Correlation between leptonic CP violation and μ-τ symmetry breaking

Teppei Baba[*] and Masaki Yasuè[†]

Department of Physics, Tokai University, 1117 Kitakaname, Hiratsuka, Kanagawa 259-1292, Japan

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Considering the μ-τ symmetry, we discuss a direct linkage between phases of flavor neutrino masses and leptonic CP violation by determining three eigenvectors associated with $\mathbf{M} = M_\nu^\dagger M_\nu$ for a complex flavor neutrino mass matrix M_ν in the flavor basis. Since the Dirac CP violation is absent in the μ-τ symmetric limit, leptonic CP violation is sensitive to the μ-τ symmetry breaking, whose effect can be evaluated by perturbation. It is found that the Dirac phase (δ) arises from the μ-τ symmetry breaking part of $\mathbf{M}_{e\mu,e\tau}$ and an additional phase (ρ) is associated with the μ-τ symmetric part of $\mathbf{M}_{e\mu,e\tau}$, where $\mathbf{M}_{ij}$ stands for an ij matrix element ($i, j = e, \mu, \tau$). The phase ρ is redundant and can be removed but leaves its effect in the Dirac CP violation characterized by $\sin(\delta + \rho)$. The perturbative results suggest the exact formula of mixing parameters including that of δ and ρ, which turns out to be free from the effects of the redundant phases. As a result, it is generally shown that the maximal atmospheric neutrino mixing necessarily accompanies either $\sin\theta_{13} = 0$ or $\cos(\delta + \rho) = 0$, the latter of which indicates maximal CP violation, where θ_{13} is the ν_e-ν_τ mixing angle.

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I. INTRODUCTION

Properties of neutrinos have been extensively studied by various experiments [1–4] since the historical confirmation of the atmospheric neutrino oscillations by the Super-Kamiokande collaboration [5]. For our understanding of hidden properties of the neutrinos it is physically significant to observe leptonic CP violation in future neutrino experiments [6] since there is no reason that prevents the appearance of CP violation in the lepton sector. Theoretically, leptonic CP violation is sensitive to phases [7] of the Pontecorvo-Maki-Nakagawa-Sakata (PMNS) unitary matrix $U_{\rm PMNS}$ that converts the flavor neutrinos $\nu_{e,\mu,\tau}$ into the massive neutrinos $\nu_{1,2,3}$: $\nu_f = (U_{\rm PMNS})_{fi}\nu_i$ ($f = e, \mu, \tau$; $i = 1, 2, 3$) [8], where a neutrino mass matrix M_ν in the flavor basis is transformed into $U^T_{\rm PMNS} M_\nu U_{\rm PMNS} = {\rm diag}(m_1, m_2, m_3)$. CP effects in neutrino reactions are known to be characterized by the Jarlskog invariant $\mathcal{J}_{CP}$ proportional to $\sin\theta_{13}\sin\delta$ [9], where θ_{13} and δ, respectively, stand for the ν_e-ν_τ mixing angle and the CP violating Dirac phase. However, these two quantities are not experimentally well known and the current data show the upper bound on θ_{13} [10]:

$$\sin^2\theta_{13} = 0.9 \begin{array}{c} +2.3 \\ -0.9 \end{array} \times 10^{-2}, \tag{1}$$

and no indication of the presence of δ. For a given value of θ_{13}, CP violation becomes maximal if $\delta = \pi/2$ (modulo π). It is of great importance to accumulate theoretical knowledge about θ_{13} and δ. We expect that it also provides useful information on the choice of CP phases in the leptogenesis [11] to create the baryon number in the Universe [12].

The leptonic CP violation is usually parametrized by phases in $U_{\rm PMNS}$ given by the particle data group [13], which we call $U^{\rm PDG}_{\rm PMNS}$, as $U^{\rm PDG}_{\rm PMNS}(\delta) = U_\nu(\delta, 0, 0)K(\beta_1, \beta_2, \beta_3)$:

$$U_\nu(\delta,0,0) = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -c_{23}s_{12} - s_{23}c_{12}s_{13}e^{i\delta} & c_{23}c_{12} - s_{23}s_{12}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{23}s_{12} - c_{23}c_{12}s_{13}e^{i\delta} & -s_{23}c_{12} - c_{23}s_{12}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix}, \qquad K(\beta_1,\beta_2,\beta_3) = {\rm diag}(e^{i\beta_1}, e^{i\beta_2}, e^{i\beta_3}), \tag{2}$$

for $c_{ij} = \cos\theta_{ij}$ and $s_{ij} = \sin\theta_{ij}$ ($i, j = 1, 2, 3$), where θ_{ij} stand for three neutrino mixing angles. The Majorana CP violation phases are determined by two combinations of $\beta_{1,2,3}$ such as $\beta_i - \beta_3$ ($i = 1, 2, 3$). Since the knowledge on flavor neutrino masses M_ν contains all information of new physics of neutrinos, to speak about CP violations, it is useful to find direct linkage between phases in M_ν and CP violating Dirac and Majorana phases. However, phases in M_ν are not uniquely determined because of the freedom of the redefinition of phases of the flavor neutrinos without affecting physical consequences and thus contain redundant phases. This freedom dictates other versions of $U_{\rm PMNS}$ than Eq. (2). The true effect of the Dirac CP violation can only be discussed after obtaining $U_{\rm PMNS}$ and is described in terms $(U_{\rm PMNS})_{fi}$ through $\mathcal{J}_{CP}$, which is free from the effect

[*]Electronic address: 5asnm016@keyaki.cc.u-tokai.ac.jp
[†]Electronic address: yasue@keyaki.cc.u-tokai.ac.jp

of the redefinition. There is a useful formula without referring to $U_{\rm PMNS}$ to determine the Jarlskog invariant in terms of $M_\nu^\dagger M_\nu(\equiv \mathbf{M})$, whose matrix element is denoted by $\mathbf{M}_{ij}$ $(i, j = e, \mu, \tau)$ [14]: $\mathcal{J}_{CP} = {\rm Im}(\mathbf{M}_{e\mu}\mathbf{M}_{\mu\tau}\mathbf{M}_{\tau e})/(\Delta m^2_{12}\Delta m^2_{23}\Delta m^2_{31})$, where $\Delta m^2_{ij} = m^2_i - m^2_j$ $(i, j = 1, 2, 3)$. However, in any case, it is not an easy task to find from $\mathcal{J}_{CP}$ which flavor neutrino mass of $\mathbf{M}$ is mainly responsible for the Dirac CP violating phase.

We have advocated the use of the μ-τ symmetry [15–19] to evaluate effects of leptonic CP violation [20–23]. It is based on the observation that the μ-τ symmetric texture gives $\sin\theta_{13} = 0$ (or $\sin\theta_{12} = 0$ [24]) giving no Dirac CP violation.[1] Therefore, leptonic CP violation is sensitive to how the μ-τ symmetry is broken. For example, if the μ-τ symmetry breaking part of M_ν is restricted to be pure imaginary and if its symmetric part is real, the Dirac CP violation becomes maximal for given mixing angles and the atmospheric neutrino mixing also becomes maximal [20–22,25]. This texture shows $M_{e\tau} = -\sigma M^*_{e\mu}$ $(\sigma = \pm 1)$ and $M_{\tau\tau} = M^*_{\mu\mu}$, where M_{ij} $(i, j = e, \mu, \tau)$ stands for the i-j matrix element of M_ν.[2] It can be further shown that this condition is extended to a more general one: $|M_{e\tau}| = |M_{e\mu}|$ and $|M_{\tau\tau}| = |M_{\mu\mu}|$ [22].[3]

It is known that the μ-τ symmetry required for neutrinos is badly broken by charged leptons. Since a pair of the charged lepton and neutrino forms an $SU(2)_L$-doublet, discussions based on the μ-τ symmetry required for neutrinos may be useless. However, it should be noted that talking about the experimental results of neutrino mixings is almost equivalent to talking about theoretical results based on the approximate μ-τ symmetry. This is based on the good guiding principle of understanding sources of small quantities in physics, which is the naturalness [26]. The naturalness dictates that the smallness of a certain physical quantity (such as $\sin^2\theta_{13} \ll 1$ and/or $\Delta m^2_\odot/|\Delta m^2_{\rm atm}| \ll 1$) implies a certain protection symmetry (such as the μ-τ symmetry). Furthermore, the similarity between charged leptons and neutrinos loses direct linkage because charged leptons are Dirac particles but neutrinos can be Majorana particles. This significant difference yields sufficient power to construct various models based on the μ-τ symmetry [17,18,27,28].

[1]For $\sin\theta_{12} = 0$, textures must contain a small parameter denoted by η and yield $\tan 2\theta_{12} \sim \varepsilon/\eta$, which vanishes as $\varepsilon \to 0$ but stays $\mathcal{O}(1)$ for $\eta \sim \varepsilon$, where ε stands for a μ-τ symmetry breaking parameter. The Dirac CP violating phase becomes $\delta + \rho$, which vanishes as will be shown in Eq. (18).

[2]The parameter σ is so chosen to yield $\sin\theta_{23} = \sigma/\sqrt{2}$ in Eq. (2) after diagonalizing the μ-τ symmetric M_ν, where $M_{e\tau} = -\sigma M_{e\mu}$ and $M_{\tau\tau} = M_{\mu\mu}$ are satisfied. This assignment of σ also appears in Eq. (A2).

[3]The condition is in fact found to be a solution to $|M_{e\mu}|^2 - |M_{e\tau}|^2 = |M_{\tau\tau}|^2 - |M_{\mu\mu}|^2$ (corresponding to $\mathbf{M}_{\mu\mu} = \mathbf{M}_{\tau\tau}$), which is the relation satisfied by a set of flavor neutrino masses discussed in Ref. [22].

In this article, along this line of thought, we discuss properties of the leptonic CP violation without entailing details of textures and examine constraints on various flavor neutrino masses to be compatible with the current neutrino oscillation data. We focus on describing the CP violating Dirac phase as well as the mixing angles in terms of flavor neutrino masses as general as possible. As a result, $U_{\rm PMNS}$ is completely expressed in terms of flavor neutrino masses and we understand which flavor neutrino masses mainly control which part of the leptonic CP violation [29]. To construct $U_{\rm PMNS}$, we must find three eigenvectors associated with $\mathbf{M}$ [24,29,30]. We first determine eigenvectors for the μ-τ symmetric part of $\mathbf{M}$. In the symmetric limit, the eigenvectors are found to generally contain a redundant phase to be denoted by ρ in the ν_e-ν_μ mixing, which serves as an additional CP violating Dirac phase after the μ-τ symmetry breaking effects are included. Next, to see the μ-τ symmetry breaking effects, we employ the usual perturbative analysis of $\mathbf{M}$, where the μ-τ symmetry breaking part of $\mathbf{M}$ is treated as a perturbation. This perturbation yields another phase to be denoted by γ associated with the ν_e-ν_τ mixing to fill $U_{\rm PMNS}$ in a consistent manner. However, this phase γ is completely removed without affecting CP violation. On the other hand, to remove the phase ρ affects CP violation and results in $\rho + \delta$ as a physical CP violating Dirac phase. We keep ρ in the PMNS unitary matrix to trace its effect. Considering the perturbative results, we are led to a general formula to express these mixings, and CP phases in terms of $\mathbf{M}$ itself.

Our formulas are shown to indeed reproduce the perturbative results and to consistently take care of the effect of the redundant phases ρ and γ. It turns out that δ and ρ are determined to be

$$\delta = -\arg(\sin\theta_{23}\mathbf{M}'_{e\mu} + \cos\theta_{23}\mathbf{M}'_{e\tau}), \tag{3}$$

$$\rho = \arg(\cos\theta_{23}\mathbf{M}'_{e\mu} - \sin\theta_{23}\mathbf{M}'_{e\tau}), \tag{4}$$

where $\mathbf{M}'_{e\mu} = e^{i\gamma}\mathbf{M}_{e\mu}$, and $\mathbf{M}'_{e\tau} = e^{-i\gamma}\mathbf{M}_{e\tau}$ are the redefined masses, which mean that the effect of γ is absorbed by the redefinition of masses. If ρ is removed, $\mathbf{M}'_{e\mu,e\tau}$ are shifted to $\mathbf{M}'_{e\mu} = e^{i(\gamma-\rho)}\mathbf{M}_{e\mu}$, and $\mathbf{M}'_{e\tau} = e^{-i(\gamma+\rho)}\mathbf{M}_{e\tau}$, which give $\delta + \rho$ in place of δ in Eq. (3) as expected. The atmospheric neutrino mixing angle is also determined and given by

$$\begin{aligned}
\theta_{23} &= \sigma\frac{\pi}{4} + \frac{\theta + \phi}{2}, \\
\sin\theta &= \frac{\sigma\sin\theta_{13}\sin 2\theta_{12}\cos(\delta + \rho)\Delta m^2_\odot}{2N}, \\
\sin\phi &= \frac{\kappa(\mathbf{M}_{\mu\mu} - \mathbf{M}_{\tau\tau})}{2N},
\end{aligned} \tag{5}$$

where $\kappa = \mathrm{Re}(\mathbf{M}'_{\mu\tau})/|\mathrm{Re}(\mathbf{M}'_{\mu\tau})|$, $N = \sqrt{\mathrm{Re}^2(\mathbf{M}'_{\mu\tau}) + (\mathbf{M}_{\mu\mu} - \mathbf{M}_{\tau\tau})^2/4}$, and $\mathbf{M}'_{\mu\tau} = e^{-2i\gamma}\mathbf{M}_{\mu\tau}$. The observed mass difference squared $\Delta m^2_\odot$ is given by $\Delta m^2_\odot = m_2^2 - m_1^2 (>0)$. For textures with $\sin\theta_{13} \neq 0$, the maximal atmospheric neutrino mixing is indeed provided by the maximal CP violating Dirac phase giving $\cos(\delta + \rho) = 0$ as long as $\mathbf{M}_{\mu\mu} = \mathbf{M}_{\tau\tau}$ is satisfied. It should be noted that the combination of $\mathbf{M}_{e\mu}\mathbf{M}_{\mu\tau}\mathbf{M}_{\tau e} (= \mathbf{M}_{e\mu}\mathbf{M}_{\mu\tau}\mathbf{M}^*_{e\tau})$ appearing in $\mathcal{J}_{CP}$ becomes independent of the redefinition of the flavor neutrinos as expected because of $\mathbf{M}'_{e\mu}\mathbf{M}'_{\mu\tau}\mathbf{M}'^*_{e\tau} = \mathbf{M}_{e\mu}\mathbf{M}_{\mu\tau}\mathbf{M}^*_{e\tau}$ for $\mathbf{M}'_{e\mu} = e^{i(\gamma-\rho)}\mathbf{M}_{e\mu}$, $\mathbf{M}'_{e\tau} = e^{-i(\gamma+\rho)}\mathbf{M}_{e\tau}$ and $\mathbf{M}'_{\mu\tau} = e^{-2i\gamma}\mathbf{M}_{\mu\tau}$.

This paper is organized as follows. In the next section, Sec. II, we discuss the physical consequence of the μ-τ symmetric part of $\mathbf{M}$. The new phase ρ is generally associated with $\mathbf{M}_{e\mu,e\tau}$. In Sec. III, the discussions are further extended to include textures without the μ-τ symmetry. We first calculate U_{PMNS} composed of three eigenvectors by the perturbative method, which treats a μ-τ symmetry breaking part of $\mathbf{M}$ as a perturbation. We next derive a set of formulas suggested by the results of the perturbation to calculate the mixing angles and phases, which can be applicable to any textures without the approximate μ-τ symmetry. In Sec. IV, properties of neutrino oscillations are discussed by using our results given by the formula. To see the power of our formula, a simple neutrino mass matrix is analyzed. The final section is devoted to summary and discussions.

II. μ-τ SYMMETRY

Let us begin with defining a neutrino mass matrix M_ν parametrized by[4]

$$M_\nu = \begin{pmatrix} M_{ee} & M_{e\mu} & M_{e\tau} \\ M_{e\mu} & M_{\mu\mu} & M_{\mu\tau} \\ M_{e\tau} & M_{\mu\tau} & M_{\tau\tau} \end{pmatrix}. \tag{6}$$

The μ-τ symmetry can be defined by the invariance of the flavor neutrino mass terms in the Lagrangian under the interchange of $\nu_\mu \leftrightarrow \nu_\tau$ or $\nu_\mu \leftrightarrow -\nu_\tau$. As a result, we obtain $M_{e\tau} = M_{e\mu}$ and $M_{\mu\mu} = M_{\tau\tau}$ for $\nu_\mu \leftrightarrow \nu_\tau$ or $M_{e\tau} = -M_{e\mu}$ and $M_{\mu\mu} = M_{\tau\tau}$ for $\nu_\mu \leftrightarrow -\nu_\tau$. We use the sign factor $\sigma = \pm 1$ to have $M_{e\tau} = -\sigma M_{e\mu}$ for the μ-τ symmetric part under the interchange of $\nu_\mu \leftrightarrow -\sigma\nu_\tau$. If M_ν has CP phases, we have to use the Hermitian matrix $\mathbf{M}$ to find eigenvectors of M_ν, which can be divided into the μ-τ symmetric part $\mathbf{M}_{\mathrm{sym}}$ and the μ-τ symmetry breaking part $\mathbf{M}_b$ as shown in the Appendix A.

The μ-τ symmetric part $\mathbf{M}_{\mathrm{sym}}$ in Eq. (A3) is analyzed in this section. We introduce a phase parameter ρ to express

$$B_+ = e^{i\rho}|B_+|, \tag{7}$$

leading to

$$\mathbf{M}_{\mathrm{sym}} = \begin{pmatrix} A & e^{i\rho}|B_+| & -\sigma e^{i\rho}|B_+| \\ e^{-i\rho}|B_+| & D_+ & E_+ \\ -\sigma e^{-i\rho}|B_+| & E_+ & D_+ \end{pmatrix}. \tag{8}$$

Three eigenvalues denoted by $\Lambda_\pm$ and Λ associated with $\mathbf{M}_{\mathrm{sym}}$ are found to be:

$$\Lambda_\pm = D_+ - \sigma E_+ + |B_+|X_\pm, \qquad \Lambda = D_+ + \sigma E_+, \tag{9}$$

where

$$X_\pm = \frac{A - D_+ + \sigma E_+ \pm \sqrt{(A - D_+ + \sigma E_+)^2 + 8|B_+|^2}}{2|B_+|}. \tag{10}$$

The corresponding eigenvectors are calculated to be

$$|\Lambda_-\rangle = N_- \begin{pmatrix} -X_- \\ -e^{-i\rho} \\ \sigma e^{-i\rho} \end{pmatrix}, \quad |\Lambda_+\rangle = N_+ \begin{pmatrix} X_+ e^{i\rho} \\ 1 \\ -\sigma \end{pmatrix}, \quad |\Lambda\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ \sigma \\ 1 \end{pmatrix}, \tag{11}$$

respectively, for Λ_-, Λ_+, and Λ, where $N_\pm = \sqrt{2 + X_\pm^2}$. There is an ambiguity to have an overall phase in each eigenvector. The orthogonality condition is obviously satisfied because of $X_+X_- = -2$.

Considering the phase ρ in Eq. (11), we parametrize the PMNS unitary matrix to be: $U^{(0)}_{\mathrm{PMNS}}(\delta, \rho) = U_\nu(\delta, \rho, 0)K(\beta_1, \beta_2, \beta_3)$: with

[4] It is understood that the charged leptons and neutrinos are rotated, if necessary, to give diagonal charged-current interactions and to define the flavor neutrinos of ν_e, ν_μ, and ν_τ.

$$U_\nu(\delta,\rho,0) = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13}e^{i\rho} & s_{13}e^{-i\delta} \\ -c_{23}s_{12}e^{-i\rho} - s_{23}c_{12}s_{13}e^{i\delta} & c_{23}c_{12} - s_{23}s_{12}s_{13}e^{i(\delta+\rho)} & s_{23}c_{13} \\ s_{23}s_{12}e^{-i\rho} - c_{23}c_{12}s_{13}e^{i\delta} & -s_{23}c_{12} - c_{23}s_{12}s_{13}e^{i(\delta+\rho)} & c_{23}c_{13} \end{pmatrix}. \tag{12}$$

It should be noted that the Dirac CP violation is characterized by $\sin\theta_{13}\sin(\delta+\rho)$ instead of $\sin\theta_{13}\sin\delta$ as indicated by $\mathcal{J}_{CP}$. It is obvious that $\mathbf{M}_{\rm sym}$ is equivalent to

$$\mathbf{M}'_{\rm sym} = \begin{pmatrix} A & e^{-i\rho}B_+ & -\sigma e^{-i\rho}B_+ \\ e^{i\rho}B_+^* & D_+ & E_+ \\ -\sigma e^{i\rho}B_+^* & E_+ & D_+ \end{pmatrix} = \begin{pmatrix} A & |B_+| & -\sigma|B_+| \\ |B_+| & D_+ & E_+ \\ -\sigma|B_+| & E_+ & D_+ \end{pmatrix}, \tag{13}$$

corresponding to Eq. (A7) with $\alpha_+ = \rho$ and $\alpha_- = 0$. Since $\mathbf{M}'_{\rm sym}$ yields $U_{\rm PMNS}^{\rm PDG}$ with $\delta = 0$, the phase ρ disappears and no CP violation exists. However, if there are μ-τ symmetry breaking effects, CP violation is active. It is anticipated that the effect of ρ remains in Dirac CP violation, where δ in $U_{\rm PMNS}^{\rm PDG}$ is replaced by $\delta+\rho$ at $U_{\rm PMNS}^{(0)}$. To trace the effect of ρ explicitly, we keep ρ in the PMNS unitary matrix hereafter.

As have been pointed out in Ref. [24], there are two categories that characterize the effect from the μ-τ symmetry. In terms of the mixing angles, the μ-τ symmetry requires either $\sin\theta_{13} = 0$ or $\sin\theta_{12} = 0$. In terms of the eigenvalues of Eq. (9), two categories depend on the order of $|\Lambda_\pm|$ and $|\Lambda|$, namely, on how these three eigenvalues are assigned to $m_{1,2,3}$. For $m_1 = \Lambda_-$, $m_2 = \Lambda_+$, and $m_3 = \Lambda$, if $\mathbf{M}$ gives $|\Lambda_-| < |\Lambda_+| < |\Lambda|$ as the normal mass ordering or $|\Lambda| < |\Lambda_-| < |\Lambda_+|$ as the inverted mass ordering, we find $U_{\rm PMNS}$ described by

$$(|\Lambda_-\rangle, |\Lambda_+\rangle, |\Lambda\rangle) = \left(\frac{1}{\sqrt{2+(X_-)^2}}\begin{pmatrix} -X_- \\ -e^{-i\rho} \\ \sigma e^{-i\rho}\end{pmatrix} \frac{1}{\sqrt{2+(X_+)^2}}\begin{pmatrix} X_+e^{i\rho} \\ 1 \\ -\sigma\end{pmatrix} \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ \sigma \\ 1\end{pmatrix}\right), \tag{14}$$

which yields

$$\tan 2\theta_{12} = \frac{2\sqrt{2}|B_+|}{D_+ - \sigma E_+ - A}, \qquad \tan 2\theta_{23} = \sigma, \qquad \sin\theta_{13} = 0. \tag{15}$$

On the other hand, for $m_1 = \Lambda_+$, $m_2 = \Lambda$, and $m_3 = \Lambda_-$, if the eigenvalues satisfy $|\Lambda_+| < |\Lambda| < |\Lambda_-|$ as the normal mass ordering or $|\Lambda_-| < |\Lambda_+| < |\Lambda|$ as the inverted mass ordering, we find that $U_{\rm PMNS}$ is described by

$$(|\Lambda_+\rangle, |\Lambda\rangle, |\Lambda_-\rangle) = \left(\frac{1}{\sqrt{2+(X_+)^2}}\begin{pmatrix} X_+ \\ \sigma e^{-i\rho} \\ -e^{-i\rho}\end{pmatrix} \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ \sigma\end{pmatrix} \frac{1}{\sqrt{2+(X_-)^2}}\begin{pmatrix} -X_-e^{i\rho} \\ -\sigma \\ 1\end{pmatrix}\right), \tag{16}$$

where $e^{-i\rho}$ is multiplied to $|\Lambda_\pm\rangle$ and the relative sign due to σ is adjusted to match $U_{\rm PMNS}^{(0)}$, which yield

$$\sin\theta_{12} = 0, \qquad \tan 2\theta_{23} = -\sigma, \qquad \tan 2\theta_{13} = \frac{2\sqrt{2}\sigma|B_+|}{A - D_+ + \sigma E_+}. \tag{17}$$

By comparing Eq. (16) with Eq. (12), we obtain

$$\delta = -\rho. \tag{18}$$

The Dirac CP violation is absent in both cases because of $\sin\theta_{13}\sin(\delta+\rho) = 0$ satisfied by either $\sin\theta_{13} = 0$ or $\delta+\rho = 0$.

We have more convenient formulas for masses and mixing angles as shown in Appendix B with $\gamma = 0$. These formulas yield the same results given by the eigenvectors and eigenvalues. Applying $\mathbf{M}_{\rm sym}$ to these relations, we find that

$$\mathrm{Re}\,(E_+)\cos 2\theta_{23} = -s_{13}c_{\rho+\delta}|X|, \qquad s_{13}s_{\rho+\delta}|X| = 0, \tag{19}$$

and

$$X = \frac{(c_{23}+\sigma s_{23})B_+}{c_{13}}, \qquad Y = (s_{23} - \sigma c_{23})B_+. \tag{20}$$

By using $\tan\theta_{13} = (\sqrt{1+\tan^2 2\theta_{13}} - 1)/\tan 2\theta_{13} = \tan 2\theta_{13}(1+\cdots) \propto |Y|$ from Eq. (B1) and noticing that $\cos 2\theta_{23} = (c_{23}+\sigma s_{23})(c_{23}-\sigma s_{23})$, we find that the solution to Eq. (19) turns out to be either

$$c_{23} = \sigma s_{23}, \tag{21}$$

leading $\sin\theta_{13} = 0$ with $\delta = 0$, or

$$c_{23} = -\sigma s_{23}, \tag{22}$$

leading $\sin\theta_{12} = 0$ with $\delta = -\rho$ because of Eq. (20) with $\tan 2\theta_{13} e^{-i\delta} \propto Y$ and $B_+ \propto e^{i\rho}$.

III. μ-τ SYMMETRY BREAKING AND *CP* PHASES

The μ-τ symmetry breaking $\mathbf{M}_b$ in Eq. (A3) is analyzed in this section. We choose a conventional perturbative analysis with $\mathbf{M}_b$ treated as a perturbation to find eigenvectors and eigenvalues associated with $\mathbf{M}$. An additional phase γ is induced by E_- in $\mathbf{M}_b$ as a main source, which is to be removed by the redefinition of $\nu_{\mu,\tau}$. The inclusion of $\mathbf{M}_b$ creates either $\sin\theta_{13} \neq 0$ or $\sin\theta_{12} \neq 0$. However, the perturbative treatment only allows to induce a tiny magnitude of $\sin\theta_{12,13}$. For $\sin\theta_{12} \sim 0$, we cannot give the consistent result with the observation, which requires that $\sin^2 2\theta_{12} = \mathcal{O}(1)$, and the physical consequence of textures with $\sin\theta_{12} = 0$ in the μ-τ symmetric limit will be discussed in a separate article.

A. Perturbative results

The starting eigenvectors and eigenvalues are provided by

$$|1^{(0)}\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} \sqrt{2}c_{12}^{(0)} \\ -s_{12}^{(0)}e^{-i\rho} \\ \sigma s_{12}^{(0)}e^{-i\rho} \end{pmatrix},$$
$$|2^{(0)}\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} \sqrt{2}s_{12}^{(0)}e^{i\rho} \\ c_{12}^{(0)} \\ -\sigma c_{12}^{(0)} \end{pmatrix}, \tag{23}$$
$$|3^{(0)}\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ \sigma \\ 1 \end{pmatrix},$$

and, from Eq. (B3),

$$m_1^{(0)2} = c_{12}^{(0)2}A + s_{12}^{(0)2}(D_+ - \sigma E_+) - 2\sqrt{2}c_{12}^{(0)}s_{12}^{(0)}|B_+|,$$
$$m_2^{(0)2} = s_{12}^{(0)2}A + c_{12}^{(0)2}(D_+ - \sigma E_+) + 2\sqrt{2}c_{12}^{(0)}s_{12}^{(0)}|B_+|,$$
$$m_3^{(0)2} = D_+ + \sigma E_+, \tag{24}$$

for textures with $\sin\theta_{13} = 0$, where we have used the superscript (0) for mixing angles determined by Eq. (15). From these results, we can find the first order results $|n^{(1)}\rangle$ and $m_n^{(1)2}$ ($n = 1, 2, 3$) as follows:

$$|1^{(1)}\rangle = a_{13}^{(1)}|3^{(0)}\rangle, \qquad |2^{(1)}\rangle = a_{23}^{(1)}|3^{(0)}\rangle,$$
$$|3^{(1)}\rangle = -(a_{13}^{(1)*}|1^{(0)}\rangle + a_{23}^{(1)*}|2^{(0)}\rangle), \tag{25}$$

where

$$a_{13}^{(1)} = \sigma\frac{\sqrt{2}c_{12}^{(0)}B_-^* - s_{12}^{(0)}(D_- - i\sigma E_-)e^{-i\rho}}{m_1^{(0)2} - m_3^{(0)2}},$$
$$a_{23}^{(1)} = \sigma\frac{\sqrt{2}s_{12}^{(0)}B_-^*e^{i\rho} + c_{12}^{(0)}(D_- - i\sigma E_-)}{m_2^{(0)2} - m_3^{(0)2}}, \tag{26}$$

and

$$m_1^{(1)2} = m_2^{(1)2} = m_3^{(1)2} = 0. \tag{27}$$

The three eigenvectors are now described by $|n\rangle = |n^{(0)}\rangle + |n^{(1)}\rangle$ ($n = 1, 2, 3$). For instance, $|3\rangle$ is calculated to be:

$$|3\rangle \approx \frac{1}{\sqrt{2}} \times \begin{pmatrix} \sigma\frac{2(2-R\cos 2\theta_{12}^{(0)})B_- + \sqrt{2}\sin 2\theta_{12}^{(0)}(D_- + \sigma iE_-)e^{i\rho}}{2\Delta m^2_{\rm atm}} \\ \sigma\left(1 + \frac{(2+R\cos 2\theta_{12}^{(0)})(D_- + \sigma iE_-) + \sqrt{2}R\sin 2\theta_{12}^{(0)}B_-e^{-i\rho}}{2\Delta m^2_{\rm atm}}\right) \\ 1 - \frac{(2+R\cos 2\theta_{12}^{(0)})(D_- + \sigma iE_-) + \sqrt{2}R\sin 2\theta_{12}^{(0)}B_-e^{-i\rho}}{2\Delta m^2_{\rm atm}} \end{pmatrix}, \tag{28}$$

where $\Delta m^2_{\rm atm} = m_3^2 - (m_1^2 + m_2^2)/2$, and $|R| \ll 1$ for $R \equiv \Delta m^2_\odot/\Delta m^2_{\rm atm}$ is used.

The PMNS unitary matrix can be constructed to be: $(|1\rangle, |2\rangle, |3\rangle)$. By using Eq. (28), we observe that the corresponding part in $U^{(0)}_{\rm PMNS}$, i.e. $(s_{13}e^{-i\delta}, s_{23}c_{13}, c_{23}c_{13})^T$, cannot be constructed from $|3\rangle$ because of the presence of the imaginary parts in all entries. It is found that a new phase γ defined in

$$P_\gamma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\gamma} & 0 \\ 0 & 0 & e^{-i\gamma} \end{pmatrix}, \tag{29}$$

is responsible for all phases in Eq. (28).[5] In fact, the PMNS unitary matrix $U_{\rm PMNS}$ that diagonalizes $\mathbf{M}_{\rm sym} + \mathbf{M}_b$ can take the form of

$$U_{\rm PMNS}(\delta, \rho, \gamma) = P_\gamma U^{(0)}_{\rm PMNS}(\delta, \rho)$$
$$= U_\nu(\delta, \rho, \gamma)K(\beta_1, \beta_2, \beta_3), \tag{30}$$

where

[5]There may be another phase (τ) associated with the 2–3 rotation as

$$\begin{pmatrix} \cos\theta_{23} & \sin\theta_{23}e^{i\tau} \\ -\sin\theta_{23}e^{-i\tau} & \cos\theta_{23} \end{pmatrix}$$

for (ν_μ, ν_τ), which is not appropriate to describe Eq. (28) in the limit of $|\tau| \ll 1$. The coexistence of γ and τ is mathematically irrelevant because three phases of the Dirac type are sufficient to determine the PMNS unitary matrix. The phase τ can be absorbed into ρ and γ. It can be proved that we have $\rho' = \rho + \tau/2$ and $\gamma' = \gamma + \tau/2$ in place of ρ and γ in Eq. (36) with $\delta' = \delta + \rho + \tau$ so that the freedom of τ expressed in terms of δ', ρ', and γ' becomes hidden.

$$U_\nu(\delta, \rho, \gamma) = P_\gamma U_\nu(\delta, \rho, 0). \tag{31}$$

We can identify $|3\rangle$ with the third row in $U_{\rm PMNS}$ and find that

$$|3\rangle \equiv \begin{pmatrix} s_{13}e^{-i\delta} \\ s_{23}e^{i\gamma} \\ c_{23}e^{-i\gamma} \end{pmatrix} \approx \frac{1}{\sqrt{2}}\begin{pmatrix} \sqrt{2}s_{13}e^{-i\delta} \\ \sigma(1-\Delta+i\gamma) \\ 1+\Delta-i\gamma \end{pmatrix}. \tag{32}$$

This identification leads to

$$\begin{aligned} s_{13}e^{-i\delta} &\approx \sigma\frac{\sqrt{2}(2-R\cos2\theta_{12})B_- + R\sin2\theta_{12}(D_- + \sigma iE_-)e^{i\rho}}{2\Delta m^2_{\rm atm}}, \\ \cos2\theta_{23}(\approx 2\Delta) &\approx -\frac{(2+R\cos2\theta_{12})D_- + \sqrt{2}R\sin2\theta_{12}\,{\rm Re}(B_-e^{-i\rho})}{\Delta m^2_{\rm atm}}, \\ \gamma &\approx \frac{(2+R\cos2\theta_{12})\sigma E_- + \sqrt{2}R\sin2\theta_{12}\,{\rm Im}(B_-e^{-i\rho})}{2\Delta m^2_{\rm atm}}, \end{aligned} \tag{33}$$

where $\theta_{12} \approx \theta_{12}^{(0)}$.

The phase γ turns out to consistently take care of other extra phases in $|1\rangle$ and $|2\rangle$. It is known that the phase γ can be removed by appropriate redefinition of the neutrinos. Let $\nu_{\rm flavor} = (\nu_e, \nu_\mu, \nu_\tau)^T$ and $\nu_{\rm mass} = (\nu_1, \nu_3, \nu_3)^T$. Because $\nu_{\rm flavor} = P_\gamma U^{(0)}_{\rm PMNS}\nu_{\rm mass}$, new flavor neutrinos given by $\nu'_{\rm flavor} = P_\gamma^{-1}\nu_{\rm flavor}$ are transformed by $U^{(0)}_{\rm PMNS}$. In this flavor base, we obtain that $-\mathcal{L}_m = \nu'^T_{\rm flavor}M'_\nu\nu'_{\rm flavor}/2$ with $\nu'_{\rm flavor} = U^{(0)}_{\rm PMNS}\nu_{\rm mass}$ and

$$M'_\nu = \begin{pmatrix} M_{ee} & e^{i\gamma}M_{e\mu} & e^{-i\gamma}M_{e\tau} \\ e^{i\gamma}M_{e\mu} & e^{2i\gamma}M_{\mu\mu} & M_{\mu\tau} \\ e^{-i\gamma}M_{e\tau} & M_{\mu\tau} & e^{2i\gamma}M_{\tau\tau} \end{pmatrix}, \tag{34}$$

which is nothing but Eq. (A7) with $\alpha_- = -\gamma$ and $\alpha_{e,+} = 0$. The phase γ is now absent in the PMNS unitary matrix for M'_ν.

If the μ-τ symmetry breaking terms have similar strengths, we expect that $|B_-| \sim |D_-| \sim |E_-|$. If this is the case, Eq. (33) can be further reduced to

$$s_{13}e^{-i\delta} \approx \frac{\sqrt{2}\sigma B_-}{\Delta m^2_{\rm atm}}, \qquad \cos2\theta_{23} \approx -\frac{2D_-}{\Delta m^2_{\rm atm}}, \qquad \gamma \approx \frac{2\sigma E_-}{2\Delta m^2_{\rm atm}}, \tag{35}$$

because of $R \ll 1$. Considering Eqs. (7) and (35) for ρ, we observe that

(i) the μ-τ symmetric B_+ is the main source of ρ,
(ii) the μ-τ symmetry breaking B_- is the main source of δ,
(iii) the μ-τ symmetry breaking D_- is the main source of $\cos2\theta_{23}$,
(iv) the μ-τ symmetry breaking E_- is the main source of γ.

These are our main results found in textures with the approximate μ-τ symmetry.

B. Exact results

As suggested from the subsection III A, we may employ M'_ν of Eq. (34) and $U_{\rm PMNS}$ of Eq. (30). The direct calculation from $U^\dagger_{\rm PMNS}\mathbf{M}U_{\rm PMNS} = {\rm diag}(m_1^2, m_2^2, m_3^2)$ yields three constraints and three masses expressed in terms of $\mathbf{M}_{ij}$ $(i, j = e, \mu, \tau)$ just corresponding to Eqs. (B1)–(B3). Let us define new flavor neutrino masses and a new Dirac phase δ':

$$\begin{aligned} B' &= e^{i(\gamma-\rho)}B, \qquad C' = e^{-i(\gamma+\rho)}C, \qquad E' = e^{-2i\gamma}E, \\ X' &= e^{-i\rho}X, \qquad Y' = e^{-i\rho}Y, \qquad \delta' = \delta + \rho, \end{aligned} \tag{36}$$

which are induced by the appropriate redefinition of the neutrinos with $\alpha_+ = \rho$ and $\alpha_- = -\gamma$ in Eq. (A7). In other words, it is equivalent to show that

$$\begin{aligned} \nu_{\rm flavor} &= U_{\rm PMNS}(\delta, \rho, \gamma)\nu_{\rm mass} \\ &= U_\nu(\delta, \rho, \gamma)K(\beta_1, \beta_2, \beta_3)\nu_{\rm mass}, \\ \nu'_{\rm flavor} &= U^{\rm PDG}_{\rm PMNS}(\delta')\nu_{\rm mass} \\ &= U_\nu(\delta', 0, 0)K(\beta_1 - \rho, \beta_2, \beta_3)\nu_{\rm mass}, \end{aligned} \tag{37}$$

as in Eq. (A5) with $\alpha_e = -\rho$. It is obvious that $BEC^*(= \mathbf{M}_{e\mu}\mathbf{M}_{\mu\tau}\mathbf{M}_{\tau e}) = B'E'C'^*(= \mathbf{M}'_{e\mu}\mathbf{M}'_{\mu\tau}\mathbf{M}'_{\tau e})$, which reassures that $\mathcal{J}_{CP}$ is a weak-base-independent quantity. The formula in Appendix B are a little bit modified and relations to be modified are expressed in terms of the new masses and phase as follows:

$$\begin{aligned} \tan2\theta_{12} &= \frac{2X'}{\Lambda_2 - \Lambda_1}, \qquad \tan2\theta_{13}e^{-i\delta'} = \frac{2Y'}{\Lambda_3 - A}, \\ {\rm Re}(E')\cos2\theta_{23} &+ D_-\sin2\theta_{23} + i\,{\rm Im}(E') = -s_{13}e^{-i\delta'}X'^*, \\ X' &= \frac{c_{23}B' - s_{23}C'}{c_{13}}(= \text{real}), \\ Y' &= s_{23}B' + c_{23}C', \\ \Lambda_2 &= c_{23}^2D + s_{23}^2F - 2s_{23}c_{23}\,{\rm Re}(E'), \\ \Lambda_3 &= s_{23}^2D + c_{23}^2F + 2s_{23}c_{23}\,{\rm Re}(E'). \end{aligned} \tag{38}$$

These relations together with other relations in Eqs. (B3) and (B4) guarantee that diagonal masses are obtained by $U_{\rm PMNS}^{\rm PDG}$, which does not contain ρ and γ. Determining ρ and δ can be rephrased as follows [31]:

$$\tan\theta_{23} = \frac{{\rm Im}(e^{-i(\rho-\gamma)}B)}{{\rm Im}(e^{-i(\rho+\gamma)}C)} = -\frac{{\rm Im}(e^{i(\delta-\gamma)}C)}{{\rm Im}(e^{i(\delta+\gamma)}B)}, \tag{39}$$

owing to the absence of the imaginary part of X' and $e^{i\delta'}Y'$. Our formula can be used to examine textures with $\sin\theta_{12}\to 0$ in the μ-τ symmetric limit.

From the relations, we find that

(i) the *CP* phase ρ is the phase of the flavor neutrino mass: $c_{23}e^{i\gamma}B - s_{23}e^{-i\gamma}$ as

$$\rho = \arg(c_{23}e^{i\gamma}B - s_{23}e^{-i\gamma}C), \tag{40}$$

(ii) the *CP* phase δ is the phase of the flavor neutrino mass: $s_{23}e^{i\gamma}B + c_{23}e^{-i\gamma}C$ as

$$\delta = -\arg(s_{23}e^{i\gamma}B + c_{23}e^{-i\gamma}C), \tag{41}$$

(iii) the real part of Eq. (B2)

$$\begin{aligned}&{\rm Re}\,(e^{-2i\gamma}E)\cos2\theta_{23} + D_-\sin2\theta_{23}\\ &\quad = -s_{13}\cos(\rho+\delta)e^{i\rho}X^*(\equiv -x),\end{aligned} \tag{42}$$

determines $\cos2\theta_{23}$, which is given by

$$\begin{aligned}\cos2\theta_{23} &= -\frac{\kappa\sigma D_-\sqrt{{\rm Re}^2(e^{-2i\gamma}E) + D_-^2 - x^2} + x\,{\rm Re}(e^{-2i\gamma}E)}{{\rm Re}^2(e^{-2i\gamma}E) + D_-^2} = \cos\left(\sigma\frac{\pi}{2} + \theta + \phi\right),\\ \cos\theta &= \sqrt{\frac{{\rm Re}^2(e^{-2i\gamma}E) + D_-^2 - x^2}{{\rm Re}^2(e^{-2i\gamma}E) + D_-^2}}, \qquad \sin\theta = \frac{\sigma x}{\sqrt{{\rm Re}^2(e^{-2i\gamma}E) + D_-^2}},\\ \cos\phi &= \frac{{\rm Re}(e^{-2i\gamma}E)}{\sqrt{{\rm Re}^2(e^{-2i\gamma}E) + D_-^2}}, \qquad \sin\phi = \frac{\kappa D_-}{\sqrt{{\rm Re}^2(e^{-2i\gamma}E) + D_-^2}},\end{aligned} \tag{43}$$

where κ is the sign of ${\rm Re}(e^{-2i\gamma}E)$, from which we obtain that $\theta_{23} = \sigma\pi/4 + (\theta+\phi)/2$.

(iv) the imaginary part of Eq. (B2)

$$\cos2\gamma\,{\rm Im}(E) - \sin2\gamma\,{\rm Re}(E) = s_{13}\sin(\rho+\delta)e^{i\rho}X^*(\equiv x'), \tag{44}$$

determines γ, which is given by

$$\begin{aligned}\sin2\gamma &= \frac{\kappa'\,{\rm Im}(E)\sqrt{|E|^2 - x'^2} - x'\,{\rm Re}(E)}{|E|^2} = \sin(\phi' - \theta'), \qquad \cos\theta' = \frac{\sqrt{|E|^2 - x'^2}}{|E|},\\ \sin\theta' &= \frac{x'}{|E|} \qquad \cos\phi' = \frac{{\rm Re}(E)}{|E|}, \qquad \sin\phi' = \frac{\kappa'|{\rm Im}(E)|}{|E|},\end{aligned} \tag{45}$$

where κ' is the sign of ${\rm Re}(E)$, from which we obtain that $\gamma = (\phi' - \theta')/2$.

These exact results can be used to examine any textures. If textures are approximately μ-τ symmetric, the perturbative result can be shown to be reproduced.

To see that the perturbative result Eq. (33) is reproduced, we introduce the approximation due to $|\gamma| \ll 1$, $|\Delta| \ll 1$, and $\sin^2\theta_{13} \ll 1$. Note that $|\gamma| \ll 1$ in Eq. (45) practically requires the suppression of ${\rm Im}(E)$. Using $|\gamma| \ll 1$, and ${\rm Re}(e^{-2i\gamma}E) \approx \sigma\Delta m^2_{\rm atm}/2$ together with $X = e^{i\rho}\sin2\theta_{12}\Delta m^2_\odot/2$ in Eq. (B6), it is readily found that

$$\begin{aligned}\cos2\theta_{23}(\approx 2\Delta) &\approx -\frac{(2 + R\cos2\theta_{12})D_- + \sigma s_{13}\cos(\rho+\delta)\sin2\theta_{12}\Delta m^2_\odot}{\Delta m^2_{\rm atm}},\\ \gamma &\approx \sigma\frac{(2 + R\cos2\theta_{12})\,{\rm Im}(E) - s_{13}\sin(\rho+\delta)\sin2\theta_{12}\Delta m^2_\odot}{2\Delta m^2_{\rm atm}},\end{aligned} \tag{46}$$

form Eqs. (43) and (45), which coincide with $\cos2\theta_{23}$ and γ in Eq. (33) because of $\sqrt{2}B_- \approx \sigma s_{13}e^{-i\delta}\Delta m^2_{\rm atm}$. The mixing angle $\tan2\theta_{13}$ in Eq. (B1) with $\Lambda_3 - A \approx (2 + R\cos2\theta_{12})\Delta m^2_{\rm atm}/2$ becomes

$$\tan2\theta_{13}e^{-i\delta} \approx 4\sqrt{2}\sigma\frac{B_- + (i\gamma - \Delta)B_+}{(2 + R\cos2\theta_{12})\Delta m^2_{\rm atm}} \approx \sigma\frac{\sqrt{2}(2 - R\cos2\theta_{12})B_- + (i\gamma - \Delta)e^{i\rho}\sin2\theta_{12}\Delta m^2_\odot}{\Delta m^2_{\rm atm}}, \tag{47}$$

where Eq. (B6) is used to give $B_+ \approx e^{i\rho}\sin 2\theta_{12}\Delta m^2_\odot/2\sqrt{2}$. Inserting Eq. (46) into Eq. (47) and ignoring terms of $\mathcal{O}(R^2)$, Eq. (47) coincides with the perturbative result Eq. (33). The conditions of $|\cos 2\theta_{23}| \ll 1$, $|\gamma| \ll 1$, and $|\sin\theta_{13}| \ll 1$ are, respectively, satisfied if $|D_-| \ll |\Delta m^2_{\rm atm}|$, $|{\rm Im}(E)| \ll |\Delta m^2_{\rm atm}|$, and $|B_-| \ll |\Delta m^2_{\rm atm}|$.

Once again, predictions of the observed quantities such as neutrino masses and mixing angles are independent of γ and ρ because they depend on the modified neutrino masses in Eq. (36), where ρ and γ are hidden by the definition of the masses. However, it should be noted that γ supplies μ-τ symmetric breaking effects because the rotation induced by γ is of the μ-τ symmetry breaking type. In approximately μ-τ symmetric textures, $|\gamma| \ll 1$ should be satisfied.

IV. PROPERTIES OF NEUTRINO OSCILLATIONS

The mixing angles and associated phases are expressed in terms of the flavor neutrino masses. The *CP* violating Dirac phase $\delta' = \delta + \rho$ is determined as $\delta = -\arg(Y)$ and $\rho = \arg(X)$, where

$$X = c_{23}e^{i\gamma}\mathbf{M}_{e\mu} - s_{23}e^{-i\gamma}\mathbf{M}_{e\tau},\quad Y = s_{23}e^{i\gamma}\mathbf{M}_{e\mu} + c_{23}e^{-i\gamma}\mathbf{M}_{e\tau}, \tag{48}$$

which also controls the mixing angles as in Eq. (B1) given by

$$\sin 2\theta_{12}e^{i\rho} = \frac{2X}{\Delta m^2_\odot}, \qquad \tan 2\theta_{13}e^{-i\delta} \approx \frac{2Y}{\Delta m^2_{\rm atm}}, \tag{49}$$

where $\Lambda_2 - \Lambda_1 = \cos 2\theta_{12}\Delta m^2_\odot$ and $\Lambda_3 - A \approx \Delta m^2_{\rm atm}$ are used. Obtaining $\sin^2\theta_{13} \lesssim 0.01$ and $\tan 2\theta_{12} = \mathcal{O}(1)$ calls for two constraints: (1) $|X| \sim \Delta m^2_\odot$ and (2) $|Y| \lesssim 0.1|\Delta m^2_{\rm atm}|$. If there is an approximate μ-τ symmetry, $X \approx \sqrt{2}(B_+ + B_-\Delta)$ and $Y \approx \sqrt{2}(B_- + B_+\Delta)$ are satisfied. Since the smallness of $|B_-|$ and Δ is a result of the approximate μ-τ symmetry, the second constraint is naturally expected. However, the smallness of $|X|$ needs that of $|B_+|$ as an additional requirement.

The deviation of the atmospheric neutrino mixing from its maximal one is estimated to be:

$$\theta_{23} = \sigma\frac{\pi}{4} + \frac{\theta + \phi}{2}, \tag{50}$$

where θ and ϕ are defined in Eq. (43). We observe that the phenomenological constraint that the atmospheric neutrino mixing is almost maximal is well satisfied if $\mathbf{M}_{\mu\mu} \sim \mathbf{M}_{\tau\tau}$ for $\phi \ll 1$ because $\theta \ll 1$ is always satisfied by $|X| = \Delta m^2_\odot/2\sin 2\theta_{12}$, and ${\rm Re}(e^{-2i\gamma}\mathbf{M}_{\mu\tau}) \approx \sigma\Delta m^2_{\rm atm}/2$ as in Eq. (B5). This constraint $\mathbf{M}_{\mu\mu} \sim \mathbf{M}_{\tau\tau}$ supplemented by $|Y| \sim 0$ indicates the presence of the approximate μ-τ symmetry if the additional constraint $|\gamma| \ll 1$ leading to the suppression of $|{\rm Im}(\mathbf{M}_{\mu\tau})|$ is satisfied.

We can also argue how the mass hierarchy of $|\Delta m^2_{\rm atm}| \gg \Delta m^2_\odot$ constrains the magnitude of $\mathbf{M}_{ee}$, $\mathbf{M}_{\mu\mu} + \mathbf{M}_{\tau\tau}$, and $\mathbf{M}_{\mu\tau}$ from Eq. (B5). The results can be summarized as follows:

(i) for the normal mass hierarchy with $|\Delta m^2_{\rm atm}| \gg \sum m^2_\odot$, $\mathbf{M}_{\mu\mu} + \mathbf{M}_{\tau\tau} \approx 2\sigma{\rm Re}(e^{-2i\gamma}\mathbf{M}_{\mu\tau}) \approx \Delta m^2_{\rm atm} \gg 2\mathbf{M}_{ee}$ are required,

(ii) for the inverted mass hierarchy ($\Lambda_3 \ll |\Delta m^2_{\rm atm}|$) and the degenerate mass pattern ($\Lambda_3 \sim |\Delta m^2_{\rm atm}|$ with $m^2_1 \sim m^2_2 \sim m^2_3 \sim |\Delta m^2_{\rm atm}|$), both with $2|\Delta m^2_{\rm atm}| \approx \sum m^2_\odot$, $3\mathbf{M}_{ee} \approx \mathbf{M}_{\mu\mu} + \mathbf{M}_{\tau\tau} \approx 6\sigma{\rm Re}(e^{-2i\gamma}\mathbf{M}_{\mu\tau}) \approx 3\Delta m^2_{\rm atm}$ are required, and

(iii) for the degenerate mass pattern ($\sum m^2_\odot \gg |\Delta m^2_{\rm atm}|$ with $m^2_1 \sim m^2_2 \sim m^2_3 \gg |\Delta m^2_{\rm atm}|$), $2\mathbf{M}_{ee} \approx \mathbf{M}_{\mu\mu} + \mathbf{M}_{\tau\tau} \approx \sum m^2_\odot \gg 2\sigma{\rm Re}(e^{-2i\gamma}\mathbf{M}_{\mu\tau})$ are required.

The set of these constraints serves as a useful guide to search for specific textures.

As an example, let us consider the simplest mass matrix discussed in Ref. [32], which is

$$M_\nu = \frac{m_0}{2}\begin{pmatrix} a\varepsilon^2 & b\varepsilon & -\sigma b\varepsilon e^{i\alpha} \\ b\varepsilon & 1+\varepsilon & \sigma \\ -\sigma b\varepsilon e^{i\alpha} & \sigma & 1+\varepsilon \end{pmatrix}, \tag{51}$$

in our assignment, where a, b are real, and $\varepsilon^2 \ll 1$. The normal mass hierarchy is realized. Because γ turns out to be $\mathcal{O}(\varepsilon^2)$, the phases ρ and δ are calculated from

$$X(\propto e^{i\rho}) \approx \frac{b\varepsilon^2 m_0^2}{2\sqrt{2}}e^{-i(\alpha/2)}\cos\frac{\alpha}{2},\quad Y(\propto e^{-i\delta}) \approx i\frac{\sigma b\varepsilon m_0^2}{\sqrt{2}}e^{-i(\alpha/2)}\sin\frac{\alpha}{2}, \tag{52}$$

to be

$$\rho \approx -\frac{\alpha}{2}, \qquad \delta \approx \frac{\alpha}{2} - \frac{\pi}{2}, \tag{53}$$

for

$$\tan 2\theta_{12} \approx \frac{b\varepsilon^2 m_0^2}{\sqrt{2}(\Lambda_2 - \Lambda_1)}\cos\frac{\alpha}{2},\quad \tan 2\theta_{13} \approx \frac{\sqrt{2}\sigma b\varepsilon m_0^2}{\Lambda_3 - A}\sin\frac{\alpha}{2}, \tag{54}$$

from which we find that $X(\propto \varepsilon^2)$ and $Y(\propto \varepsilon)$ satisfy the condition $|X| \ll |Y|$. From X, $\Delta m^2_\odot$ is computed to be:

$$\Delta m^2_\odot \approx \frac{b\varepsilon^2 m_0^2}{\sqrt{2}\sin 2\theta_{12}}\cos\frac{\alpha}{2}. \tag{55}$$

The dependence of ε is different for $\tan 2\theta_{12}$ and $\tan 2\theta_{13}$. As a result, $\tan 2\theta_{12} = \mathcal{O}(1)$ is satisfied because $\Lambda_2 - \Lambda_1 = \cos 2\theta_{12}\Delta m^2_\odot$ is proportional to ε^2 while $\tan 2\theta_{13} \sim \varepsilon$ is satisfied because of $\Lambda_3 - A \approx \Delta m^2_{\rm atm}$ from Eq. (B5).

Since the Dirac *CP* violating phase is given by $\delta + \rho$, we find that $\delta + \rho \approx -\pi/2$. This texture shows

(i) almost maximal *CP* violation

irrespective of the size of α. The same conclusion can be obtained by evaluating $\mathcal{J}_{CP}$, which is given by

$$\mathcal{J}_{CP} = \frac{\mathrm{Im}(\mathbf{M}_{e\mu}\mathbf{M}_{\mu\tau}\mathbf{M}_{\tau e})}{\Delta m_{12}^2 \Delta m_{23}^2 \Delta m_{31}^2} \approx -\frac{\sqrt{2}b\varepsilon}{8}\sin 2\theta_{12}\sin\frac{\alpha}{2}, \tag{56}$$

$$\begin{aligned}\mathcal{J}_{CP} &= \frac{1}{8}\sin 2\theta_{12}\sin 2\theta_{23}\sin 2\theta_{13}\cos\theta_{13}\sin\delta \\ &\approx \frac{\sqrt{2}b\varepsilon}{8}\sin 2\theta_{12}\sin\frac{\alpha}{2}\sin\delta,\end{aligned} \tag{57}$$

where we have used $\Delta m_{12}^2 = -\Delta m_\odot^2$ and $\Delta m_{31}^2 \approx -\Delta m_{23}^2 \approx \Delta m_{\rm atm}^2$ in Eq. (56), and $\sin 2\theta_{23} \approx \sigma$, $\sin 2\theta_{13}$ from Eq. (54), and $\cos\theta_{13} \approx 1$ in Eq. (57). These two results indicate that $\sin\delta \approx -1$ corresponding to $\delta + \rho \approx -\pi/2$ in our estimation, thus confirming the conclusion: the emergence of almost maximal *CP* violation, which differs from the conclusion in Ref. [32]. The atmospheric neutrino mixing is almost maximal because of $\theta \approx 0$ from $\delta + \rho \approx -\pi/2$ and $\phi = 0$ from $\mathbf{M}_{\mu\mu} = \mathbf{M}_{\tau\tau}$ in Eq. (43), and is also consistent with Eq. (39) giving $\mathrm{Im}(e^{-i\rho}\mathbf{M}_{e\mu}) \approx \sigma\,\mathrm{Im}(e^{-i\rho}\mathbf{M}_{e\tau}) \approx 2a\varepsilon\sin(\alpha/2)$. The advantage of our method lies in the fact that the source of the *CP* violating phase can be directly related to $\mathbf{M}_{e\mu,e\tau}$ in X for ρ and in Y for δ. The appearance of the almost maximal *CP* violation can also be read off from the redefined texture given by Eq. (A6) with $\alpha_+ = -\alpha_e = \rho \approx -\alpha/2$, and $\alpha_- = -\gamma(\approx 0)$:

$$M'_\nu \approx \frac{m_0}{2}\begin{pmatrix} a\varepsilon^2 e^{-i\alpha} & b\varepsilon e^{-i((\alpha/2)-\gamma)} & -\sigma b\varepsilon e^{i((\alpha/2)-\gamma)} \\ b\varepsilon e^{-i((\alpha/2)-\gamma)} & (1+\varepsilon)e^{2i\gamma} & \sigma \\ -\sigma b\varepsilon e^{i((\alpha/2)-\gamma)} & \sigma & (1+\varepsilon)e^{-2i\gamma}\end{pmatrix}. \tag{58}$$

It yields the almost maximal *CP* violation because this texture approximately satisfies the condition on the maximal *CP* violation: $M'_{e\tau} = -\sigma M'^*_{e\mu}$, $M'_{\tau\tau} = M'^*_{\mu\mu}$, and $M'_{ee} + \sigma M'_{\mu\tau} = \text{real}$ [20–22,25], where the last condition is valid up to the order ε.

V. SUMMARY AND DISCUSSIONS

We have clarified the connection between the *CP* violating phases and phases of the flavor neutrino masses. In terms of the PMNS unitary matrix, any textures can provide $U_{\rm PMNS}^{\rm PDG}$ constructed from their three eigenvectors if the flavor neutrinos are appropriately rotated. However, in terms of the flavor neutrino masses M_ν, appropriate rotations modify the phase structure of M_ν and the amount of the rotations can be absorbed in the redefinition of the flavor neutrino masses. Our formulas automatically take care of the effect of these rotations, which yield

$$\begin{aligned}\mathbf{M} &= \begin{pmatrix} A & B & C \\ B^* & D & E \\ C^* & E^* & F\end{pmatrix} \to \mathbf{M}' = \begin{pmatrix} A & B' & C' \\ B'^* & D' & E' \\ C'^* & E'^* & F'\end{pmatrix} \\ &= \begin{pmatrix} A & e^{i(\gamma-\rho)}B & e^{-i(\gamma+\rho)}C \\ e^{-i(\gamma-\rho)}B^* & D & e^{-2i\gamma}E \\ e^{i(\gamma+\rho)}C^* & e^{2i\gamma}E^* & F\end{pmatrix}.\end{aligned} \tag{59}$$

If $\mathbf{M}$ is employed, the PMNS unitary matrix is $U_{\rm PMNS}$ of Eq. (30) while, if $\mathbf{M}'$ is employed, it is $U_{\rm PMNS}^{\rm PDG}$ of Eq. (2), where ρ and γ are absent. The phases ρ and γ are thus redundant. Namely, for

$$\nu'_{\rm flavor} = \begin{pmatrix} e^{-i\rho} & 0 & 0 \\ 0 & e^{-i\gamma} & 0 \\ 0 & 0 & e^{i\gamma}\end{pmatrix}\nu_{\rm flavor}, \tag{60}$$

we have found that

$$\begin{aligned}\nu_{\rm flavor} &= U_{\rm PMNS}(\delta,\rho,\gamma)\nu_{\rm mass} \\ &= U_\nu(\delta,\rho,\gamma)K(\beta_1,\beta_2,\beta_3)\nu_{\rm mass} \quad \text{for } \mathbf{M}, \\ \nu'_{\rm flavor} &= U_{PMNS}^{\rm PDG}(\delta+\rho)\nu_{\rm mass} \\ &= U_\nu(\delta+\rho,0,0)K(\beta_1-\rho,\beta_2,\beta_3)\nu_{\rm mass} \quad \text{for } \mathbf{M}'.\end{aligned} \tag{61}$$

For both cases, we obtain $\mathcal{J}_{CP}$ proportional to $\sin(\delta+\rho)$. It should be noted that the Majorana phase for ν_1 is shifted by the rotation of the flavor neutrinos. The required amount of the rotations to remove ρ and γ in the PMNS unitary matrix can be determined from $\rho = \arg(X)$ with $X = (c_{23}e^{i\gamma}\mathbf{M}_{e\mu} - s_{23}e^{-i\gamma}\mathbf{M}_{e\tau})/c_{13}$, and $\gamma = (\phi' - \theta')/2$, where θ' and ϕ' are defined in Eq. (45). The Dirac *CP* phase δ is determined by $\delta = -\arg(Y)$ with $Y = s_{23}e^{i\gamma}\mathbf{M}_{e\mu} + c_{23}e^{-i\gamma}\mathbf{M}_{e\tau}$. It should be emphasized that the phases ρ and δ are, respectively, associated with $\tan 2\theta_{12}$ and $\tan 2\theta_{13}$:

$$\tan 2\theta_{12}e^{i\rho} \propto X, \qquad \tan 2\theta_{13}e^{-i\delta} \propto Y. \tag{62}$$

We have also found that $\mathcal{J}_{CP}$ is independent of the redefinition of masses in Eq. (59) because of $\mathbf{M}'_{e\mu}\mathbf{M}'_{\mu\tau}\mathbf{M}'^*_{e\tau} = \mathbf{M}_{e\mu}\mathbf{M}_{\mu\tau}\mathbf{M}^*_{e\tau}$.

The conditions satisfied by approximately μ-τ symmetric textures are equivalent to those for textures without *CP* violation, and are described by

(i) $s_{23}e^{i\gamma}\mathbf{M}_{e\mu} + c_{23}e^{-i\gamma}\mathbf{M}_{e\tau} \sim 0$,
(ii) $\mathbf{M}_{\mu\mu} \sim \mathbf{M}_{\tau\tau}$.

The additional condition is required for textures with *CP* violation:
(i) $\mathrm{Im}(\mathbf{M}_{\mu\tau}) \sim 0$ corresponding to $\gamma \sim 0$,

where γ is calculated in Eq. (45). If the μ-τ symmetry breaking parts of $\mathbf{M}$ have similar strengths, we conclude that
(i) the μ-τ symmetric B_+ is the main source of ρ,
(ii) the μ-τ symmetry breaking B_- is the main source of δ,
(iii) the μ-τ symmetry breaking D_- is the main source of $\cos 2\theta_{23}$,
(iv) the μ-τ symmetry breaking E_- is the main source of γ,

where $B_\pm$, D_-, and E_- have been defined in Eq. (A2). As one of the exact results, we have obtained that

$$\theta_{23} = \pm\pi/4 + (\theta + \phi)/2, \tag{63}$$

where θ and ϕ are defined in Eq. (43). The maximal atmospheric neutrino mixing is realized only if $\cos(\delta + \rho) = 0$ and $\mathbf{M}_{\mu\mu} = \mathbf{M}_{\tau\tau}$ are satisfied. Therefore, we observe that
(i) the maximal Dirac *CP* violation is linked to the maximal atmospheric neutrino mixing,

for $\sin\theta_{13} \neq 0$. This condition is satisfied in the textures discussed in Refs. [20–22,25].

To give complete discussions on the leptonic *CP* violation, we have to consider the Majorana *CP* violation. As in Eq. (61), the redundant phase ρ remains in the Majorana phase because Majorana masses directly receive the effect of phases in M_ν. Since the present method is based on the Hermite matrix $M_\nu^\dagger M_\nu$, the *CP* violating Majorana phases disappear in $M_\nu^\dagger M_\nu$. To obtain effects of the Majorana phases, the simplest way is to perform the transformation $U_{\rm PMNS}^T M_\nu U_{\rm PMNS} = \mathrm{diag}(m_1, m_2, m_3)$, which is expected to lead to a similar result to the one shown in the appendix of Ref. [21]. We will discuss this extension of the present method in a future publication.

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APPENDIX A: μ-τ SYMMETRIC TEXTURE AND BREAKING PART

In this appendix, we show the parametrization of our neutrino mass matrix. For a given M_ν, we can formally divide M_ν into the μ-τ symmetric part $M_{\rm sym}$ and its breaking part M_b expressed in terms of $M_{e\mu}^{(\pm)} = (M_{e\mu} \pm (-\sigma M_{e\tau}))/2$ $(\sigma = \pm 1)$ and $M_{\mu\mu}^{(\pm)} = (M_{\mu\mu} \pm M_{\tau\tau})/2$ [20,21]:

$$M_\nu = \begin{pmatrix} M_{ee} & M_{e\mu} & -\sigma M_{e\mu} \\ M_{e\mu} & M_{\mu\mu} & M_{\mu\tau} \\ -\sigma M_{e\mu} & M_{\mu\tau} & M_{\mu\mu} \end{pmatrix} = M_{\rm sym} + M_b \tag{A1}$$

with

$$M_{\rm sym} = \begin{pmatrix} M_{ee} & M_{e\mu}^{(+)} & -\sigma M_{e\mu}^{(+)} \\ M_{e\mu}^{(+)} & M_{\mu\mu}^{(+)} & M_{\mu\tau} \\ -\sigma M_{e\mu}^{(+)} & M_{\mu\tau} & M_{\mu\mu}^{(+)} \end{pmatrix},$$
$$M_b = \begin{pmatrix} 0 & M_{e\mu}^{(-)} & \sigma M_{e\mu}^{(-)} \\ M_{e\mu}^{(-)} & M_{\mu\mu}^{(-)} & 0 \\ \sigma M_{e\mu}^{(-)} & 0 & -M_{\mu\mu}^{(-)} \end{pmatrix}. \tag{A2}$$

The Lagrangian for $M_{\rm sym}$: $-\mathcal{L}_{\rm mass} = \nu^T M_{\rm sym}\nu/2$ with $\nu = (\nu_e, \nu_\mu, \nu_\tau)^T$ turns out to be invariant under the exchange of $\nu_\mu \leftrightarrow -\sigma\nu_\tau$. We parametrize $\mathbf{M}(= M_\nu^\dagger M_\nu)$ as $\mathbf{M} = \mathbf{M}_{\rm sym} + \mathbf{M}_b$:

$$\mathbf{M}_{\rm sym} = \begin{pmatrix} A & B_+ & -\sigma B_+ \\ B_+^* & D_+ & E_+ \\ -\sigma B_+^* & E_+ & D_+ \end{pmatrix},$$
$$\mathbf{M}_b = \begin{pmatrix} 0 & B_- & \sigma B_- \\ B_-^* & D_- & iE_- \\ \sigma B_-^* & -iE_- & -D_- \end{pmatrix}, \tag{A3}$$

where

$$\begin{aligned}
A &= |M_{ee}|^2 + 2(|M_{e\mu}^{(+)}|^2 + |M_{e\mu}^{(-)}|^2),\\
B_+ &= M_{ee}^* M_{e\mu}^{(+)} + M_{e\mu}^{(+)*}(M_{\mu\mu}^{(+)} - \sigma M_{\mu\tau}) + M_{e\mu}^{(-)*}M_{\mu\mu}^{(-)},\\
B_- &= M_{ee}^* M_{e\mu}^{(-)} + M_{e\mu}^{(-)*}(M_{\mu\mu}^{(+)} + \sigma M_{\mu\tau}) + M_{e\mu}^{(+)*}M_{\mu\mu}^{(-)},\\
D_+ &= |M_{e\mu}^{(+)}|^2 + |M_{e\mu}^{(-)}|^2 + |M_{\mu\mu}^{(+)}|^2 + |M_{\mu\mu}^{(-)}|^2 + |M_{\mu\tau}|^2,\\
D_- &= 2\,\mathrm{Re}(M_{e\mu}^{(-)*}M_{e\mu}^{(+)} + M_{\mu\mu}^{(-)*}M_{\mu\mu}^{(+)}),\\
E_+ &= \mathrm{Re}(E) = \sigma(|M_{e\mu}^{(-)}|^2 - |M_{e\mu}^{(+)}|^2) + 2\,\mathrm{Re}(M_{\mu\mu}^{(+)*}M_{\mu\tau}),\\
E_- &= \mathrm{Im}(E) = 2\,\mathrm{Im}(M_{\mu\mu}^{(-)*}M_{\mu\tau} - \sigma M_{e\mu}^{(-)*}M_{e\mu}^{(+)}),
\end{aligned} \tag{A4}$$

for $E = E_+ + iE_-$. Similarly, we define $B = B_+ + B_-$, $C = -\sigma(B_+ - B_-)$, $D = D_+ + D_-$, and $F = D_+ - D_-$ to describe matrix elements of $\mathbf{M}$.

By the redefinition of the flavor neutrino $\nu_{\rm flavor} = (\nu_e, \nu_\mu, \nu_\tau)^T$ as

$$\nu'_{\rm flavor} = \begin{pmatrix} e^{i\alpha_e} & 0 & 0 \\ 0 & e^{i\alpha_\mu} & 0 \\ 0 & 0 & e^{i\alpha_\tau} \end{pmatrix} \nu_{\rm flavor} = e^{i\alpha_e} \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i(\alpha_+ + \alpha_-)} & 0 \\ 0 & 0 & e^{i(\alpha_+ - \alpha_-)} \end{pmatrix} \nu_{\rm flavor}, \tag{A5}$$

for $\alpha_+ = (\alpha_\mu + \alpha_\tau)/2 - \alpha_e$ and $\alpha_- = (\alpha_\mu - \alpha_\tau)/2$, this M_ν becomes equivalent to

$$M'_\nu = e^{-2i\alpha_e} \begin{pmatrix} M_{ee} & e^{-i(\alpha_+ + \alpha_-)} M_{e\mu} & e^{-i(\alpha_+ - \alpha_-)} M_{e\tau} \\ e^{-i(\alpha_+ + \alpha_-)} M_{e\mu} & e^{-2i(\alpha_+ + \alpha_-)} M_{\mu\mu} & e^{-2i\alpha_+} M_{\mu\tau} \\ e^{-i(\alpha_+ - \alpha_-)} M_{e\tau} & e^{-2i\alpha_+} M_{\mu\tau} & e^{-2i(\alpha_+ - \alpha_-)} M_{\tau\tau} \end{pmatrix}. \tag{A6}$$

The relevant interactions are kept invariant as $g\bar{\ell}\gamma^\mu W^{(-)}_\mu \nu_{\rm flavor}/\sqrt{2} - \nu^T_{\rm flavor} M_\nu \nu_{\rm flavor}/2 = g\bar{\ell}'\gamma^\mu W^{(-)}_\mu \nu'_{\rm flavor}/\sqrt{2} - \nu'^T_{\rm flavor} M'_\nu \nu'_{\rm flavor}/2$, where $\ell = (\ell_e, \ell_\mu, \ell_\tau)^T \equiv (e, \mu, \tau)^T$ and $\ell'_f = e^{i\alpha_f}\ell_f$ for $f = e, \mu, \tau$. Similarly,

$$\mathbf{M}' = \begin{pmatrix} A & e^{-i(\alpha_+ + \alpha_-)} B & e^{-i(\alpha_+ - \alpha_-)} C \\ e^{i(\alpha_+ + \alpha_-)} B^* & D & e^{2i\alpha_-} E \\ e^{i(\alpha_+ - \alpha_-)} C^* & e^{-2i\alpha_-} E^* & F \end{pmatrix}, \tag{A7}$$

is equivalent to $\mathbf{M}$ in Eq. (A3).

It is instructive to note that there are three typical forms of the PMNS unitary matrix depending on how the flavor neutrinos are redefined:

(1) $U_{\rm PMNS}$ with δ, ρ, and γ

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\gamma} & 0 \\ 0 & 0 & e^{-i\gamma} \end{pmatrix} \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13}e^{i\rho} & s_{13}e^{-i\delta} \\ -c_{23}s_{12}e^{-i\rho} - s_{23}c_{12}s_{13}e^{i\delta} & c_{23}c_{12} - s_{23}s_{12}s_{13}e^{i(\delta+\rho)} & s_{23}c_{13} \\ s_{23}s_{12}e^{-i\rho} - c_{23}c_{12}s_{13}e^{i\delta} & -s_{23}c_{12} - c_{23}s_{12}s_{13}e^{i(\delta+\rho)} & c_{23}c_{13} \end{pmatrix} \cdot \begin{pmatrix} e^{i\beta_1} & 0 & 0 \\ 0 & e^{i\beta_2} & 0 \\ 0 & 0 & e^{i\beta_3} \end{pmatrix}, \tag{A8}$$

for

$$\begin{pmatrix} A & B & C \\ B^* & D & E \\ C^* & E^* & F \end{pmatrix}, \tag{A9}$$

(2) $U_{\rm PMNS}$ with δ and ρ

$$\begin{pmatrix} c_{12}c_{13} & s_{12}c_{13}e^{i\rho} & s_{13}e^{-i\delta} \\ -c_{23}s_{12}e^{-i\rho} - s_{23}c_{12}s_{13}e^{i\delta} & c_{23}c_{12} - s_{23}s_{12}s_{13}e^{i(\delta+\rho)} & s_{23}c_{13} \\ s_{23}s_{12}e^{-i\rho} - c_{23}c_{12}s_{13}e^{i\delta} & -s_{23}c_{12} - c_{23}s_{12}s_{13}e^{i(\delta+\rho)} & c_{23}c_{13} \end{pmatrix} \begin{pmatrix} e^{i\beta_1} & 0 & 0 \\ 0 & e^{i\beta_2} & 0 \\ 0 & 0 & e^{i\beta_3} \end{pmatrix}, \tag{A10}$$

for

$$\begin{pmatrix} A & e^{i\gamma}B & e^{-i\gamma}C \\ e^{-i\gamma}B^* & D & e^{-2i\gamma}E \\ e^{i\gamma}C^* & e^{2i\gamma}E^* & F \end{pmatrix}, \tag{A11}$$

(3) $U_{\rm PMNS}$ with $\delta' (= \delta + \rho)$

$$\begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta'} \\ -c_{23}s_{12} - s_{23}c_{12}s_{13}e^{i\delta'} & c_{23}c_{12} - s_{23}s_{12}s_{13}e^{i\delta'} & s_{23}c_{13} \\ s_{23}s_{12} - c_{23}c_{12}s_{13}e^{i\delta'} & -s_{23}c_{12} - c_{23}s_{12}s_{13}e^{i\delta'} & c_{23}c_{13} \end{pmatrix} \begin{pmatrix} e^{i(\beta_1-\rho)} & 0 & 0 \\ 0 & e^{i\beta_2} & 0 \\ 0 & 0 & e^{i\beta_3} \end{pmatrix}, \tag{A12}$$

for

$$\begin{pmatrix} A & e^{-i(\rho-\gamma)}B & e^{-i(\rho+\gamma)}C \\ e^{i(\rho-\gamma)}B^* & D & e^{-2i\gamma}E \\ e^{i(\rho+\gamma)}C^* & e^{2i\gamma}E^* & F \end{pmatrix}. \tag{A13}$$

APPENDIX B: FORMULA FOR MASSES, MIXINGS, AND PHASES

By adopting $U_{\rm PMNS}$ of Eq. (30) to diagonalize $\mathbf{M}$, which $U^\dagger_{\rm PMNS}\mathbf{M}U_{\rm PMNS} = {\rm diag}(m_1^2, m_2^2, m_3^2)$ is satisfied, we then obtain that

$$\tan 2\theta_{12}e^{i\rho} = \frac{2X}{\Lambda_2 - \Lambda_1}, \qquad \tan 2\theta_{13}e^{-i\delta} = \frac{2Y}{\Lambda_3 - A}, \tag{B1}$$

$$\mathrm{Re}\,(e^{-2i\gamma}E)\cos 2\theta_{23} + D_-\sin 2\theta_{23} + i\,\mathrm{Im}(e^{-2i\gamma}E) = -s_{13}e^{-i(\delta+\rho)}(e^{-i\rho}X)^*, \tag{B2}$$

for three mixing angles and three phases, and

$$m_1^2 = c_{12}^2\Lambda_1 + s_{12}^2\Lambda_2 - 2c_{12}s_{12}e^{-i\rho}X, \qquad m_2^2 = s_{12}^2\Lambda_1 + c_{12}^2\Lambda_2 + 2c_{12}s_{12}e^{-i\rho}X, \qquad m_3^2 = \frac{c_{13}^2\Lambda_3 - s_{13}^2 A}{c_{13}^2 - s_{13}^2}, \tag{B3}$$

for three masses, where

$$\begin{aligned} X &= \frac{c_{23}e^{i\gamma}B - s_{23}e^{-i\gamma}C}{c_{13}}, \qquad Y = s_{23}e^{i\gamma}B + c_{23}e^{-i\gamma}C, \qquad \Lambda_1 = \frac{c_{13}^2 A - s_{13}^2\Lambda_3}{c_{13}^2 - s_{13}^2}, \\ \Lambda_2 &= c_{23}^2 D + s_{23}^2 F - 2s_{23}c_{23}\,\mathrm{Re}(e^{-2i\gamma}E), \qquad \Lambda_3 = s_{23}^2 D + c_{23}^2 F + 2s_{23}c_{23}\,\mathrm{Re}(e^{-2i\gamma}E). \end{aligned} \tag{B4}$$

Note that the Dirac CP violation involves the angle $\delta + \rho$. There are useful relations:

$$\begin{aligned} A &\approx \frac{\Sigma m_\odot^2 - \cos 2\theta_{12}\Delta m_\odot^2 + s_{13}^2(2\Delta m_{\rm atm}^2 + \cos 2\theta_{12}\Delta m_\odot^2)}{2}, \\ D_+ &\approx \frac{1}{2}\left(\Delta m_{\rm atm}^2 + \sum m_\odot^2 + \frac{\cos 2\theta_{12}\Delta m_\odot^2 - s_{13}^2(2\Delta m_{\rm atm}^2 + \cos 2\theta_{12}\Delta m_\odot^2)}{2}\right), \\ \mathrm{Re}(e^{-2i\gamma}E) - 2D_-\Delta &\approx \frac{1}{2}\sigma\left(\Delta m_{\rm atm}^2 - \frac{\cos 2\theta_{12}\Delta m_\odot^2 + s_{13}^2(2\Delta m_{\rm atm}^2 + \cos 2\theta_{12}\Delta m_\odot^2)}{2}\right), \\ \Lambda_1 &= \frac{\Sigma m_\odot^2 - \cos 2\theta_{12}\Delta m_\odot^2}{2}, \qquad \Lambda_2 = \frac{\Sigma m_\odot^2 + \cos 2\theta_{12}\Delta m_\odot^2}{2}, \\ \Lambda_3 &\approx \frac{2\Delta m_{\rm atm}^2 + \Sigma m_\odot^2 - s_{13}^2(2\Delta m_{\rm atm}^2 + \cos 2\theta_{12}\Delta m_\odot^2)}{2}, \end{aligned} \tag{B5}$$

up to $\mathcal{O}(s_{13}^2)$, where $\sum m_\odot^2 = m_1^2 + m_2^2$ and $|\Delta m_{\rm atm}^2| \gg \Delta m_\odot^2$ is used to neglect other terms. The term $\tan 2\theta_{12}$ in Eq. (B1) is identically satisfied by these $\Lambda_{1,2}$ because

$$X = \tfrac{1}{2}e^{i\rho}\sin 2\theta_{12}\Delta m_\odot^2, \tag{B6}$$

is obtained from Eq. (B3).

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