_L' \oplus P_\ell' \oplus P_R')$, where

$$\begin{aligned} P_q &= \mathrm{diag}(1, 1, 1), & P_L &= \mathrm{diag}(1, 1, -1), \\ P_\ell &= \mathrm{diag}(1, 1, -1), & P_R &= \mathrm{diag}(1, 1, -1), \\ P_q' &= \mathrm{diag}(1, 1, 1), & P_L' &= \mathrm{diag}(1, 1, -1), \\ P_\ell' &= \mathrm{diag}(1, 1, 1), & P_R' &= \mathrm{diag}(1, 1, 1). \end{aligned} \tag{3.6}$$

Under this action, the parity assignments of the vector components of the gauge multiplets are

$$\begin{aligned} A_q^\mu &= \begin{pmatrix} (+,+) & (+,+) & (+,+) \\ (+,+) & (+,+) & (+,+) \\ (+,+) & (+,+) & (+,+) \end{pmatrix}, \\ A_L^\mu &= \begin{pmatrix} (+,+) & (+,+) & (-,-) \\ (+,+) & (+,+) & (-,-) \\ (-,-) & (-,-) & (+,+) \end{pmatrix}, \\ A_\ell^\mu &= \begin{pmatrix} (+,+) & (+,+) & (-,+) \\ (+,+) & (+,+) & (-,+) \\ (-,+) & (-,+) & (+,+) \end{pmatrix}, \\ A_R^\mu &= \begin{pmatrix} (+,+) & (+,+) & (-,+) \\ (+,+) & (+,+) & (-,+) \\ (-,+) & (-,+) & (+,+) \end{pmatrix}. \end{aligned} \tag{3.7}$$

The wave functions of these component fields have the Kaluza-Klein (KK) decompositions

$$A^{\mu}_{++}(x^\nu, y) = \frac{1}{\sqrt{2\pi R}} A^{\mu(0)}_{++}(x^\nu) + \frac{1}{\sqrt{\pi R}} \sum_{n=1}^{\infty} A^{\mu(2n)}_{++}(x^\nu) \times \cos\left(\frac{2ny}{R}\right), \quad (3.8)$$

$$A^{\mu}_{-+}(x^\nu, y) = \frac{1}{\sqrt{\pi R}} \sum_{n=0}^{\infty} A^{\mu(2n+1)}_{-+}(x^\nu) \sin\left(\frac{(2n+1)y}{R}\right), \quad (3.9)$$

$$A^{\mu}_{+-}(x^\nu, y) = \frac{1}{\sqrt{\pi R}} \sum_{n=0}^{\infty} A^{\mu(2n+1)}_{+-}(x^\nu) \cos\left(\frac{(2n+1)y}{R}\right), \quad (3.10)$$

$$A^{\mu}_{--}(x^\nu, y) = \frac{1}{\sqrt{\pi R}} \sum_{n=0}^{\infty} A^{\mu(2n+2)}_{--}(x^\nu) \sin\left(\frac{(2n+2)y}{R}\right). \quad (3.11)$$

The towers of four-dimensional fields $A^{\mu(2n)}_{++}(x^\nu)$, $A^{\mu(2n+1)}_{-+}(x^\nu)$, $A^{\mu(2n+1)}_{+-}(x^\nu)$, $A^{\mu(2n+2)}_{--}(x^\nu)$ upon compactification acquire KK masses $2nM_c$, $(2n+1)M_c$, $(2n+1)M_c$, $(2n+2)M_c$, respectively, where $M_c \equiv 1/R$ is defined to be the compactification scale. Only the $(+,+)$ fields possess massless zero modes, and hence the massless sector is restricted to a subset of the full five-dimensional theory. The towers $(-,+)$ and $(+,-)$ have degenerate mass spectra, as do the towers $(-,-)$ and $(+,+)$ excluding the zero mode.

We can now discern the nature of the theory on the branes. In general, only those fields which are odd under $P(P')$ vanish at $y = 0$ $(y' = 0)$. Thus, one can see that at $y = 0$, the gauge symmetry $SU(3)_q \otimes SU(2)_L \otimes SU(2)_\ell \otimes SU(2)_R \otimes U(1)^3$ is respected, while at the fixed point $y = \pi R/2$ the symmetry is $SU(3)_q \otimes SU(2)_L \otimes SU(3)_\ell \otimes SU(3)_R \otimes U(1)_L$. The cumulative effect of the orbifold compactification is the reduction of the overall symmetry to the intersection of these two subgroups, $G \equiv G_{y=0} \cap G_{y=\pi R/2} = SU(3)_q \otimes SU(2)_L \otimes SU(2)_\ell \otimes SU(2)_R \otimes U(1)^3$. Clearly, further symmetry breaking is required to first reproduce, then break, the SM.

As is evident in Eq. (3.5), the scalar components A^5_i necessarily have opposite parities to their vector counterparts allowing one to determine the scalar content at the fixed points. The transformation of A^{5a}_L importantly reveals a scalar doublet which has a massless zero mode at the four-dimensional level, and this possesses the appropriate hypercharge to be identified as a SM Higgs doublet. This type of identification is referred to as gauge-Higgs unification and has been employed extensively as an origin for the electroweak (EW) Higgs in orbifold GUTs [17]. Attention to this sector will be given in Sec. IV B.

B. Brane breaking

After the orbifold compactification, the symmetry of the four-dimensional effective theory is $SU(3)_q \otimes SU(2)_L \otimes SU(2)_\ell \otimes SU(2)_R \otimes U(1)^3$, while on the matter brane the gauge symmetry is $G_{\pi R/2} \equiv SU(3)_q \otimes SU(2)_L \otimes SU(3)_\ell \otimes SU(3)_R \otimes U(1)_L$. The usual approach to breaking this remaining symmetry is to implement boundary conditions on the compactified space for the gauge fields [22]. This method is more generalized than the orbifold mechanism, leading to a greater set of symmetry breaking opportunities.

The structure of the boundary conditions are realized in a UV completion of our GUT theory in which localized scalar fields χ_i prescribe the breaking. As the fields are localized on the matter brane where only the reduced $G_{\pi R/2}$ symmetry is operative, the fields are housed in incomplete quartification multiplets, with transformation properties

$$\chi_{1,2} \sim (\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{3}), \qquad \chi_3 \sim (\mathbf{1}, \mathbf{1}, \mathbf{3}, \mathbf{1}), \qquad \chi_4 \sim (\mathbf{1}, \mathbf{1}, \mathbf{3}, \bar{\mathbf{3}}). \quad (3.12)$$

The three fields $\chi_{1,2,3}$ carry $U(1)_L$ charge $-2/\sqrt{3}$ and are necessary for the $SU(2)_R \otimes U(1)^3 \to U(1)_Y$ breaking. The final Higgs field, χ_4 is neutral under $U(1)_L$. Its VEVs do not increase the symmetry breaking but they are important contributors to fermion mass generation. As will be detailed later, we shall be considering the limit in which this sector decouples entirely from the brane, and so the addition of these scalars does not pose a complication.

The effect of these fields can be ascertained from the kinetic sector of our action. The five-dimensional action assumes the form

$$\mathcal{S} \supset \int d^5x \left(-\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} - \frac{1}{2} F^a_{5\nu} F^{a5\nu} + (D_\mu \chi_i^\dagger D^\mu \chi_i - V(\chi)) \delta(y - \pi R/2) \right), \quad (3.13)$$

and the relevant terms to consider are

$$\int_0^{\pi R/2} dy \left(-\frac{1}{2} \partial_5 A_\mu(x^\nu, y) \partial^5 A^\mu(x^\nu, y) + \frac{g_5^2}{4} \langle\chi\rangle^\dagger \langle\chi\rangle A^\mu(x^\nu, y) A_\mu(x^\nu, y) \delta(y - \pi R/2) \right). \quad (3.14)$$

g_5 is the five-dimensional gauge coupling constant defined by $g_5 \equiv g_4 \sqrt{\frac{\pi R}{2}}$ and has mass dimension $-1/2$. Variation in $A^\mu(x^\nu, y)$ gives, after integration over the extra dimension, the surface terms

$$-\partial_5 A_\mu(x^\nu, y)|_{y=0},$$
$$\left(-\partial_5 A_\mu(x^\nu, y) + \frac{g_5^2}{2} \langle\chi\rangle^\dagger \langle\chi\rangle A_\mu(x^\nu, y) \right) \delta A^\mu(x^\nu, y) \Bigg|_{y=\pi R/2}, \quad (3.15)$$

which must vanish. Making the definition

$$V \equiv \frac{g_5^2}{2} \langle \chi \rangle^\dagger \langle \chi \rangle, \tag{3.16}$$

we obtain the boundary conditions

$$\partial_5 A_\mu(x^\nu, 0) = 0, \tag{3.17}$$

$$\partial_5 A_\mu(x^\nu, \pi R/2) = V A_\mu(x^\nu, \pi R/2), \tag{3.18}$$

which illustrate the constraints imposed on the gauge fields by the boundary Higgs sector. V, having dimensions of mass, reflects the order parameter of the symmetry breaking. If $V = 0$, Eq. (3.18) reduces to Neumann boundary conditions and returns the usual orbifold behavior. If $V \neq 0$, then the zero mode for A_μ is no longer massless, and one has gauge boson mass terms localized at the matter brane as we shall now clarify.

These boundary conditions affect only the fields which are even at $y = \pi R/2$, with the towers corresponding to the parities $(\pm, -)$ disappearing on the matter brane. The nonvanishing fields $A_\mu^{(+,+)}$ have the generic profile

$$A_\mu(x^\nu, y) = N_n \cos(M_n y) A_\mu^{(n)}(x^\nu) \tag{3.19}$$

with KK mass M_n. Enforcement of Eq. (3.18) appreciably modifies the wave functions of these gauge bosons, with their KK masses shifted by an amount governed by

$$M_n \tan(M_n \pi R/2) = -V. \tag{3.20}$$

For large values of V, Eq. (3.20) shows that the KK tower has a mass spectrum approximated by

$$M_n \approx M_c(2n+1)\left(1 + \frac{M_c}{\pi V} + \ldots\right), \qquad n = 0, 1, 2, \ldots \tag{3.21}$$

giving a tower with the lowest-lying states $M_c, 3M_c, 5M_c, \ldots$. This represents an offset of M_c relative to the $V = 0$ tower, with the field no longer retaining a massless zero mode. As $V \to \infty$, the brane localized mass terms for the gauge fields become larger and their wave functions are eventually expelled from the brane. A similar mass shift can be induced by the boundary conditions on the $(-, +)$ fields.

From Eq. (3.16), the association of V with the VEVs of the boundary scalar sector implies that the limit $V \to \infty$ is attained when $\langle \chi \rangle \to \infty$. However, when the VEVs of the Higgs fields are taken to infinity, the shift in the KK masses of the gauge fields is finite, giving the exotic gauge fields masses dependent only upon the compactification scale M_c. Consequently, these fields remain as ingredients in the effective theory while the boundary Higgs sector decouples entirely, and we can view our reduced symmetry theory in an effective Higgless limit. Interestingly, in this limit also, the high-energy behavior of the massive gauge boson scattering remains unspoilt as shown in [22].

Depending on the exact nature of V, a shift can be induced in the KK towers of the gauge fields generating a greater symmetry breaking than the orbifold compactification. The gauge fields are decomposed into one of six possible KK towers depending on the orbifold and boundary conditions, and we need to choose the latter such that the $G_{\pi R/2}/H$ gauge fields are exiled from the brane, while those corresponding to the H symmetry remain unperturbed. Given that the structure of the breaking parameter V is predetermined by the Higgs fields, the desired boundary constraints can be satisfied with our scalar fields acquiring VEVs of the form

$$\langle \chi_1 \rangle = \begin{pmatrix} 0 \\ 0 \\ v_1 \end{pmatrix}, \qquad \langle \chi_2 \rangle = \begin{pmatrix} v_2 \\ 0 \\ v_3 \end{pmatrix}, \tag{3.22}$$

$$\langle \chi_3 \rangle = \begin{pmatrix} 0 \\ 0 \\ v_4 \end{pmatrix}, \qquad \langle \chi_4 \rangle = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ v_5 & 0 & v_6 \end{pmatrix}. \tag{3.23}$$

The fields $\chi_{1,2,3}$ and χ_4 have a $U(1)_L$ charge of $-2/\sqrt{3}$ and 0, respectively, where these assignments and the VEV structure are motivated by the embedding of the fields in the conventional quartification models [14].

These fields transform trivially under $SU(3)_q \otimes SU(2)_L$ and thus the gauge fields defining these symmetries remain unaffected by the scalars. The $SU(3)_\ell \otimes SU(3)_R$ gauge fields, however, are subject to the constraint

$$\partial_5 A_\mu^i(x^\nu, \pi R/2) = V_{ij} A_\mu^j(x^\nu, \pi R/2), \tag{3.24}$$

where V_{ij} is a matrix in the space of these fields. For the non-Abelian components this can be decomposed into the form

$$\partial_5 A_\ell^{\mu a}(x^\nu, \pi R/2) = 0, \qquad a = 1, 2, 3, \tag{3.25}$$

$$\partial_5 A_\ell^{\mu \hat{a}}(x^\nu, \pi R/2) = V_\ell A_\ell^{\mu \hat{a}}(x^\nu, \pi R/2), \qquad \hat{a} = 4, 5, 6, 7, \tag{3.26}$$

$$\partial_5 A_R^{\mu \hat{a}}(x^\nu, \pi R/2) = V_R A_R^{\mu \hat{a}}(x^\nu, \pi R/2), \quad \hat{a} = 1, 2, 4, 5, 6, 7, \tag{3.27}$$

where the a's refer to the unbroken group indices and $\hat{a}$ the broken indices. The $V_{\ell,R}$ entries have the general form $\sum_i c_i g_5^2 v_i^2$, for some constant c_i, with their precise values determined by the dynamics of the UV completed theory.

The constraints on the $U(1)$ factors are more nontrivial. The hypercharge gauge field is

$$A_Y^\mu = \left(\frac{1}{\sqrt{3}}(A_L^8 + A_\ell^8 + A_R^8) + A_R^3\right)^\mu, \tag{3.28}$$

and this must remain massless in the low-energy theory. Subsequently, this linear combination must correspond to the sole zero eigenvalue of the relevant subset of the generalized V matrix of Eq. (3.24) defined by

$$\partial_5 \hat{A}_i^{\mu}(x^\nu, \pi R/2) = V_{ij}\hat{A}_j^{\mu}, \qquad i, j = 1, 2, 3, 4, \tag{3.29}$$

where

$$\hat{A}^{\mu} \equiv (A_L^{8,\mu}(x^\nu, \pi R/2), A_\ell^{8,\mu}(x^\nu, \pi R/2), A_R^{3,\mu}(x^\nu, \pi R/2), A_R^{8,\mu}(x^\nu, \pi R/2))^T. \tag{3.30}$$

The explicit form of V_{ij} was determined in Ref. [19] for a model described by the trinification gauge group, with the entries functions of $g_5^2 v_i^2$ as defined by the VEV structure. This parametrization generalizes to the quartification case and reveals a zero eigenvalue for the vector $1/\sqrt{3}\{1, 1, \sqrt{3}, 1\}$ and nonzero eigenvalues for the remaining three eigenvectors, denoted as $A_{X_1}^{\mu}$, $A_{X_2}^{\mu}$, and $A_{X_3}^{\mu}$. Consequently, the $(+,+)$ towers corresponding to these latter three physical fields have their mass spectra shifted by M_c and they are expelled from the low-energy theory.

The effect of the implementation of these boundary conditions on the 32 quartification gauge fields is summarized in Table I where the fields have been listed in terms of their $G_{\pi R/2}$ representations. The mass spectra shown incorporates any induced shifts due to nontrivial values of V, from which it can be deciphered that only the vector fields respecting $G_{\rm SM} \otimes SU(2)_\ell$ possess a massless mode, with the other $(+,+)$ towers misaligned. The new mass spectra of these shifted towers is now degenerate with the $(-,+)$ towers. Furthermore, the only scalar that has a massless mode transforms as an $SU(2)_L$ doublet (and its conjugate). This mode has a component that cannot be gauged away, giving us a physical electroweak Higgs field.

To summarize, the orbifold compactification alone reduces the gauge symmetry on the two fixed points to

$$G_0 = SU(3)_q \otimes SU(2)_L \otimes SU(2)_\ell \otimes SU(2)_R \otimes U(1)^3, \tag{3.31}$$

$$G_{\pi R/2} = SU(3)_q \otimes SU(2)_L \otimes SU(3)_\ell \otimes SU(3)_R \otimes U(1)_L, \tag{3.32}$$

which reduces the overall symmetry of the low-energy four-dimensional theory to $SU(3)_q \otimes SU(2)_L \otimes SU(2)_\ell \otimes SU(2)_R \otimes U(1)^3$. The boundary scalar sector then instigates the further breaking

$$G_{\pi R/2} \rightarrow SU(3)_q \otimes SU(2)_L \otimes SU(2)_\ell \otimes U(1)_Y \tag{3.33}$$

TABLE I. The decomposition of the 32 gauge fields into their respective KK towers at the fixed point $y = \pi R/2$. Their representations are given in terms of the brane symmetry before the additional breaking via the localized Higgs sector. The towers $(\pm, -)$ have wave functions which are odd on this brane, and thus are vanishing.

Fields	Representation	KK tower	Mass
$A_q^{\mu a}$, $a = 1, 2, \ldots, 8$	$(\mathbf{8}, \mathbf{1}, \mathbf{1}, \mathbf{1})$	$(+, +)$	$2nM_c$
A_q^{5a}, $a = 1, 2, \ldots, 8$	$(\mathbf{8}, \mathbf{1}, \mathbf{1}, \mathbf{1})$	$(-, -)$	$2(n+1)M_c$
$A_L^{\mu a}$, $a = 1, 2, 3$	$(\mathbf{1}, \mathbf{3}, \mathbf{1}, \mathbf{1})$	$(+, +)$	$2nM_c$
A_L^{5a}, $a = 1, 2, 3$	$(\mathbf{1}, \mathbf{3}, \mathbf{1}, \mathbf{1})$	$(-, -)$	$2(n+1)M_c$
$A_L^{\mu\hat{a}}$, $\hat{a} = 4, 5, 6, 7$	$(\mathbf{1}, \mathbf{2}, \mathbf{1}, \mathbf{1}) + (\mathbf{1}, \mathbf{2}, \mathbf{1}, \mathbf{1})$	$(-, -)$	$2(n+1)M_c$
$A_L^{5\hat{a}}$, $\hat{a} = 4, 5, 6, 7$	$(\mathbf{1}, \mathbf{2}, \mathbf{1}, \mathbf{1}) + (\mathbf{1}, \mathbf{2}, \mathbf{1}, \mathbf{1})$	$(+, +)$	$2nM_c$
$A_\ell^{\mu a}$, $a = 1, 2, 3$	$(\mathbf{1}, \mathbf{1}, \mathbf{3}, \mathbf{1})$	$(+, +, V = 0)$	$2nM_c$
A_ℓ^{5a}, $a = 1, 2, 3$	$(\mathbf{1}, \mathbf{1}, \mathbf{3}, \mathbf{1})$	$(-, -)$	$2(n+1)M_c$
$A_\ell^{\mu\hat{a}}$, $\hat{a} = 4, 5, 6, 7$	$(\mathbf{1}, \mathbf{1}, \mathbf{2}, \mathbf{1}) + (\mathbf{1}, \mathbf{1}, \mathbf{2}, \mathbf{1})$	$(+, -)$	$(2n+1)M_c$
$A_\ell^{5\hat{a}}$, $\hat{a} = 4, 5, 6, 7$	$(\mathbf{1}, \mathbf{1}, \mathbf{2}, \mathbf{1}) + (\mathbf{1}, \mathbf{1}, \mathbf{2}, \mathbf{1})$	$(-, +)$	$(2n+1)M_c$
$A_R^{\mu a}$, $a = 1, 2$	$\subset (\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{3})$	$(+, +, V \rightarrow \infty)$	$(2n+1)M_c$
A_R^{5a}, $a = 1, 2$	$\subset (\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{3})$	$(-, -)$	$2(n+1)M_c$
$A_R^{\mu a}$, $a = 4, 5, 6, 7$	$(\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{2}) + (\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{2})$	$(-, +)$	$(2n+1)M_c$
A_R^{5a}, $a = 4, 5, 6, 7$	$(\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{2}) + (\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{2})$	$(+, -)$	$(2n+1)M_c$
A_Y^{μ}	$(\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1})$	$(+, +, V = 0)$	$2nM_c$
$A_{X_1}^{\mu}$	$(\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1})$	$(+, +, V \rightarrow \infty)$	$(2n+1)M_c$
$A_{X_2}^{\mu}$	$(\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1})$	$(+, +, V \rightarrow \infty)$	$(2n+1)M_c$
$A_{X_3}^{\mu}$	$(\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1})$	$(+, +, V \rightarrow \infty)$	$(2n+1)M_c$

returning the desired $G_{\rm SM} \otimes SU(2)_\ell$ symmetry operative at the electromagnetic level.

IV. FERMION MASSES

A. GUT scale masses

We now introduce matter into our scheme. Because of the identification of the Higgs field as a remnant zero mode arising from the extra-dimensional component of the gauge fields, if the fermions were bulk fields, their coupling with the Higgs would be universal. Thus, it would prove difficult to recover the detailed structure of Yukawa couplings in the low-energy theory. As an aside we note that it may be interesting to study a higher-dimensional quartification model employing split fermions. The high degree of symmetry present in the quartification model can be utilized to motivate the fermion localization pattern of a split fermion model [23], though presumably a different symmetry breaking mechanism to that employed here would be required. In this work we shall assume that the fermions are localized on the $y = \pi R/2$ brane where they are housed in representations that need only respect the $SU(3)_q \otimes SU(2)_L \otimes SU(3)_\ell \otimes SU(3)_R \otimes U(1)_L$ symmetry.

Although blind to the entire $SU(3)^4$ symmetry, we consider the full fermion content of minimal quartification so that we can achieve anomaly cancellation. The exotic content, it transpires, will become massive through couplings with the boundary Higgs sector. The fermion representations under the brane symmetry are defined as follows:

$$q_1 \sim (\mathbf{3}, \mathbf{2}, \mathbf{1}, \mathbf{1})\left(\frac{1}{\sqrt{3}}\right), \qquad q_2 \sim (\mathbf{3}, \mathbf{1}, \mathbf{1}, \mathbf{1})\left(-\frac{2}{\sqrt{3}}\right), \qquad q^c \sim (\bar{\mathbf{3}}, \mathbf{1}, \mathbf{1}, \mathbf{3})(0), \tag{4.1}$$

$$\ell_1 \sim (\mathbf{1}, \mathbf{2}, \bar{\mathbf{3}}, \mathbf{1})\left(-\frac{1}{\sqrt{3}}\right), \qquad \ell_2 \sim (\mathbf{1}, \mathbf{1}, \bar{\mathbf{3}}, \mathbf{1})\left(\frac{2}{\sqrt{3}}\right), \qquad \ell^c \sim (\mathbf{1}, \mathbf{1}, \mathbf{3}, \bar{\mathbf{3}})(0), \tag{4.2}$$

whose matrix representations can be deduced from Eq. (2.3).

The boundary Higgs fields χ_i couple to the localized fermions giving six invariant Yukawa interaction terms

$$\begin{matrix} Y_1 q^c q_2 \bar{\chi}_1 & Y_2 q^c q_2 \bar{\chi}_2, & Y_3 \ell_1 \ell_1 \bar{\chi}_3, \\ Y_4 \ell_2 \ell^c \chi_1, & Y_5 \ell_2 \ell^c \chi_2, & Y_6 \ell^c \ell^c \chi_4. \end{matrix} \tag{4.3}$$

Equation (4.3) gives the quark mass terms

$$(Y_1 v_1 + Y_2 v_3) h^c h + Y_2 v_2 d^c h, \tag{4.4}$$

revealing mixing between the h^c and d^c, giving only one linear combination that is massless. This we identify to be the physical left-handed d antiquark. Similarly, the exotic leptonic components all mix. These mixings are sufficient to generate GUT scale Dirac masses for all but one exotic lepton, leaving only the SM fields and the right-handed neutrino as the low-energy massless fermion spectrum. As these mass terms are proportional to v_i, in the decoupling limit $v_i \to \infty$, all the exotic fermions are removed from the low-energy theory along with the χ_i fields.

B. Electroweak scale masses and the Higgs

In addition to the fermion spectrum, we have an electroweak scalar doublet arising from the fields $A_L^{5\hat{a}}$ which has a zero mode on the brane. We denote this doublet as $\Phi \sim (\mathbf{1}, \mathbf{2}, \mathbf{1}, \mathbf{1})(-1)$. What remains now is to induce electroweak symmetry breaking and to generate the correct Yukawa structure of these fields with the fermions. However, given that the SM Higgs is a remnant of the higher-dimensional gauge sector, its dynamics are still dictated by the bulk theory. Local couplings between Φ and the fermion fields are restricted by the higher-dimensional gauge invariance. This can be seen explicitly by looking at the gauge transformation parameters. A general $SU(3)^4$ gauge transformation is defined by

$$U = e^{i[\eta(x^\nu)\xi(y)]_j^a T_j^a}, \qquad j = q, L, \ell, R, \tag{4.5}$$

where we have assumed a separable form for the gauge parameters. At the fixed points the y-dependent parameters corresponding to the broken generators satisfy $\xi_j^{\hat{a}} = 0$ and $\partial_5 \xi_j^{\hat{a}}|_{y_f} \neq 0$. A remnant of the bulk symmetry at $y = \pi R/2$, however, is the condition $A_i^{5\hat{a}} \to A_i^{5\hat{a}} + \partial_5 \xi_i^{\hat{a}}$. This shift is broken by any Yukawa couplings directly involving Φ. As a result, the only gauge invariant interactions between the brane fermions and the SM Higgs field are nonlocal, involving Wilson lines.

The Wilson line operator is defined as

$$W = \mathcal{P}\exp\left(i\int_{y_i}^{y_f} \Phi dy\right)\bigg|_{\mathbf{R}}, \tag{4.6}$$

in the representation $\mathbf{R}$ where $\mathcal{P}$ is the path-ordered product, and y_i and y_f are fixed points of the orbifold. If $y_1 = y_2 = \pi R/2$, this operator is linear under the gauge transformations and thus can couple to the fermion fields. Its nonlocality means that the Higgs potential generated at the quantum level is also nonlocal and thus insensitive to the UV physics. We assume that the generation of the Wilson lines is due to some dynamics in the UV completion of the theory. Irrespective of their origin, these operators are natural entities to consider, satisfying invariance under both the gauge symmetry and the orbifold projections.

The nonlocal Lagrangian has the general form

$$\mathcal{L}^{\rm NL} = \lambda f_1|_{y=y_i} W \chi|_{y=y_f} + \lambda' f_2|_{y=y_i} \bar{W}\,\bar{\chi}\,|_{y=y_f} + \dots, \tag{4.7}$$

where $f_1, f_2, \dots$ are localized fermion bilinears, and each term is suppressed by the appropriate power of the fundamental scale Λ. If the Higgs field acquires a nontrivial VEV, then its nonlocal interaction with the brane fields generates the effective four-dimensional local Yukawa terms

$$\frac{1}{\Lambda}\lambda_1\bar{\chi}_1 q_1 q^c\Phi, \quad \frac{1}{\Lambda}\lambda_2\bar{\chi}_2 q_1 q^c\Phi, \quad \frac{1}{\Lambda^2}\lambda_3\chi_1\chi_2 q_1 q^c\bar{\Phi}$$
$$\frac{1}{\Lambda}\lambda_4\bar{\chi}_4\ell_1\ell_2\Phi, \quad \frac{1}{\Lambda^2}\lambda_5\bar{\chi}_1\bar{\chi}_4\ell_1\ell_2\Phi,$$
$$\frac{1}{\Lambda^2}\lambda_6\bar{\chi}_2\bar{\chi}_4\ell_1\ell_2\Phi, \quad \frac{1}{\Lambda}\lambda_7\chi_1\ell_1\ell^c\bar{\Phi}, \quad \frac{1}{\Lambda}\lambda_8\chi_2\ell_1\ell^c\bar{\Phi},$$
$$\frac{1}{\Lambda^2}\lambda_9\chi_3\bar{\chi}_4\ell_1\ell^c\bar{\Phi}, \quad \frac{1}{\Lambda^2}\lambda_{10}\bar{\chi}_1\bar{\chi}_2\ell_1\ell^c\Phi. \tag{4.8}$$

These terms arise after the boundary Higgs sector has developed its VEV structure. As commented in Ref. [18], these terms remain at the electroweak order as $\chi/\Lambda \sim \mathcal{O}(1)$ in the decoupling limit $v_i \to \Lambda \to \infty$, with their interactions dictated only by $G_{\rm SM} \otimes SU(2)_\ell$ not the UV physics.

These couplings return electroweak scale Dirac masses to the SM quarks and GUT scale masses to the exotic quarks, with mixing between the d and h quarks heavily suppressed. Analysis of the leptonic sector reveals that the electron does not mix at all with the exotic particles, and the mixing between the neutrinos and exotic neutral singlets is significantly suppressed. There are GUT scale Dirac masses for all the exotic particles, while the electron and neutrino acquire electroweak scale Dirac masses. We shall not concern ourselves with naturally generating seesaw suppressed Majorana masses for the neutrinos. We suspect that the inclusion of gauge singlet fermions with Majorana masses at the unification scale will invoke a seesaw style suppression of the electroweak scale Dirac masses, as occurs in four-dimensional models [13]. However, we shall settle for highly tuned Dirac Yukawa coupling constants in what follows. Nevertheless, the Yukawa interactions that are induced on the brane by the Wilson line operators yield an appealing fermion mass spectrum. We note that as our construct is nonsupersymmetric the usual fine-tuning is required to stabilize the electroweak scale relative to the unification scale.

V. GAUGE COUPLING UNIFICATION

We turn now to calculating the unification scale of our scheme. To evaluate the evolution of the gauge coupling constants, we need to ascertain how exactly the presence of the extra dimension affects the renormalization group equations. We essentially have two mass scales to consider: the compactification scale which characterizes the size of the extra dimension, and the cutoff or unification scale Λ.

Corrections due to the physics in the UV regime arise in the form of brane localized operators. These respect the $G_{\rm SM} \otimes SU(2)_\ell$ symmetry, and even if set to zero at tree-level can be generated by radiative corrections. The effects of these operators, however, can be tamed, and much effort has been expended on ensuring that their presence does not destroy the higher-dimensional unification [16,24,25]. The zero-mode coupling in the effective four-dimensional theory is obtained by integrating the five-dimensional action for the gauge fields, and this estimates the value at the cutoff to be

$$\frac{1}{g_i^2(\Lambda)} = \frac{2\pi R}{g_5^2(\Lambda)} + \frac{1}{\tilde{g}_{4i}^2(\Lambda)}. \tag{5.1}$$

g_5 is the $SU(3)^4$-invariant coupling parameter while $g_{\tilde{4}i}$ is the dimensionless coupling constant arising from these brane localized operators. If the volume factor of the extra dimension is sufficiently large then the sensitivity to these brane corrections is diluted by the bulk contribution [24]. If $g_5^2 = \zeta/\Lambda$, $\tilde{g}_4^2 = \zeta_a$ then the suppression is given by $(\zeta/\zeta_a)(1/\Lambda\pi R)$. So as long as the extra dimension is sufficiently large in extent, the volume factor $\Lambda\pi R$ is sufficient to negate the brane modification of $g_i(\Lambda)$.

What remains is to determine the threshold contributions which arise at the compactification scale M_c. At energies below M_c, the extra dimension is not observable. The zero modes of the KK gauge boson towers and brane localized fermions define our effective theory to be that of the SM supplemented with the additional $SU(2)_\ell$ symmetry. Within this regime, the evolution of the coupling constants proceeds via the usual renormalization group equations

$$\frac{1}{\alpha_i(M_c)} = \frac{1}{\alpha_i(M_{\rm EW})} + \frac{b_i}{2\pi}\ln\frac{M_{\rm EW}}{M_c}. \tag{5.2}$$

$\alpha_i = g_i^2/4\pi$ and the b_i are the beta coefficients which enumerate both the number and type of particles which contribute to the evolution.

Above M_c the extra dimension is manifest through the appearance of the infinite towers of KK modes with increasing mass, and our effective theory is the 5-dimensional GUT with symmetry group $SU(3)^4$. The towers, however, do not have a universal contribution to the running of the coupling constants. This is in part because they do not fill complete $SU(3)^4$ representations, and also due to the misalignment of the towers resulting from the brane breaking. At M_c the $G = SU(3)_q \otimes SU(2)_L \otimes SU(3)_\ell \otimes SU(3)_R \otimes U(1)$ symmetry is valid, with the modes corresponding to the G/H fields starting to appear. These states contribute at each $(2n+1)M_c$ level. As the evolution proceeds from M_c to $2M_c$ the full quartification symmetry emerges with the $SU(3)^4/G$ states inputting at the $(2n+2)M_c$ energy levels. At each nth level, new excitation modes contribute until the couplings merge at $M_{\rm GUT}$. It has been shown that if the difference on the runnings $\delta_i(\mu) \equiv \alpha_i^{-1}(\mu) - \alpha_j^{-1}(\mu)$ is considered then these KK modes dominate the evolution above M_c, with the differential running logarithmically sensitive to the cutoff as per [16,24,25]

$$\delta_i(M_{\rm GUT}) = \delta_i(M_{\rm EW}) + \frac{(\beta_i - \beta_j)}{2\pi} \ln\frac{M_{\rm EW}}{M_c} - \frac{1}{2\pi}\Delta_i(M_{\rm GUT}), \tag{5.3}$$

where

$$\Delta_i(M_{\rm GUT}) = (\beta_i - \beta_j)\ln\frac{M_{\rm GUT}}{M_c} + (\gamma_i - \gamma_j) \times \sum_{n=0}^{N_k} \ln\frac{M_{\rm GUT}}{(2n+1)M_c} + (\eta_i - \eta_j) \times \sum_{n=0}^{N_k} \ln\frac{M_{\rm GUT}}{(2n+2)M_c}. \tag{5.4}$$

Here, β_i are the zero-mode beta coefficients, γ_i are the beta coefficients of the modes with mass $(2n+1)M_c$ and η_i those of the modes with mass $(2n+2)M_c$. These factors compensate for the varying contributions of the KK levels. N_k is the level at which the KK towers are truncated, with $(2N_k+1)M_c \lesssim M_{\rm GUT}$. We choose to explore the running with respect to the evolution of α_Y, taking $j = Y$.

Table II lists the beta coefficients for all the KK modes, where the gauge bosons have been relabeled in terms of their $G_{\rm SM} \otimes SU(2)_\ell$ representations. The multiplet $A_{\rm SM'}$ consists of the $G_{\rm SM} \otimes SU(2)_\ell$ gauge fields, while the EW Higgs is contained in $A_{(\mathbf{1},\mathbf{2},\mathbf{1})(\pm 1)}$. The remaining two multiplets represent exotic fields contained in G/H. These arise from the A_R^μ fields aligned along the T^4, T^5 directions and those corresponding to $A_R^{\mu a=1,2}$ whose KK masses have received a shift.

Given that the exotic states begin to emerge at M_c, the compactification scale has the lower bound $M_c \gtrsim 1$ TeV. If unification could result at this compactification energy, then there would be a promising spectrum of exotic particles within reach of the LHC. Unfortunately the three SM coupling parameters do not unify with a low M_c. The

TABLE II. The enumeration of the KK modes and their contribution to the renormalization group equations. The b's represent the zero-mode beta coefficients, while $\tilde{b}_i$ reflect the excitation modes. Here the decomposition of the five-dimensional gauge fields is in terms of their H representation.

Multiplet	(b_Y, b_L, b_q, b_ℓ)	$(\tilde{b}_Y, \tilde{b}_L, \tilde{b}_q, \tilde{b}_\ell)$
$A_{\rm SM'}$	$(0, -\frac{22}{3}, -11, -\frac{22}{3})$	$(0, -\frac{20}{3}, -10, -\frac{20}{3})$
$A_{(\mathbf{1},\mathbf{2},\mathbf{1})(-1)}$	$(\frac{1}{12}, \frac{1}{6}, 0, 0)$	$(-\frac{5}{6}, -\frac{7}{2}, 0, 0)$
$(\eta_Y, \eta_L, \eta_q, \eta_\ell)$		$(-\frac{5}{6}, -\frac{61}{6}, -10, -\frac{20}{3})$
$A_{(\mathbf{1},\mathbf{1},\mathbf{2})(-1)}$	$\cdots$	$(-\frac{5}{6}, 0, 0, -\frac{7}{2})$
$A_{(\mathbf{1},1,1)(\pm 2)}$	$\cdots$	$(-\frac{10}{3}, 0, 0, 0)$
$(\gamma_Y, \gamma_L, \gamma_q, \gamma_\ell)$		$(-\frac{25}{6}, 0, 0, -\frac{7}{2})$
Matter	$(\frac{10}{3}, 4, 4, 0)$	$\cdots$
Total	$(\frac{41}{12}, -\frac{19}{6}, -7, -\frac{22}{3})$	$(-5, -\frac{61}{6}, -10, -\frac{61}{6})$

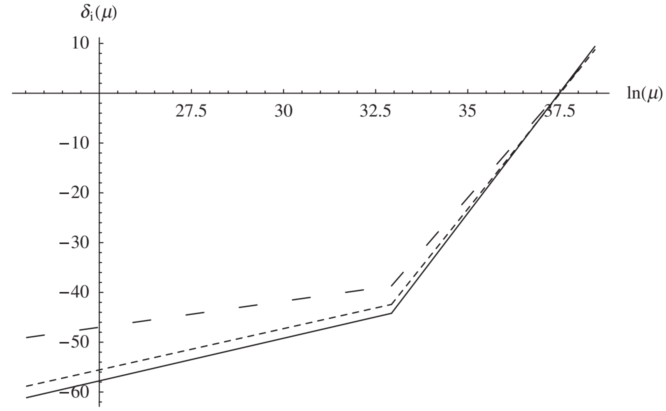

FIG. 1. The differential running of the SM gauge coupling constants for a compactification scale of $M_c \sim 2 \times 10^{14}$ GeV as a function of $\ln(\mu)$. The solid line is δ_ℓ, the short-dashed line is δ_q, the long-dashed line is δ_L, and the condition for unification is $\delta_\ell = \delta_q = \delta_L = 0$. This occurs at $M_{\rm GUT} \sim 1.9 \times 10^{16}$ GeV.

lowest value of the compactification scale which is consistent with approximate unification is $M_c \sim 10^{10}$ GeV, with unification occurring at $M_{\rm GUT} \sim 10^{12}$ GeV. As we increase the compactification scale further, the energy interval between M_c and $M_{\rm GUT}$ decreases slightly, and fewer KK modes contribute to the running. The most favorable unification scenario occurs for $M_c \sim 2 \times 10^{14}$ GeV. Here we have ~50 KK states for each tower in the summation with the coupling constants intersecting at $M_{\rm GUT} \sim 1.9 \times 10^{16}$ GeV as shown in Fig. 1. In this case, the leptonic color coupling parameter must be $\alpha_\ell \sim 0.19$ at the electroweak level, surpassing the electroweak scale value of the strong coupling constant. We reiterate that the differential running has been defined relative to the hypercharge coupling constant so that

$$\delta_{q,L,\ell}(\mu) = \alpha^{-1}_{q,L,\ell}(\mu) - \alpha_Y^{-1}(u). \tag{5.5}$$

To complete the unification analysis we determine the largest possible compactification scale which allows for unification. As M_c increases in energy, necessarily so too does the unification scale. Thus the upper bound on M_c will result when the GUT scale is identified as the five-dimensional Planck scale. This is attained for $M_c \sim 10^{16}$ GeV which gives unification at $M_{\rm GUT} \sim 8.2 \times 10^{17}$ GeV. To summarize we find that the relation $\delta_i(M_{\rm GUT}) = 0$, $\forall i = q, L, \ell, Y$ is satisfied only for a compactification scale of order 10^{10} GeV $< M_c < 10^{16}$ GeV, with the corresponding unification scale lying in the range 10^{12} GeV $< M_{\rm GUT} < 8 \times 10^{17}$ GeV. Within these bounds, the amalgamation of the strong and electroweak couplings constrains the value of the leptonic color coupling parameter to be $0.08 \lesssim \alpha_\ell(M_{\rm EW}) \lesssim 0.25$ at the electroweak scale.

VI. A MODEL WITHOUT $SU(2)_\ell$

We now briefly comment on five-dimensional quartified models which do not have a residual leptonic $SU(2)_\ell$ symmetry, i.e. $H = G_{SM}$. There exists only a limited approach to extend upon the previous scheme in which this symmetry remained in the low-energy theory. We require the breaking of the leptonic symmetry to be achieved by a combination of the orbifolding and brane localized scalar fields.

To obtain a consistent low-energy model, the orbifold projections must be

$$\begin{aligned} P_q &= \mathrm{diag}(1, 1, 1), & P_L &= \mathrm{diag}(1, 1, -1),\\ P_\ell &= \mathrm{diag}(1, 1, 1), & P_R &= \mathrm{diag}(1, 1, -1),\\ P'_q &= \mathrm{diag}(1, 1, 1), & P'_L &= \mathrm{diag}(1, 1, -1),\\ P'_\ell &= \mathrm{diag}(1, 1, -1), & P'_R &= \mathrm{diag}(1, 1, 1), \end{aligned} \tag{6.1}$$

which break the quartification symmetry to $SU(3)_q \otimes SU(2)_L \otimes SU(2)_\ell \otimes SU(3)_R \otimes U(1)_L \otimes U(1)_\ell$ on the matter brane. As the quarks are singlets under leptonic color, we still require the full $SU(3)_R$ symmetry on the brane and the fields χ_1 and χ_2 to generate mass terms. Hence, our quark sector will be identical to that of the previous model. To instigate the breaking to the SM gauge group we need also the additional boundary scalar fields

$$\begin{aligned} \chi_3 &\sim (\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{3})(0, -2/\sqrt{3}),\\ \chi_4 &\sim (\mathbf{1}, \mathbf{1}, \mathbf{2}, \mathbf{1})(-2/\sqrt{3}, -1/\sqrt{3}),\\ \chi_5 &\sim (\mathbf{1}, \mathbf{1}, \mathbf{2}, \mathbf{3})(0, 1/\sqrt{3}), \end{aligned} \tag{6.2}$$

with the VEV patterns

$$\langle\chi_3\rangle = \begin{pmatrix} 0\\ 0\\ v_4 \end{pmatrix}, \qquad \langle\chi_4\rangle = \begin{pmatrix} v_5\\ 0 \end{pmatrix}$$
$$\langle\chi_5\rangle = \begin{pmatrix} v_6 & 0 & v_7\\ 0 & v_8 & 0 \end{pmatrix}. \tag{6.3}$$

We now have only four charged lepton fields per family and they all mix. The Yukawa interaction terms generated locally through $\chi_1 - \chi_4$ and those induced nonlocally with Φ impart an electroweak scale Dirac mass to the electron, muo,n and tauon particles. The remaining three eigenvalues per family are all of GUT scale. The neutral particles again have Dirac masses, with four of the eigenvalues at the GUT scale, and the other of EW order which we identify with the neutrino. Thus the brane localized Higgs fields break the symmetry down to the SM gauge group and deliver mass terms to the exotic particles. At the electroweak scale, the theory comprises of the minimal SM particle spectrum and the right-handed neutrino. The exotic fermions all have heavy masses and are decoupled from the theory in the Higgsless limit.

Above the compactification scale, the excitation modes of the SM gauge fields and the multiplet containing the SM Higgs field appear at each $(2n + 2)M_c$ level. Four exotic singlets with hypercharge $Y = \pm 2$ also emerge at the $(2n + 1)M_c$ energy levels, with three of these arising from the shifted $(+, +)$ towers of A^μ_ℓ and A^μ_R.

We have investigated the prospects for unification with this new symmetry framework. We find that the fewer states contributing at each $(2n + 1)M_c$ level fail to sway the coupling constants to unify in a phenomenologically consistent fashion. The couplings converge only when $M_c \geq 10^{16}$ GeV. However, for compactification scales this large, the unification always occurs at energies greater than the five-dimensional Planck scale and so cannot result. It is interesting that unification within our five-dimensional framework requires $SU(2)_\ell$ to remain as an exact symmetry.

VII. COMPARISON WITH CONVENTIONAL MODELS

We have shown that unification can be achieved within a five-dimensional quartification model. Before concluding we shall contrast the five-dimensional construct with the conventional quartification models of [13,14]. Our intention is to draw the reader's attention to the advantages and disadvantages of each approach.

Let us first consider the case of unbroken $SU(2)_\ell$. In the four-dimensional case unification could be achieved with four distinct models, each differing in their symmetry breaking pattern. No predictions regarding the favorability of the distinct models was possible with all of these models permitting a flexible range of symmetry breaking scales consistent with unification. However, intermediate symmetry breaking scales were required, otherwise unification mandates the inclusion of 14–37 SM Higgs doublets. The models with intermediate breaking scales required eight distinct scalar $SU(3)^4$ multiplets (related by the cyclic Z_4 symmetry), and these naturally returned seven SM Higgs doublets. Furthermore, this results in complicated Higgs potentials with many arbitrary parameters. These parameters must be tuned to generate the desired stages of symmetry breaking, with further assumptions required to ensure that the masses of the scalars were consistent with unification. The GUT scale was found to lie in the range 10^{12}–10^{18} GeV and proton decay inducing scalars were required to be of order the GUT scale to ensure proton longevity. Finally we note that several of the four-dimensional models predicted that the $SU(2)_R$ W and Z' bosons should be order TeV and are thus observable at the LHC.

We contrast this with the five-dimensional case. Here the demands of a realistic SM fermion spectrum restricted the intermediate symmetry group on the matter brane $G_{\pi R/2}$, resulting in just one feasible choice, namely, $G_{\pi R/2} \equiv SU(3)_q \otimes SU(2)_L \otimes SU(3)_\ell \otimes SU(3)_R \otimes U(1)_L$. We have shown that unification can be achieved via this route, occurring for a range of compactification scales, $10^{10} \lesssim M_c \lesssim 10^{16}$ GeV. Only one SM Higgs doublet arose in the five-dimensional construct and this was adequate to achieve unification. The reduced symmetry operative on the matter brane meant that brane localized scalars need not fill out entire $SU(3)^4$ representations. Thus no colored scalars were required and the proton was found to be stable. This has the advantage of permitting unification at low scales (relative to typical four-dimensional unification scales). The five-dimensional framework naturally motivates the intermediate mass scales necessary for unification. These are introduced in a somewhat *ad hoc* way in four-dimensional constructs, but here they arise as KK excitations of bulk fields. Thus all intermediate mass scales are set by the one parameter, M_c, and the inclusion of intermediate mass scales does not imply an increase in parameter numbers. As we consider the Higgsless limit, the complications which arise in the Higgs sector of four-dimensional models essentially disappear. However, no new phenomenology appears in our model until the scale M_c and given the large M_c values required to achieve unification this prohibits the observation of any new particles at the LHC. This is similar to the five-dimensional trinification models of [18] which reproduce the minimal supersymmetric SM at low energies but predict no additional phenomenology.

The four-dimensional models with broken $SU(2)_\ell$ found to unify in [14] possessed essentially the same features as those outlined above for the unbroken $SU(2)_\ell$ case and thus provide no real advantage over these models. Given that unification cannot be achieved without a remnant $SU(2)_\ell$ symmetry in our five-dimensional model we shall not comment further on these.

VIII. CONCLUSION

We have investigated five-dimensional quartification models where the symmetry breaking is achieved by a combination of orbifold compactification and the introduction of a boundary Higgs sector. We have shown that the SM Higgs doublet may be identified as the fifth component of a higher-dimensional gauge field. This forces matter to be brane localized, with the SM Yukawa structure arising from fermion couplings with Wilson line operators. The models may be considered in a Higgsless limit wherein all gauge fields corresponding to generators broken above the electroweak scale have their mass set by the compactification scale. As in four-dimensional models, intermediate mass scales were required to ensure unification. However, only one arbitrary scale, namely, the compactification scale, was required, with the embedding of the quartification model in a higher-dimensional framework naturally introducing a new class of threshold corrections to the running coupling constants, corresponding to KK excitations of bulk gauge fields. Surprisingly we found that a unique symmetry breaking pattern consistent with both unification and phenomenological demands emerged from our framework. This required $SU(2)_\ell$ to remain as an exact low-energy symmetry and differed markedly from the four-dimensional case where multiple symmetry breaking routes consistent with unification have been found for both broken and unbroken $SU(2)_\ell$ models. Importantly the higher-dimensional model alleviates the complications arising from the Higgs sector of four-dimensional models by rendering this sector supplementary.

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