Statistics and UV-IR mixing with twisted Poincaré invariance

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(Received 15 October 2006; revised manuscript received 28 November 2006; published 12 February 2007)

We elaborate on the role of quantum statistics in twisted Poincaré invariant theories. It is shown that, in order to have twisted Poincaré group as the symmetry of a quantum theory, statistics must be twisted. It is also confirmed that the removal of UV-IR mixing (in the absence of gauge fields) in such theories is a natural consequence.

DOI: 10.1103/PhysRevD.75.045009 PACS numbers: 11.10.Nx, 11.30.Cp

I. INTRODUCTION

Following the application of Drinfel'd's twist for the Poincaré group on the noncommutative Groenewold-Moyal (GM) plane [1,2], much interest has been generated in the study of its physical consequences. One such consequence pointed out in [3,4] is that the usual statistics are not compatible with the twisted action of the Poincaré group. This is in agreement with what is already known in quantum group theory. Among the consequences of this result is the removal of UV-IR mixing [5] in the S-matrix in the absence of gauge fields.

Recently there have been claims that this twisting of statistics is unnecessary or even wrong, and that the removal of UV-IR mixing is the result of a wrong choice of interaction. In this paper we explain our point of view more clearly, demonstrating that if one wants to retain the twisted Poincaré symmetry in a quantum theory, then one is forced to implement twisted statistics. Secondly, the form of the interaction is dictated by quantum symmetry as well.

The paper is organized as follows. After briefly reviewing the Drinfel'd twist for Poincaré group in the Sec. II, we elaborate on its implications for quantum statistics in Sec. III. Section IV discusses the choice of the correct twisted Lorentz-invariant interaction Hamiltonian. In Sec. V, we show by an explicit calculation that the correlation functions and hence the S-matrix of the noncommutative quantum field theory (NCQFT) with usual statistics are not invariant under the twisted symmetry, while the same are manifestly so for the theory with twisted statistics. Section VI discusses some issues related to the functional integral for theories with twisted Poincaré symmetry. Section VII describes the notion of locality in the twisted statistics approach and Sec. VIII addresses some general issues regarding the tensor products of fields.

II. THE TWIST

The action of a symmetry group on the tensor product of representation spaces carrying the same representation ρ is given by a coproduct Δ:

$$g \triangleright (\phi \otimes \chi) = (\rho \otimes \rho)\Delta(g)(\phi \otimes \chi). \qquad (2.1)$$

If the representation space happens to be an algebra as well, we further have the compatibility condition

$$m((\rho \otimes \rho)\Delta(g)(\phi \otimes \chi)) = \rho(g)m(\phi \otimes \chi) \qquad (2.2)$$

where m is the multiplication map.

The GM plane is the algebra $\mathcal{A}_\theta$ of functions $f \in \mathbb{R}^n$ with the product defined by

$$f * g = m_\theta(f \otimes g) = m_0 \mathcal{F}(f \otimes g) \qquad (2.3)$$

where m_0 is the usual pointwise multiplication,

$$\mathcal{F} = e^{-(i/2)\theta^{\mu\nu} P_\mu \otimes P_\nu}, \qquad P_\mu = -i\partial_\mu, \qquad (2.4)$$

is called the *twist* element, and this rule for multiplication is often called the *star* product. Explicitly (2.3) gives

$$(f * g)(x) = \exp\left(\frac{i}{2}\theta^{\mu\nu}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial y^\nu}\right) f(x) g(y)\bigg|_{x=y}. \qquad (2.5)$$

The usual coproduct Δ_0 on the Poincaré group,

$$\Delta_0(\Lambda) = \Lambda \times \Lambda, \qquad \Lambda \in \text{Poincaré group}, \qquad (2.6)$$

is not compatible with the star product. But a new coproduct Δ_θ obtained using the twist is compatible, where

$$\Delta_\theta(\Lambda) = \mathcal{F}^{-1}\Delta_0(\Lambda)\mathcal{F}. \qquad (2.7)$$

For details see [1,2]. Note that $\Delta_\theta(a) = \Delta_0(a)$ if a is a translation.

III. TWISTED STATISTICS

Twisting the coproduct implies twisting of statistics in quantum theory, as we will argue in this section. This result

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holds for an n-particle quantum mechanical system and also for quantum field theory.

A. Quantum mechanics

The wave function of a two-particle system for $\theta^{\mu\nu} = 0$ in position representation is a function of two variables, hence it lives in $\mathcal{A}_0 \otimes \mathcal{A}_0$, the tensor product of two copies of the algebra of functions on commutative $\mathbb{R}^n$, and transforms according to the usual coproduct Δ_0. Similarly in noncommutative case, the wavefunction lives in $\mathcal{A}_\theta \otimes \mathcal{A}_\theta$ and transforms according to the twisted coproduct Δ_θ.

A general element of the tensor product has no particular symmetry. Usually we require that the physical wave functions describing identical particles are either symmetric (bosons) or antisymmetric (fermions). This requires us to work with either the symmetrized or antisymmetrized tensor product

$$\phi \otimes_S \chi \equiv \frac{1}{2}(\phi \otimes \chi + \chi \otimes \phi), \tag{3.1}$$

$$\phi \otimes_A \chi \equiv \frac{1}{2}(\phi \otimes \chi - \chi \otimes \phi) \tag{3.2}$$

which satisfy

$$\phi \otimes_S \chi = +\chi \otimes_S \phi, \tag{3.3}$$

$$\phi \otimes_A \chi = -\chi \otimes_A \phi. \tag{3.4}$$

In a Lorentz-invariant theory, these relations have to hold in all frames of reference. In other words, performing a Lorentz transformation on $\phi \otimes \chi$ and then (anti-)symmetrizing has to be the same as (anti-)symmetrization followed by the Lorentz transformation.

It is not difficult to show that the twisted coproduct (2.7) is not compatible with usual symmetrization/antisymmetrization (3.1) and (3.2). To see this, let us write $\mathcal{F}^{-1}$ and $\mathcal{F}$ in the Sweedler notation (see for e.g. pg. 5 of [6]) as

$$\mathcal{F}^{-1} = \sum_\alpha f^{(1)\alpha} \otimes f^{(2)}_\alpha, \qquad \mathcal{F} = \sum_\alpha \tilde{f}^{(1)\alpha} \otimes \tilde{f}^{(2)}_\alpha, \quad \text{with} \tag{3.5}$$

$$\mathcal{F}^{-1}\mathcal{F} = \mathbf{1} \otimes \mathbf{1} = \sum_{\alpha,\beta} f^{(1)\alpha}\tilde{f}^{(1)\beta} \otimes f^{(2)}_\alpha \tilde{f}^{(2)}_\beta. \tag{3.6}$$

Under a Lorentz transformation Λ,

$$\begin{aligned}\Lambda\colon \phi \otimes \chi &\longrightarrow (\rho \otimes \rho)\Delta_\theta(\Lambda)(\phi \otimes \chi)\\ &= \sum_{\alpha,\beta} \rho(f^{(1)\alpha}\Lambda\tilde{f}^{(1)\beta})\phi \otimes \rho(f^{(2)}_\alpha \Lambda \tilde{f}^{(2)}_\beta)\chi.\end{aligned} \tag{3.7}$$

Subsequent symmetrization/antisymmetrization gives us

$$\begin{aligned}&\sum_{\alpha,\beta}(\rho(f^{(1)\alpha}\Lambda\tilde{f}^{(1)\beta})\phi \otimes \rho(f^{(2)}_\alpha\Lambda\tilde{f}^{(2)}_\beta)\chi\\ &\quad \pm \rho(f^{(2)}_\alpha\Lambda\tilde{f}^{(2)}_\beta)\chi \otimes \rho(f^{(1)\alpha}\Lambda\tilde{f}^{(1)\beta})\phi)\end{aligned} \tag{3.8}$$

whereas

$$\begin{aligned}(\rho \otimes \rho)\Delta_\theta(\Lambda)(\phi \otimes_{S,A} \chi) = \sum_{\alpha'\beta}(&\rho(f^{(1)\alpha}\Lambda\tilde{f}^{(1)\beta})\phi\\ &\otimes \rho(f^{(2)}_\alpha\Lambda\tilde{f}^{(2)}_\beta)\chi\\ &\pm \rho(f^{(1)\alpha}\Lambda\tilde{f}^{(1)\beta})\chi\\ &\otimes \rho(f^{(2)}_\alpha\Lambda\tilde{f}^{(2)}_\beta)\phi)\end{aligned} \tag{3.9}$$

which is not the same as (3.8) [See [4] for the same proof which avoids Sweedler notation.]. The origin of this difference can be traced to the fact that the coproduct is not cocommutative except when $\theta^{\mu\nu} = 0$.

There is another way to phrase this compatibility (or lack thereof) of Lorentz transformations and symmetrization. Let τ_0 be the statistics (flip) operator associated with exchange:

$$\tau_0(\phi \otimes \chi) = \chi \otimes \phi. \tag{3.10}$$

In usual quantum theory, we have the axiom that τ_0 is superselected, i.e., all the observables commute with τ_0. What this means is that no operator in the physical Hilbert space can change statistics. In particular the quantum operators that implement Lorentz symmetry must commute with the statistics operator. Also all the states in a given superselection sector are eigenstates of τ_0 with the same eigenvalue. Given an element $\phi \otimes \chi$ of the tensor product, the physical Hilbert spaces can be constructed from the elements

$$\left(\frac{1 \pm \tau_0}{2}\right)(\phi \otimes \chi). \tag{3.11}$$

As is obvious from Eq. (3.8) and (3.9),

$$\tau_0\Delta_\theta(\Lambda) \neq \Delta_\theta(\Lambda)\tau_0 \tag{3.12}$$

showing that the usual statistics is not compatible with the coproduct. But notice that the new statistics operator

$$\tau_\theta \equiv \mathcal{F}^{-1}\tau_0\mathcal{F}, \qquad \tau_\theta^2 = \mathbf{1} \otimes \mathbf{1} \tag{3.13}$$

does commute with the twisted coproduct. The states constructed according to

$$\begin{aligned}\phi \otimes_{S_\theta} \chi &\equiv \left(\frac{1 + \tau_\theta}{2}\right)(\phi \otimes \chi),\\ \phi \otimes_{A_\theta} \chi &\equiv \left(\frac{1 - \tau_\theta}{2}\right)(\phi \otimes \chi)\end{aligned} \tag{3.14}$$

form the physical two-particle Hilbert spaces of (generalized) bosons and fermions and obey twisted statistics.

For plane waves $e_p(x) = e^{-ip\cdot x}$ we get

$$\begin{aligned}\left(\frac{\mathbf{1} \pm \tau_\theta}{2}\right)(e_p \otimes e_q) &\equiv e_p \otimes_{S_\theta,A_\theta} e_q\\ &= \pm e^{-ip_\mu\theta^{\mu\nu}q_\nu} e_q \otimes_{S_\theta,A_\theta} e_p,\end{aligned} \tag{3.15}$$

$$(e_p \otimes_{S_\theta,A_\theta} e_q)(x_1,x_2) = \pm e^{-i(\partial/\partial x_1^\mu)\theta^{\mu\nu}(\partial/\partial x_2^\nu)}(e_p \otimes_{S_\theta,A_\theta} e_q) \times (x_2,x_1). \quad (3.16)$$

Using the antisymmetry of $\theta^{\mu\nu}$, τ_θ may also be equivalently written as

$$\tau_\theta = \mathcal{F}^{-2}\tau_0. \quad (3.17)$$

This form of τ_θ allows to make contact with quantum group theory, and identifies $\mathcal{F}^{-2}$ as the corresponding R-matrix.

B. Statistics of quantum fields

A quantum field on evaluation at a spacetime point (or more generally on pairing with a test function) gives an operator acting on a Hilbert space. A field at x_1 acting on the vacuum gives a one-particle state centered at x_1. When we write $\Phi(x_1)\Phi(x_2)$ we mean $(\Phi \otimes \Phi)(x_1, x_2)$. Acting on the vacuum we generate a two-particle state, where one particle is centered at x_1 and the other at x_2. (We retain just the creation operator part of Φ here.) Notice that it just involves evaluation of the two functions in the tensor product and *not* a multiplication map as we get a function of two variables. On the other hand the star product is a map from $\mathcal{A}_\theta \otimes \mathcal{A}_\theta$ to $\mathcal{A}_\theta$ and gives a function of a single variable. Hence there is no reason *a priori* to put a starlike operation between $\Phi(x_1)\Phi(x_2)$. We will have more to say about this in Sec. VIII.

If a_p is the annihilation operator of the second-quantized field $\Phi(x)$, we want, as in standard quantum field theory,

$$\langle 0|\Phi^{(-)}(x)a_p^\dagger|0\rangle = e_p(x), \quad (3.18)$$

$$\frac{1}{2}\langle 0|\Phi^{(-)}(x_1)\Phi^{(-)}(x_2)a_q^\dagger a_p^\dagger|0\rangle = \left(\frac{\mathbf{1} \pm \tau_\theta}{2}\right)(e_p \otimes e_q)(x_1,x_2) \equiv (e_p \otimes_{S_\theta,A_\theta} e_q)(x_1,x_2) \quad (3.19)$$

[We suppress spin indices. Also here we retain only the annihilation part of the field in $\Phi^{(-)}$]. Note the reversal of p and q from LHS to RHS of (3.19). This is the standard prescription used to establish the connection between quantum field operators and (multi-)particle wavefunctions. The correctness of this prescription can be verified by applying it to the fermionic case, for $\theta^{\mu\nu} = 0$.

This compatibility between twisted statistics and Poincaré invariance has profound consequences for commutation relations. For example when the states are labeled by momenta, we have, from exchanging p and q in (3.19)

$$|p,q\rangle_{S_\theta,A_\theta} = \pm e^{i\theta^{\mu\nu}p_\mu q_\nu}|q,p\rangle_{S_\theta,A_\theta} \quad (3.20)$$

This is the origin of the commutation relation

$$a_p^\dagger a_q^\dagger = \pm e^{i\theta^{\mu\nu}p_\mu q_\nu}a_q^\dagger a_p^\dagger. \quad (3.21)$$

The adjoint relation

$$a_p a_q = \pm e^{i\theta^{\mu\nu}p_\mu q_\nu}a_q a_p \quad (3.22)$$

also follows from the complex conjugate of (3.19) on using (3.16).

The statistics induced on the free quantum fields by (3.19) is given, on using (3.16), by

$$\Phi^{(-)}(x_1)\Phi^{(-)}(x_2) = \pm e^{i\theta^{\mu\nu}(\partial/\partial x_2^\mu)(\partial/\partial x_1^\nu)}\Phi^{(-)}(x_2)\Phi^{(-)}(x_1). \quad (3.23)$$

Any quantization has to be compatible with the above statistics of the fields.

So far we have had no occasion to use the algebraic properties of $\mathcal{A}_\theta$. All we have used is the assumption that the symmetry of the theory is the twisted Poincaré group symmetry. That, of course, was forced on us from automorphism properties of $\mathcal{A}_\theta$.

IV. CHOICE OF INTERACTION HAMILTONIAN

It was claimed by [7] that the absence of UV-IR mixing in noncommutative theories may be due to a specific choice of the interaction Hamiltonian. Here we point out that our choice of the Hamiltonian is forced on us from the requirement of twisted Poincaré invariance.

The interaction Hamiltonian is built out of fields. We need a multiplication map to write down a Hamiltonian density starting from fields, as it is a scalar function of just one variable. Also in order to have twisted Poincaré invariance, one has to ensure that the Hamiltonian density transforms like a scalar field. This will only happen if we choose a star product (twisted multiplication map) between the fields to write down the Hamiltonian density. Hence a generic interaction Hamiltonian density involving just one hermitean spin zero field (for simplicity) is

$$\mathcal{H}_I(x) = \Phi(x) * \Phi(x) * \cdots * \Phi(x) \quad (4.1)$$

where $\Phi(x)$ obeys twisted statistics. This form of Hamiltonian and the twisted statistics of the fields is all that is needed to show that there is no UV-IR mixing in this theory [3,5].

We remark that the Hamiltonian used in [7] which uses conventional statistics, is not invariant under the twisted action of the Poincaré group.

The "S-operator" of the theory is the same as the "S-operator" of the corresponding commutative model, while the asymptotic fields obey the twisted statistics. (But this statement is not true when gauge fields are present [8].)

We again emphasize that the above form of the interaction Hamiltonian density is the only choice possible if the theory is to be twisted Lorentz invariant. The Hamiltonian must commute with the generators of the symmetry in order for the dynamics to be invariant. In this regard, both the star-product and twisted statistics are essential and an interaction with only the star-product and usual statistics of fields is not invariant under the twisted Poincaré symmetry.

V. ON THE INVARIANCE OF CORRELATION FUNCTIONS

A. The twisted action on the tensor product of plane waves

As a preliminary to the calculations, we first consider the actions of the twisted coproduct of the Poincaré group on the tensor products of plane waves.

On a single plane wave, the Lorentz transformation Λ and translation P_μ acts according to

$$(\Lambda e_p)(x) = e_p(\Lambda^{-1}x) = e_{\Lambda p}(x),\qquad (P_\mu e_p)(x) = -p_\mu e_p(x) \tag{5.1}$$

where we used $\Lambda^{-1} = \Lambda^T$ and $P_\mu = -i\partial_\mu$. Hence

$$\Lambda e_p = e_{\Lambda p}, \qquad \partial_\mu e_p = -ip_\mu e_p. \tag{5.2}$$

Let U denote the representation of the (enveloping algebra of the) Poincaré group on arbitrary tensor products of plane waves. The latter respond to translations in the usual manner, so we focus on Lorentz transformations Λ. On e_k, the action of $U(\Lambda)$ is as in (5.1):

$$U(\Lambda)e_k = e_{\Lambda k}. \tag{5.3}$$

On $e_{k_1} \otimes e_{k_2}$, we must find the action using the coproduct:

$$\begin{aligned} U(\Lambda)e_{k_1}\otimes e_{k_2} &= \Delta_\theta(\Lambda)e_{k_1}\otimes e_{k_2} \\ &= e^{-(i/2)\partial_\mu\theta^{\mu\nu}\otimes\partial_\nu}(\Lambda\otimes\Lambda)e^{(i/2)\partial_\mu\theta^{\mu\nu}\otimes\partial_\nu}e_{k_1}\otimes e_{k_2} \\ &= e_{\Lambda k_1}\otimes \underbrace{e^{-(1/2)(\Lambda k_1)_\mu\theta^{\mu\nu}\partial_\nu}\Lambda e^{(1/2)k_{1\mu}\theta^{\mu\nu}\partial_\nu}}_{\Lambda_1}e_{k_2} \\ &= e^{(i/2)k_1\cdot\delta_\Lambda\theta\cdot k_2}e_{\Lambda k_1}\otimes e_{\Lambda k_2}, \end{aligned} \tag{5.4}$$

where

$$k_1\cdot\delta_\Lambda\theta\cdot k_2 \equiv k_{1\mu}(\delta_\Lambda\theta)^{\mu\nu}k_{2\nu}, \qquad \delta_\Lambda\theta \equiv \Lambda^{-1}\theta\Lambda - \theta. \tag{5.5}$$

The action on $e_{k_1}\otimes e_{k_2}\otimes e_{k_3}$ is found using the coproduct on Λ_1:

$$\begin{aligned} \Delta_\theta(\Lambda_1) = {}& (e^{-(1/2)(\Lambda k_1)_\mu\theta^{\mu\nu}\partial_\nu}\otimes e^{-(1/2)(\Lambda k_1)_\mu\theta^{\mu\nu}\partial_\nu}) \\ &\times(e^{-(i/2)\partial_\mu\theta^{\mu\nu}\otimes\partial_\nu}\Lambda\otimes\Lambda e^{(i/2)\partial_\mu\theta^{\mu\nu}\otimes\partial_\nu}) \\ &\times(e^{(1/2)k_{1\mu}\theta^{\mu\nu}\partial_\nu}\otimes e^{(1/2)k_{1\mu}\theta^{\mu\nu}\partial_\nu}). \end{aligned} \tag{5.6}$$

It gives

$$U(\Lambda)e_{k_1}\otimes e_{k_2}\otimes e_{k_3} = e_{\Lambda k_1}\otimes\Delta_\theta(\Lambda_1)(e_{k_2}\otimes e_{k_3}) \tag{5.7}$$

where

$$\begin{aligned} \Delta_\theta(\Lambda_1)(e_{k_2}\otimes e_{k_3}) &= e^{(i/2)k_1\cdot\delta_\Lambda\theta\cdot k_2}e_{\Lambda k_2}\otimes\Lambda_2 e_{k_3}, \\ \Lambda_2 &= e^{-(1/2)(\Lambda k_1+\Lambda k_2)_\mu\theta^{\mu\nu}\partial_\nu} \\ &\quad\times\Lambda e^{(1/2)(k_1+k_2)_\mu\theta^{\mu\nu}\partial_\nu}. \end{aligned} \tag{5.8}$$

Thus

$$U(\Lambda)e_{k_1}\otimes e_{k_2}\otimes e_{k_3} = e^{(i/2)k_1\cdot\delta_\Lambda\theta\cdot k_2+(i/2)(k_1+k_2)\cdot\delta_\Lambda\theta\cdot k_3}e_{\Lambda k_1}\otimes e_{\Lambda k_2}\otimes e_{\Lambda k_3}. \tag{5.9}$$

The action on $e_{k_1}\otimes e_{k_2}\otimes e_{k_3}\otimes e_{k_4}$ is found by splitting Λ_2 again with a Δ_θ. In this way we see that in general,

$$U(\Lambda)e_{k_1}\otimes e_{k_2}\ldots\otimes e_{k_N} = e^{(i/2)k_1\cdot\delta_\Lambda\theta\cdot k_2+(i/2)(k_1+k_2)\cdot\delta_\Lambda\theta\cdot k_3+\ldots(k_1+k_2\ldots+k_{N-1})\cdot\delta_\Lambda\cdot\theta k_N}e_{\Lambda k_1}\otimes e_{\Lambda k_2}\ldots\otimes e_{\Lambda k_N}. \tag{5.10}$$

B. Correlation functions of NCQFT with untwisted statistics

Consider the scalar field theory on the GM plane with the Lagrangian (density)

$$\mathcal{L}_* = \frac{1}{2}\partial_\mu\Phi * \partial^\mu\Phi - \frac{1}{2}m^2\Phi*\Phi - \frac{\lambda}{4!}\Phi*\Phi*\Phi*\Phi, \tag{5.11}$$

where $\Phi^\dagger = \Phi$. Since statistics is not twisted, the annihilation and creation operators c_p, $c_p^\dagger$ of Φ are those for $\theta^{\mu\nu} = 0$.

The correlation functions of (5.11) are not Lorentz-invariant under the twisted coproduct. It is enough to prove this result for the free field theory where $\lambda = 0$.

The correlation functions for the product of an odd number of fields is zero. We show now that the four-point function is not Lorentz-invariant under the twisted coproduct. That can be adapted to show that the two-point function *is* Lorentz invariant. (Translational invariance is preserved by both untwisted and twisted statistics.)

The scalar field has the mode expansion

$$\Phi(x) = \int\frac{d^3p}{(2\pi)^{3/2}(2p_0)}(c_p e_p(x) + c_p^\dagger e_{-p}(x)) \tag{5.12}$$

where $p_0 = +\sqrt{|\vec{p}|^2+m^2}$ and c_p and $c_p^\dagger$ are the annihilation-creation operators for $\theta^{\mu\nu} = 0$:

$$[c_p, c_k] = 0 = [c_p^\dagger, c_k^\dagger], \qquad [c_k, c_k^\dagger] = 2p_0\delta^3(p-k). \tag{5.13}$$

The four-point function in this case, with no statistics twist, is

$$\langle 0|\Phi(x_1)\Phi(x_2)\Phi(x_3)\Phi(x_4)|0\rangle = D(x_1-x_2)D(x_3-x_4) + D(x_1-x_3)D(x_2-x_4) + D(x_1-x_4)D(x_2-x_3) \equiv I + II + III, \tag{5.14}$$

$$D(x) = \int \frac{d^3p}{(2\pi)^3(2p_0)} e^{-ip\cdot x} = D(\Lambda x). \tag{5.15}$$

We now show that I and III are invariant (for the twisted coproduct), but not II.

Consider I:

$$I = \frac{1}{(2\pi)^6}\int\Big(\prod_i \frac{d^3p_i}{(2p_{i0})}\Big) e_{p_1}(x_1)e_{-p_2}(x_2)e_{p_3}(x_3)e_{-p_4}(x_4) \times (2p_{10})(2p_{30})\delta^3(p_1-p_2)\delta^3(p_3-p_4). \tag{5.16}$$

Applying (5.10) with $k_1 = p_1$, $k_2 = -p_2$, $k_3 = p_3$, $k_4 = -p_4$, we find that the phase in (5.10) becomes 1 because of the δ-functions and that

$$\Lambda\colon I \to D(\Lambda^{-1}(x_1-x_2))D(\Lambda^{-1}(x_3-x_4)) = I. \tag{5.17}$$

A similar calculation shows the Lorentz invariance of III.

Now consider II:

$$II = \frac{1}{(2\pi)^6}\int\Big(\prod_i \frac{d^3p_i}{(2p_{i0})}\Big) e_{p_1}(x_1)e_{p_2}(x_2)e_{-p_3}(x_3)e_{-p_4}(x_4) \times (2p_{10})(2p_{20})\delta^3(p_1-p_3)\delta^3(p_2-p_4). \tag{5.18}$$

So with $k_1 = p_1$, $k_2 = p_2$, $k_3 = -p_3$, $k_4 = -p_4$ the phase becomes $e^{(i/2)p_1\cdot\delta_\Lambda\theta\cdot p_2}$ and

$$\Lambda\colon II \to \int \frac{d^3p_1 d^3p_2}{(2\pi)^6(2p_{10})(2p_{20})} e^{ip_1\cdot\delta_\Lambda\theta\cdot p_2} e^{i(\Lambda p_1)\cdot(x_1-x_3)} \times e^{i(\Lambda p_2)\cdot(x_2-x_4)} \neq II. \tag{5.19}$$

It is not Lorentz-invariant.

C. Correlation functions of NCQFT with twisted statistics

In this case the free field is

$$\Phi(x) = \int \frac{d^3p}{(2\pi)^{3/2}(2p_0)}(a_p e_p(x) + a_p^\dagger e_{-p}(x)). \tag{5.20}$$

Let $\mathcal{P}_\mu$ be the Fock space momentum operator:

$$\mathcal{P}_\mu = \int \frac{d^3p}{2p_0} p_\mu c_p^\dagger c_p. \tag{5.21}$$

Then, as shown in [4,9], the operators a_p, $a_p^\dagger$ can be written as follows:

$$a_p = c_p e^{-(i/2)p_\mu\theta^{\mu\nu}\mathcal{P}_\nu}, \qquad a_p^\dagger = c_p^\dagger e^{+(i/2)p_\mu\theta^{\mu\nu}\mathcal{P}_\nu}. \tag{5.22}$$

Using (5.16) and (5.18), we calculate the four-point function with twisted statistics:

$$\langle 0|\Phi(x_1)\Phi(x_2)\Phi(x_3)\Phi(x_4)|0\rangle = I + III + \frac{1}{(2\pi)^6}\int\prod_i \frac{d^3p_i}{(2p_{i0})} e^{ip_{1\mu}\theta^{\mu\nu}p_{2\nu}} e_{p_1}(x_1) \times e_{-p_2}(x_2)e_{p_3}(x_3)e_{-p_4}(x_4) \times (2p_{10})(2p_{20}) \times \delta^3(p_1-p_3)\delta^3(p_2-p_4). \tag{5.23}$$

$$\equiv I + III + II' \tag{5.24}$$

where I and III are Poincaré invariant as shown before. As for II', we find, using (5.10) with $k_1 = p_1$, $k_2 = -p_2$, $k_3 = p_3$, $k_4 = -p_4$ and the δ-functions,

$$\Lambda\colon II' \to \frac{1}{(2\pi)^6}\int\prod_i \frac{d^3p_i}{(2p_{i0})} e_{\Lambda p_1}(x_1)e_{-\Lambda p_2}(x_2)e_{\Lambda p_3}(x_3) \times e_{-\Lambda p_4}(x_4) e^{ip_{1\mu}\theta^{\mu\nu}p_{2\nu}} e^{ip_1\cdot\delta_\Lambda\theta\cdot p_2}(2p_{10}) \times (2p_{20})\delta^3(p_1-p_3)\delta^3(p_2-p_4). \tag{5.25}$$

Since

$$e^{ip_{1\mu}\theta^{\mu\nu}p_{2\nu}} e^{ip_1\cdot\delta_\Lambda\theta\cdot p_2} = e^{i(\Lambda p_1)_\mu\theta^{\mu\nu}(\Lambda p_2)_\nu} \tag{5.26}$$

the Poincaré invariance of II' also follows. The phase $e^{ip_{1\mu}\theta^{\mu\nu}p_{2\nu}}$ in (5.23) which comes from twisted statistics is essential to reach this conclusion.

VI. FUNCTIONAL INTEGRAL

We saw above that in order to have twisted Poincaré invariance in a quantum theory, we must also have twisted statistics. This has implications for a functional integral formulation of the quantum theory too. This is because statistics of the fields is an input in a functional integral. For example, in the case of usual fermions, statistics is not derived from functional integral, but is rather inferred from other considerations and then built into the functional integral by use of anticommuting classical fields.

Similarly, in order to construct a functional integral which gives a twisted Poincaré invariant quantum theory, we must use the correct statistics as an input and construct the functional integral out of classical fields which obey the twisted statistics. In particular its full measure consists of tensor products of individual measures at different points and the individual measures must obey twisted statistics among themselves in order for the total measure to be Poincaré invariant. This again is in analogy to the case of fermions, where individual measures anticommute among themselves. We will not go here into the full details of the construction of the functional integral which gives the twisted quantum field theory. It has been done by Oeckl [3]. It will suffice here to show that the conventional functional integral does not give a twisted Poincaré invariant theory.

The following functional integral was considered by [10] and claimed to be twist-Poincaré invariant:

$$W = \int \prod_x \mathcal{D}(\phi(x)) e^{i \int d^4x \mathcal{L}_*(x)} \tag{6.1}$$

where $\mathcal{L}_*$ is, for example, the star-Lagrangian (density)

$$\mathcal{L}_*(x) = \frac{1}{2}\partial_\mu \phi(x) * \partial^\mu \phi(x) - \frac{1}{2} m^2 \phi(x) * \phi(x) - \frac{\lambda}{4!} \phi(x) * \phi(x) * \phi(x) * \phi(x) \tag{6.2}$$

and $\mathcal{D}(\phi(x))$ is the usual measure.

With the functional integral defined with this measure, we obtain conventional quantization of noncommutative field theory with no statistics twist, and its Feynman rules.

But this measure is not invariant under the twisted Poincaré group. We can show this by a simple argument. Consider

$$\int \prod_x \mathcal{D}(\phi(x)) \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) e^{i\int d^4x \mathcal{L}_*(x)} = \langle 0|T\{\Phi(x_1)\Phi(x_2)\Phi(x_3)\Phi(x_4)\}|0\rangle. \tag{6.3}$$

It is enough to consider $\lambda = 0$. It is clear that if the measure is not invariant for $\lambda = 0$, turning on the interactions cannot suddenly make it invariant. Let us suppose for convenience that $x_1^0 > x_2^0 > x_3^0 > x_4^0$. Then

$$\langle 0|T\{\Phi(x_1)\Phi(x_2)\Phi(x_3)\Phi(x_4)\}|0\rangle = \langle 0|\Phi(x_1)\Phi(x_2)\Phi(x_3)\Phi(x_4)|0\rangle. \tag{6.4}$$

which is the same as (5.14). But we saw above that

$$\langle 0|\Phi(x_1)\Phi(x_2)\Phi(x_3)\Phi(x_4)|0\rangle \neq \Lambda \triangleright \langle 0|\Phi(x_1)\Phi(x_2)\Phi(x_3)\Phi(x_4)|0\rangle. \tag{6.5}$$

Hence it follows that

$$\int \prod_x \mathcal{D}(\phi(x)) \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) e^{i\int d^4x \mathcal{L}_*(x)} \neq \int \prod_x \mathcal{D}(\phi(x)) \Lambda \triangleright (\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)) e^{i\int d^4x \mathcal{L}_*(x)} \tag{6.6}$$

showing that the measure is not twist-Poincaré invariant.

VII. LOCALITY

A. $\theta^{\mu\nu} \neq 0$, untwisted statistics

The conventional quantization of a scalar field on the noncommutative plane leads to nonlocal physics. However this nonlocality is due to nonlocal interaction terms and does not show up in the free theory. As remarked earlier the free theory is identical to the scalar field theory for $\theta^{\mu\nu} = 0$.

B. $\theta^{\mu\nu} \neq 0$, twisted statistics

The situation is quite different when one quantizes using twisted statistics. In this case, even the free theory is nonlocal. We have

$$\begin{aligned}
[\Phi(x), \Phi(y)] = \int \frac{d^3p d^3k}{(2\pi)^3 (2p_0)(2k_0)} &\times [e^{-i(p\cdot x + k\cdot y)}(1 - e^{-i\theta^{\mu\nu}p_\mu k_\nu}) a_p a_k \\
&+ e^{i(p\cdot x + k\cdot y)}(1 - e^{-i\theta^{\mu\nu}p_\mu k_\nu}) a_p^\dagger a_k^\dagger \\
&+ e^{-i(p\cdot x - k\cdot y)}\{(1 - e^{i\theta^{\mu\nu}p_\mu k_\nu}) a_p a_k^\dagger \\
&- (2p_0)\delta^3(p-k)\} \\
&+ e^{i(p\cdot x - k\cdot y)}\{(1 - e^{i\theta^{\mu\nu}p_\mu k_\nu}) a_p^\dagger a_k \\
&+ (2p_0)\delta^3(p-k)\}]
\end{aligned} \tag{7.1}$$

This operator is not zero when x and y are spacelike separated. For example, we can calculate it between two single-particle momentum eigenstates $|q\rangle$ and $|r\rangle$. We have

$$\begin{aligned}
\langle q|[\Phi(x), \Phi(y)]|r\rangle = (e^{i\theta^{\mu\nu}q_\mu r_\nu} - 1)(e^{-ir\cdot x + iq\cdot y} - e^{iq\cdot x - ir\cdot y}) \\
+ (2q_0)\delta^3(q-r) \\
\times [D(x-y) - D(y-x)]
\end{aligned} \tag{7.2}$$

where $D(x-y)$ was defined in (5.15). The last two terms together vanish for spacelike separations, but the first term is in general nonzero for $q \neq r$.

Although the free theory is (twisted) Poincaré invariant, it is nonlocal. Hence the spin-statistics theorem does not apply to it and there is no internal inconsistency coming from this theorem.

VIII. ON TWISTED TENSOR PRODUCT

In [10], it has been suggested that the $*$-product and the twist of statistics are one and the same.

We feel that this remark is incorrect. It is well-known in Hopf algebra theory [6] that the coproduct on a (quasitriangular) Hopf algebra is associated with an "R-matrix" and that the latter fixes statistics. In our case, $R = \mathcal{F}^{-2}$ and that gives the representation of the permutation group via (3.13).

Incidentally, a "twisted" tensor product has been used in [10] in connection with the Drinfel'd twist. Its connection to the $*$-product is vague at best. It leads to twisted statistics, but not the correct one. We can see this as follows.

The twisted tensor product considered is

$$\Phi_0^{(+)} \otimes_T \Phi_0^{(+)} \equiv e^{(i/2)\partial_\mu \otimes \theta^{\mu\nu}\partial_\nu} \Phi_0^{(+)} \otimes \Phi_0^{(+)} \tag{8.1}$$

where the field $\Phi_0^{(+)}$ is the creation part (say) of a free field constructed from the standard creation and annihilation operators in the usual manner. We have,

$$\Phi_0^{(+)}(x)\Phi_0^{(+)}(y) = \Phi_0^{(+)}(y)\Phi_0^{(+)}(x) \tag{8.2}$$

so that

$$(\Phi_0^{(+)} \otimes_T \Phi_0^{(+)})(x, y) = (e^{(i/2)\partial_\mu \otimes \theta^{\mu\nu}\partial_\nu})(\Phi_0^{(+)} \otimes \Phi_0^{(+)})(x, y) \tag{8.3}$$

$$= \exp\left(\frac{i}{2}\frac{\partial}{\partial x^\mu}\theta^{\mu\nu}\frac{\partial}{\partial y^\nu}\right)\Phi_0^{(+)}(x)\Phi_0^{(+)}(y) \tag{8.4}$$

$$= \exp\left(-\frac{i}{2}\frac{\partial}{\partial y^\mu}\theta^{\mu\nu}\frac{\partial}{\partial x^\nu}\right)\Phi_0^{(+)}(y)\Phi_0^{(+)}(x) \tag{8.5}$$

$$= e^{-i\partial_\mu\otimes\theta^{\mu\nu}\partial_\nu}(\Phi_0^{(+)}\otimes_T\Phi_0^{(+)})(y, x) \tag{8.6}$$

This does not agree with (3.23).

ACKNOWLEDGMENTS

The work of A. P. B., B. Q. and A. P. is supported in part by the DOE under grant number DE-FG02-85ER40231.

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