Propagator Dyson-Schwinger equations of Coulomb gauge Yang-Mills theory within the first order formalism

P. Watson and H. Reinhardt

Institut für Theoretische Physik, Universität Tübingen, Auf der Morgenstelle 14, D-72076 Tübingen, Germany

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Coulomb gauge Yang-Mills theory within the first order formalism is considered with a view of deriving the propagator Dyson-Schwinger equations. The first order formalism is studied with special emphasis on the Becchi-Rouet-Stora (BRS) invariance and it is found that there exists two forms of invariance—invariance under the standard BRS transform and under a second, nonstandard transform. The field equations of motion and symmetries are derived explicitly and certain exact relations that simplify the formalism are presented. It is shown that the Ward-Takahashi identity arising from invariance under the nonstandard part of the BRS transform is guaranteed by the functional equations of motion. The Feynman rules and the general decomposition of the two-point Green's functions are derived. The propagator Dyson-Schwinger equations are derived and certain aspects (energy independence of ghost Green's functions and the cancellation of energy divergences) are discussed.

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I. INTRODUCTION

Whilst there is little doubt that quantum chromodynamics (QCD) is the theory of the strong interaction, despite four decades of intense effort the genuine solution of the confinement puzzle and the hadron spectrum remains elusive. That is not to say that no progress has been made—our understanding of QCD is being steadily augmented in many ways; for example, with lattice Monte Carlo techniques [1,2] and effective theories ([3] and references therein). One way to understand the problem of confinement and the hadron spectrum from *ab initio* principles is to study the Dyson-Schwinger equations. These equations are the central equations to the Lagrange formulation of a field theory. They are in the continuum and embody all symmetries of the system at hand.

Dyson-Schwinger studies of QCD in Landau gauge have enjoyed a renaissance in the last ten years. A consistent picture of how the important degrees of freedom in the infrared (i.e., those responsible for confinement and the hadron spectrum) stem from the ghost sector of the theory has emerged [4–6] and this has led to increasingly sophisticated calculations of QCD properties, culminating recently in hadron observables [7] and possible explanations of confinement (see the recent review [8] for a discussion of this topic). Landau gauge, in addition to having the appealing property of covariance, has a distinct advantage when searching for practical approximation schemes that allow one to extract information about the infrared behavior of QCD Green's functions, namely, that the ghost-gluon vertex remains UV finite to all orders in perturbation theory [9]. For this reason, it has been possible to extract unambiguous information about the system [6].

Despite the calculational advantages enjoyed by Landau gauge, it is perhaps not the best choice of gauge to study the infrared physics of QCD. In this respect, Coulomb gauge is perhaps more advantageous. There exists a natural picture (though not a proof) of confinement in Coulomb gauge [10] and phenomenological applications guide the way to understanding the spectrum of hadrons, for example, in [11,12] (as indeed they have done in Landau gauge [8,13]). This is not to say that one choice of gauge is better than another—it is crucial to our understanding of the problem that more than one gauge is considered: first, because the physical observables are gauge invariant and it is a test of our approximations that the results respect this and second, because while confinement is a gauge invariant reality, its mechanism may be manifested differently in different gauges such that we will learn far more by studying the different gauges.

Recently, progress has been made in studying Yang-Mills theory in Coulomb gauge within the Hamiltonian approach [14–17]. Here, the advantage is that Gauß' law can be explicitly resolved (such that, in principle, gauge invariance is fully accounted for) and this results in an explicit expression for the static potential between color charges. In [14–17], the Yang-Mills Schroedinger equation was solved variationally for the vacuum state using Gaussian type ansätze for the wave functional. Minimizing the energy density results in a coupled set of Dyson-Schwinger equations which have been solved analytically in the infrared [18] and numerically in the entire momentum regime. If the geometric structure of the space of gauge orbits reflected by the nontrivial Faddeev-Popov determinant is properly included [16], one finds an infrared divergent gluon energy and a linear rising static quark potential—both signals of confinement. Furthermore, these confinement properties have been shown to be not dependent on the specific ansatz for the vacuum wave functional but result from the geometric structure of the space of gauge orbits [17]. However, in spite of this success, one should bear in mind that an ansatz for the wave

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functional is always required and so the approach does not *a priori* provide a systematic expansion or truncation scheme as, for example, the loop expansion scheme used in the common Dyson-Schwinger approach. A study of the Dyson-Schwinger equations in Coulomb gauge will hopefully shed some light on the problem.

Given the appealing properties of Coulomb gauge, it is perhaps surprising that no pure Dyson-Schwinger study exists in the literature. However, there is a good reason for this: in Coulomb gauge, closed ghost loops give rise to unregulated divergences—the energy divergence problem. It has been found only relatively recently how one may circumvent this problem such that a Dyson-Schwinger study may be attempted [10]. The key lies in using the first order formalism. There is not yet a complete proof that the local formulation of Coulomb gauge Yang-Mills theory within the first order formalism is renormalizable but significant progress has been made [10,19]. There is one undesirable feature to the first order formalism and this is that the number of fields proliferates. As will be seen in this paper, this does have serious implications for the Dyson-Schwinger equations.

The purpose of the present work is to derive the Dyson-Schwinger equations for Coulomb gauge Yang-Mills theory within the first order formalism. These equations will form the basis for an extended program studying QCD in Coulomb gauge. The paper is organized as follows. We begin in Sec. II by introducing Yang-Mills theory in Coulomb gauge and the first order formalism. In particular, we consider the BRS invariance of the system. Having introduced the first order formalism, we then motivate the reasons for considering it (the cancellation of the energy divergent sector and the reduction to physical degrees of freedom) in Sec. III. Section IV is then concerned with the derivation of the equations of motion and the equations that stem from the BRS invariance. There exists certain relationships that give rise to exact statements about the Green's functions that enter the system and these are detailed in Sec. V. The Feynman rules and the general decomposition of the two-point Green's functions are derived and discussed in Secs. VI and VII. In Sec. VIII, the Dyson-Schwinger equations are derived in some detail and are discussed. Finally, we summarize and give an outlook of future work in Sec. IX.

II. FIRST ORDER FORMALISM AND BRS INVARIANCE

Throughout this work, we work in Minkowski space and with the following conventions. The metric is $g_{\mu\nu} = \mathrm{diag}(1, -\vec{1})$. Greek letters $(\mu, \nu, \ldots)$ denote Lorentz indices, roman subscripts $(i, j, \ldots)$ denote spatial indices, and superscripts $(a, b, \ldots)$ denote color indices. We will sometimes also write configuration space coordinates $(x, y, \ldots)$ as subscripts where no confusion arises.

The Yang-Mills action is defined as

$$\mathcal{S}_{\mathrm{YM}} = \int d^4x\left[-\frac{1}{4}F_{\mu\nu}^a F^{a\mu\nu}\right], \tag{2.1}$$

where the (antisymmetric) field strength tensor F is given in terms of the gauge field A_μ^a:

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + gf^{abc}A_\mu^b A_\nu^c. \tag{2.2}$$

In the above, the f^{abc} are the structure constants of the $SU(N_c)$ group whose generators obey $[T^a, T^b] = \imath f^{abc}T^c$. The Yang-Mills action is invariant under a local $SU(N_c)$ gauge transform characterized by the parameter θ_x^a:

$$U_x = \exp\{-\imath\theta_x^a T^a\}. \tag{2.3}$$

The field strength tensor can be expressed in terms of the chromo-electric and chromo-magnetic fields ($\sigma = A^0$)

$$\begin{aligned} \vec{E}^a &= -\partial^0\vec{A}^a - \vec{\nabla}\sigma^a + gf^{abc}\vec{A}^b\sigma^c, \\ B_i^a &= \epsilon_{ijk}\left[\nabla_j A_k^a - \tfrac{1}{2}gf^{abc}A_j^b A_k^c\right] \end{aligned} \tag{2.4}$$

such that $\mathcal{S}_{\mathrm{YM}} = \int(E^2 - B^2)/2$. The electric and magnetic terms in the action do not mix under the gauge transform which for the gauge fields is written

$$A_\mu \to A_\mu' = U_x A_\mu U_x^\dagger - \frac{\imath}{g}(\partial_\mu U_x)U_x^\dagger. \tag{2.5}$$

Given an infinitessimal transform $U_x = 1 - \imath\theta_x^a T^a$ the variation of the gauge field is

$$\delta A_\mu^a = -\frac{1}{g}\hat{D}_\mu^{ac}\theta^c, \tag{2.6}$$

where the covariant derivative in the adjoint representation is given by

$$\hat{D}_\mu^{ac} = \delta^{ac}\partial_\mu + gf^{abc}A_\mu^b. \tag{2.7}$$

Let us consider the functional integral

$$Z = \int \mathcal{D}\Phi\exp\{\imath\mathcal{S}_{\mathrm{YM}}\}, \tag{2.8}$$

where Φ denotes the collection of all fields. Since the action is invariant under gauge transformations, Z is divergent by virtue of the zero mode. To overcome this problem we use the Faddeev-Popov technique and introduce a gauge-fixing term along with an associated ghost term [20]. Using a Lagrange multiplier field to implement the gauge fixing, in Coulomb gauge ($\vec{\nabla}\cdot\vec{A} = 0$) we can then write

$$\begin{aligned} Z &= \int \mathcal{D}\Phi\exp\{\imath\mathcal{S}_{YM} + \imath\mathcal{S}_{fp}\}, \\ \mathcal{S}_{fp} &= \int d^4x[-\lambda^a\vec{\nabla}\cdot\vec{A}^a - \bar{c}^a\vec{\nabla}\cdot\vec{D}^{ab}c^b]. \end{aligned} \tag{2.9}$$

The new term in the action is invariant under the standard BRS transform whereby the infinitessimal parameter θ^a is factorized into two Grassmann-valued components $\theta^a = c^a\delta\lambda$, where $\delta\lambda$ is the infinitessimal variation (not to be confused with the colored Lagrange multiplier field λ^a). The BRS transform of the new fields reads

$$\delta\bar{c}^a = \frac{1}{g}\lambda^a\delta\lambda, \qquad \delta c^a = -\frac{1}{2}f^{abc}c^b c^c\delta\lambda, \qquad \delta\lambda^a = 0. \tag{2.10}$$

For reasons that will become clear in the next section (expounded in [10]) we convert to the first order (or phase space) formalism by splitting the Yang-Mills action into chromo-electric and chromo-magnetic terms and introducing an auxiliary field ($\vec{\pi}$) via the following identity

$$\exp\left\{\imath\int d^4x\frac{1}{2}\vec{E}^a\cdot\vec{E}^a\right\} = \int\mathcal{D}\vec{\pi}\exp\left\{\imath\int d^4x\left[-\frac{1}{2}\vec{\pi}^a\cdot\vec{\pi}^a - \vec{\pi}^a\cdot\vec{E}^a\right]\right\}. \tag{2.11}$$

Classically, the $\vec{\pi}$-field would be the momentum conjugate to $\vec{A}$. In order to maintain BRS invariance, we require that

$$\int d^4x[\delta\vec{\pi}^a\cdot(\vec{\pi}^a + \vec{E}^a) + \vec{\pi}^a\cdot\delta\vec{E}^a] = 0. \tag{2.12}$$

Given that the variation of $\vec{E}$ under the infinitessimal gauge transformation is $\delta\vec{E}^a = f^{abc}\vec{E}^c\theta^b$, then the general solution to Eq. (2.12) is

$$\delta\vec{\pi}^a = f^{abc}\theta^b[(1-\alpha)\vec{\pi}^c - \alpha\vec{E}^c], \tag{2.13}$$

where α is some noncolored constant, but which in general could be some function of position x. The $\vec{\pi}$-field is split into transverse and longitudinal components using the identity

$$\begin{aligned}\text{const} &= \int\mathcal{D}\phi\delta(\vec{\nabla}\cdot\vec{\pi} + \nabla^2\phi)\\ &= \int\mathcal{D}\{\phi,\tau\}\exp\left\{-\imath\int d^4x\tau^a(\vec{\nabla}\cdot\vec{\pi}^a + \nabla^2\phi^a)\right\}.\end{aligned} \tag{2.14}$$

This constant is gauge invariant and this means that the new fields ϕ and τ must transform as

$$\delta\phi^a = \frac{\vec{\nabla}}{(-\nabla^2)}\cdot\delta\vec{\pi}^a, \qquad \delta\tau^a = 0. \tag{2.15}$$

If we make the change of variables $\vec{\pi}\to\vec{\pi} - \vec{\nabla}\phi$ then collecting together all the parts of Z that contain $\vec{\pi}$, we can write

$$Z_\pi = \int\mathcal{D}\{\vec{\pi},\phi,\tau\}\exp\left\{\imath\int d^4x\left[-\tau^a\vec{\nabla}\cdot\vec{\pi}^a - \frac{1}{2}(\vec{\pi}^a - \vec{\nabla}\phi^a)\cdot(\vec{\pi}^a - \vec{\nabla}\phi^a) - (\vec{\pi}^a - \vec{\nabla}\phi^a)\cdot\vec{E}^a\right]\right\} \tag{2.16}$$

which is now invariant under

$$\begin{aligned}\delta\vec{E}^a &= f^{abc}\vec{E}^c\theta^b,\\ \delta\vec{\pi}^a &= f^{abc}\theta^b[(1-\alpha)(\vec{\pi}^c - \vec{\nabla}\phi^c) - \alpha\vec{E}^c] + \vec{\nabla}\delta\phi^a,\\ \delta\phi^a &= f^{abc}\left\{\frac{\vec{\nabla}}{(-\nabla^2)}\cdot[(1-\alpha)(\vec{\pi}^c - \vec{\nabla}\phi^c) - \alpha\vec{E}^c]\theta^b\right\},\\ \delta\tau^a &= 0.\end{aligned} \tag{2.17}$$

We notice that the parts of the transform that are proportional to α are independent of the rest of the BRS transform and can thus be regarded as a separate invariance. In particular, since it is independent of the Faddeev-Popov components, then we may regard it quite generally as a local transform parametrized by θ^a. This new invariance stems from the arbitrariness in introducing the $\vec{\pi}$-field. If we expand the chromo-electric field into its component form then in summary we can write our full functional integral as

$$Z = \int\mathcal{D}\Phi\exp\{\imath\mathcal{S}_B + \imath\mathcal{S}_{fp} + \imath\mathcal{S}_\pi\} \tag{2.18}$$

with

$$\begin{aligned}\mathcal{S}_B &= \int d^4x\left[-\frac{1}{2}\vec{B}^a\cdot\vec{B}^a\right],\\ \mathcal{S}_{fp} &= \int d^4x[-\lambda^a\vec{\nabla}\cdot\vec{A}^a - \bar{c}^a\vec{\nabla}\cdot\vec{D}^{ab}c^b],\\ \mathcal{S}_\pi &= \int d^4x\left[-\tau^a\vec{\nabla}\cdot\vec{\pi}^a - \frac{1}{2}(\vec{\pi}^a - \vec{\nabla}\phi^a)\cdot(\vec{\pi}^a - \vec{\nabla}\phi^a) + (\vec{\pi}^a - \vec{\nabla}\phi^a)\cdot(\partial^0\vec{A}^a + \vec{D}^{ab}\sigma^b)\right],\end{aligned} \tag{2.19}$$

and which is invariant under *two* sets of transforms: the BRS

$$\begin{aligned}\delta\vec{A}^a &= \frac{1}{g}\vec{D}^{ac}c^c\delta\lambda, \qquad \delta\sigma^a = -\frac{1}{g}D^{0ac}c^c\delta\lambda,\\ \delta\bar{c}^a &= \frac{1}{g}\lambda^a\delta\lambda, \qquad \delta c^a = -\frac{1}{2}f^{abc}c^b c^c\delta\lambda,\\ \delta\vec{\pi}^a &= f^{abc}c^b\delta\lambda(\vec{\pi}^c - \vec{\nabla}\phi^c) + \vec{\nabla}\delta\phi^a,\\ \delta\phi^a &= f^{abc}\left\{\frac{\vec{\nabla}}{(-\nabla^2)}\cdot(\vec{\pi}^c - \vec{\nabla}\phi^c)c^b\delta\lambda\right\},\\ \delta\lambda^a &= 0, \qquad \delta\tau^a = 0,\end{aligned} \tag{2.20}$$

and the new transform, which we denote the α-transform,

$$\begin{aligned}\delta\vec{\pi}^a &= f^{abc}\theta^b(\vec{\pi}^c - \vec{\nabla}\phi^c - \partial^0\vec{A} - \vec{D}^{cd}\sigma^d) + \vec{\nabla}\delta\phi^a,\\ \delta\phi^a &= f^{abc}\left\{\frac{\vec{\nabla}}{(-\nabla^2)}\cdot(\vec{\pi}^c - \vec{\nabla}\phi^c - \partial^0\vec{A}^c - \vec{D}^{cd}\sigma^d)\theta^b\right\}\end{aligned} \tag{2.21}$$

(all other fields being unchanged). It useful for later to denote the combination of fields and differential operators occurring in Eq. (2.21) as

$$\vec{X}^c = \vec{\pi}^c - \vec{\nabla}\phi^c - \partial^0\vec{A} - \vec{D}^{cd}\sigma^d. \quad (2.22)$$

III. FORMAL REDUCTION TO "PHYSICAL" DEGREES OF FREEDOM

There are two factors that motivate our use of the first order formalism. The first lies in the ability, albeit formally, to reduce the functional integral previously considered (and hence the generating functional) to "physical" degrees of freedom [10]. These are the transverse gluon and transverse $\vec{\pi}$ fields which in classical terms would be the configuration variables and their momentum conjugates. We keep the term physical in quotation marks because it is realized that in Yang-Mills theory, the true physical objects would be the color singlet glueballs, their observables being the mass spectrum and the decay widths. The second factor concerns the well-known energy divergence problem of Coulomb gauge QCD [21–23]. In Coulomb gauge, the Faddeev-Popov operator involves only spatial derivatives and the spatial components of the gauge fields, but these fields are themselves dependent on the spacetime position. This leads to the ghost propagator and ghost-gluon vertex being independent of the energy whereas loops involving pure ghost components are integrated over both 3-momentum *and* energy which gives an ill-defined integration. In the usual, second order, formulation of the theory these energy divergences do in principle cancel order by order in perturbation theory (tested up to two loops [23]) but this cancellation is difficult to isolate. Within the first order formalism the cancellation is made manifest such that the problem of ill-defined integrals can be circumvented.

Given the functional integral, Eq. (2.18), and the action, Eq. (2.19), we rewrite the Lagrange multiplier terms as δ-function constraints and the ghost terms as the original Faddeev-Popov determinant. Since the δ-function constraints are now exact we can automatically eliminate any $\vec{\nabla}\cdot\vec{A}$ and $\vec{\nabla}\cdot\vec{\pi}$ terms in the action. This is clearly at the expense of a local formulation and the BRS invariance of the theory is no longer manifest. The functional integral is now

$$Z = \int \mathcal{D}\Phi \det[-\vec{\nabla}\cdot\vec{D}\delta^4(x-y)]\delta(\vec{\nabla}\cdot\vec{A}) \times \delta(\vec{\nabla}\cdot\vec{\pi})\exp\{\imath\mathcal{S}\} \quad (3.1)$$

with

$$\mathcal{S} = \int d^4x\Big[-\frac{1}{2}\vec{B}^a\cdot\vec{B}^a - \frac{1}{2}\vec{\pi}^a\cdot\vec{\pi}^a + \frac{1}{2}\phi^a\nabla^2\phi^a + \vec{\pi}^a\cdot\partial^0\vec{A}^a + \sigma^a(\vec{\nabla}\cdot\vec{D}^{ab}\phi^b + g\hat{\rho}^a)\Big], \quad (3.2)$$

where we have defined an effective charge $\hat{\rho}^a = f^{ade}\vec{A}^d\cdot\vec{\pi}^e$. The integral over σ can also be written as a δ-function constraint and is the implementation of the chromodynamical equivalent of Gauß' law giving

$$Z = \int \mathcal{D}\Phi \det[-\vec{\nabla}\cdot\vec{D}\delta^4(x-y)]\delta(\vec{\nabla}\cdot\vec{A})\delta(\vec{\nabla}\cdot\vec{\pi}) \times \delta(-\vec{\nabla}\cdot\vec{D}^{ab}\phi^b - g\hat{\rho}^a)\exp\{\imath\mathcal{S}\} \quad (3.3)$$

with

$$\mathcal{S} = \int d^4x\Big[-\frac{1}{2}\vec{B}^a\cdot\vec{B}^a - \frac{1}{2}\vec{\pi}^a\cdot\vec{\pi}^a + \frac{1}{2}\phi^a\nabla^2\phi^a + \vec{\pi}^a\cdot\partial^0\vec{A}^a\Big]. \quad (3.4)$$

Let us define the inverse Faddeev-Popov operator M:

$$[-\vec{\nabla}\cdot\vec{D}^{ab}]M^{bc} = \delta^{ac}. \quad (3.5)$$

With this definition we can factorize the Gauß law δ-function constraint as

$$\delta(-\vec{\nabla}\cdot\vec{D}^{ab}\phi^b - g\hat{\rho}^a) = \det[-\vec{\nabla}\cdot\vec{D}\delta^4(x-y)]^{-1} \times \delta(\phi^a - M^{ab}g\hat{\rho}^b). \quad (3.6)$$

Crucially, the inverse functional determinant cancels the original Faddeev-Popov determinant, leaving us with

$$Z = \int \mathcal{D}\Phi\delta(\vec{\nabla}\cdot\vec{A})\delta(\vec{\nabla}\cdot\vec{\pi})\delta(\phi^a - M^{ab}g\hat{\rho}^b)\exp\{\imath\mathcal{S}\}. \quad (3.7)$$

We now use the δ-function constraint to eliminate the ϕ-field. Recognizing the Hermitian nature of the inverse Faddeev-Popov operator M we can reorder the operators in the action to give us

$$Z = \int \mathcal{D}\Phi\delta(\vec{\nabla}\cdot\vec{A})\delta(\vec{\nabla}\cdot\vec{\pi})\exp\{\imath\mathcal{S}\} \quad (3.8)$$

with

$$\mathcal{S} = \int d^4x\Big[-\frac{1}{2}\vec{B}^a\cdot\vec{B}^a - \frac{1}{2}\vec{\pi}^a\cdot\vec{\pi}^a - \frac{1}{2}g\hat{\rho}^bM^{ba}(-\nabla^2)M^{ac}g\hat{\rho}^c + \vec{\pi}^a\cdot\partial^0\vec{A}^a\Big]. \quad (3.9)$$

The above action is our desired form, with only transverse $\vec{A}$ and $\vec{\pi}$ fields present. All other fields, especially those responsible for the Faddeev-Popov determinant (i.e., the certainly unphysical ghosts) have been formally eliminated. However, the appearance of the functional δ-functions and the inverse Faddeev-Popov operator M have led to a nonlocal formalism. It is not known how to use forms such as the above in calculational schemes. The issue of renormalizability is certainly unclear and one does not have a Ward identity in the usual sense.

The nonlocal nature of the above result may not lend itself to calculational devices but does serve as a guide to the local formulation. In particular, it is evident that decomposition of degrees of freedom, both physical and unphysical, inherent to the first order formalism leads more naturally to the cancellation of the unphysical components in the description of physical phenomena than perhaps other choices such as Landau gauge. The task ahead is to identify, within the local formulation, how these cancellations arise and to ensure that approximation schemes respect such cancellations. For example, the cancellation of the Faddeev-Popov determinant and the appearance of the inverse Faddeev-Popov operator should lead to the separation of the physical gluon dynamics contained within the ghost sector and the unphysical ghosts themselves, i.e., the unphysical ghost loop of the gluon polarization should be cancelled while another loop containing only physical information will take its place. Also, it should be evident that the energy divergences associated with ghost loops are explicitly cancelled such that ill-defined integrals do not occur.

IV. FIELD EQUATIONS OF MOTION AND CONTINUOUS SYMMETRIES

The generating functional of the theory is given by our previously considered functional integral in the presence of sources. Explicitly, given the action, Eq. (2.19), we have

$$Z[J] = \int \mathcal{D}\Phi \exp\{\imath\mathcal{S}_B + \imath\mathcal{S}_{fp} + \imath\mathcal{S}_\pi + \imath\mathcal{S}_s\} \tag{4.1}$$

with sources defined by

$$\mathcal{S}_s = \int d^4x[\rho^a\sigma^a + \vec{J}^a\cdot\vec{A}^a + \bar{c}^a\eta^a + \bar{\eta}^a c^a + \kappa^a\phi^a + \vec{K}^a\cdot\vec{\pi}^a + \xi_\lambda^a\lambda^a + \xi_\tau^a\tau^a]. \tag{4.2}$$

It is useful to introduce a compact notation for the sources and fields and we denote a generic field Φ_α with source J_α such that the index α stands for all attributes of the field in question (including its type) such that, for instance, we could write

$$\mathcal{S}_s = J_\alpha\Phi_\alpha, \tag{4.3}$$

where summation over all discrete indices and integration over all continuous arguments is implicitly understood.

The field equations of motion are derived from the observation that the integral of a total derivative vanishes up to boundary terms. The boundary terms vanish but this is not so trivial in the light of the Gribov problem. Perturbatively (expanding around the free field), there are certainly no boundary terms to be considered, however, nonperturbatively the presence of so-called Gribov copies does complicate the picture somewhat.

In [24], Gribov showed that the Faddeev-Popov technique does not uniquely fix the gauge and showed that even after gauge fixing there are (physically equivalent) gauge configurations related by finite gauge transforms still present. It was proposed to restrict the space of gauge field configurations (A) to the so-called Gribov region Ω defined by

$$\Omega \equiv \{A\colon \vec{\nabla}\cdot\vec{A} = 0; -\vec{\nabla}\cdot\vec{D} \geq 0\}. \tag{4.4}$$

Ω is a region where the Coulomb gauge condition holds and furthermore, the eigenvalues of the Faddeev-Popov operator are all positive. It contains the element $\vec{A} = 0$ and is bounded in every direction [25]. However, as explained in [26] and references therein, the Gribov region is not entirely free of Gribov copies and one should more correctly consider the fundamental modular region Λ which is defined as the region free of Gribov copies. It turns out though that the functional integral is dominated by configurations on the common boundary of Λ and Ω so that, in practice, restriction to the Gribov region is sufficient.

Given that nontrivial boundary conditions are being imposed (i.e., restricting to the Gribov region Ω), the question of the boundary terms in the derivation of the field equations of motion now becomes extremely relevant. However, by definition on the boundary of Ω, the Faddeev-Popov determinant *vanishes* such that the boundary terms are identically zero. The form of the field equations of motion is therefore equivalent as if we had extended the integration region to the full configuration space [26].

Writing $\mathcal{S} = \mathcal{S}_B + \mathcal{S}_{fp} + \mathcal{S}_\pi$ we have that

$$0 = \int \mathcal{D}\Phi \frac{\delta}{\delta\imath\Phi_\alpha}\exp\{\imath\mathcal{S} + \imath\mathcal{S}_s\} = \int \mathcal{D}\Phi\left\{\frac{\delta\mathcal{S}}{\delta\Phi_\alpha} + \frac{\delta\mathcal{S}_s}{\delta\Phi_\alpha}\right\}\exp\{\imath\mathcal{S} + \imath\mathcal{S}_s\} \tag{4.5}$$

and so, taking advantage of the linearity in the fields of the source term of the action we have

$$J_\alpha Z = -\int \mathcal{D}\Phi\left\{\frac{\delta\mathcal{S}}{\delta\Phi_\alpha}\right\}\exp\{\imath\mathcal{S} + \imath\mathcal{S}_s\}. \tag{4.6}$$

We use the convention that all Grassmann-valued derivatives are left derivatives and so in the above there will be an additional minus sign on the left-hand side when α refers to derivatives with respect to either the c-field or the η-source. The explicit form of the various field equations of motion are given in Appendix A.

Continuous transforms, under which the action is invariant, can be regarded as changes of variable and providing that the Jacobian is trivial, one is left with an equation relating the variations of the source terms in the action. We consider the two invariances derived explicitly in the previous section and that the Jacobian factors are trivial is shown in Appendix B. In the case of the BRS transform, Eq. (2.20), we have

$$0 = \int \mathcal{D}\Phi \frac{\delta}{\delta\imath\delta\lambda} \exp\{\imath\mathcal{S} + \imath\mathcal{S}_s + \imath\delta\mathcal{S}_s\}_{\delta\lambda=0}$$
$$= \int \mathcal{D}\Phi \int d^4x \Big\{ -\frac{1}{g}\rho^a D^{0ab}c^b + \frac{1}{g}\vec{J}^a\cdot\vec{D}^{ab}c^b - \frac{1}{g}\lambda^a\eta^a - \frac{1}{2}f^{abc}\bar{\eta}^a c^b c^c + f^{abc}c^b(\vec{\pi}^c - \vec{\nabla}\phi^c) \cdot \Big[\vec{K}^a - \frac{\vec{\nabla}}{(-\nabla^2)}(\kappa^a - \vec{\nabla}\cdot\vec{K}^a)\Big]\Big\} \exp\{\imath\mathcal{S} + \imath\mathcal{S}_s\}. \quad (4.7)$$

Notice that the infinitessimal variation $\delta\lambda$ that parametrizes the BRS transform is a global quantity, leading to the overall integral over x. The equation for the α-transform is:

$$0 = \int \mathcal{D}\Phi \frac{\delta}{\delta\imath\theta_x^a} \exp\{\imath\mathcal{S} + \imath\mathcal{S}_s + \imath\delta\mathcal{S}_s\}_{\theta=0}$$
$$= \int \mathcal{D}\Phi f^{abc}\vec{X}_x^c \cdot \Big[\vec{K}_x^a - \frac{\vec{\nabla}_x}{(-\nabla_x^2)}(\kappa_x^a - \vec{\nabla}_x\cdot\vec{K}_x^a)\Big] \times \exp\{\imath\mathcal{S} + \imath\mathcal{S}_s\} \quad (4.8)$$

where $\vec{X}$ is given by Eq. (2.22). Constraints imposed by the discrete symmetries of time-reversal and parity will be discussed later.

The above equations of motion and symmetries refer to functional derivatives of the full generating functional. In practice we are concerned with connected two-point (propagator) and one-particle irreducible n-point (proper) Green's functions since these comprise the least sophisticated common building blocks from which all other amplitudes may be constructed. The generating functional of connected Green's functions is $W[J]$ where

$$Z[J] = e^{W[J]}. \quad (4.9)$$

We introduce a bracket notation for functional derivatives of W such that

$$\langle \imath J_1 \rangle = \frac{\delta W}{\delta \imath J_1}. \quad (4.10)$$

The classical field Φ_α is defined as

$$\Phi_\alpha = \frac{1}{Z}\int \mathcal{D}\Phi\Phi_\alpha \exp\{\imath\mathcal{S} + \imath\mathcal{S}_s\} = \frac{1}{Z}\frac{\delta Z}{\delta\imath J_\alpha} \quad (4.11)$$

(the classical field is distinct from the quantum fields which are functionally integrated over, but for convenience we use the same notation). The generating functional of proper Green's functions is the effective action Γ, which is a function of the classical fields and is defined via a Legendre transform of W:

$$\Gamma[\Phi] = W[J] - \imath J_\alpha \Phi_\alpha. \quad (4.12)$$

We use the same bracket notation to denote derivatives of Γ with respect to fields—no confusion arises since we never mix derivatives with respect to sources and fields.

Let us now present the equations of motion in terms of proper functions (from which we will derive the Dyson-Schwinger equations). Using the equations of motion listed in Appendix A we have the following equations:

(i) σ-based. This is the functional form of Gauß' law.

$$\langle \imath\sigma_x^a\rangle - \langle \imath\tau_x^a\rangle = \nabla_x^2\phi_x^a + gf^{abc}A_{ix}^b\pi_{ix}^c - gf^{abc}A_{ix}^b\nabla_{ix}\phi_x^c + gf^{abc}\langle \imath J_{ix}^b \imath K_{ix}^c\rangle - gf^{abc}\int d^4y\,\delta(x-y)\nabla_{ix}\langle \imath J_{iy}^b \imath\kappa_x^c\rangle. \quad (4.13)$$

Note that we have implicitly used the τ equation of motion, Eq. (A8), in order to eliminate terms involving $\vec{\nabla}\cdot\vec{\pi}$ in favor of the source ξ_τ.

(ii) $\vec{A}$-based. We write this in such a way as to factorize the functional derivatives and the kinematical factors. The equation reads:

$$\langle \imath A_{ix}^a\rangle = \nabla_{ix}\lambda_x^a - \partial_x^0\pi_{ix}^a + \partial_x^0\nabla_{ix}\phi_x^a + [\delta_{ij}\nabla_x^2 - \nabla_{ix}\nabla_{jx}]A_{jx}^a + gf^{abc}\int d^4yd^4z\,\delta(y-x)\delta(z-x)[\nabla_{iz}\bar{c}_z^b c_y^c + \pi_{iz}^b\sigma_y^c - \nabla_{iz}\phi_z^b\sigma_y^c] + gf^{abc}\int d^4yd^4z\,\delta(y-x)\delta(z-x)[\nabla_{iz}\langle \imath\bar{\eta}_y^c \imath\eta_z^b\rangle + \langle \imath K_{iz}^b \imath\rho_y^c\rangle - \nabla_{iz}\langle \imath\kappa_z^b \imath\rho_y^c\rangle] + gf^{abc}\int d^4yd^4z\,\delta(y-x)\delta(z-x)\{\delta_{jk}\nabla_{iz} + 2\delta_{ij}\nabla_{ky} - \delta_{ik}\nabla_{jy}\}[\langle \imath J_{jy}^b \imath J_{kz}^c\rangle + A_{jy}^b A_{kz}^c] - \frac{1}{4}g^2 f^{fbc}f^{fde}\delta_{jk}\delta_{li}[\delta^{cg}\delta^{eh}(\delta^{ab}\delta^{di} + \delta^{ad}\delta^{bi}) + \delta^{bg}\delta^{dh}(\delta^{ac}\delta^{ie} + \delta^{ae}\delta^{ic})][\langle \imath J_{jx}^g \imath J_{kx}^h \imath J_{lx}^i\rangle + A_{jx}^g\langle \imath J_{kx}^h \imath J_{lx}^i\rangle + A_{kx}^h\langle \imath J_{jx}^g \imath J_{lx}^i\rangle + A_{lx}^i\langle \imath J_{jx}^g \imath J_{kx}^h\rangle + A_{jx}^g A_{kx}^h A_{lx}^i]. \quad (4.14)$$

(iii) ghost-based. The ghost and the antighost equations provide the same information. The two fields are complimentary and derivatives must come in pairs if the expression is to survive when sources are set to zero. The antighost equation is

$$\langle \imath\bar{c}_x^a\rangle = -\nabla_x^2 c_x^a - gf^{abc}\nabla_{ix}[\langle \imath\bar{\eta}_x^b \imath J_{ix}^c\rangle + c_x^b A_{ix}^a]. \quad (4.15)$$

(iv) $\vec{\pi}$-based.

$$\langle \imath \pi^a_{ix} \rangle = \nabla_{ix}\tau^a_x - \pi^a_{ix} + \nabla_{ix}\phi^a_x + \partial^0_x A^a_{ix} + \nabla_{ix}\sigma^a_x + gf^{abc}[\langle \imath \rho^b_x \imath J^c_{ix}\rangle + \sigma^b_x A^c_{ix}]. \quad (4.16)$$

(v) ϕ-based. We notice that the interaction terms in the equation of motion for the ϕ-field, Eq. (A5) are, up to a derivative, *identical* to those of the $\vec{\pi}$-based equation, Eq. (A6). This arises since $-\vec{\nabla}\phi$ is nothing more than the longitudinal part of $\vec{\pi}$ and means that there is a redundancy in the formalism that can be exploited to simplify proceedings. We can write

$$(\vec{\nabla}_x \cdot \vec{K}^a_x - \kappa^a_x)Z = -\int \mathcal{D}\Phi \nabla^2_x \tau^a_x \exp\{\imath\mathcal{S} + \imath\mathcal{S}_s\}, \quad (4.17)$$

from which it follows that

$$\vec{\nabla}_x \cdot \vec{K}^a_x - \kappa^a_x = -\nabla^2_x \langle \imath \xi^a_{\tau x}\rangle, \quad (4.18)$$

$$\langle \imath \phi^a_x \rangle - \vec{\nabla}_x \cdot \langle \imath \vec{\pi}^a_x \rangle = -\nabla^2_x \tau^a_x. \quad (4.19)$$

(vi) λ- and τ-based. It will be useful to have these equations written in terms of both connected and proper Green's functions.

$$\xi^a_{\lambda x} = \vec{\nabla}_x \cdot \langle \imath \vec{J}^a_x \rangle, \qquad \langle \imath \lambda^a_x \rangle = -\vec{\nabla}_x \cdot \vec{A}^a_x, \quad (4.20)$$

$$\xi^a_{\tau x} = \vec{\nabla}_x \cdot \langle \imath \vec{K}^a_x \rangle, \qquad \langle \imath \tau^a_x \rangle = -\vec{\nabla}_x \cdot \vec{\pi}^a_x. \quad (4.21)$$

The BRS transform gives rise to the following equation (the Ward-Takahashi identity):

$$0 = \int d^4x \bigg\{ \frac{1}{g}(\partial^0_x \rho^a_x)\langle \imath \bar{\eta}^a_x \rangle - f^{acb}\rho^a_x[\langle \imath \rho^c_x \imath \bar{\eta}^b_x\rangle + \langle \imath \rho^c_x\rangle\langle \imath \bar{\eta}^b_x\rangle] - \frac{1}{g}(\nabla_{ix}J^a_{ix})\langle \imath \bar{\eta}^a_x\rangle - f^{acb}J^a_{ix}[\langle \imath J^c_{ix}\imath \bar{\eta}^b_x\rangle + \langle \imath J^c_{ix}\rangle\langle \imath \bar{\eta}^b_x\rangle]$$
$$- \frac{1}{g}\eta^a_x\langle \imath \xi^a_{\lambda x}\rangle - \frac{1}{2}f^{abc}\bar{\eta}^a_x[\langle \imath \bar{\eta}^b_x \imath \bar{\eta}^c_x\rangle + \langle \imath \bar{\eta}^b_x\rangle\langle \imath \bar{\eta}^c_x\rangle] + f^{abc}\bigg[K^a_{ix} - \frac{\nabla_{ix}}{(-\nabla^2_x)}(\kappa^a_x - \nabla_{jx}K^a_{jx})\bigg]$$
$$\times \bigg[\langle \imath K^c_{ix}\imath \bar{\eta}^b_x\rangle - \int d^4y\, \delta(x-y)\nabla_{ix}\langle \imath \kappa^c_x \imath \bar{\eta}^b_y\rangle + \langle \imath K^c_{ix}\rangle\langle \imath \bar{\eta}^b_x\rangle - \langle \imath \bar{\eta}^b_x\rangle \nabla_{ix}\langle \imath \kappa^c_x\rangle\bigg]\bigg\}. \quad (4.22)$$

We consider for now only the form of the equation relating connected Green's functions. As will be seen in the next section, it will not be necessary to consider the equation generated by the invariance under the α-transform.

V. EXACT RELATIONS FOR GREEN'S FUNCTIONS

Given the set of "master" field equations of motion and symmetries, it is pertinent to find out if any of the constraints can be combined to give unambiguous information about the eventual Green's functions of the theory. We find that such simplifications do in fact exist.

Let us start by discussing the functional equation generated by α-invariance, Eq. (4.8). It is not necessary here to consider functional derivatives of either the generating functional of connected Green's functions (W) or the effective action (Γ) since the derivation applies to the functional integrals directly. From Appendix A, the $\vec{\pi}$-based field equation of motion, Eq. (A6) is

$$K^a_{ix}Z[J] = -\int \mathcal{D}\Phi\{\nabla_{ix}\tau^a_x - X^a_{ix}\}\exp\{\imath\mathcal{S} + \imath\mathcal{S}_s\}. \quad (5.1)$$

Using Eq. (5.1) we can rewrite Eq. (4.8) as

$$0 = f^{abc}\int \mathcal{D}\Phi\bigg[\vec{K}^a_x - \frac{\vec{\nabla}_x}{(-\nabla^2_x)}(\kappa^a_x - \vec{\nabla}_x \cdot \vec{K}^a_x)\bigg] \cdot [\vec{K}^c_x + \vec{\nabla}_x \tau^c_x]\exp\{\imath\mathcal{S} + \imath\mathcal{S}_s\}. \quad (5.2)$$

Since f^{abc} is antisymmetric and noting Eq. (4.17), the above is now an almost trivial identity. We have thus shown that the $\vec{\pi}$- and ϕ-based equations of motion, Eqs. (4.17) and (5.1), guarantee that α-invariance is respected. Conversely, approximations to the equations of motion will destroy the symmetry. This is a concrete example of a general feature of any physical field theory—the full solutions of the field equations of motion (and the subsequent functional derivatives which comprise the Dyson-Schwinger equations) contain all the information given by the symmetry considerations. In this case, we have the ambiguity associated with introducing the $\vec{\pi}$-field and assigning its properties under the BRS transform encoded within the invariance under the α-transform and the field equations of motion are "aware" of this. What is unusual about this, however, is that the equivalence of the field equations of motion and the equations generated by invariance under a symmetry is invariably impossible to show (except order by order in perturbation theory)—full gauge invariance being the archetypal example.

Let us now continue the discussion by considering those equations of motion which do not contain interaction terms. In the absence of interactions, the solutions to these equations can be written down without difficulty. In terms of connected Green's function, the only nonzero functional derivative of the λ-equation, Eq. (4.20) is

$$\vec{\nabla}_x \cdot \langle \imath \xi^b_{\lambda y}\imath \vec{J}^a_x\rangle = -\imath\delta^{ba}\delta(y-x), \quad (5.3)$$

the right-hand side vanishing for all other derivatives. Separating the configuration space arguments and setting our conventions for the Fourier transform, we have for a general two-point function (connected or proper) which obeys translational invariance:

$$\langle \imath J_\alpha(y) \imath J_\beta(x)\rangle = \langle \imath J_\alpha(y-x)\imath J_\beta(0)\rangle = \int \text{đ}k W_{\alpha\beta}(k) e^{-\imath k\cdot(y-x)}, \tag{5.4}$$

where $\text{đ}k = d^4k/(2\pi)^4$ and it is implicitly understood that the relevant prescription to avoid integration over poles is present such that the analytic continuation to Euclidean space may be performed. We can immediately write down the functional derivatives of $\langle \imath J_{ix}^q\rangle$ using Eq. (4.20):

$$\begin{aligned} \langle \imath J_{jy}^b \imath J_{ix}^a\rangle &= \int \text{đ}k W_{AAji}^{ba}(k) t_{ji}(\vec{k}) e^{-\imath k\cdot(y-x)},\\ \langle \imath K_{jy}^b \imath J_{ix}^a\rangle &= \int \text{đ}k W_{\pi Aji}^{ba}(k) t_{ji}(\vec{k}) e^{-\imath k\cdot(y-x)},\\ \langle \imath \xi_{\lambda y}^b \imath J_{ix}^a\rangle &= \int \text{đ}k \delta^{ba} \frac{k_i}{\vec{k}^2} e^{-\imath k\cdot(y-x)},\\ \langle \imath \xi_{\tau y}^b \imath J_{ix}^a\rangle &= \langle \imath \rho_y^b \imath J_{ix}^a\rangle = \langle \imath \kappa_y^b \imath J_{ix}^a\rangle = 0. \end{aligned} \tag{5.5}$$

Similarly, for the τ-equation, Eq. (4.21), we have

$$\begin{aligned} \langle \imath J_{jy}^b \imath K_{ix}^a\rangle &= \int \text{đ}k W_{A\pi ji}^{ba}(k) t_{ji}(\vec{k}) e^{-\imath k\cdot(y-x)},\\ \langle \imath K_{jy}^b \imath K_{ix}^a\rangle &= \int \text{đ}k W_{\pi\pi ji}^{ba}(k) t_{ji}(\vec{k}) e^{-\imath k\cdot(y-x)},\\ \langle \imath \xi_{\tau y}^b \imath K_{ix}^a\rangle &= \int \text{đ}k \delta^{ba} \frac{k_i}{\vec{k}^2} e^{-\imath k\cdot(y-x)},\\ \langle \imath \rho_y^b \imath K_{ix}^a\rangle &= \langle \imath \kappa_y^b \imath K_{ix}^a\rangle = \langle \imath \xi_{\lambda y}^b \imath K_{ix}^a\rangle = 0. \end{aligned} \tag{5.6}$$

We see that, as expected, the propagators involving only the vector fields are transverse and the only other contributions relate to the Lagrange multiplier fields and are purely kinematical in nature. There is one subtlety to the above and that is that while the equations for $\langle \imath\xi_\lambda \imath J\rangle$ and $\langle \imath\xi_\tau \imath K\rangle$ are exact in the presence of sources, all other equations refer implicitly to the case where the sources are set to zero and the discrete parity symmetry has been applied (see later for a more complete discussion).

In the same fashion, let us consider the λ-equation, Eq. (4.20), (and similarly the τ-equation, Eq. (4.21)) in terms of proper Green's functions. This equation does contain the same information as its counterpart for connected Green's functions, but clearly has a different character. The only nonzero functional derivative is

$$\langle \imath A_{iy}^b \imath \lambda_x^a\rangle = \int \text{đ}k \delta^{ba} k_i e^{-\imath k\cdot(y-x)}, \tag{5.7}$$

all others vanishing, *even in the presence of sources*. This applies to all proper n-point functions involving the λ-field. Similarly, we have the only nonvanishing proper function involving the τ-field:

$$\langle \imath \pi_{iy}^b \imath \tau_x^a\rangle = \int \text{đ}k \delta^{ba} k_i e^{-\imath k\cdot(y-x)}. \tag{5.8}$$

That there are no proper n-point functions involving functional derivatives with respect to the Lagrange multiplier fields apart from the two special cases above leads to an important facet concerning the Dyson-Schwinger equations—there will be no self-energy terms involving derivatives with respect to the λ- or τ-fields since they have no proper vertices, despite the fact that the propagators associated with these fields may be nontrivial.

Next, let us turn to the ϕ-based equation of motion in terms of connected Green's functions, Eq. (4.18). There are only two nonvanishing functional derivatives and we can write down the solutions as before

$$\begin{aligned} \langle \imath K_{iy}^b \imath \xi_{\tau x}^a\rangle &= \int \text{đ}k \delta^{ba} \frac{(-k_i)}{\vec{k}^2} e^{-\imath k\cdot(y-x)},\\ \langle \imath \kappa_y^b \imath \xi_{\tau x}^a\rangle &= \int \text{đ}k \delta^{ba} \frac{\imath}{\vec{k}^2} e^{-\imath k\cdot(y-x)},\\ \langle \imath J_{iy}^b \imath \xi_{\tau x}^a\rangle &= \langle \imath \rho_y^b \imath \xi_{\tau x}^a\rangle = \langle \imath \xi_{\lambda x}^b \imath \xi_{\tau x}^a\rangle = \langle \imath \xi_{\tau x}^b \imath \xi_{\tau x}^a\rangle = 0. \end{aligned} \tag{5.9}$$

Notice that *all* of the connected Green's functions involving the τ-field are now known and are purely kinematical in nature. In terms of proper Green's functions, we consider the ϕ-based equation of motion, Eq. (4.19). Recognizing that functional derivatives with respect to the λ- and τ-fields yield no more information, we can omit them from the current discussion. The equation tells us that given a proper Green's function involving π, we can immediately construct the corresponding functional derivative with respect to ϕ. We can thus conclude that as far as the proper Green's functions are concerned, derivatives with respect to the ϕ-field are redundant.

Finally, let us consider the equation derived from the BRS transform in terms of connected Green's functions, Eq. (4.22). Since the ghost/antighost fields must come in pairs, we may take the functional derivative of this with respect to $\imath\eta_z^d$ and subsequently set the ghost sources to zero while considering only the rest. We get:

$$\frac{\imath}{g}\langle \imath\xi^d_{\lambda z}\rangle = \int d^4x\bigg\{\frac{1}{g}(\partial^0_x\rho^a_x)\langle\imath\bar\eta^a_x\imath\eta^d_z\rangle - f^{acb}\rho^a_x[\langle\imath\rho^c_x\imath\bar\eta^b_x\imath\eta^d_z\rangle + \langle\imath\rho^c_x\rangle\langle\imath\bar\eta^b_x\imath\eta^d_z\rangle] - \frac{1}{g}(\nabla_{ix}J^a_{ix})\langle\imath\bar\eta^a_x\imath\eta^d_z\rangle - f^{acb}J^a_{ix}[\langle\imath J^c_{ix}\imath\bar\eta^b_x\imath\eta^d_z\rangle + \langle\imath J^c_{ix}\rangle\langle\imath\bar\eta^b_x\imath\eta^d_z\rangle] + f^{abc}\bigg[K^a_{ix} - \frac{\nabla_{ix}}{(-\nabla^2_x)}(\kappa^a_x - \nabla_{jx}K^a_{jx})\bigg]\bigg[\langle\imath K^c_{ix}\imath\bar\eta^b_x\imath\eta^d_z\rangle - \int d^4y\,\delta(x-y)\nabla_{ix}\langle\imath\kappa^c_x\imath\bar\eta^b_y\imath\eta^d_z\rangle + \langle\imath K^c_{ix}\rangle\langle\imath\bar\eta^b_x\imath\eta^d_z\rangle - \langle\imath\bar\eta^b_x\imath\eta^d_z\rangle\nabla_{ix}\langle\imath\kappa^c_x\rangle\bigg]\bigg\}. \tag{5.10}$$

For now, the pertinent information from this identity comes from taking the functional derivative with respect to the source ξ_λ and setting all sources to zero. The result is

$$\langle\imath\xi^e_{\lambda\omega}\imath\xi^d_{\lambda z}\rangle = 0. \tag{5.11}$$

We notice that all other functional derivatives lead to nontrivial relations involving interaction terms. This includes both the $\langle\imath\rho\imath\xi_\lambda\rangle$ and the $\langle\imath\kappa\imath\xi_\lambda\rangle$ connected Green's functions and we conclude that these functions are not merely kinematical factors as one might expect from quantities involving Lagrange multiplier fields. We shall return to this topic at a later stage.

VI. FEYNMAN (AND OTHER) RULES

Whilst it is entirely possible to deduce the complete set of Feynman rules directly from the action, we shall follow a slightly less obvious path here. We derive not only the basic Feynman rules but collect all the tree-level (and incidentally primitively divergent) quantities that will be of interest. This means that in addition to the tree-level propagators (i.e., connected two-point Green's functions) and proper vertices (i.e., proper three- and four-point functions) we derive also the proper two-point functions. The reason for this is that (as will be discussed in some detail later) the connected and proper two-point functions are not related in the usual way as inverses of one another. The tree-level quantities of interest can be easily derived from the respective equations of motion. Indeed, recalling the previous section, some are already known exactly and we will not need to discuss them further.

Before beginning, let us highlight a basic feature of the Fourier transform to momentum space. We know the commutation or anticommutation rules for our fields/sources and this will lead to the simplification that we need only consider combinations of fields/sources and let the commutation rules take care of the permutations. However, momentum assignments must be uniformly applied and this leads to some nontrivial relations. Consider first the generic proper two-point function $\langle\imath\Phi_\alpha(x)\imath\Phi_\beta(y)\rangle$ where we have $\Phi_\beta(y)\Phi_\alpha(x) = \eta\Phi_\alpha(x)\Phi_\beta(y)$ with $\eta = \pm 1$. We then have

$$\langle\imath\Phi_\alpha(x)\imath\Phi_\beta(y)\rangle = \eta\langle\imath\Phi_\beta(y)\imath\Phi_\alpha(x)\rangle \tag{6.1}$$

such that in momentum space

$$\Gamma_{\alpha\beta}(k) = \eta\Gamma_{\beta\alpha}(-k). \tag{6.2}$$

A similar argument applies for connected two-point functions. The situation for proper three-point functions is slightly less complicated since all momenta are defined as incoming. Indeed, we have (the δ-function expressing momentum conservation comes about because of translational invariance)

$$\langle\imath\Phi_\alpha\imath\Phi_\beta\imath\Phi_\gamma\rangle = \int \bar{d}k_\alpha \bar{d}k_\beta \bar{d}k_\gamma (2\pi)^4\delta(k_\alpha + k_\beta + k_\gamma) \times \Gamma_{\alpha\beta\gamma}(k_\alpha, k_\beta, k_\gamma)e^{-\imath k_\alpha\cdot x_\alpha - \imath k_\beta\cdot x_\beta - \imath k_\gamma\cdot x_\gamma} \tag{6.3}$$

such that, for example

$$\Gamma_{\beta\alpha\gamma}(k_\beta, k_\alpha, k_\gamma) = \eta_{\alpha\beta}\Gamma_{\alpha\beta\gamma}(k_\alpha, k_\beta, k_\gamma), \tag{6.4}$$

where $\eta_{\alpha\beta}$ refers to the sign incurred when swapping α and β.

Let us now consider the connected two-point functions. Setting the coupling to zero in the equations of motion (listed in Appendix A) that involve interaction terms gives us the following nontrivial relations (the superscript $\langle\rangle^{(0)}$ denotes the tree-level quantity)

$$\begin{aligned}
\rho^a_x - \xi^a_{\tau x} &= -\nabla^2_x\langle\imath\kappa^a_x\rangle^{(0)},\\
J^a_{ix} &= -\nabla_{ix}\langle\imath\xi^a_{\lambda x}\rangle^{(0)} + \partial^0_x\langle\imath K^a_{ix}\rangle^{(0)} - \partial^0_x\nabla_{ix}\langle\imath\kappa^a_x\rangle^{(0)}\\
&\quad - [\delta_{ij}\nabla^2_x + \nabla_{ix}\nabla_{jx}]\langle\imath J^a_{jx}\rangle^{(0)},\\
\eta^a_x &= \nabla^2_x\langle\imath\bar\eta^a_x\rangle^{(0)},\\
K^a_{ix} &= -\nabla_{ix}\langle\imath\xi^a_{\tau x}\rangle^{(0)} + \langle\imath K^a_{ix}\rangle^{(0)} - \nabla_{ix}\langle\imath\kappa^a_x\rangle^{(0)}\\
&\quad - \partial^0_x\langle\imath J^a_{ix}\rangle^{(0)} - \nabla_{ix}\langle\imath\rho^a_x\rangle^{(0)}.
\end{aligned} \tag{6.5}$$

Clearly, the ghost propagator is distinct from the rest, since the ghost field must appear with its antighost counterpart. The tree-level ghost propagator is

$$W^{(0)ab}_{\bar cc}(k) = -\delta^{ab}\frac{\imath}{\vec k^2}. \tag{6.6}$$

The remaining tree-level propagators in momentum space (without the common color factor δ^{ab}) are summarized in Table I. Those entries that are underlined are the exact relations considered previously.

We can repeat the analysis for the tree-level proper two-point functions. The relevant equations are:

TABLE I. Tree-level propagators (without color factors) in momentum space. Underlined entries denote exact results.

W	A_j	π_j	σ	ϕ	λ	τ
A_i	$t_{ij}(k)\frac{i}{(k_0^2-\vec{k}^2)}$	$t_{ij}(k)\frac{(-k^0)}{(k_0^2-\vec{k}^2)}$	$\underline{0}$	$\underline{0}$	$\underline{\frac{(-k_i)}{\vec{k}^2}}$	$\underline{0}$
π_i	$t_{ij}(k)\frac{k^0}{(k_0^2-\vec{k}^2)}$	$t_{ij}(k)\frac{\imath\vec{k}^2}{(k_0^2-\vec{k}^2)}$	$\underline{0}$	$\underline{0}$	$\underline{0}$	$\underline{\frac{(-k_i)}{\vec{k}^2}}$
σ	$\underline{0}$	$\underline{0}$	$\frac{\imath}{\vec{k}^2}$	$\frac{(-\imath)}{\vec{k}^2}$	$\frac{(-k^0)}{\vec{k}^2}$	$\underline{0}$
ϕ	$\underline{0}$	$\underline{0}$	$\frac{(-\imath)}{\vec{k}^2}$	0	0	$\underline{\frac{\imath}{\vec{k}^2}}$
λ	$\underline{\frac{k_j}{\vec{k}^2}}$	$\underline{0}$	$\frac{k^0}{\vec{k}^2}$	0	$\underline{0}$	$\underline{0}$
τ	$\underline{0}$	$\underline{\frac{k_j}{\vec{k}^2}}$	$\underline{0}$	$\underline{\frac{\imath}{\vec{k}^2}}$	$\underline{0}$	$\underline{0}$

$$\langle \imath\sigma_x^a\rangle^{(0)} - \langle \imath\tau_x^a\rangle^{(0)} = \nabla_x^2\phi_x^a,$$
$$\langle \imath A_{ix}^a\rangle^{(0)} = \nabla_{ix}\lambda_x^a - \partial_x^0\pi_{ix}^a + \partial_x^0\nabla_{ix}\phi_x^a + [\delta_{ij}\nabla_x^2 - \nabla_{ix}\nabla_{jx}]A_{jx}^a,$$
$$\langle \imath\bar{c}_x^a\rangle^{(0)} = -\nabla_x^2 c_x^a,$$
$$\langle \imath\pi_{ix}^a\rangle^{(0)} = \nabla_{ix}\tau_x^a - \pi_{ix}^a + \nabla_{ix}\phi_x^a + \partial_x^0 A_{ix}^a + \nabla_{ix}\sigma_x^a. \tag{6.7}$$

The ghost proper two-point function is

$$\Gamma_{\bar{c}c}^{(0)ab}(k) = \delta^{ab}\imath\vec{k}^2. \tag{6.8}$$

The remaining proper two-point functions are summarized in Table II where the reader is reminded that all proper functions involving derivatives with respect to the ϕ-field can be constructed from the corresponding π derivative.

Determining the tree-level vertices (three- and four-point proper Green's functions) follows the same pattern as for the two-point functions. They follow by isolating the parts of the equations of motion that have explicit factors of the coupling g and functionally differentiating. In momentum space (defining all momenta to be incoming), we have

TABLE II. Tree-level proper two-point functions (without color factors) in momentum space. Underlined entries denote exact results. Bracketed quantities refer to functions that are fully determined by others.

Γ	A_j	π_j	σ	ϕ	λ	τ
A_i	$t_{ij}(k)\imath\vec{k}^2$	$\delta_{ij}k^0$	0	$\{-\imath k^0 k_i\}$	$\underline{k_i}$	$\underline{0}$
π_i	$-k^0\delta_{ij}$	$\imath\delta_{ij}$	k_i	$\{k_i\}$	$\underline{0}$	$\underline{k_i}$
σ	0	$-k_j$	0	$\{\imath\vec{k}^2\}$	$\underline{0}$	$\underline{0}$
ϕ	$\{-\imath k^0 k_j\}$	$\{-k_j\}$	$\{\imath\vec{k}^2\}$	$\{\imath\vec{k}^2\}$	$\underline{0}$	$\underline{0}$
λ	$\underline{-k_j}$	$\underline{0}$	$\underline{0}$	$\underline{0}$	$\underline{0}$	$\underline{0}$
τ	$\underline{0}$	$\underline{-k_j}$	$\underline{0}$	$\underline{0}$	$\underline{0}$	$\underline{0}$

$$\Gamma_{\pi\sigma Aij}^{(0)abc} = -gf^{abc}\delta_{ij},$$
$$\Gamma_{3Aijk}^{(0)abc}(p_a, p_b, p_c) = -\imath gf^{abc}[\delta_{ij}(p_a - p_b)_k + \delta_{jk}(p_b - p_c)_i + \delta_{ki}(p_c - p_a)_j],$$
$$\Gamma_{4Aijkl}^{(0)abcd} = -\imath g^2\{\delta_{ij}\delta_{kl}[f^{ace}f^{bde} - f^{ade}f^{cbe}] + \delta_{ik}\delta_{jl}[f^{abe}f^{cde} - f^{ade}f^{bce}] + \delta_{il}\delta_{jk}[f^{ace}f^{dbe} - f^{abe}f^{cde}]\},$$
$$\Gamma_{\bar{c}cAi}^{(0)abc}(p_{\bar{c}}, p_c, p_A) = -\imath gf^{abc}p_{\bar{c}i}. \tag{6.9}$$

We notice that all the tree-level vertices are independent of the energy. In addition, there is a tree-level vertex involving ϕ that can be constructed from its counterpart involving $\vec{\pi}$ and that reads:

$$\Gamma_{\phi\sigma Ai}^{(0)abc}(p_\phi, p_\sigma, p_A) = \imath p_{\phi j}\Gamma_{\pi\sigma Aji}^{(0)abc} = -\imath gf^{abc}p_{\phi i}. \tag{6.10}$$

This vertex has exactly the same form as the ghost-gluon vertex with the incoming ϕ-momentum playing the same role as the incoming $\bar{c}$-momentum. It is worth mentioning that the ghost-gluon, and three- and four-gluon vertices are identical to the Landau gauge forms except that only the spatial components of the vectors are present.

Let us now discuss the cancellation of the ghost (energy divergent) sector. In any Feynman diagram containing a closed ghost loop, there will be an associated energy divergence. It is a general result that associated with any closed loop involving Grassmann-valued fields (ghosts or fermions) there will be a factor of (-1). However, Green's functions are given by the sum of all possible contributing Feynman diagrams. Since the Feynman rules for $W_{\sigma\phi}$ and $\Gamma_{\phi\sigma A}$ are identical to $W_{\bar{c}c}$ and $\Gamma_{\bar{c}cA}$ we will have, for each closed ghost loop, another loop involving scalar fields without the factor (-1). Even before performing the loop integration (and regularisation) the integrands of the two diagrams will cancel exactly. In this way we see that the energy divergences coming from the ghost sector will be eliminated, as expected given that the Faddeev-Popov determinant can formally be cancelled. There is one caveat to this. Whilst we have shown that the energy divergences coming from the ghost sector have been eliminated, we have not shown that the remaining loops involving scalar fields are free of energy divergences (although a quick glance at the form of the Dyson-Schwinger equations later will suffice to see that this is the case at leading order). We propose to look further into this in a future publication.

VII. DECOMPOSITION OF TWO-POINT FUNCTIONS

In order to constrain the possible form of the two-point functions under investigation we can utilize information about discrete symmetries. We consider time-reversal and parity and we know that Yang-Mills theory respects both. Under time-reversal the generic field $\Phi_\alpha(x^0, \vec{x})$ is transformed as follows:

$$\Phi_\alpha(x^0, \vec{x}) = \eta_\alpha \Phi_\alpha(-x^0, \vec{x}), \tag{7.1}$$

where $\eta_\alpha = \pm 1$. Since the action, Eq. (2.19), is invariant under time-reversal (it is a pure number) then by considering each term in turn, we deduce that

$$\eta_A = \eta_\lambda = 1, \qquad \eta_\pi = \eta_\tau = \eta_\phi = \eta_\sigma = -1, \qquad \eta_{\bar{c}} = \eta_c = \pm 1. \tag{7.2}$$

The sources have the same transformation properties as the field. These properties allow us to extract information about the energy dependence of Green's functions. For instance, we have that

$$\Gamma^{ab}_{A\pi ij}(k^0, \vec{k}) = -\Gamma^{ab}_{A\pi ij}(-k^0, \vec{k}), \tag{7.3}$$

from which one can infer that

$$\Gamma^{ab}_{A\pi ij}(k^0, \vec{k}) = \delta^{ab} k^0 \Gamma_{A\pi ij}(k_0^2, \vec{k}) \tag{7.4}$$

(the sign convention is chosen to match the perturbative results). Aside from $\Gamma_{A\phi}$ which is unambiguously related to $\Gamma_{A\pi}$, the only other proper two-point function that carries the external factor k^0 is

$$\Gamma^{ab}_{A\sigma i}(k^0, \vec{k}) = \delta^{ab} k^0 \Gamma_{A\sigma i}(k_0^2, \vec{k}). \tag{7.5}$$

Having extracted the explicit factors of k^0 in the proper two-point functions, the (as yet) unknown functions that multiply them are functions of k_0^2. Turning to the propagators, we assign the factor $-k^0$ to $W_{A\pi}$, $W_{\sigma\lambda}$, and $W_{\phi\lambda}$.

The second discrete symmetry of interest is parity whereby

$$\Phi_\alpha(x^0, \vec{x}) = \eta_\alpha \Phi_\alpha(x^0, -\vec{x}), \tag{7.6}$$

where again, $\eta_\alpha = \pm 1$. Again the action is invariant and we deduce that

$$\eta_A = \eta_\pi = -1, \qquad \eta_\sigma = \eta_\phi = \eta_\lambda = \eta_\tau = 1, \qquad \eta_{\bar{c}} = \eta_c = \pm 1 \tag{7.7}$$

with the sources transforming as the fields. This symmetry is rather more obvious than time-reversal. The physical sense is that for every vector field (and with an associated spatial index) we have some explicit vector factor (again with the associated spatial index). Where the vector fields $\vec{A}$ and $\vec{\pi}$ occur in the propagators, we use the transversality conditions from earlier to see that the vector-scalar propagators must vanish, except those involving the appropriate Lagrange multiplier field.

What the above tells us is how to construct the most general allowed forms of the two-point functions. The dressing functions are scalar functions of the positive, scalar arguments k_0^2 and $\vec{k}^2$. We summarize the results in Tables III and IV. The ghost propagator is written $W^{ab}_{\bar{c}c}(k) = -\delta^{ab} \imath D_c / \vec{k}^2$. We have nine unknown propagator dressing functions. Including the proper ghost two-point function $\Gamma^{ab}_{\bar{c}c}(k) = \delta^{ab} \imath \vec{k}^2 \Gamma_c$ we see that there are ten proper two-point dressing functions. The extra functions come about because we have used only the propagator form of the identity equation (5.10) to eliminate $W_{\lambda\lambda}$.

Obviously the propagator and proper two-point dressing functions are related via the Legendre transform. Whereas in covariant gauges this relationship is merely an inversion, in our case there is considerably more detail. The connection between the connected and proper two-point functions stems from the observation that

$$\frac{\delta \imath J_\beta}{\delta \imath J_\alpha} = \delta_{\alpha\beta} = -\imath \frac{\delta}{\delta \imath J_\alpha} \langle \imath \Phi_\beta \rangle = \frac{\delta \Phi_\gamma}{\delta \imath J_\alpha} \langle \imath \Phi_\gamma \imath \Phi_\beta \rangle = \langle \imath J_\alpha \imath J_\gamma \rangle \langle \imath \Phi_\gamma \imath \Phi_\beta \rangle. \tag{7.8}$$

(Recall here that there is an implicit summation over all discrete indices and integration over continuous variables labeled by γ.) The ghost two-point functions are somewhat special in that once sources are set to zero, only ghost-antighost pairs need be considered. The above relation becomes

$$\int d^4 z \langle \imath \bar{\eta}^a_x \imath \eta^c_z \rangle \langle \imath \bar{c}^c_z \imath c^b_y \rangle = \delta^{ab} \delta(x - y). \tag{7.9}$$

Fourier transforming to momentum space and using the decomposition from above, we get that

$$D_c(k_0^2, \vec{k}^2) \Gamma_c(k_0^2, \vec{k}^2) = 1 \tag{7.10}$$

showing that the ghost propagator dressing function is simply the inverse of the ghost proper two-point function. Turning to the rest, we are faced with a problem akin to matrix inversion in order to see the connection since the sum over all the different possible sources/fields labeled by γ is nontrivial in the above general formula. The decompositions of the two-point function do however mitigate the complexity somewhat. We tabulate the possible combinations of terms in Table V.

We start by considering the top-left components of Table V involving only $\vec{A}$, $\vec{\pi}$ and the known functions

TABLE III. General form of propagators in momentum space. The global color factor δ^{ab} has been extracted. All unknown functions $D_{\alpha\beta}$ are dimensionless, scalar functions of k_0^2 and $\vec{k}^2$.

W	A_j	π_j	σ	ϕ	λ	τ
A_i	$t_{ij}(k) \frac{\imath D_{AA}}{(k_0^2 - \vec{k}^2)}$	$t_{ij}(k) \frac{(-k^0) D_{A\pi}}{(k_0^2 - \vec{k}^2)}$	0	0	$\frac{(-k_i)}{\vec{k}^2}$	0
π_i	$t_{ij}(k) \frac{k^0 D_{A\pi}}{(k_0^2 - \vec{k}^2)}$	$t_{ij}(k) \frac{\imath \vec{k}^2 D_{\pi\pi}}{(k_0^2 - \vec{k}^2)}$	0	0	0	$\frac{(-k_i)}{\vec{k}^2}$
σ	0	0	$\frac{\imath D_{\sigma\sigma}}{\vec{k}^2}$	$\frac{-\imath D_{\sigma\phi}}{\vec{k}^2}$	$\frac{(-k^0) D_{\sigma\lambda}}{\vec{k}^2}$	0
ϕ	0	0	$\frac{-\imath D_{\sigma\phi}}{\vec{k}^2}$	$\frac{-\imath D_{\phi\phi}}{\vec{k}^2}$	$\frac{(-k^0) D_{\phi\lambda}}{\vec{k}^2}$	$\frac{\imath}{\vec{k}^2}$
λ	$\frac{k_j}{\vec{k}^2}$	0	$\frac{k^0 D_{\sigma\lambda}}{\vec{k}^2}$	$\frac{k^0 D_{\phi\lambda}}{\vec{k}^2}$	0	0
τ	0	$\frac{k_j}{\vec{k}^2}$	0	$\frac{\imath}{\vec{k}^2}$	0	0

TABLE IV. General form of the proper two-point functions in momentum space. The global color factor δ^{ab} has been extracted. All unknown functions $D_{\alpha\beta}$ are dimensionless, scalar functions of k_0^2 and $\vec{k}^2$. Bracketed quantities refer to functions that are fully determined by others.

Γ	A_j	π_j	σ	ϕ	λ	τ
A_i	$t_{ij}(\vec{k})\imath\vec{k}^2\Gamma_{AA} + \imath k_i k_j \bar{\Gamma}_{AA}$	$k^0(\delta_{ij}\Gamma_{A\pi} + l_{ij}(\vec{k})\bar{\Gamma}_{A\pi})$	$-\imath k^0 k_i \Gamma_{A\sigma}$	$\{-\imath k^0 k_i(\Gamma_{A\pi} + \bar{\Gamma}_{A\pi})\}$	k_i	0
π_i	$-k^0(\delta_{ij}\Gamma_{A\pi} + l_{ij}(\vec{k})\bar{\Gamma}_{A\pi})$	$\imath\delta_{ij}\Gamma_{\pi\pi} + \imath l_{ij}(\vec{k})\bar{\Gamma}_{\pi\pi}$	$k_i\Gamma_{\pi\sigma}$	$\{k_i(\Gamma_{\pi\pi} + \bar{\Gamma}_{\pi\pi})\}$	0	k_i
σ	$-\imath k^0 k_j \Gamma_{A\sigma}$	$-k_j\Gamma_{\pi\sigma}$	$\imath\vec{k}^2\Gamma_{\sigma\sigma}$	$\{\imath\vec{k}^2\Gamma_{\pi\sigma}\}$	0	0
ϕ	$\{-\imath k^0 k_j(\Gamma_{A\pi} + \bar{\Gamma}_{A\pi})\}$	$\{-k_j(\Gamma_{\pi\pi} + \bar{\Gamma}_{\pi\pi})\}$	$\{\imath\vec{k}^2\Gamma_{\pi\sigma}\}$	$\{-\imath\vec{k}^2(\Gamma_{\pi\pi} + \bar{\Gamma}_{\pi\pi})\}$	0	0
λ	$-k_j$	0	0	0	0	0
τ	0	$-k_j$	0	0	0	0

with λ and τ. After decomposition, we have (suppressing the common argument k)

$$\begin{aligned} k_0^2 D_{A\pi}\Gamma_{A\pi} - \vec{k}^2 D_{AA}\Gamma_{AA} &= k_0^2 - \vec{k}^2, \\ k_0^2 D_{A\pi}\Gamma_{A\pi} - \vec{k}^2 D_{\pi\pi}\Gamma_{\pi\pi} &= k_0^2 - \vec{k}^2, \\ D_{AA}\Gamma_{A\pi} - D_{A\pi}\Gamma_{\pi\pi} &= 0, \\ D_{A\pi}\Gamma_{AA} - D_{\pi\pi}\Gamma_{A\pi} &= 0. \end{aligned} \tag{7.11}$$

We can thus express the propagator functions D in terms of the proper two-point functions Γ and we have

$$\begin{aligned} D_{AA} &= \frac{(k_0^2 - \vec{k}^2)\Gamma_{\pi\pi}}{(k_0^2\Gamma_{A\pi}^2 - \vec{k}^2\Gamma_{AA}\Gamma_{\pi\pi})}, \\ D_{\pi\pi} &= \frac{(k_0^2 - \vec{k}^2)\Gamma_{AA}}{(k_0^2\Gamma_{A\pi}^2 - \vec{k}^2\Gamma_{AA}\Gamma_{\pi\pi})}, \\ D_{A\pi} &= \frac{(k_0^2 - \vec{k}^2)\Gamma_{A\pi}}{(k_0^2\Gamma_{A\pi}^2 - \vec{k}^2\Gamma_{AA}\Gamma_{\pi\pi})}. \end{aligned} \tag{7.12}$$

Clearly, these expressions can be inverted to give the functions Γ in terms of the functions D. Next, let us consider the central components of Table V involving only σ and ϕ. We get the following equations:

TABLE V. Possible terms for the equations relating propagator and proper two-point functions stemming from the Legendre transform. Entries denote the allowed field types γ in Eq. (7.8).

α, β	$\vec{A}$	$\vec{\pi}$	σ	ϕ	λ	τ
$\vec{A}$	$\vec{A}, \vec{\pi}, \lambda$	$\vec{A}, \vec{\pi}$	$\vec{A}, \vec{\pi}$	$\vec{A}, \vec{\pi}$	$\vec{A}$	$\vec{\pi}$
$\vec{\pi}$	$\vec{A}, \vec{\pi}$	$\vec{A}, \vec{\pi}, \tau$	$\vec{A}, \vec{\pi}$	$\vec{A}, \vec{\pi}$	$\vec{A}$	$\vec{\pi}$
σ	σ, ϕ, λ	σ, ϕ	σ, ϕ	σ, ϕ	—	—
ϕ	σ, ϕ, λ	σ, ϕ, τ	σ, ϕ	σ, ϕ	—	—
λ	$\vec{A}, \sigma, \phi$	$\vec{A}, \sigma, \phi$	$\vec{A}, \sigma, \phi$	$\vec{A}, \sigma, \phi$	$\vec{A}$	—
τ	$\vec{\pi}, \phi$	$\vec{\pi}, \phi$	$\vec{\pi}, \phi$	$\vec{\pi}, \phi$	—	$\vec{\pi}$

$$\begin{aligned} -D_{\sigma\sigma}\Gamma_{\sigma\sigma} + D_{\sigma\phi}\Gamma_{\pi\sigma} &= 1, \\ D_{\sigma\phi}\Gamma_{\pi\sigma} + D_{\phi\phi}(\Gamma_{\pi\pi} + \bar{\Gamma}_{\pi\pi}) &= 1, \\ -D_{\sigma\sigma}\Gamma_{\pi\sigma} + D_{\sigma\phi}(\Gamma_{\pi\pi} + \bar{\Gamma}_{\pi\pi}) &= 0, \\ D_{\sigma\phi}\Gamma_{\sigma\sigma} + D_{\phi\phi}\Gamma_{\pi\sigma} &= 0. \end{aligned} \tag{7.13}$$

The propagator functions in terms of the proper two-point functions are then

$$\begin{aligned} D_{\sigma\sigma} &= \frac{(\Gamma_{\pi\pi} + \bar{\Gamma}_{\pi\pi})}{\Gamma_{\pi\sigma}^2 - \Gamma_{\sigma\sigma}(\Gamma_{\pi\pi} + \bar{\Gamma}_{\pi\pi})}, \\ D_{\phi\phi} &= -\frac{\Gamma_{\sigma\sigma}}{\Gamma_{\pi\sigma}^2 - \Gamma_{\sigma\sigma}(\Gamma_{\pi\pi} + \bar{\Gamma}_{\pi\pi})}, \\ D_{\sigma\phi} &= \frac{\Gamma_{\pi\sigma}}{\Gamma_{\pi\sigma}^2 - \Gamma_{\sigma\sigma}(\Gamma_{\pi\pi} + \bar{\Gamma}_{\pi\pi})}. \end{aligned} \tag{7.14}$$

There are three more equations that are of interest. These are the $\sigma - A$, $\phi - A$, and $\lambda - A$ entries of Table V and they read:

$$D_{\sigma\sigma}\Gamma_{A\sigma} - D_{\sigma\phi}(\Gamma_{A\pi} + \bar{\Gamma}_{A\pi}) + D_{\sigma\lambda} = 0, \tag{7.15}$$

$$-D_{\sigma\phi}\Gamma_{A\sigma} - D_{\phi\phi}(\Gamma_{A\pi} + \bar{\Gamma}_{A\pi}) + D_{\phi\lambda} = 0, \tag{7.16}$$

$$\bar{\Gamma}_{AA} - \frac{k_0^2}{\vec{k}^2}[D_{\sigma\lambda}\Gamma_{A\sigma} + D_{\phi\lambda}(\Gamma_{A\pi} + \bar{\Gamma}_{A\pi})] = 0. \tag{7.17}$$

What these equations tell us is that $D_{\sigma\lambda}$ and $D_{\phi\lambda}$ are related to $\bar{\Gamma}_{A\pi}$ and $\Gamma_{A\sigma}$ with all other coefficients being determined. $\bar{\Gamma}_{AA}$ is then given as a specific combination and is the "extra" proper two-point function alluded to earlier. However, these functions will not be of any real concern since $D_{\sigma\lambda}$ and $D_{\phi\lambda}$ do not enter any loop diagrams of the Dyson-Schwinger equations. In effect, $\bar{\Gamma}_{A\pi}$, $\Gamma_{A\sigma}$, and $\bar{\Gamma}_{AA}$ form a consistency check on the truncation of the Dyson-Schwinger equations since we have that

$$\begin{aligned} \bar{\Gamma}_{AA} = \frac{k_0^2}{\vec{k}^2}[&-D_{\sigma\sigma}\Gamma_{A\sigma}^2 + 2D_{\sigma\phi}\Gamma_{A\sigma}(\Gamma_{A\pi} + \bar{\Gamma}_{A\pi}) \\ &+ D_{\phi\phi}(\Gamma_{A\pi} + \bar{\Gamma}_{A\pi})^2]. \end{aligned} \tag{7.18}$$

It is apparent that unlike covariant gauges, the proper two-point function for the gluon is not necessarily transverse.

In summary, leaving the problem of the vertices aside, in order to solve the two-point Dyson-Schwinger equations we need to calculate seven proper two-point functions:

$$\Gamma_c, \Gamma_{AA}, \Gamma_{A\pi}, \Gamma_{\pi\pi}, \bar{\Gamma}_{\pi\pi}, \Gamma_{\sigma\sigma}, \Gamma_{\sigma\pi}, \tag{7.19}$$

which will give us the required propagator functions:

$$D_c, D_{AA}, D_{A\pi}, D_{\pi\pi}, D_{\sigma\sigma}, D_{\sigma\phi}, D_{\phi\phi}. \tag{7.20}$$

The three proper two-point functions $\bar{\Gamma}_{AA}$, $\bar{\Gamma}_{A\pi}$, and $\Gamma_{A\sigma}$ give a consistency check on any truncation scheme but do not directly contribute further.

VIII. DERIVATION OF THE PROPAGATOR DYSON-SCHWINGER EQUATIONS

In this section, we present the explicit derivation of the relevant Dyson-Schwinger equations for proper two-point functions.

A. Ghost equations

As will be shown in this subsection, the ghost sector of the theory plays a rather special role. We will begin by deriving the ghost Dyson-Schwinger equation (this will serve as a template for the derivation of the other Dyson-Schwinger equations). With this it is possible to point out two particular features of the ghost sector: that the ghost-gluon vertex is UV finite and that the energy (k^0 component) argument of any ghost line is irrelevant, i.e., that any proper function involving ghost fields is independent of the ghost energy.

The derivation of the Dyson-Schwinger equation for the ghost proper two-point function begins with Eq. (4.15). Taking the functional derivative with respect to $\imath c_w^d$, using the configuration space definition of the tree-level ghost-gluon vertex, and omitting terms which will eventually vanish when sources are set to zero, we have

$$\langle \imath c_w^d \imath \bar{c}_x^a \rangle = \imath \delta^{da} \nabla_x^2 \delta(w - x) + \int d^4y d^4z \Gamma_{\bar{c}cAi}^{(0)abc}(x, y, z) \times \frac{\delta}{\delta \imath c_w^d} \langle \imath \bar{\eta}_y^b \imath J_{iz}^c \rangle. \tag{8.1}$$

Using partial differentiation we see that

$$\frac{\delta}{\delta \imath c_w^d} \langle \imath \bar{\eta}_y^b \imath J_{iz}^c \rangle = -\imath \int d^4 \upsilon \langle \imath J_{iz}^c \imath \bar{\eta}_y^b \imath \eta_\upsilon^e \rangle \langle \imath \bar{c}_\upsilon^e \imath c_w^d \rangle. \tag{8.2}$$

In the above we have again used the fact that when sources are set to zero, the only ghost functions that survive are those with pairs of ghost-antighost fields. Since the ghost fields anticommute, we get that

$$\langle \imath \bar{c}_x^a \imath c_w^d \rangle = -\imath \delta^{ad} \nabla_x^2 \delta(x - w) + \imath \int d^4y d^4z d^4\upsilon \Gamma_{\bar{c}cAi}^{(0)abc}(x, y, z) \times \langle \imath J_{iz}^c \imath \bar{\eta}_y^b \imath \eta_\upsilon^e \rangle \langle \imath \bar{c}_\upsilon^e \imath c_w^d \rangle. \tag{8.3}$$

Taking the partial derivative of Eq. (7.9) with respect to $\imath J_{iz}^c$ we have (notice that when using partial derivatives here, we must include all possible contributions which for clarity are included explicitly here):

$$\int d^4\upsilon \langle \imath J_{iz}^c \imath \bar{\eta}_y^b \imath \eta_\upsilon^e \rangle \langle \imath \bar{c}_\upsilon^e \imath c_w^d \rangle = -\imath \int d^4\upsilon d^4u \langle \imath \bar{\eta}_y^b \imath \eta_\upsilon^e \rangle \times \{ \langle \imath J_{iz}^c \imath J_{ju}^f \rangle \langle \imath A_{ju}^f \imath \bar{c}_\upsilon^e \imath c_w^d \rangle + \langle \imath J_{iz}^c \imath K_{ju}^f \rangle \langle \imath \pi_{ju}^f \imath \bar{c}_\upsilon^e \imath c_w^d \rangle \}. \tag{8.4}$$

Our ghost Dyson-Schwinger equation in configuration space is thus

$$\langle \imath \bar{c}_x^a \imath c_w^d \rangle = -\imath \delta^{ad} \nabla_x^2 \delta(x - w) + \int d^4y d^4z d^4\upsilon d^4u \Gamma_{\bar{c}cAi}^{(0)abc}(x, y, z) \times \langle \imath \bar{\eta}_y^b \imath \eta_\upsilon^e \rangle \{ \langle \imath J_{iz}^c \imath J_{ju}^f \rangle \langle \imath A_{ju}^f \imath \bar{c}_\upsilon^e \imath c_w^d \rangle + \langle \imath J_{iz}^c \imath K_{ju}^f \rangle \langle \imath \pi_{ju}^f \imath \bar{c}_\upsilon^e \imath c_w^d \rangle \}. \tag{8.5}$$

We Fourier transform this result to get the Dyson-Schwinger equation for the proper two-point ghost function in momentum space:

$$\Gamma_c^{ad}(k) = \delta^{ad} \imath \vec{k}^2 - \int (-\mathit{đ}\omega) \Gamma_{\bar{c}cAi}^{(0)abc}(k, -\omega, \omega - k) \times W_c^{be}(\omega) \{ W_{AAij}^{cf}(k - \omega) \Gamma_{\bar{c}cAj}^{edf}(\omega, -k, k - \omega) + W_{A\pi ij}^{cf}(k - \omega) \Gamma_{\bar{c}c\pi j}^{edf}(\omega, -k, k - \omega) \}. \tag{8.6}$$

With the convention that the self-energy term on the right-hand side has an overall minus sign, we identify $(-\mathit{đ}\omega)$ as the loop integration measure in momentum space.

With any two-point Dyson-Schwinger equation, it is clear that there are two orderings for the functional derivatives on the left-hand side. In the same way, there are three orderings for three-point functions and so on. This means that there are n different equations for the n-point proper Green's functions, although obviously they all have the same solution and must be related in some way. It is therefore instructive to consider also the equation generated by the reverse ordering to see if this will have any consequence. In the ghost case, this means repeating the above analysis but starting with the second ghost equation of motion, Eq. (A4). The corresponding Dyson-Schwinger equation in momentum space is

$$\Gamma_c^{ad}(k) = \delta^{ad} \imath \vec{k}^2 - \int (-\mathit{đ}\omega) \{ W_{AAij}^{fc}(\omega) \Gamma_{\bar{c}cAj}^{abc}(k, -k - \omega, \omega) + W_{A\pi ij}^{fc}(\omega) \Gamma_{\bar{c}c\pi j}^{abc}(k, -k - \omega, \omega) \} \times W_c^{be}(k + \omega) \Gamma_{\bar{c}cAi}^{(0)edf}(k + \omega, -k, -\omega). \tag{8.7}$$

This equation is formally equivalent to Eq. (8.6) but we notice that the ordering of the dressed vertices is different. (It is useful to check that the two equations are the same by taking both vertices to be bare such that the equivalence is manifest.)

Notice that one of the vertices that form the loop term(s) must be bare. This arises naturally through the derivation above and if one considers a perturbative expansion it is crucial to avoid overcounting of graphs. The choice of which vertex is bare is arbitrary and related to the fact that there are n ways of writing the equation for an n-point function. Given that for any loop term we can extract a single bare vertex, for any three-point function involving a ghost-antighost pair we will have a loop term with the following structure (see also Fig. 1):

$$\int đ\omega \Gamma^{dgac}_{\alpha\beta\bar{c}cj}(\omega, p, k-p, -k-\omega) W^{ce}_{c}(k+\omega) \times W^{fd}_{A\alpha ij}(\omega)\Gamma^{(0)ebf}_{\bar{c}cAi}(k+\omega, -k, -\omega). \tag{8.8}$$

Now, since the only propagators involving A are transverse (the $W_{A\lambda}$ propagator is disallowed since no proper vertex function with λ-derivative exists) the loop term must vanish as $k \to 0$ for finite p [27]. Since the loop term vanishes under some finite, kinematical configuration, a UV divergence (which is independent of the kinematical configuration) cannot occur and we can say that this vertex is UV finite. It is tempting to think that such an argument applies to the two-point ghost equation, however this is false since while the loop term vanishes, so does the $\vec{k}^2$ factor that multiplies the rest of the equation.

Let us now show that any Green's function involving a ghost-antighost pair is independent of the ghost and antighost energies. The proof of this is perturbative in nature. We notice that both the tree-level ghost propagator and the ghost-gluon vertex are independent of the energy. This means that in any one-loop diagram which has at least one internal ghost propagator (and hence at least two ghost-gluon vertices) the energy scale associated with the ghost propagator is absent. Using energy conservation another energy scale can be eliminated and we choose this to be the antighost energy. At two loops, we now have the situation whereby the dressed internal ghost propagator is again independent of the energy and the dressed ghost-gluon vertex only depends on the gluon energy and so the argument can be repeated. This can be applied to all orders in the perturbative expansion which completes the proof. We thus have that in particular

$$D_c(k_0^2, \vec{k}^2) = D_c(\vec{k}^2), \quad \Gamma_{A\bar{c}ci}(k_1, k_2, k_3) = \Gamma_{A\bar{c}ci}(k_1, \vec{k}_2, \vec{k}_3). \tag{8.9}$$

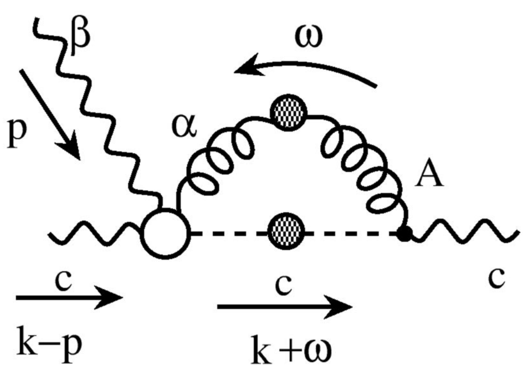

FIG. 1. A diagrammatical representation of the $\Gamma_{\bar{c}c\beta}(k-p, -k, p)$ proper vertex dressing. Because of the form of the tree-level ghost-gluon vertex and the tranversality of the vector propagator, the dressing function vanishes in the limit $k \to 0$. Filled blobs denote dressed propagators and empty circles denote dressed proper vertex functions. Wavy lines denote proper functions, springs denote connected (propagator) functions, and dashed lines denote the ghost propagator.

B. σ-based equation

Given the discussion in the previous section about which proper two-point functions are relevant, there are only two proper two-point functions involving derivatives with respect to σ to consider—$\langle \imath\sigma\imath\sigma\rangle$ and $\langle \imath\sigma\imath\pi\rangle$. Since the σ-based equation of motion, Eq. (4.13), involves two interaction terms whereas the π-based equation, Eq. (4.16), has only one, we use the derivatives of Eq. (4.16) to derive the Dyson-Schwinger equation for $\langle \imath\sigma\imath\pi\rangle$ (see next subsection). We therefore consider the functional derivative of Eq. (4.13) with respect to $\imath\sigma_w^d$, after which the sources will be set to zero. We have, again identifying the tree-level vertices,

$$\langle \imath\sigma_w^d \imath\sigma_x^a\rangle = -\int d^4y d^4z \Gamma^{(0)cab}_{\pi\sigma Aij}(z,x,y)\frac{\delta}{\delta \imath\sigma_w^d}\langle \imath J^b_{jy}\imath K^c_{iz}\rangle - \int d^4y d^4z \Gamma^{(0)cab}_{\phi\sigma Ai}(z,x,y)\frac{\delta}{\delta \imath\sigma_w^d}\langle \imath J^b_{iy}\imath\kappa^c_z\rangle. \tag{8.10}$$

Using partial differentiation, and with compact notation,

$$\frac{\delta}{\delta \imath\sigma_w^d}\langle \imath J^b_{jy}\imath K^c_{iz}\rangle = -\langle \imath K^c_{iz}\imath J_\alpha\rangle\langle \imath J^b_{jy}\imath J_\beta\rangle\langle \imath\Phi_\beta \imath\Phi_\alpha \imath\sigma_w^d\rangle \tag{8.11}$$

(similarly for the second term). This gives the Dyson-Schwinger equation in configuration space:

$$\langle \imath\sigma_w^d \imath\sigma_x^a\rangle = \int d^4y d^4z \Gamma^{(0)cab}_{\pi\sigma Aij}(z,x,y)\langle \imath K^c_{iz}\imath J_\alpha\rangle\langle \imath J^b_{jy}\imath J_\beta\rangle \times \langle \imath\Phi_\beta \imath\Phi_\alpha \imath\sigma_w^d\rangle + \int d^4y d^4z \Gamma^{(0)cab}_{\phi\sigma Ai}(z,x,y) \times \langle \imath\kappa^c_z \imath J_\alpha\rangle\langle \imath J^b_{iy}\imath J_\beta\rangle\langle \imath\Phi_\beta \imath\Phi_\alpha \imath\sigma_w^d\rangle. \tag{8.12}$$

Taking the Fourier transform and tidying-up indices, the Dyson-Schwinger equation in momentum space is thus

$$\Gamma^{ad}_{\sigma\sigma}(k) = -\int (-đ\omega)\Gamma^{(0)cab}_{\pi\sigma Aij}(\omega - k, k, -\omega) \times W^{be}_{A\beta jl}(\omega)\Gamma^{efd}_{\beta\alpha\sigma lk}(\omega, k-\omega, -k)W^{fc}_{\alpha\pi ki}(\omega - k) - \int (-đ\omega)\Gamma^{(0)cab}_{\phi\sigma Ai}(\omega - k, k, -\omega) \times W^{be}_{A\beta ij}(\omega)\Gamma^{efd}_{\beta\alpha\sigma j}(\omega, k-\omega, -k)W^{fc}_{\alpha\phi}(\omega - k). \tag{8.13}$$

A couple of remarks are in order here. First, there is no bare term on the right-hand side because the action, under the first order formalism, is linear in σ. Second, the implicit summation over the terms labeled by α and β means that in fact there are eight possible loop terms comprising the self-energy. However, only two of these involve a primitively divergent vertex. It is an uncomfortable truth that the formal, nonlocal delta function constraint arising from the linearity of the action in σ blossoms into a large set of local self-energy integrals.

C. π-based equations

Since the π-based equation of motion, Eq. (4.16), contains only a single interaction term, we favor it to calculate the $\langle \imath A \imath \pi\rangle$ and $\langle \imath\pi\imath\sigma\rangle$ proper two-point functions (as well as $\langle \imath\pi\imath\pi\rangle$). As discussed previously, we could in principle calculate these from the A-based and σ-based equations as well in order to check the veracity of any truncations used and in fact, this connection may serve useful in elucidating constraints on the form of the truncated vertices used. Perturbatively, all equations will provide the same result at any given order.

Using the same techniques as in the last subsection, we get the following Dyson-Schwinger equations in momentum space:

$$\Gamma^{ad}_{\pi\sigma i}(k) = \delta^{ad}k_i - \int (-đ\omega)\Gamma^{(0)abc}_{\pi\sigma Aij}(k, -\omega, \omega - k) \times W^{be}_{\sigma\beta}(\omega)\Gamma^{efd}_{\beta\alpha\sigma l}(\omega, k-\omega, -k) \times W^{fc}_{\alpha Alj}(\omega - k), \tag{8.14}$$

$$\Gamma^{ad}_{\pi Aik}(k) = -\delta^{ad}k^0\delta_{ik} - \int (-đ\omega)\Gamma^{(0)abc}_{\pi\sigma Aij}(k, -\omega, \omega - k) \times W^{be}_{\sigma\beta}(\omega)\Gamma^{efd}_{\beta\alpha Alk}(\omega, k-\omega, -k) \times W^{fc}_{\alpha Alj}(\omega - k), \tag{8.15}$$

$$\Gamma^{ad}_{\pi\pi ik}(k) = \imath\delta^{ad}\delta_{ik} - \int (-đ\omega)\Gamma^{(0)abc}_{\pi\sigma Aij}(k, -\omega, \omega - k) \times W^{be}_{\sigma\beta}(\omega)\Gamma^{efd}_{\beta\alpha\pi lk}(\omega, k-\omega, -k) \times W^{fc}_{\alpha Alj}(\omega - k). \tag{8.16}$$

Again, notice that the summation over the allowed types of fields indicated by α and β leads to multiple possibilities.

D. A-based equation

Using the tree-level forms for the vertices and discarding those terms which will eventually vanish when sources are set to zero, it is possible to rewrite Eq. (4.14) as

$$\langle \imath A^a_{ix}\rangle = [\delta_{ij}\nabla^2_x - \nabla_{ix}\nabla_{jx}]A^a_{jx} + \int d^4y d^4z \Gamma^{(0)bca}_{\bar{c}cAi}(z,y,x)\langle \imath\bar{\eta}^c_y \imath\eta^b_z\rangle - \int d^4y d^4z \Gamma^{(0)bca}_{\phi\sigma Ai}(z,y,x)\langle \imath\rho^c_y\imath\kappa^b_z\rangle - \int d^4y d^4z \Gamma^{(0)bca}_{\pi\sigma Aji}(z,y,x)\langle \imath\rho^c_y \imath K^b_{jz}\rangle - \int d^4y d^4z \frac{1}{2}\Gamma^{(0)bca}_{3Akji}(z,y,x)\langle \imath J^c_{jy}\imath J^b_{kz}\rangle - \int d^4y d^4z d^4w \frac{1}{6}\Gamma^{(0)dcba}_{4Alkji}(w,z,y,x)[3\imath A^b_{jy}\langle \imath J^c_{kz}\imath J^d_{lw}\rangle + \imath\langle \imath J^b_{jy}\imath J^c_{kz}\imath J^d_{lw}\rangle]. \tag{8.17}$$

Functionally differentiating this with respect to A and proceeding as before, noting the following for the four-gluon connected vertex

$$\imath\frac{\delta}{\delta\imath A^e_{mv}}\langle \imath J^b_{jy}\imath J^c_{kz}\imath J^d_{lw}\rangle = -\langle \imath J^c_{kz}\imath J_\nu\rangle\langle \imath A^e_{mv}\imath\Phi_\nu\imath\Phi_\mu\rangle\langle \imath J_\mu\imath J_\gamma\rangle\langle \imath J^b_{jy}\imath J_\lambda\rangle\langle \imath\Phi_\lambda\imath\Phi_\gamma\imath\Phi_\delta\rangle\langle \imath J_\delta\imath J^d_{lw}\rangle - \langle \imath J^c_{kz}\imath J_\gamma\rangle\langle \imath J^b_{jy}\imath J_\nu\rangle\langle \imath A^e_{mv}\imath\Phi_\nu\imath\Phi_\mu\rangle\langle \imath J_\mu\imath J_\lambda\rangle\langle \imath\Phi_\lambda\imath\Phi_\gamma\imath\Phi_\delta\rangle\langle \imath J_\delta\imath J^d_{lw}\rangle - \langle \imath J^c_{kz}\imath J_\gamma\rangle\langle \imath J^b_{jy}\imath J_\lambda\rangle\langle \imath\Phi_\lambda\imath\Phi_\gamma\imath\Phi_\delta\rangle\langle \imath J_\delta\imath J_\mu\rangle\langle \imath A^e_{mv}\imath\Phi_\mu\imath\Phi_\nu\rangle\langle \imath J_\nu\imath J^d_{lw}\rangle + \langle \imath J^c_{kz}\imath J_\gamma\rangle\langle \imath J^b_{jy}\imath J_\lambda\rangle\langle \imath A^e_{mv}\imath\Phi_\lambda\imath\Phi_\gamma\imath\Phi_\delta\rangle\langle \imath J_\delta\imath J^d_{lw}\rangle \tag{8.18}$$

gives the gluon Dyson-Schwinger equation which in momentum space reads:

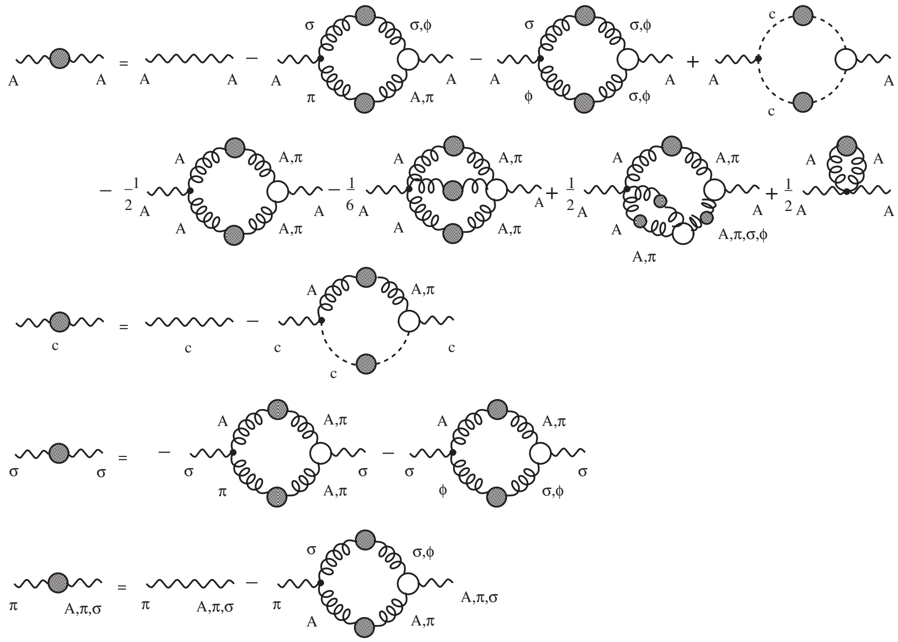

FIG. 2. A diagrammatical representation of the coupled system of Dyson-Schwinger equations. Filled blobs denote dressed propagators and empty circles denote dressed proper vertex functions. Wavy lines denote proper functions, springs denote connected (propagator) functions, and dashed lines denote the ghost propagator. Labels indicate the various possible propagator and vertex combinations that comprise the self-energy terms.

$$
\begin{aligned}
\Gamma^{ae}_{AAim}(k) &= \imath\delta^{ae}[\vec{k}^2\delta_{im} - k_i k_m] + \int(-\text{đ}\omega)\Gamma^{(0)bca}_{\bar{c}cAi}(\omega-k,-\omega,k)W^{cd}_{c}(\omega)\Gamma^{dfe}_{\bar{c}cAm}(\omega,k-\omega,-k)W^{fb}_{c}(\omega-k)\\
&\quad-\int(-\text{đ}\omega)\Gamma^{(0)bca}_{\phi\sigma Ai}(\omega-k,-\omega,k)W^{cd}_{\sigma\beta}(\omega)\Gamma^{dfe}_{\beta\alpha Am}(\omega,k-\omega,-k)W^{fb}_{\alpha\phi}(\omega-k)\\
&\quad-\int(-\text{đ}\omega)\Gamma^{(0)bca}_{A\pi\sigma ij}(\omega-k,-\omega,k)W^{cd}_{\sigma\beta}(\omega)\Gamma^{dfe}_{\beta\alpha Akm}(\omega,k-\omega,-k)W^{fb}_{\alpha\pi kj}(\omega-k)\\
&\quad-\frac{1}{2}\int(-\text{đ}\omega)\Gamma^{(0)bca}_{3Akji}(\omega-k,-\omega,k)W^{cd}_{A\beta jl}(\omega)\Gamma^{dfe}_{\beta\alpha Alnm}(\omega,k-\omega,-k)W^{fb}_{\alpha Ank}(\omega-k)\\
&\quad-\frac{1}{6}\int(-\text{đ}\omega)(-\text{đ}\upsilon)\Gamma^{(0)dcba}_{4Alkji}(-\upsilon,-\omega,\upsilon+\omega-k,k)W^{bf}_{A\lambda jn}(k-\upsilon-\omega)W^{cg}_{A\gamma ko}(\omega)W^{dh}_{A\delta lp}(\upsilon)\\
&\quad\times\Gamma^{fghe}_{\lambda\gamma\delta Anopm}(k-\omega-\upsilon,\omega,\upsilon,-k)+\frac{1}{2}\int(-\text{đ}\omega)\Gamma^{(0)aecd}_{4Aimlk}(k,-k,\omega,-\omega)W^{cd}_{AAkl}(-\omega)\\
&\quad+\frac{1}{2}\int(-\text{đ}\omega)(-\text{đ}\upsilon)\Gamma^{(0)dcba}_{4Alkji}(-\upsilon,-\omega,\upsilon+\omega-k,k)W^{df}_{A\delta ln}(\upsilon)W^{cg}_{A\gamma ko}(\omega)\Gamma^{fgh}_{\delta\gamma\lambda nop}(\upsilon,\omega,-\upsilon-\omega)\\
&\quad\times W^{hi}_{\lambda\mu pq}(\upsilon+\omega)\Gamma^{ije}_{\mu\nu Aqrm}(\upsilon+\omega,k-\upsilon-\omega,-k)W^{jd}_{\nu Arj}(\omega+\upsilon-k).
\end{aligned}
\tag{8.19}
$$

Again, the occurrence of the summation over $\alpha, \ldots, \lambda$ leads to many different possible loop terms.

We present the complete set of Dyson-Schwinger equations in Fig. 2.

IX. SUMMARY AND OUTLOOK

In this work, we have derived the Dyson-Schwinger equations for Coulomb gauge Yang-Mills theory within the first order formalism. In discussing the first order formalism it was noted that the standard BRS transform is supplemented by a second transform which arises from the ambiguity in setting the gauge transform properties of the π and ϕ fields. The motivation behind the use of the first order formalism is twofold: the energy divergent ghost sector can be formally eliminated and the system can be formally reduced to physical degrees of freedom, formal here meaning that the resulting expressions are nonlocal and not useful for practical studies. The cancellation of the ghost sector is seen within the context of the Dyson-Schwinger equations and the Green's functions stemming from the local action. It remains to be seen how the physical degrees of freedom emerge.

Given that the boundary conditions imposed by considering the Gribov problem and that the Jacobians of both the standard BRS transform and its supplemental transform within the first order formalism remain trivial, the field equations of motion and the Ward-Takahashi identity have been explicitly derived. The supplemental part of the BRS transform has been shown to be equivalent to the equations of motion at the level of the functional integral and as such is more or less trivial. Certain exact (i.e., not containing interaction terms) relations for the Green's functions of the theory have been discussed and their solutions presented. These relations serve to simplify the framework considerably. The propagators pertaining to vector fields are shown to be transverse, the proper functions involving Lagrange multiplier fields reduce to kinematical factors or vanish and the proper functions involving functional derivatives with respect to the ϕ-field can be explicitly derived from those involving the corresponding π-field derivatives.

The full set of Feynman rules for the system has been derived along with the tree-level proper two-point functions and the general form of the two-point functions (connected and proper) has been discussed. The relationship between the (connected) propagators and the proper two-point functions, stemming from the Legendre transform, has been studied. The resulting equations show that within the first order formalism the dressing functions of the two types of two-point Green's functions are nontrivially related to each other. In addition, given that there are no vertices involving derivatives with respect to the Lagrange multiplier fields, the set of Dyson-Schwinger equations needed to study the two-point functions of the theory is reduced.

The relevant Dyson-Schwinger equations for the system have been derived in some detail. It is shown how the number of self-energy terms is considerably amplified by the introduction of the various fields inherent to the first order formalism. The Dyson-Schwinger equations arising from the ghost fields are shown to be independent of the ghost energy and the vertices involving the ghost fields are UV finite.

Despite the complexity of dealing with a noncovariant system with many degrees of freedom, the outlook is positive and the rich structure of the Dyson-Schwinger equations is not as intimidating as it might initially appear. The noncovariance of the setting means that all the dressing functions that must be calculated are generally functions of two variables. However, as seen from the Feynman rules, the energy dependence of the theory stems from the tree-level propagators alone and not from vertices (a consequence of the fact that the only explicit time derivative in the action occurs within a kinetic term). The time dependence of the integral kernels will therefore be significantly less complicated than perhaps would otherwise occur. Given the experience in Landau gauge adapting the techniques, both analytical and numerical, to solve the Dyson-Schwinger equations in Coulomb gauge seems eminently possible though certainly challenging. The results of such a study should provide a better understanding of the issues of confinement, and with the inclusion of quarks, the hadron spectrum.

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APPENDIX A: EXPLICIT FORM OF THE EQUATIONS OF MOTION

For completeness we list the explicit equations of motion for the various fields represented by Eq. (4.6):

$$\rho_x^a Z[J] = \int \mathcal{D}\Phi\{\vec{D}_x^{ab}\cdot(\vec{\pi}_x^b - \vec{\nabla}_x\phi_x^b)\}\exp\{\imath\mathcal{S}+\imath\mathcal{S}_s\}, \tag{A1}$$

$$\begin{aligned} J_{ix}^a Z[J] = -\int \mathcal{D}\Phi\Big\{&\nabla_{ix}\lambda_x^a - gf^{bac}(\nabla_{ix}\bar{c}_x^b)c_x^c \\ &- \partial_x^0(\pi_{ix}^a - \nabla_{ix}\phi_x^a) - gf^{bac}(\pi_{ix}^b - \nabla_{ix}\phi_x^b)\sigma_x^c \\ &+ [\delta_{ij}\nabla_x^2 - \nabla_{ix}\nabla_{jx}]A_{jx}^a + gf^{abc}[A_{jx}^b\nabla_{ix}A_{jx}^c \\ &+ 2A_{jx}^c\nabla_{jx}A_{ix}^b - A_{ix}^c\nabla_{jx}A_{jx}^b] \\ &- \frac{1}{4}g^2 f^{fbc}f^{fde}[\delta^{ab}A_{jx}^cA_{jx}^eA_{ix}^d + \delta^{ad}A_{jx}^cA_{jx}^eA_{ix}^b \\ &+ \delta^{ac}A_{jx}^bA_{jx}^dA_{ix}^e + \delta^{ae}A_{jx}^bA_{jx}^dA_{ix}^c]\Big\}\exp\{\imath\mathcal{S}+\imath\mathcal{S}_s\}, \end{aligned} \tag{A2}$$

$$\eta_x^a Z[J] = \int \mathcal{D}\Phi\{\vec{\nabla}_x\cdot\vec{D}_x^{ab}c_x^b\}\exp\{\imath\mathcal{S}+\imath\mathcal{S}_s\}, \tag{A3}$$

$$\bar{\eta}_x^a Z[J] = \int \mathcal{D}\Phi\{\vec{D}_x^{ab}\cdot\vec{\nabla}_x\bar{c}_x^b\}\exp\{\imath\mathcal{S}+\imath\mathcal{S}_s\}, \tag{A4}$$

$$\kappa_x^a Z[J] = -\int \mathcal{D}\Phi\{-\vec{\nabla}_x\cdot\vec{X}_x^a\}\exp\{\imath\mathcal{S}+\imath\mathcal{S}_s\}, \tag{A5}$$

$$K_{ix}^a Z[J] = -\int \mathcal{D}\Phi\{\nabla_{ix}\tau_x^a - X_{ix}^a\}\exp\{\imath\mathcal{S}+\imath\mathcal{S}_s\}, \tag{A6}$$

$$\xi_{\lambda x}^a Z[J] = \int \mathcal{D}\Phi\{\vec{\nabla}_x\cdot\vec{A}_x^a\}\exp\{\imath\mathcal{S}+\imath\mathcal{S}_s\}, \tag{A7}$$

$$\xi_{\tau x}^a Z[J] = \int \mathcal{D}\Phi\{\vec{\nabla}_x\cdot\vec{\pi}_x^a\}\exp\{\imath\mathcal{S}+\imath\mathcal{S}_s\}, \tag{A8}$$

with X defined in Eq. (2.22).

APPENDIX B: JACOBIAN FACTORS

The BRS and α-transforms, Eqs. (2.20) and (2.21), respectively, can be regarded as changes of variable in the functional integral and that the action is invariant will lead to Ward-Takahashi identities. However, as with all changes of integration variable, we must consider the relevant Jacobian factors. These turn out to be trivial, although this is not immediately obvious. Starting with the BRS transform, we begin with the observation that $\delta\lambda^2 = 0$ (since $\delta\lambda$ is Grassmann-valued) which means that in the Jacobian determinant only the diagonal elements will contribute since all off-diagonal elements are $\mathcal{O}(\delta\lambda)$. Besides the trivial unit terms of the form $\delta^{ab}\delta(x-y)$ (with an extra δ_{ij} factor for vector fields) all diagonal terms have the color structure $f^{abc}H^c$, as can be seen in the form of the transform. Given that

$$\det(\delta^{ab} + f^{abc}H^c) = \exp\{\mathrm{Tr}\log(\delta^{ab} + f^{abc}H^c)\} \quad \text{(B1)}$$

only the first term proportional to f^{abc} survives when the logarithm is expanded ($H \sim \delta\lambda$). We then see that $\mathrm{Tr} f^{abc}H^c = 0$ which leaves only the unit term of the exponential. The Jacobian for the BRS transform is thus trivial.

To see that the Jacobian involved for the α-transform is trivial is slightly more involved. First we note that the only nontrivial part of the matrix of variations involves only the $\vec{\pi}$ and ϕ fields, all other rows or columns reducing to trivial identity contributions. The submatrix of variations for the $\vec{\pi}$ and ϕ fields can be written $\mathbb{1} + K$ where $K \sim f^{abc}\theta_x^c$ and is independent of the fields. Since the transform is a change of variables, the functional integral is independent of θ_x^c and we can write

$$0 = \frac{\delta Z}{\delta\theta_x^c}\bigg|_{\theta=0} = \frac{\delta}{\delta\theta_x^c} J[\theta]\int \mathcal{D}\Phi \exp\{\imath\mathcal{S} + \imath\mathcal{S}_s[\theta]\}|_{\theta=0}. \quad \text{(B2)}$$

With this in mind, it suffices to show that

$$\frac{\delta J[\theta]}{\delta\theta_x^c}\bigg|_{\theta=0} = 0 \quad \text{(B3)}$$

in order for the actual form of the Jacobian to be irrelevant (note that $J[\theta=0]=1$). We can write

$$J[\theta] = \exp\{\mathrm{Tr}\log(1 + K[\theta])\} \quad \text{(B4)}$$

and so

$$\frac{\delta J[\theta]}{\delta\theta_x^c}\bigg|_{\theta=0} = \mathrm{Tr}\bigg[\frac{\delta K[\theta]}{\delta\theta_x^c} - K[\theta]\frac{\delta K[\theta]}{\delta\theta_x^c} + \ldots\bigg]\bigg|_{\theta=0} \times J[\theta=0]. \quad \text{(B5)}$$

Since K is linear in θ, only the first term of the expansion is present when $\theta = 0$. This term has the color structure f^{abc}, which vanishes under the trace operation and so, indeed

$$\frac{\delta J[\theta]}{\delta\theta_x^c}\bigg|_{\theta=0} = 0 \quad \text{(B6)}$$

and the Jacobian, although itself not unity is trivial and does not further enter the discussion of the functional integral under the transform.

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