Application of the renormalization group to the calculation of the vacuum decay rate in flat and curved space-time

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I show that an application of renormalization group arguments may lead to significant corrections to the vacuum decay rate for phase transitions in flat and curved space-time. It can also give some information regarding its dependence on the parameters of the theory, including the cosmological constant in the case of decay in curved space-time.

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I. INTRODUCTION

The calculation of the false vacuum decay rate in a first-order phase transition in quantum field theory is based on methods developed in [1]. One proceeds by finding a bounce solution to the classical Euclidean field equations. Then the bubble nucleation rate per unit volume, Γ, can be written in the form

$$\Gamma = Ae^{-B}, \tag{1}$$

where B is the Euclidean action of the bounce, and A is an expression involving functional determinants calculated in the background of the bounce, which is usually considered to be of order 1 times a dimensionful parameter of the theory. However, this procedure is not valid in the cases where quantum effects become important. For example, there are cases where even the metastability of the false vacuum is induced by quantum or high temperature effects. One then uses the effective potential in order to calculate the bounce solution. The use of the running effective couplings and renormalization effects in B leads to significant corrections [2,3], as does the numerical calculation of A in various cases [4–6] using a method described in [7]. An analytical method that was developed for the treatment of radiative corrections [8] leads to significant contributions to (1) in a consistent manner [8,9]. The method described in [8] is interesting because it provides a perturbative calculation of corrections to quantities that are nonperturbative in nature. However, this method applies to vacuum decay at zero temperature, in flat space-time, in theories that are not plagued by infrared divergencies [4,8–10].

The situation is less clear if one wishes to include gravitational effects [11]. Many formal results regarding the symmetry of the bounce and the existence of a negative eigenvalue of the functional determinants in A, that exist for flat space, have only recently been addressed in curved space-time [12–14] and there are many new features [15–18] due mainly to the finiteness of the compactified de Sitter space.

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A better knowledge of the decay rate (1) in flat and curved space-time is therefore important, in view of the numerous cosmological applications (see, for example, [19] and references therein).

Here I show that the application of renormalization group arguments to (1) may lead to important corrections to the prefactor A, that is corrections larger than order one, and also to some clues regarding the dependence of the bubble nucleation rate on the parameters of the theory (masses, coupling constants). In the case of vacuum decay with gravity these parameters include the vacuum energy or cosmological constant, and this dependence may be significant for cosmological applications [19].

In Sec. II, I will apply the standard renormalization group arguments to the calculation of the decay rate, Γ, in flat space-time. Similar arguments have been applied for the calculation of corrections to the effective potential and the bounce action, B [2,20–24]; they have not however been applied to the prefactor, A. I show that, depending on the parameters of the theory, these effects may become important. In Sec. III, I will apply these arguments in the case of vacuum decay in curved space-time, including the renormalization group dependence of the vacuum energy or cosmological constant [20–22]. I conclude with some comments in Sec. IV.

II. RENORMALIZATION GROUP FOR VACUUM DECAY IN FLAT SPACE-TIME

I consider a theory with a single scalar field ϕ with mass m and coupling λ, with an effective potential $U(\phi)$ that has a relative minimum at $\phi = \phi_f$ and an absolute minimum at $\phi = \phi_t$. In order for this to happen usually there are additional fields, with additional couplings and masses, that I will not denote explicitly. The Euclidean action

$$S = \int d^4x \left(\frac{1}{2} (\partial\phi)^2 + U(\phi) \right) \tag{2}$$

is minimized at the solution of

$$\Box\phi = \frac{\partial U}{\partial \phi} \tag{3}$$

with appropriate boundary conditions, which gives the bounce solution $\phi_b(x)$. The effective potential satisfies the renormalization group (RG) equation, for a renormalization scale μ,

$$\mathcal{D}U = 4\gamma U \tag{4}$$

$$\mathcal{D} = \mu\frac{\partial}{\partial\mu} + \beta_\lambda\frac{\partial}{\partial\lambda} + \gamma_m m\frac{\partial}{\partial m} \tag{5}$$

with β_λ, γ_m, γ the modified RG functions [22], with solution

$$U = \xi^4(t)U(\phi, \lambda(t), m(t), \mu(t)), \tag{6}$$

where $\mu(t) = \mu e^t$,

$$\xi(t) = e^{-\int_0^t \gamma(t')dt'} \tag{7}$$

and $\lambda(t)$, $m(t)$ the running coupling and mass.

Using the scaling properties of the bounce equations (3) and (6) we see that the relative equation for the bounce is

$$\phi_b = \phi_b(\xi^2(t)x, \lambda(t), m(t), \mu(t)). \tag{8}$$

Then the bounce action becomes

$$S = \xi^{-4}(t)S(\lambda(t), m(t), \mu(t)) \tag{9}$$

and satisfies

$$\mathcal{D}S = -4\gamma S. \tag{10}$$

We write a general expression for the false vacuum decay rate:

$$\Gamma = m^4 A(\lambda, m, \mu)e^{-S(\lambda,m,\mu)}, \tag{11}$$

where A is a dimensionless function that comes from the functional determinants involved in the calculation of the vacuum decay rate and S is the bounce action with the false vacuum contribution subtracted [1]. Then imposing $\mathcal{D}\Gamma = 0$ gives

$$\mathcal{D}A = -4(\gamma S + \gamma_m)A. \tag{12}$$

This is the equation that expresses the renormalization group invariance of the physical quantity Γ, with solution

$$A = A(\lambda(t), m(t), \mu(t))e^{4\int_0^t(\gamma S+\gamma_m)dt'}. \tag{13}$$

We see that the prefactor A has a dependence on the parameters of the theory that may be larger than order one, depending on γ, γ_m, S. For the simple case of a $\lambda\phi^4$ theory (for negative λ), where $\gamma_m \sim \lambda$, $\gamma \sim \lambda^2$, $S \sim 1/\lambda$, it is easy to see that the correction is smaller than order one (turns out to be of order λ), hence insignificant for the calculation of the decay rate. However, for other theories where S scales differently with λ this gives corrections that are exponentially large, although smaller than the exponential of the bounce action. This is the case, for example, for theories with radiative symmetry breaking where $\gamma \sim g^2$, $S \sim 1/g^4$, with g the gauge coupling and the corrections turn out to be of order $1/g^2$, that is larger than order one, although smaller than the bounce action. This is expected from the results in [8,9], although one does not expect the contributions from the renormalization group to provide the complete higher order corrections that were calculated there.

This is, however, a general argument that applies to any model, and it can give some information concerning the higher order corrections and their dependence on the parameters involved. In the case of vacuum decay in curved space-time, treated in the next section, it is more useful, since higher order calculations may involve quantum gravity effects that cannot be seen in perturbation theory.

III. RENORMALIZATION GROUP FOR VACUUM DECAY IN CURVED SPACE-TIME

Unlike vacuum decay in flat space-time, in the case of tunneling in de Sitter space-time the exponent B in (1) is the difference of two finite actions: the bounce action and the finite Euclidean action of de Sitter space-time that depends solely on the cosmological constant. The different renormalization group behavior of these two gives additional contributions to the previous estimates that also depend on the cosmological constant.

For definiteness I will consider a potential term (with coupling constant λ and mass parameter m) that is everywhere positive and has two minima, a relative minimum at ϕ_f with $U(\phi_f) = \Lambda$, and an absolute minimum at ϕ_t. I will also consider the thin wall approximation [1] where the difference ε between the two minima satisfies

$$\lambda\varepsilon \ll m^4. \tag{14}$$

Then the flat space bounce solution has radius

$$R_b \sim \frac{m^3}{\lambda\varepsilon}. \tag{15}$$

The curved space-time formalism of [11] calculates the gravitational contributions to the decay rate for this case, which corresponds to tunneling from a de Sitter space with cosmological constant Λ to a de Sitter space with a slightly smaller cosmological constant $\Lambda - \varepsilon$.

In order to apply the dilute instanton gas arguments of [1,7], however, we need to work in the approximation where the bounce radius is much smaller than the radius of the compactified de Sitter space

$$R_{\rm dS} = \left(\frac{3M_P^2}{8\pi\Lambda}\right)^{1/2}. \tag{16}$$

For $R_b \ll R_{\rm dS}$ the gravitational corrections to the flat space results are of higher order. It is this approximation (together with the thin wall approximation mentioned before) that I will use here in order to apply the arguments of the previous section. In this case we can use the renormalization group coefficients for flat space as a first approxima-

tion. Higher (gravitational) corrections to these terms will be suppressed by powers of $R_b/R_{\rm dS}$.

The arguments of the previous section can be extended to the case of false vacuum decay in curved space-time if we include the running of the vacuum energy in the effective potential [21,22]. In order to describe tunneling in curved space-time [11], one considers an $O(4)$-invariant metric

$$ds^2 = d\tau^2 + \rho(\tau)^2 (d\Omega)^2. \tag{17}$$

The Euclidean equations of motion are

$$\phi'' + \frac{3\rho'}{\rho}\phi' = \frac{\partial U}{\partial \phi} \tag{18}$$

$$\rho'^2 = 1 + \frac{8\pi G}{3}\rho^2\left(\frac{1}{2}\phi'^2 - U\right), \tag{19}$$

where the prime is $d/d\tau$, and the Euclidean action is

$$S = 4\pi^2 \int d\tau \left(\rho^3 U - \frac{3\rho}{8\pi G}\right). \tag{20}$$

We write the effective potential as $U = U_0 + \Lambda$, where U_0 is the part that vanishes at ϕ_f, the relative minimum, and Λ is the cosmological constant term, with $d\Lambda/dt = \beta_\Lambda$. As was mentioned before, I will consider the flat space value for β_Λ [21,22], which is of order m^4, and neglect the gravitational corrections to β_Λ which are expected to be of order $R_b/R_{\rm dS}$.

The exponent B for the tunneling rate is $B = S_b - S_f$ with S_b the bounce action and S_f the false vacuum action

$$S_f = -\frac{8\pi M_P^4}{3\Lambda}. \tag{21}$$

In order to exploit the scaling properties of the bounce solution, we write the solution of the renormalization group equation for U as

$$U = \xi^4(t) U(\phi, \lambda(t), m(t), \mu(t), \bar{\Lambda}(t)), \tag{22}$$

where $\bar{\Lambda} = \xi^{-4}\Lambda$. Then the solutions ϕ_b, ρ_b, of (18) and (19) scale as

$$\phi_b = \phi_b(\xi^2(t)\tau, \lambda(t), m(t), \mu(t), \bar{\Lambda}(t)) \tag{23}$$

$$\rho_b = \frac{1}{\xi^2(t)}\rho_b(\xi^2(t)\tau, \lambda(t), m(t), \mu(t), \bar{\Lambda}(t)) \tag{24}$$

and the bounce action similarly to the previous case

$$S_b = \xi(t)^{-4} S(\lambda(t), m(t), \mu(t), \bar{\Lambda}(t)) \tag{25}$$

$$\mathcal{D} S_b = -4\gamma S_b(\bar{\Lambda}), \tag{26}$$

where $\mathcal{D}$ includes the running of $\bar{\Lambda}$. For the false vacuum action we write

$$\frac{d}{dt}S_f = -\frac{8\pi M_P^4}{3\bar{\Lambda}}\left(4\gamma - 4\gamma - \frac{1}{\xi^4\Lambda}\frac{d\Lambda}{dt}\right) \tag{27}$$

to get

$$\mathcal{D} B = -4\gamma B_{CdL}(\bar{\Lambda}) + \frac{8\pi M_P^4}{3\Lambda}\left(\frac{\beta_\Lambda}{\Lambda} + 4\gamma\xi^4\right), \tag{28}$$

where $B_{CdL}(\bar{\Lambda})$ is the exponent for decay in curved space-time [11], more precisely the one derived in [25] for arbitrary values of the cosmological constant, with the modified running for Λ. Using similar arguments as in the previous section we get for the prefactor

$$A = A(t)\exp\left(4\int(\gamma B_{CdL}(\bar{\Lambda}) + \gamma_m)dt' - \int\frac{8\pi M_P^4}{3\Lambda}\left(\frac{\beta_\Lambda}{\Lambda} + 4\gamma\xi^4\right)dt'\right). \tag{29}$$

We see again that this gives a correction to the prefactor which may be smaller than the exponential but also, depending on the specifics of the model, may be larger than order one, like in the flat case, hence important for the calculation of the decay rate. Again, this contribution has a nontrivial dependence on the parameters of the theory, including the cosmological constant.

In summary our approximations are $R_b \ll R_{\rm dS}$ together with the thin wall approximation (14). They hold for a large range of the parameters of the model, provided that the cosmological constant term (and the mass parameters of the theory) are not of Planck scale value. That is, we are not calculating quantum gravitational contributions, some of which may be estimated by extending this method with the renormalization group in curved space-time [26–29]. It is interesting, however, that a nontrivial dependence of the prefactor on the cosmological constant term arises even at this scale.

IV. COMMENTS

We see that an application of the usual renormalization group arguments can give some contributions to the calculation of the false vacuum decay rate, especially in theories where the quantum or gravitational effects are important. In the case of tunneling in curved space-time, this may give some insight to the dependence of the tunneling rate on the parameters of the theory, such as the cosmological constant. One would like to have a more detailed description of tunneling between different vacua and show that the most probable is the one with the observed values of the physical parameters and the cosmological constant. The complete description may involve contributions from quantum gravity effects that are not treatable perturbatively. The information that can be obtained from the renormalization group, as described in this work or otherwise, is therefore important.

I should also note that the renormalization group arguments that I used here are the usual ones of quantum field theory in flat space-time, together with the running of the vacuum energy, interpreted as a cosmological constant. They correspond, therefore, to the limit of weak gravity, or equivalently, to the cases where the radius of the bounce solution is much smaller than the radius of the compactified de Sitter space, which is, anyway, the situation where the arguments of [1,7] can be straightforwardly applied. Some ways to improve the present results may involve the use of multiscale renormalization group arguments [30,31] and the use of renormalization group in curved space-time [26–29].

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