# Improved approximations for fermion pair production in inhomogeneous electric fields

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Reformulating the instantons in a complex plane for tunneling or transmitting states, we calculate the pair-production rate of charged fermions in a spatially localized electric field, illustrated by the Sauter electric field $E_0 \mathrm{sech}^2(z/L)$, and in a temporally localized electric field such as $E_0 \mathrm{sech}^2(t/T)$. The integration of the quadratic part of WKB instanton actions over the frequency and transverse momentum leads to the pair-production rate obtained by the worldline-instanton method, including the prefactor, of Phys. Rev. D **72**, 105004 (2005) and **73**, 065028 (2006). It is further shown that the WKB instanton action plus the next-to-leading-order contribution in spinor QED equals the WKB instanton action in scalar QED, thus justifying why the WKB instanton in scalar QED can work for the pair production of fermions. Finally we obtain the pair-production rate in a spatially localized electric field together with a constant magnetic field in the same direction.

## I. INTRODUCTION

The physics of strong electromagnetic fields, in particular, vacuum polarization and pair production, has been studied by Sauter [1], Heisenberg, Euler [2], and Weisskopf [3] even before the advent of quantum electrodynamics (QED). Using the proper-time path integral, Schwinger obtained the one-loop QED effective action in a constant electromagnetic field [4]. The imaginary part of the effective action under the influence of electric fields leads to the decay rate of the vacuum and thereby the pair-production rate of charged particles. (For a recent review and references on QED and pair production, see Ref. [5].) One approach to pair production is tunneling of virtual pairs from the Dirac sea [6–13], in which instantons determine the tunneling probability and thereby the pair-production rate. Another approach is the recent worldline method [14–22].

On the other hand, physics of strong fields in laboratory has recently attracted much attention due to the rapid development of laser technology. In the near future the x-ray free electron lasers from Linac Coherent Light Source at SLAC [23] and TESLA at DESY [24] are expected to produce at the focus an electric field near the critical strength $E_c = m^2 c^3 / e\hbar (1.3 \times 10^{18}$ V/m) for electron-position pair production. Also ultrahigh intense lasers have been developed using a technique of amplifying pulses of picosecond and few femtosecond time scales (for a review and references, see Ref. [25]). The focal region of colliding lasers may correspond to a strong

QED regime, in which not only vacuum polarization but also pair production can be tested [26] (see also Ref. [27] for other QED related physics). The localized beam in space and time does necessarily imply inhomogeneous electromagnetic fields. Thus pair production of charged particles by inhomogeneous electric fields is not only a theoretical issue but also an experimental concern.

In previous papers [12,13], we formulated the pair-production rate of charged particles in inhomogeneous electric electric fields in terms of the instantons for tunneling states in the space-dependent (Coulomb) gauge, and worked out explicitly the rate for the Sauter electric field [28] in the WKB approximation. Recently, using the worldline path integral, Dunne and Schubert [15] obtained worldline instantons in a gauge-independent way, and Dunne, Wang, Gies, and Schubert [16] further found the prefactor to the worldline instantons to calculate the pair-production rate by inhomogeneous fields.

In this paper, using the phase-integral method [29,30], we further elaborate the instanton method of Refs. [12,13] by defining the instanton action in a gauge-independent way as a contour integral in the complex space or time plane. The divergence problem at turning points in the WKB method can be naturally avoided by the contour integral excluding the branch cut connecting two complex roots corresponding to turning points. We use the new instanton method to calculate the leading-order (LO) and next-to-leading-order (NLO) WKB instanton actions both in a spatially localized electric field $E_0 \mathrm{sech}^2(z/L)$ and in a temporally localized electric field $E_0 \mathrm{sech}^2(t/T)$ of Sauter type [28], and then obtain the pair-production rate for charged fermions. We then compare the pair-production rate with that obtained by the worldline-instanton method [15,16] and with the exact result by Nikishov [31].

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We show that for the Sauter electric field [28], the minimum value of our WKB instanton action $\mathcal{S}_{\mathbf{k}_\perp}^{(0)}(\omega) = 2S_{\mathbf{k}_\perp}^{(0)}(\omega)$, as a function of the transverse momentum $\mathbf{k}_\perp$ and the frequency $\omega$, is the same as that of the single-worldline-instanton action of Ref. [15], and that the Gaussian approximation for the integrals over $\omega$ and $\mathbf{k}_\perp$ in our approach using $\mathcal{S}_{\mathbf{k}_\perp}^{(0)}(\omega)$ gives the same pair-production rate, including the prefactor [16], as the worldline-instanton method. We further calculate the NLO WKB actions $\mathcal{S}_{sp,\mathbf{k}_\perp}^{(2)}(\omega)$ in spinor QED and show that the sum of the WKB and NLO instanton actions in spinor QED is equal to the WKB instanton action in scalar QED. Finally we study pair production in the Sauter electric field together with a constant magnetic field in the same direction.

The organization of this paper is as follows. In Sec. II, we reformulate the instanton method in a gauge-independent way in the complex plane of space or time. We also show that the terms in the WKB approximation for a constant electric field vanish beyond the leading order, and the WKB approximation thus reproduces the well-known exact one-loop result from the point of view of the instanton method. In Sec. III, we calculate within the leading-order WKB approximation the instanton actions in scalar QED for a localized electric field of spatial or temporal Sauter-type and obtain the pair-production rate for both cases. We find that the WKB instanton action in scalar QED gives a pair-production rate closer to the exact result for spinor QED up to higher order in an adiabaticity parameter. In Sec. IV, we calculate the WKB and NLO instanton actions in spinor QED and find that the instanton action up through the NLO contribution for each frequency and transverse momentum equals the WKB instanton action in scalar QED, thus justifying the reason why the WKB instanton action in scalar QED gives a good result. In Sec. V, we find the WKB instanton action in the Sauter electric field together with a constant magnetic field in the same direction.

## II. GAUGE INDEPENDENCE OF INSTANTON ACTIONS

The worldline-instanton method is based on Feynman's worldline path integral [32], manifestly a gauge invariant formalism, and calculates the instanton action from a closed instanton trajectory [15–22]. Affleck, Alvarez, and Manton found the instanton action in a constant electric field from the closed instanton trajectory [33]. In this paper we show how the instanton method of Refs. [12,13] can be reformulated by the same contour integral for two different gauges. For the sake of simplicity we first focus on the constant electric field, which can be treated exactly, and then extend to inhomogeneous fields in the next section. The electric field can be written in terms of two simple alternative gauges: the space-dependent

(Coulomb) gauge and the time-dependent gauge. We shall first work on scalar QED, described by the Klein-Gordon equation, and then discuss spinor QED, described by the Dirac equation, in Sec. IV. The Klein-Gordon equation for charged particles with charge $q$ ($q > 0$) and mass $m$ takes the form [in units with $\hbar = c = 1$ and with metric signature $(+, -, -, -)$]

$$[\eta^{\mu\nu}(\partial_\mu + iqA_\mu)(\partial_\nu + iqA_\nu) + m^2]\Phi = 0. \quad (1)$$

In the space-dependent gauge, the Klein-Gordon equation, after being decomposed into Fourier modes, becomes a tunneling problem, the so-called sub-barrier penetration. The essence of the instanton method is that the tunneling states of this equation lead to the vacuum decay and pair production of charged particles. On the other hand, in the time-dependent gauge, the mode-decomposed equation resembles a one-dimensional scattering problem over a potential barrier, the so-called superbarrier transmission. The positive-frequency modes define the vacuum state at past infinity, and the negative-frequency modes at future infinity lead to the number of created pairs [34,35]. Fröman and Fröman [29] showed that the probability amplitudes of both the tunneling and the transmitted states could be found in terms of a contour integral in the complex plane of space or time. This contour integral yields the same WKB instanton action in both gauges for a constant electric field.

### A. Constant *E*-field in the space-dependent gauge

In the space-dependent (Coulomb) gauge, a constant electric field along the $z$-direction has the potential $A_\mu = (-Ez, 0, 0, 0)$. The mode-decomposed Klein-Gordon equation in (1) now takes the form

$$[-\partial_z^2 + Q(z)]\phi_{\omega\mathbf{k}_\perp}(z) = 0, \quad (2)$$

where

$$Q(z) = m^2 + \mathbf{k}_\perp^2 - (\omega + qEz)^2. \quad (3)$$

Each mode equation now describes the tunneling problem under the upside-down harmonic potential barrier and has two real turning points

$$z_\pm = -\frac{\omega}{qE} \pm \sqrt{\frac{m^2 + \mathbf{k}_\perp^2}{(qE)^2}}. \quad (4)$$

The exact wave function is given by the complex parabolic cylindrical function $E(S_{\mathbf{k}_\perp}/\pi, \sqrt{2/qE}(qzE + \omega))$ in terms of the instanton action $S_{\mathbf{k}_\perp} = \pi(m^2 + \mathbf{k}_\perp^2)/(2qE)$ [12].

Using the phase-integral formula [29,30], the wave function can be written in one asymptotic region $z \ll z_-$ as

$$\phi_{\omega\mathbf{k}_\perp}(z) = A(e^{i\pi/4}\varphi_{\omega\mathbf{k}_\perp}(z)) + B(e^{i\pi/4}\varphi_{\omega\mathbf{k}_\perp}(z))^*, \quad (5)$$

where

$$A = (e^{2S_{\mathbf{k}}} + 1)^{1/2}, \qquad B = e^{S_{\mathbf{k}}}, \quad (6)$$

and $\varphi_{\omega\mathbf{k}_\perp}(z)$ has unit incoming flux (with the group velocity away from the barrier on the back side). Here the leading-order WKB instanton action is given by

$$S^{(0)}_{\mathbf{k}_\perp} = \int_{z_-}^{z_+} \sqrt{Q(z)}\,dz = \frac{\pi(m^2 + \mathbf{k}_\perp^2)}{2qE}, \qquad (7)$$

which is the exact action for the constant $E$-field, that is, $S_{\mathbf{k}_\perp} = S^{(0)}_{\mathbf{k}_\perp}$. In the other asymptotic region $z \gg z_+$, the tunneling wave function is given by

$$\phi_{\omega\mathbf{k}_\perp}(z) = (e^{-(i\pi/4)}\varphi_{\omega\mathbf{k}_\perp}(z))^*. \qquad (8)$$

Now the action can also be defined in the complex $z$ plane by the contour integral [29,30]

$$\mathcal{S}_{\mathbf{k}_\perp} = 2S_{\mathbf{k}_\perp} = -i\oint_{\Gamma_K} q(z)\,dz, \qquad (9)$$

where the integral is along the contour in Fig. 1. Here,

$$q(z) = \sqrt{-Q(z)} \sum_{n=0}^{\infty} Y_{2n}(z), \qquad (10)$$

and the first two $Y_{2n}$ are [29,30]

$$Y_0(z) = 1, \qquad Y_2(z) = -\frac{1}{32Q^3}\left[5\left(\frac{dQ}{dz}\right)^2 - 4Q\frac{d^2Q}{dz^2}\right]. \qquad (11)$$

Note that the complex function $\sqrt{-Q}$ has a branch cut along the real line segment connecting two roots $z_\pm$ and is single-valued outside the closed loop of Fig. 1 [36]. From the Laurent expansion for large $z$,

$$\sqrt{-Q} = qE\left[z - \frac{z_+ + z_-}{2} - \frac{(z_+ - z_-)^2}{8z} - \cdots\right], \qquad (12)$$

we find the residue $qE(z_+ - z_-)^2/8$ at the simple pole at $z = \infty$ [37]. Hence the contour integral in the exterior region of the closed loop in Fig. 1 leads to the leading-order WKB instanton action

$$\mathcal{S}^{(0)}_{\mathbf{k}_\perp} = 2\pi i\left[-iqE\frac{(z_+ - z_-)^2}{8}\right] = \frac{\pi(m^2 + \mathbf{k}_\perp^2)}{qE}. \qquad (13)$$

The WKB instanton action agrees with Eq. (13) of Ref. [12] and Eq. (21) of Ref. [13], and also with Eq. (27) of Ref. [15] for the action of the worldline instanton.

## B. Constant $E$-field in the time-dependent gauge

In the time-dependent gauge, the potential is given by $A_\mu = (0, 0, 0, -Et)$. The mode-decomposed Klein-Gordon equation of (1) takes the form

$$[-\partial_t^2 + Q_t(t)]\phi_{\omega\mathbf{k}}(t) = 0, \qquad (14)$$

where

$$Q_t(t) = -[m^2 + \mathbf{k}_\perp^2 + (k_z - qEt)^2]. \qquad (15)$$

The problem now becomes the superbarrier transmission over the upside-down harmonic potential. The vacuum state is defined in terms of the positive-frequency (adiabatic) solution at past infinity ($t = -\infty$). Particle production is ascribed to the negative-frequency (adiabatic) solution at future infinity ($t = \infty$).

The function $\sqrt{-Q_t}$ now has a branch cut along a line segment parallel to the imaginary axis in the complex $t$ plane, which connects two complex roots

$$t_\pm = \frac{k_z}{qE} \pm i\sqrt{\frac{m^2 + \mathbf{k}_\perp^2}{(qE)^2}}. \qquad (16)$$

Then the transmission probability is determined by the contour integral [29]

$$\mathcal{S}_{\mathbf{k}_\perp} = 2S_{\mathbf{k}_\perp} = i\oint_{\Gamma_K} \sqrt{-Q_t(t)}\,dt, \qquad (17)$$

along the contour in Fig. 2. As $\sqrt{-Q_t}$ has a simple pole at $t = \infty$,

$$\sqrt{-Q_t} = qE\left[t - \frac{t_+ + t_-}{2} - \frac{(t_+ - t_-)^2}{8t} - \cdots\right], \qquad (18)$$

and the residue is $qE(t_+ - t_-)^2/8$, we obtain the WKB action

$$\mathcal{S}_{\mathbf{k}_\perp} = 2\pi i\left[iqE\frac{(t_+ - t_-)^2}{8}\right] = \frac{\pi(m^2 + \mathbf{k}_\perp^2)}{qE}. \qquad (19)$$

Though the Klein-Gordon equation in the time-dependent gauge potential for the constant $E$-field is equivalent to the scattering problem over a potential barrier, the transmission probability is determined by the action now defined in the complex time plane. This holds
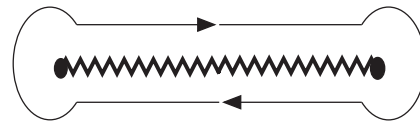


FIG. 2. The contour integral in the complex plane $t$ excluding a branch cut connecting two complex roots $z_\pm$.



FIG. 1. The contour integral in the complex plane $z$ excluding a branch cut connecting two real roots $z_\pm$.

true for general time-dependent electric fields. In this sense, the action defined by the contour integral (9) or (17) in the complex plane of space or time is the same for both gauge potentials.

### C. $S^{(2)} = 0$ for constant $E$-field

In the space-dependent gauge, the next-to-leading-order (NLO) WKB contribution to the instanton action takes the form

$$S_{\mathbf{k}_\perp}^{(2)} = i \oint_{\Gamma_K} \sqrt{-Q(z)} \frac{1}{32Q^3} \left[ 5\left(\frac{dQ}{dz}\right)^2 - 4Q\frac{d^2Q}{dz^2} \right] dz \tag{20}$$

for the same contour in Fig. 1. The simple roots $z_\pm$ of $Q(z)$ cannot give simple poles, as they are excluded by the contour. Further the integrand does not have a simple pole at infinity, since it has the expansion of the form

$$\frac{1}{32Q^3}\left[5\left(\frac{dQ}{dz}\right)^2 - 4Q\frac{d^2Q}{dz^2}\right] = \sum_{n=4}^{\infty} \frac{a_n}{z^n}. \tag{21}$$

Therefore, the contour integral (20) vanishes, and $S_{\mathbf{k}_\perp}^{(2)} = 0$ for the constant $E$-field. As all higher order actions vanish by the same argument, the WKB instanton actions agree with the result from the exact solution of the Klein-Gordon equation. Also the same argument holds for the time-dependent gauge. The integral over all the transverse momentum recovers the well-known pair-production rate in a constant electric field [4].

## III. LOCALIZED ELECTRIC FIELDS

Consider a static plane-symmetric but $z$-dependent electric field $E(z)$ in the $z$-direction of maximum value $E_0$ and of effective length $L$ defined so that

$$E_0 L \equiv \frac{1}{2} \int_{-\infty}^{\infty} E(z) dz. \tag{22}$$

It may be characterized by the parameter

$$\epsilon \equiv \frac{m}{qE_0 L}. \tag{23}$$

Similarly, a homogeneous time-dependent electric field $E(t)$ in the $z$-direction of maximum value $E_0$ and of effective time $T$ defined so that

$$E_0 T \equiv \frac{1}{2} \int_{-\infty}^{\infty} E(t) dt \tag{24}$$

may be characterized by the parameter

$$\epsilon_t \equiv \frac{m}{qE_0 T}. \tag{25}$$

Pair production is energetically allowed for $\epsilon < 1$ and for any $\epsilon_t$, though it is strongly suppressed for $\epsilon_t \gg 1$. Pair production by localized electric fields significantly differs

from that by the constant electric field due to a size effect [38]. The finite size effect on pair production was also shown in many inhomogeneous electric fields [12,13,15,16].

We shall also define another parameter,

$$\delta \equiv \frac{qE_0}{\pi m^2}, \tag{26}$$

which (for $\epsilon < 1$ or $\epsilon_t < 1$) is small when the pair-production rate is small. A third useful parameter is the following combination of the previous two:

$$b \equiv \frac{\delta}{\sqrt{1 - \epsilon^2}} \quad \text{or} \quad b_t \equiv \frac{\delta}{\sqrt{1 + \epsilon_t^2}}. \tag{27}$$

When the WKB approximation is good, then $b \ll 1$ or $b_t \ll 1$, so $b$ or $b_t$ serves as an adiabaticity parameter.

In this section we consider the spatially localized electric field $E(z) = E_0 \text{sech}^2(z/L)$ in the $z$-direction given by the Sauter potential [28]

$$A_0(z) = -E_0 L \tanh\left(\frac{z}{L}\right). \tag{28}$$

From now on $E(z) = E_0 \text{sech}^2(z/L)$ will be called the Sauter electric field [28]. Pairs are produced only when $qE_0 L - m \geq \omega \geq -(qE_0 L - m)$. This electric field varies slowly over the effective length $L$. The other field is a uniform but temporally localized electric field $E(t) = E_0 \text{sech}^2(t/T)$, effectively lasting over the period $T$.

### A. Sauter electric field

In the first case of the Sauter potential $A_0(z) = -E_0 L \tanh(z/L)$ [28], we change the variable $\zeta = L \tanh(z/L)$ to write the leading-order WKB action as

$$S_{\mathbf{k}_\perp}^{(0)} = 2S_{\mathbf{k}_\perp} = -i \oint_{\Gamma_K} \frac{\sqrt{-Q(\zeta)}}{1 - \frac{\zeta^2}{L^2}} d\zeta, \tag{29}$$

where

$$Q(\zeta) = m^2 + \mathbf{k}_\perp^2 - (\omega + qE_0\zeta)^2. \tag{30}$$

As $|\zeta| \leq L$, we expand the integrand as a Laurent series

$$\frac{\sqrt{-Q(\zeta)}}{1 - \frac{\zeta^2}{L^2}} = qE_0 \sum_{n=0}^{\infty} \sum_{l=0}^{\infty} \frac{C_l}{L^{2n}} \zeta^{2n+1-l}, \tag{31}$$

where we also expand the square root for large $\zeta$ as

$$f^{(0)}(\zeta) \equiv \sqrt{-\frac{Q(\zeta)}{(qE_0\zeta)^2}} = \sum_{l=0}^{\infty} \frac{C_l}{\zeta^l}. \tag{32}$$

The integrand of (29) has simple poles at large $\zeta$ for $l = 2n + 2$ ($\geq 2$), and the sum of residues is

$$\sum_{\text{residue}} = -qE_0 L^2 \sum_{n=1}^{\infty} \frac{C_{2n}}{L^{2n}}. \tag{33}$$

Noting that

$$f^{(0)}(L) + f^{(0)}(-L) = 2\sum_{n=0}^{\infty} \frac{C_{2n}}{L^{2n}}, \qquad (34)$$

we finally obtain the leading-order WKB action

$$\mathcal{S}_{\mathbf{k}_\perp}^{(0)} = \pi q E_0 L^2 [2 - f^{(0)}(L) - f^{(0)}(-L)]. \qquad (35)$$

Therefore, the leading-order WKB action in scalar QED can be written as

$$\mathcal{S}_{\mathbf{k}_\perp}^{(0)} = \frac{Z}{2}[2 - \sqrt{(1+\mu)^2 - \epsilon^2(1+\kappa^2)}$$
$$- \sqrt{(1-\mu)^2 - \epsilon^2(1+\kappa^2)}], \qquad (36)$$

in terms of the dimensionless scaled variables

$$\mu \equiv \frac{\omega}{qEL}, \qquad \kappa \equiv \frac{k_\perp}{m}, \qquad (37)$$

and the dimensionless parameters

$$Z = 2\pi q E_0 L^2 = \frac{2}{\delta \epsilon^2}, \qquad \epsilon = \frac{m}{qE_0L}, \qquad \delta = \frac{qE_0}{\pi m^2}. \qquad (38)$$

We shall first compare the scalar QED instanton actions with other results. As $\mathcal{S}_{\mathbf{k}_\perp}^{(0)}$ is an even function of $\mu$ and $\kappa$, we may expand the action (36) as a power series in $\mu$ and $\kappa$,

$$\mathcal{S}_{\mathbf{k}_\perp}^{(0)} = \mathcal{S}_{\{0\}}^{(0)} + \mathcal{S}_{\{2\}}^{(0)} + \mathcal{S}_{\{4\}}^{(0)} + \sum_{n=3}^{\infty} \mathcal{S}_{\{2n\}}^{(0)}, \qquad (39)$$

where the first few $\mathcal{S}_{\{2n\}}^{(0)}$ are

$$\mathcal{S}_{\{0\}}^{(0)} = Z(1 - \sqrt{1-\epsilon^2}),$$

$$\mathcal{S}_{\{2\}}^{(0)} = \frac{\mu^2}{\delta(1-\epsilon^2)^{3/2}} + \frac{\kappa^2}{\delta(1-\epsilon^2)^{1/2}},$$

$$\mathcal{S}_{\{4\}}^{(0)} = \frac{1}{4\delta(1-\epsilon^2)^{7/2}}[(4+\epsilon^2)\mu^4$$
$$+ 2(1-\epsilon^2)(2+\epsilon^2)\mu^2\kappa^2 + \epsilon^2(1-\epsilon^2)^2\kappa^4]. \qquad (40)$$

In the special case of $\omega = 0$ and $\mathbf{k}_\perp = 0$, we obtain the action

$$\mathcal{S}_{\{0\}}^{(0)} = \frac{1}{\delta}\frac{2}{1+\sqrt{1-\epsilon^2}} = Z(1-\sqrt{1-\epsilon^2})$$
$$= \frac{2}{\sqrt{1-\epsilon^2}+1-\epsilon^2}\frac{1}{b} > \frac{1}{b} \gg 1, \qquad (41)$$

agreeing with the action of the worldline instanton (62) of Ref. [15]. This is also true for spinor QED as will be shown in Sec. IV.

Weighting the Gaussian integral of $e^{-(\mathcal{S}_{\{0\}}^{(0)}+\mathcal{S}_{\{2\}}^{(0)})}$ over $\omega$ and $\mathbf{k}_\perp$ by a power series expansion of $e^{-\sum_{n=2}\mathcal{S}_{\{2n\}}^{(0)}}$ through

8th order in $\omega$ and $\mathbf{k}_\perp$, we obtain a WKB approximation for the pair-production rate of charged particles per unit time and unit cross-sectional area,

$$\mathcal{N}^{(0)} = \frac{2}{(2\pi)^3} \int d\omega \int d^2\mathbf{k}_\perp e^{-\mathcal{S}_{\mathbf{k}}^{(0)}}$$
$$\approx \frac{(qE_0)^{5/2}L}{4\pi^3 m}(1-\epsilon^2)^{5/4}e^{-Z(1-\sqrt{1-\epsilon^2})}$$
$$\times \left[1 - \frac{5}{16}(4+3\epsilon^2)b + \frac{105}{512}(4-\epsilon^2)^2 b^2 \right.$$
$$\left. - \frac{315}{8192}(320 - 432\epsilon^2 + 124\epsilon^4 - \epsilon^6)b^3\right]. \qquad (42)$$

Here and hereafter we insert a factor of 2, for two charged scalar fields, to compare with the results of spinor QED with two spins for each spin-$1/2$ fermion state. Thus all single-scalar QED results would be half of those listed here.

The contribution up to the quadratic terms of the WKB instanton action, the factor in front of the square bracket, agrees with Eq. (4.7) of Ref. [16] from the worldline-instanton approximation. The 2nd, 3rd, and 4th terms in the square bracket are the contributions from the 4th, 6th, and 8th order terms of the WKB instanton action, though these terms in the action are intertwined in the pair-production rate because of the exponentiation of the action.

It is interesting to compare this leading-order WKB result in scalar QED with the exact result in spinor QED [31], given in terms of $Z$ and $\epsilon$ by the double integral in Eq. (62) of Ref. [13]. For $b \ll 1$, so that one may drop terms that are exponentially smaller than the dominant terms by factors like $e^{-\mathcal{S}_{\{0\}}^{(0)}} < e^{-1/b}$, the leading approximation to the double integral, given by Eq. (64) of Ref. [13], agrees with the leading term of Eq. (42), the coefficient in front of the square bracket. We have checked that taking additional dominant terms of the double integral (suppressed not exponentially but only by powers of $\mu$ and $\kappa$) indeed gives precisely the same total result as the extreme right hand side of Eq. (42). Indeed, one can show that the entire leading-order WKB approximation for scalar QED pair production in the Sauter potential, given by the first right hand side of Eq. (42), immediately after the $=$ sign, is equal to the total power series of all dominant terms of the exact result for spinor QED when one drops only terms that are exponentially suppressed.

### B. Temporally localized field

In the second case of the temporally localized electric field, we change the variable $\tau = T\tanh(t/T)$ to write the WKB actions as

$$\mathcal{S}_{\mathbf{k}_\perp}^{(0)} = 2S_{\mathbf{k}_\perp} = i \oint_{\Gamma_K} \frac{\sqrt{-Q_t(\tau)}}{1 - \frac{\tau^2}{T^2}} d\tau, \qquad (43)$$

where

$$Q_t(\tau) = -[m^2 + \mathbf{k}_\perp^2 + (k_z + qE_0\tau)^2]. \qquad (44)$$

Here and hereafter the subscript $t$ denotes any quantity in the time-dependent gauge. The integral (43) can be obtained from Eq. (29) for the spatially localized electric field by replacing $L$ by $T$ and $Q$ from Eq. (3) by $Q_t$ from Eq. (15). In terms of the scaled variables

$$\mu_t = \frac{k_z}{qE_0T}, \qquad \kappa = \frac{k_\perp}{m}, \qquad (45)$$

and parameters

$$Z_t = 2\pi qE_0T^2 = \frac{2}{\delta\epsilon_t^2}, \qquad \epsilon_t = \frac{m}{qE_0T}, \qquad \delta = \frac{qE_0}{\pi m^2}, \qquad (46)$$

we have

$$\mathcal{S}_{\mathbf{k}_\perp}^{(0)} = \frac{Z_t}{2}[\sqrt{(1+\mu_t)^2 + \epsilon_t^2(1+\kappa^2)}$$
$$+ \sqrt{(1-\mu_t)^2 + \epsilon_t^2(1+\kappa^2)} - 2]. \qquad (47)$$

In the special case of $k_z = 0$ and $\mathbf{k}_\perp = 0$, the action is

$$\mathcal{S}_{\{0\}}^{(0)} = \frac{1}{\delta}\frac{2}{1+\sqrt{1+\epsilon_t^2}} = Z_t(\sqrt{1+\epsilon_t^2} - 1)$$
$$= \frac{2}{\sqrt{1+\epsilon_t^2}+1+\epsilon_t^2}\frac{1}{b_t} < \frac{1}{b_t}, \qquad (48)$$

which agrees with Eq. (38) of Ref. [15].

As in the case of the Sauter electric field, we expand the instanton action in power of $\mu_t$ and $\kappa$ as

$$\mathcal{S}_{\mathbf{k}_\perp}^{(0)} = \mathcal{S}_{\{0\}}^{(0)} + \mathcal{S}_{\{2\}}^{(0)} + \mathcal{S}_{\{4\}}^{(0)} + \sum_{n=3}^{\infty}\mathcal{S}_{\{2n\}}^{(0)}, \qquad (49)$$

where

$$\mathcal{S}_{\{0\}}^{(0)} = Z_t(\sqrt{1+\epsilon_t^2} - 1),$$
$$\mathcal{S}_{\{2\}}^{(0)} = \frac{\mu_t^2}{\delta(1+\epsilon_t^2)^{3/2}} + \frac{\kappa^2}{\delta(1+\epsilon_t^2)^{1/2}},$$
$$\mathcal{S}_{\{4\}}^{(0)} = \frac{1}{4\delta(1+\epsilon_t^2)^{7/2}}[(4-\epsilon_t^2)\mu_t^4$$
$$+ 2(1+\epsilon_t^2)(2-\epsilon_t^2)\mu_t^2\kappa^2 - \epsilon_t^2(1+\epsilon_t^2)^2\kappa^4]. \quad (50)$$

The pair-production density (per unit spatial volume) from the WKB actions up through quartic terms is given by

$$\mathcal{N}^{(0)} \approx \frac{(q_0E)^{5/2}T}{4\pi^3 m}(1+\epsilon_t^2)^{5/4}e^{-Z_t(\sqrt{1+\epsilon_t^2}-1)}$$
$$\times\left[1 - \frac{5}{16}(4-3\epsilon_t^2)b_t\right]. \qquad (51)$$

where $b_t$ is the adiabaticity parameter in Eq. (27). The pair-production density up to the quadratic terms, the factor in front of the square bracket, agrees with Eq. (3.40) of

Ref. [16]. We note that Eq. (51) for the temporal Sauter-type electric field [28] can be obtained by analytically continuing $\epsilon^2$ to $-\epsilon_t^2$ in Eq. (42) for the Sauter electric field.

## IV. SPINOR QED

The eigen-component of the Dirac equation for spin-1/2 fermions with charge $q$ ($q > 0$) and mass $m$ [in units with $\hbar = c = 1$ and with metric signature $(+, -, -, -)$] takes the form [31,39]

$$[\eta^{\mu\nu}(\partial_\mu + iqA_\mu)(\partial_\nu + iqA_\nu) + m^2 + 2i\sigma qE]\Phi_\sigma = 0. \qquad (52)$$

The Dirac equation ($\sigma = 1/2$) is the relativistic field equation for spinor QED, whereas in scalar QED the charged spinless bosons are described by the Klein-Gordon equation ($\sigma = 0$), the single component equation in (52) without the imaginary term.

A comment on the complex instanton actions in spinor QED is in order. Each eigen-component of the Dirac equation in Eq. (52) satisfies the Klein-Gordon equation with a complex potential and thus leads to complex instanton actions. For fermions, if we impose the boundary condition from causality that a particle moves with a group velocity away from the barrier on the back side, the expected number of pairs produced per mode is then given by one minus the reflection probability [13,31],

$$\mathcal{N}(\mathbf{k}_\perp) = 1 - \left|\frac{A}{B}\right|^2 = e^{-(S_\mathbf{k}+S_\mathbf{k}^*)} = e^{-(\mathcal{S}_{\mathbf{k}_\perp}+\mathcal{S}_{\mathbf{k}_\perp}^*)/2}. \qquad (53)$$

As the imaginary part of an instanton action does *not* contribute to the pair-production rate, from now on we shall refer to only real part of the instanton action unless stated otherwise. For a constant electric field, the WKB action in spinor QED is

$$\mathcal{S}_{\mathrm{sp},\mathbf{k}_\perp}^{(0)} = \frac{\pi(m^2 + \mathbf{k}_\perp^2)}{qE} + i\pi, \qquad (54)$$

so the real part agrees with Eq. (13) of Ref. [12] and Eq. (21) of Ref. [13], and also with Eq. (27) of Ref. [15] for the action of the worldline instanton.

For the Sauter potential $A_0(z) = -E_0L\tanh(z/L)$ [28], we change variables to $\zeta = L\tanh(z/L)$ to write the leading-order WKB instanton action as

$$\mathcal{S}_{\mathrm{sp},\mathbf{k}_\perp}^{(0)} = 2S_{\mathbf{k}_\perp} = -i\oint_{\Gamma_K}\frac{\sqrt{-Q_{\mathrm{sp}}(\zeta)}}{1 - \frac{\zeta^2}{L^2}}d\zeta, \qquad (55)$$

where

$$Q_{\mathrm{sp}}(\zeta) = m^2 + \mathbf{k}_\perp^2 - (\omega + qE_0\zeta)^2 + iqE_0\left(1 - \frac{\zeta^2}{L^2}\right). \qquad (56)$$

As the contour integral (55) does not depend on whether $Q_{sp}(z)$ is real or complex, we can repeat the same procedure as in scalar QED in Sec. III A. We thus obtain the leading-order WKB complex action in spinor QED

$$\mathcal{S}^{(0)}_{sp,\mathbf{k}_\perp} = \mathcal{S}^{(0)}_{sc,\mathbf{k}_\perp} + Z[\sqrt{1 + i\pi\delta\epsilon^2} - 1], \qquad (57)$$

where $\mathcal{S}^{(0)}_{sc,\mathbf{k}_\perp}$ is the WKB action (36) in scalar QED. The real part of the spinor action, which determines the pair-production rate

$$\Re e(\mathcal{S}^{(0)}_{sp,\mathbf{k}_\perp}) = \mathcal{S}^{(0)}_{sc,\mathbf{k}_\perp} + \frac{Z}{2}[\sqrt{1 + i\pi\delta\epsilon^2} + \sqrt{1 - i\pi\delta\epsilon^2} - 2] \qquad (58)$$

is larger than the scalar QED action by $\pi^2\delta\epsilon^2/4$ up to order $\mathcal{O}(\delta^3\epsilon^6)$. We may write the NLO WKB contribution to the action as

$$\mathcal{S}^{(2)}_{sp,\mathbf{k}_\perp} = -i\oint_{\Gamma_K} \frac{f^{(2)}(\zeta)}{1 - \frac{\zeta^2}{L^2}} d\zeta, \qquad (59)$$

where

$$f^{(2)}(\zeta) \equiv -\frac{\sqrt{-Q_{sp}}}{32 Q_{sp}^3}\left[5\left(\frac{dQ_{sp}}{dz}\right)^2 - 4Q_{sp}\frac{d^2 Q_{sp}}{dz^2}\right]. \qquad (60)$$

The NLO action is given by

$$\mathcal{S}^{(2)}_{sp,\mathbf{k}_\perp} = -\frac{\pi^2\delta\epsilon^2}{4\sqrt{1 + i\pi\delta\epsilon^2}}, \qquad (61)$$

so the real part cancels out the excess in the leading-order real spinor action, at least up to order of $\mathcal{O}(\delta^3\epsilon^6)$. Therefore the spinor instanton action up through the NLO contribution approximately equals the scalar action,

$$\Re e(\mathcal{S}^{(0)}_{sp,\mathbf{k}_\perp} + \mathcal{S}^{(2)}_{sp,\mathbf{k}_\perp}) \approx \mathcal{S}^{(0)}_{sc,\mathbf{k}_\perp}, \qquad (62)$$

to the same order. This explains the reason why the leading-order WKB action in scalar QED gives the pair-production rate closer to the exact result in spinor QED as explained in Sec. III.

Similarly, for the temporally localized electric field, we obtain the WKB real action in spinor QED as

$$\mathcal{S}^{(0)}_{sp,\mathbf{k}_\perp} = \mathcal{S}^{(0)}_{sc,\mathbf{k}_\perp} + \frac{Z_t}{2}[2 - \sqrt{1 + i\pi\delta\epsilon^2} - \sqrt{1 - i\pi\delta\epsilon^2}]. \qquad (63)$$

The WKB and NLO actions in spinor QED for the temporal electric field also leads to the WKB action in scalar QED as given by Eq. (62).

## V. LOCALIZED ELECTRIC FIELD AND CONSTANT MAGNETIC FIELD

Finally we study the effect of a constant magnetic field $B$, in the same direction as the electric field, on the production of charged fermion pairs in the Sauter electric field [28] with $\epsilon \ll 1$ and $\epsilon\delta \ll 1$. Note that the instantons exist when the Landau levels are limited to

$$qB(2j_{max} + 1) = \min\{(\omega + qE_0L)^2 - m^2, (\omega - qE_0L)^2 - m^2\}. \qquad (64)$$

In terms of the potential energy difference

$$V = qA_0(-\infty) - qA_0(+\infty) = 2qE_0L, \qquad (65)$$

the highest Landau level is

$$j_{max} = \frac{1}{2qB}\left[\left(\frac{V}{2} - |\omega|\right)^2 - m^2\right] - \frac{1}{2}. \qquad (66)$$

The pair-production rate per unit area and unit time for spinor QED is now

$$\mathcal{N} = \frac{(qB)}{(2\pi)^2}\int_{-((V/2)-m)}^{(V/2)-m} d\omega \sum_{j=0}^{j_{max}} \sum_{\sigma_\pm = \pm 1/2} e^{-\mathcal{S}_{\sigma_\pm j}}, \qquad (67)$$

where $(qB)/(2\pi)$ is the number of Landau levels and another factor $1/(2\pi)$ is from the $\omega$ integration. Here the instanton actions are determined by

$$\mathcal{S}_{\sigma_\pm j} = \sum_{n=0}^{\infty} \mathcal{S}^{(2n)}_{\sigma_\pm j}, \qquad (68)$$

where the dominant contribution comes from the WKB instantons with action

$$\mathcal{S}^{(0)}_{\sigma_\pm j} = -i\oint_{\Gamma_K} dz\sqrt{\omega + qE_0L\tanh(Z/L))^2 - m^2 - qb(2j + 1 - 2\sigma_\pm)}$$
$$= \frac{Z}{2}[2 - \sqrt{(1+\mu)^2 - \epsilon^2(1+\tilde{\kappa}^2)} - \sqrt{(1-\mu)^2 - \epsilon^2(1+\tilde{\kappa}^2)}], \qquad (69)$$

where $\sigma_\pm = \pm 1/2$ and

$$\tilde{\kappa}^2 = \frac{qB(2j + 1 - 2\sigma_\pm)}{m^2}. \qquad (70)$$

Expanding the WKB instanton action up to $\mu^2$ and $\tilde{\kappa}^2$, summing over $j$ and $\sigma_\pm$, and integrating over $\omega$, we obtain

the pair-production rate per unit area and unit time

$$\mathcal{N}^{(0)} \approx \frac{qBL(qE_0)^{3/2}}{4\pi^2 m}(1 - \epsilon^2)^{3/4}$$
$$\times e^{-Z(1-\sqrt{1-\epsilon^2})}\coth\left(\frac{\pi B}{E_0(1-\epsilon^2)^{1/2}}\right). \qquad (71)$$

When $B = 0$, we recover the pair-production rate of Eq. (42), as expected.

## VI. CONCLUSION

In this paper we have further elaborated the previous instanton method [12,13] by reformulating the instanton action as a contour integral in the complex plane of space or time. For a general electric field with the gauge potential $A_0(z) = -E_0 f(z)$ or $A_3(t) = -E_0 f(t)$, $\zeta = f(z)$ or $f(t)$ being an analytical function, we may express the instanton action as

$$S^{(0)} = -i \oint_{\Gamma_K} \sqrt{(\omega + qE_0\zeta)^2 - (m^2 + \mathbf{k}_\perp^2) - iqE_0\left(\frac{df}{dz}\right)}$$
$$\times \frac{d\zeta}{(df/dz)}, \tag{72}$$

and a similar formula holds for the time-dependent gauge potential. As for the Sauter potential, the stratagem is to find the inverse function $z = f^{-1}(\zeta)$, expand the integrand for large $\zeta$, and then calculate the contour integral. The inverse function can be found at least as a power series. This formulation is gauge independent in the sense that the same instanton action in the complex plane determines the transmission probability or the tunneling probability for fermion pair production either in the time-dependent gauge for time-dependent electric fields or in the space-dependent gauge for space-dependent electric fields. Furthermore, the new formulation allows one to calculate the instanton action beyond the WKB approximation without encountering the divergence problem at turning points, since it excludes the branch cut connecting two turning points.

We first applied the instanton method to a constant electric field both in the space-dependent gauge and in the time-dependent gauge. In both gauges the instantons recovered the well-known exact result for the constant electric field. It is shown that the contributions beyond the leading-order WKB approximation vanish for a constant electric field, and thus the leading-order WKB approximation is exact as expected in the one-loop effective action.

We then applied the formulation to a spatially or temporally localized electric field of Sauter type [28]. We showed that our actions agree with the action of the worldline instantons of Dunne and Schubert [15] in the special case of zero frequency and transverse momentum for a spatially localized electric field and of zero three-momentum for a temporally localized electric field. The pair-production rates obtained by using our leading-order WKB approximations that are expanded up to quadratic terms of frequency and transverse momentum or three-momentum also agree with those by the worldline-instanton method, including the prefactor [16]. Furthermore, the exact pair-production rate in spinor QED by Nikishov [31] is better approximated by the leading-order WKB instanton action in scalar QED. We show that the cancellation of the leading-order and the next-to-leading-order actions in spinor QED yields the leading-order scalar QED action and resolves this apparent dilemma. Finally, we calculate the actions and the pair-production rate in the Sauter electric field in the presence of a constant magnetic field.

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