

Spin-2 particles in gravitational fields

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We give a solution of the wave equation for massless, or massive spin-2 particles propagating in a gravitational background. The solution is covariant, gauge-invariant and exact to first order in the background gravitational field. The background contribution is confined to a phase factor from which geometrical and physical optics can be derived. The phase also describes Mashhoon's spin-rotation coupling and, in general, the spin-gravity interaction.

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I. INTRODUCTION

In a recent paper, Shen [1] has extended Mashhoon's spin-rotation coupling [2,3] to include the coupling of a graviton's spin to a weak gravitomagnetic field and has stressed the role of the self-interaction of the gravitational field itself. Ramos and Mashhoon [4] have then studied the effect of the helicity-gravitomagnetic field coupling on weak gravitational waves and shown that a gravitational Skrotskii effect [5] exists. They further show that the Skrotskii rotation angle is twice that expected for electromagnetic waves.

The purpose of this paper is to extend some of the results of these authors to include any type of weak inertial and gravitational fields and massive as well as massless spin-2 particles and to provide a general framework for the study of spin-2 particles in external inertial and gravitational fields. This is accomplished by solving the equation

$$\nabla_\alpha \nabla^\alpha \Phi_{\mu\nu} + m^2 \Phi_{\mu\nu} = 0, \quad (1.1)$$

where $m \geq 0$ is the mass of the particle and ∇_α indicates covariant differentiation. We use units $\hbar = c = 1$.

The solution of covariant wave equations for scalar [6], spin-1/2 and spin-1 particles [7–9] yields in general meaningful insights into aspects of the interaction of quantum systems with gravity whenever the gravitational field need not be quantized. The interaction of quantum systems with inertial and gravitational fields produces quantum phases. Though these are in general path-dependent, phase differences are observable, in principle, by means of Earth bound, or space interferometers, or by gravitational lensing.

The choice of (1.1) requires some justification. The propagation equations of higher spin fields contain in general curvature dependent terms that make the formulation of these fields particularly difficult when $m = 0$ [10]. For spin-2 fields, the simplest equation of propagation used in lensing is derived in [11] and is given by

$$\nabla_\alpha \nabla^\alpha \Phi_{\mu\nu} + 2R_{\alpha\mu\beta\nu} \Phi^{\alpha\beta} = 0. \quad (1.2)$$

The second term in (1.2) is localized in a region surrounding the lens that is small relative to the distances between lens, source and observer and is neglected when the wavelength λ associated with $\Phi_{\mu\nu}$ is smaller than the typical radius of curvature of the gravitational background [12]. For the metric used in Secs. III and IV, we find, in particular, that the curvature term may be neglected when $\lambda \ll \sqrt{\frac{\rho^3}{r_g}}$, where ρ is the distance from the lens and r_g its Schwarzschild radius. This condition is satisfied in most lensing problems. It also is adequate to treat the problems discussed in [1,4].

Equation (1.2) can be generalized to include a mass term

$$\nabla_\alpha \nabla^\alpha \Phi_{\mu\nu} + 2R_{\alpha\mu\beta\nu} \Phi^{\alpha\beta} + m^2 \Phi_{\mu\nu} = 0. \quad (1.3)$$

Here too the curvature term is smaller than the mass term whenever $m > 1/r_g$. For Earth bound experiments $r_g = 2GM_\oplus/R_\oplus$ and the curvature term becomes negligible for $m > 2.5 \cdot 10^{-6}$ GeV. In view of the applications discussed below, the curvature term is therefore neglected and (1.3) reduces to our initial equation (1.1).

The background gravitational field is represented by the metric deviation $\gamma_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu}$, and the Minkowski metric $\eta_{\mu\nu}$ has signature -2 . To first order in $\gamma_{\mu\nu}$, (1.1) can be written in the form

$$(\eta^{\alpha\beta} - \gamma^{\alpha\beta}) \partial_\alpha \partial_\beta \Phi_{\mu\nu} + R_{\sigma\mu} \Phi_\nu^\sigma + R_{\sigma\nu} \Phi_\mu^\sigma - 2\Gamma_{\mu\alpha}^\sigma \partial^\alpha \Phi_{\nu\sigma} - 2\Gamma_{\nu\alpha}^\sigma \partial^\alpha \Phi_{\mu\sigma} + m^2 \Phi_{\mu\nu} = 0, \quad (1.4)$$

where $R_{\mu\beta} = -(1/2) \partial_\alpha \partial^\alpha \gamma_{\mu\beta}$ is the linearized Ricci tensor of the background metric and $\Gamma_{\sigma\mu,\alpha} = 1/2(\gamma_{\alpha\sigma,\mu} + \gamma_{\alpha\mu,\sigma} - \gamma_{\sigma\mu,\alpha})$ is the corresponding Christoffel symbol of the first kind.

The plan of the paper is as follows. We give the solution of (1.1) and illustrate some of its properties in Sec. II. In Sec. III we discuss spin-gravity coupling and geometrical

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optics. Section IV is concerned with wave effects in particle optics. In Sec. V we summarize and discuss the results.

II. SOLUTION OF THE SPIN-2 WAVE EQUATION

It is easy to prove, by direct substitution, that a solution of (1.4), exact to first order in $\gamma_{\mu\nu}$, is represented by

$$\begin{aligned} \Phi_{\mu\nu} = & \phi_{\mu\nu} - \frac{1}{4} \int_P^x dz^\lambda (\gamma_{\alpha\lambda,\beta}(z) - \gamma_{\beta\lambda,\alpha}(z)) \\ & \times [(x^\alpha - z^\alpha) \partial^\beta \phi_{\mu\nu}(x) - (x^\beta - z^\beta) \partial^\alpha \phi_{\mu\nu}(x)] \\ & + \frac{1}{2} \int_P^x dz^\lambda \gamma_{\alpha\lambda}(z) \partial^\alpha \phi_{\mu\nu}(x) \\ & + \int_P^x dz^\lambda \Gamma_{\mu\lambda,\sigma}(z) \phi_\nu^\sigma(x) + \int_P^x dz^\lambda \Gamma_{\nu\lambda,\sigma}(z) \phi_\mu^\sigma(x), \end{aligned} \quad (2.1)$$

where $\phi_{\mu\nu}$ satisfies the field-free equation

$$(\partial_\alpha \partial^\alpha + m^2) \phi_{\mu\nu}(x) = 0, \quad (2.2)$$

and the Lanczos-DeDonder gauge condition $\gamma_{\alpha\nu,\nu} - \frac{1}{2} \gamma_{\sigma,\alpha}^\sigma = 0$ has been used. In (2.1) P is a fixed reference point and x a generic point along the particle's worldline.

The particular case $m = 0$ yields the solution $\Phi_{\mu\nu}$ for a linearized gravitational field $\phi_{\mu\nu}$ propagating in a background gravitational field $\gamma_{\mu\nu}$.

The solution (2.1) applies equally well when $\phi_{\mu\nu}$ is a plane wave or a wave packet solution of (2.2). No additional approximations are made regarding $\Phi_{\mu\nu}$ that obviously satisfies the Eq. (2.2) when $\gamma_{\mu\nu}$ vanishes.

We show below that, as in [6,7], the solution is manifestly covariant. It also is completely gauge invariant and the effect of gravitation is entirely contained in the phase of the wave function. In fact, (2.1) can be written in the form $\Phi_{\mu\nu} = \exp(-i\xi) \phi_{\mu\nu} \simeq (1 - i\xi) \phi_{\mu\nu}$ or, explicitly, as

$$\begin{aligned} \Phi_{\mu\nu}(x) = & \phi_{\mu\nu}(x) + \frac{1}{2} \int_P^x dz^\lambda \gamma_{\alpha\lambda}(z) \partial^\alpha \phi_{\mu\nu}(x) \\ & - \frac{1}{2} \int_P^x dz^\lambda (\gamma_{\alpha\lambda,\beta}(z) - \gamma_{\beta\lambda,\alpha}(z)) \\ & \times [(x^\alpha - z^\alpha) \partial^\beta + iS^{\alpha\beta}] \phi_{\mu\nu}(x) \\ & - \frac{i}{2} \int_P^x dz^\lambda \gamma_{\beta\sigma,\lambda}(z) T^{\beta\sigma} \phi_{\mu\nu}(x), \end{aligned} \quad (2.3)$$

where

$$\begin{aligned} S^{\alpha\beta} \phi_{\mu\nu} \equiv & \frac{i}{2} (\delta_\sigma^\alpha \delta_\nu^\beta \delta_\mu^\tau - \delta_\sigma^\beta \delta_\mu^\alpha \delta_\nu^\tau + \delta_\sigma^\alpha \delta_\nu^\beta \delta_\mu^\tau \\ & - \delta_\sigma^\beta \delta_\nu^\alpha \delta_\mu^\tau) \phi_\tau^\sigma \\ T^{\beta\sigma} \phi_{\mu\nu} \equiv & i(\delta_\mu^\beta \delta_\nu^\sigma + \delta_\nu^\beta \delta_\mu^\sigma) \phi_\tau^\sigma. \end{aligned} \quad (2.4)$$

From $S^{\alpha\beta}$ one constructs the rotation matrices $S_i = -2i\epsilon_{ijk} S^{jk}$ that satisfy the commutation relations $[S_i, S_j] = i\epsilon_{ijk} S_k$. The spin-gravity interaction is therefore

contained in the term

$$\begin{aligned} \Phi'_{\mu\nu} \equiv & -\frac{i}{2} \int_P^x dz^\lambda (\gamma_{\alpha\lambda,\beta} - \gamma_{\beta\lambda,\alpha}) S^{\alpha\beta} \phi_{\mu\nu}(x) \\ = & \frac{1}{2} \int_P^x dz^\lambda [(\gamma_{\sigma\lambda,\mu} - \gamma_{\mu\lambda,\sigma}) \phi_\nu^\sigma \\ & + (\gamma_{\sigma\lambda,\nu} - \gamma_{\nu\lambda,\sigma}) \phi_\mu^\sigma]. \end{aligned} \quad (2.5)$$

The solution (2.1) is invariant under the gauge transformations $\gamma_{\mu\nu} \rightarrow \gamma_{\mu\nu} - \xi_{\mu,\nu} - \xi_{\nu,\mu}$, where ξ_μ are small quantities of the first order. If, in fact, we choose a closed integration path Γ , Stokes theorem transforms the first three integrals of (2.3) into $1/4 \int_\Sigma d\sigma^{\lambda\kappa} R_{\lambda\kappa\alpha\beta} (L^{\alpha\beta} + S^{\alpha\beta}) \phi_{\mu\nu}$, where Σ is the surface bound by Γ , $J^{\alpha\beta} = L^{\alpha\beta} + S^{\alpha\beta}$ is the total angular momentum of the particle and $R_{\lambda\kappa\alpha\beta} = 1/2(\gamma_{\lambda\beta,\kappa\alpha} + \gamma_{\kappa\alpha,\lambda\beta} - \gamma_{\lambda\alpha,\kappa\beta} - \gamma_{\kappa\beta,\lambda\alpha})$ is the linearized Riemann tensor. For the same path Γ the integral involving $T^{\beta\sigma}$ in (2.3) vanishes. It behaves like a gauge term and may therefore be dropped. For the same closed paths, (2.3) gives

$$\Phi_{\mu\nu} \simeq (1 - i\xi) \phi_{\mu\nu} = \left(1 - \frac{i}{4} \int_\Sigma d\sigma^{\lambda\kappa} R_{\lambda\kappa\alpha\beta} J^{\alpha\beta}\right) \phi_{\mu\nu}, \quad (2.6)$$

which obviously is covariant and gauge invariant. For practical applications (2.1) is easier to use.

The phase ξ is sometimes referred to as gravitational Berry phase [13] because space-time plays in it the role of Berry's parameter space [14].

III. HELICITY-GRAVITY COUPLING AND GEOMETRICAL OPTICS

The helicity-rotation coupling for massless, or massive spin-2 particles follows immediately from the $S^{\alpha\beta}$ term in (2.3). In fact, the particle energy is changed by virtue of its spin by an amount given by the time integral of this spin term

$$\xi^{hr} = -\frac{1}{2} \int_P^x dz^0 (\gamma_{\alpha 0,\beta} - \gamma_{\beta 0,\alpha}) S^{\alpha\beta}, \quad (3.1)$$

that must then be applied to a solution of (2.2). For rotation about the x^3 -axis, $\gamma_{0i} = \Omega(y, -x, 0)$, we find $\xi^{hr} = -\int_P^x dz^0 2\Omega S^3$ and the energy of the particle therefore changes by $\pm 2\Omega$, where the factor ± 2 refers to the particle's helicity, as discussed by Ramos and Mashhoon [4]. Equation (3.1) extends their result to any weak gravitational, or inertial field.

The effect of (2.5) on $\phi_{\mu\nu}$ can be easily seen in the case of a gravitational wave propagating in the x -direction and represented by the components $\phi_{22} = -\phi_{33} = \epsilon_{23} \exp[ik(t-x)]$ and $\phi_{23} = \epsilon_{23} \exp[ik(t-x)]$. For an observer rotating about the x -axis the metric is $\gamma_{00} = -\Omega^2 r^2$, $\gamma_{11} = \gamma_{22} = \gamma_{33} = -1$, $\gamma_{0i} = \Omega(0, z, -y)$.

Then the two independent polarizations ϕ_{23} and $\phi_{22} - \phi_{33}$ are transformed by $S_{\alpha\beta}$ into $\Phi_{23} = -2\Omega(x^0 - x_P^0) \times (\phi_{22} - \phi_{33})/2$ and $1/2(\Phi_{22} - \Phi_{33}) = 2\Omega(x^0 - x_P^0)\phi_{23}$.

For closed integration paths and vanishing spin, (2.3) coincides with the solution of a scalar particle in a gravitational field, as expected. This proves the frequently quoted statement [15] that gravitational radiation propagating in a gravitational background is affected by gravitation in the same way that electromagnetic radiation is (when the photon spin is neglected).

The geometrical optics approximation follows immediately from (2.3) with $S_{\alpha\beta} = 0$, $T_{\alpha\beta} = 0$. We obtain from (2.1) and (2.3)

$$\begin{aligned} \Phi_{\mu\nu}^0 &= \phi_{\mu\nu}(x) + \frac{1}{2} \int_P^x dz^\lambda \gamma_{\alpha\lambda}(z) \partial^\alpha \phi_{\mu\nu}(x) \\ &\quad - \frac{1}{2} \int_P^x dz^\lambda (\gamma_{\alpha\lambda,\beta}(z) - \gamma_{\beta\lambda,\alpha}(z)) \\ &\quad \times [(x^\alpha - z^\alpha) \partial^\beta] \phi_{\mu\nu}(x). \end{aligned} \quad (3.2)$$

We first calculate the general relativistic deflection of a spin-2 particle in a gravitational field. It follows immediately from $\Phi_{\mu\nu}^0$. Assuming, for simplicity, that the spin-2 particles are massless and propagate along the z -direction, so that $k^\alpha \simeq (k, 0, 0, k)$, and $ds^2 = 0$ or $dt = dz$, using plane waves for $\phi_{\mu\nu}$ and writing

$$\begin{aligned} \chi &= k_\sigma x^\sigma - \frac{1}{4} \int_P^x dz^\lambda (\gamma_{\alpha\lambda,\beta}(z) - \gamma_{\beta\lambda,\alpha}(z)) \\ &\quad \times [(x^\alpha - z^\alpha) k^\beta - (x^\beta - z^\beta) k^\alpha] \\ &\quad + \frac{1}{2} \int_P^x dz^\lambda \gamma_{\alpha\lambda}(z) k^\alpha, \end{aligned} \quad (3.3)$$

we obtain the particle momentum

$$\begin{aligned} \tilde{k}_\sigma &= \frac{\partial \chi}{\partial x^\sigma} \equiv \chi_{,\sigma} \\ &= k_\sigma - \frac{1}{2} \int_P^x dz^\lambda (\gamma_{\sigma\lambda,\beta} - \gamma_{\beta\lambda,\sigma}) k^\beta + \frac{1}{2} \gamma_{\alpha\sigma} k^\alpha. \end{aligned} \quad (3.4)$$

It is easy to show from (3.4) that χ satisfies the eikonal equation $g^{\alpha\beta} \chi_{,\alpha} \chi_{,\beta} = 0$.

The calculation of the deflection angle is particularly simple if we choose the background metric

$$\gamma_{00} = 2U(r), \quad \gamma_{ij} = 2U(r) \delta_{ij}, \quad (3.5)$$

where $U(r) = -GM/r$ and $r = \sqrt{x^2 + y^2 + z^2}$, which is frequently used in gravitational lensing. For this metric, χ is given by

$$\begin{aligned} \chi &\simeq -\frac{k}{2} \int_P^x [(x - x') \phi_{,z'} dx' + (y - y') \phi_{,z'} dy' \\ &\quad - 2[(x - x') \phi_{,x'} + (y - y') \phi_{,y'}] dz'] + k \int_P^x dz' \phi. \end{aligned} \quad (3.6)$$

The space components of the momentum are therefore

$$\tilde{k}_1 = 2k \int_P^x \left(-\frac{1}{2} \frac{\partial U}{\partial z} dx + \frac{\partial U}{\partial x} dz \right), \quad (3.7)$$

$$\tilde{k}_2 = 2k \int_P^x \left(-\frac{1}{2} \frac{\partial U}{\partial z} dy + \frac{\partial U}{\partial y} dz \right), \quad (3.8)$$

$$\tilde{k}_3 = k(1 + U). \quad (3.9)$$

We then have

$$\tilde{\mathbf{k}} = \tilde{\mathbf{k}}_\perp + k_3 \mathbf{e}_3, \quad \tilde{\mathbf{k}}_\perp = k_1 \mathbf{e}_1 + k_2 \mathbf{e}_2, \quad (3.10)$$

where $\tilde{\mathbf{k}}_\perp$ is the component of the momentum orthogonal to the direction of propagation of the particles.

Since only phase differences are physical, it is convenient to choose the space-time path by placing the particle source at a distance very large relative to the dimensions of M , while the generic point is located at z along the z direction and $z \gg x, y$. Equations (3.7), (3.8), and (3.9) simplify to

$$\tilde{k}_1 = 2k \int_{-\infty}^z \frac{\partial U}{\partial x} dz = k \frac{2GM}{R^2} x \left(1 + \frac{z}{r} \right), \quad (3.11)$$

$$\tilde{k}_2 = 2k \int_{-\infty}^z \frac{\partial U}{\partial y} dz = k \frac{2GM}{R^2} y \left(1 + \frac{z}{r} \right), \quad (3.12)$$

$$\tilde{k}_3 = k(1 + U), \quad (3.13)$$

where $R = \sqrt{x^2 + y^2}$. By defining the deflection angle as

$$\tan \theta = \frac{\tilde{k}_\perp}{\tilde{k}_3}, \quad (3.14)$$

we find

$$\tan \theta \sim \theta \sim \frac{2GM}{R} \left(1 + \frac{z}{r} \right), \quad (3.15)$$

and, in the limit $z \rightarrow \infty$, we obtain the usual Einstein result

$$\theta_M \sim \frac{4GM}{R}. \quad (3.16)$$

The index of refraction can be derived from the known equation $n = \tilde{k}/\tilde{k}_0$. Choosing the direction of propagation of the particle along the $x^3 -$ axis, and using (3.4), we find

$$n \simeq 1 + \frac{1}{k_0} (\chi_{,3} - \chi_{,0}) - \frac{m^2}{2k_0^2} \left(1 - \frac{1}{k_0} \chi_{,0} \right) \quad (3.17)$$

and, again, for $k_0 \gg m$, or for vanishing m ,

while $k^2 = k \sin\theta^+$, $k^3 = k \cos\theta^+$, $R_1 = \sqrt{r_0^2 + b^{+2}}$, and the change in phase is

$$\begin{aligned}
 -\frac{\Delta\tilde{\phi}_{LO}}{GM} &= 2 \int_{r_L}^{r_L+R_1} dz^0 q_{LO}(z^0)^{-3/2} \\
 &\times (R_1 - z^0 + r_L + r_0 \cos\theta^+)(r_L + R_1 - z^0) \\
 &\times \left[k^2 \frac{\cos\theta^{+2}}{\sin\theta^+} + k^3 \frac{\sin\theta^{+2}}{\cos\theta^+} - k \right] \\
 &+ \int_{r_L}^{r_L+R_1} dz^0 q_{LO}^{-1/2} \\
 &\times [-k - k^2 \sin\theta^+ - k^3 \cos\theta^+]. \quad (4.5)
 \end{aligned}$$

For SP we find

$$\begin{aligned}
 U &= -\frac{GM}{q_{SP}(z^0)^{1/2}}, \\
 q_{SP}(z^0) &\equiv b^{-2} + (R - z^0)^2 - 2(R - z^0)b^- \cos\gamma, \quad (4.6)
 \end{aligned}$$

$k^2 = k \cos\gamma$, $k^3 = k \sin\gamma$, $\tan\gamma = D_{ds}/(s + b^-)$, $R = \sqrt{D_{ds}^2 + (s + b^-)^2}$, and the corresponding change in phase is

$$\begin{aligned}
 -\frac{\Delta\tilde{\phi}_{SP}}{GM} &= 2 \int_0^R dz^0 q_{SP}(z^0)^{-3/2} (R - z^0 - b^- \cos\gamma) \\
 &\times (R - z^0) \left[k^2 \frac{\sin\gamma^2}{\cos\gamma} + k^3 \frac{\cos\gamma^2}{\sin\gamma} + k \right] \\
 &+ \int_0^R dz^0 q_{SP}(z^0)^{-1/2} [-k - k^2 \cos\gamma - k^3 \sin\gamma]. \quad (4.7)
 \end{aligned}$$

Finally, for PO we get

$$\begin{aligned}
 U &= -\frac{GM}{q_{PO}(z^0)^{1/2}}, \\
 q_{PO}(z^0) &\equiv r_0^2 + (R_2 + R - z^0)^2 \\
 &- 2(R_2 + R - z^0)r_0 \cos\theta^-, \quad (4.8)
 \end{aligned}$$

$k^2 = -k \sin\theta^-$, $k^3 = k \cos\theta^-$, $R_2 = \sqrt{r_0^2 + b^{-2}}$, and the relative change in phase is

$$\begin{aligned}
 -\frac{\Delta\tilde{\phi}_{PO}}{GM} &= 2 \int_R^{R+R_2} dz^0 q_{PO}(z^0)^{-3/2} \\
 &\times (z^0 - R_2 - R + r_0 \cos\theta^-)(R + R_2 - z^0) \\
 &\times \left[-k^2 \frac{\cos\theta^{-2}}{\sin\theta^-} + k^3 \frac{\sin\theta^{-2}}{\cos\theta^-} - k \right] \\
 &+ \int_R^{R+R_2} dz^0 q_{PO}^{-1/2} \\
 &\times [-k + k^2 \sin\theta^- - k^3 \cos\theta^-]. \quad (4.9)
 \end{aligned}$$

The total change in phase is therefore $\Delta\tilde{\phi} = \Delta\tilde{\phi}_{SL} + \Delta\tilde{\phi}_{LO} - \Delta\tilde{\phi}_{SP} - \Delta\tilde{\phi}_{PO}$. All integrations in (4.3), (4.5), (4.7), and (4.9) can be performed exactly. All results can be expressed in terms of physical variables r_s , r_0 , b^+ , b^- , and s and lensing variables D_s , D_{ds} , D_d , θ^+ , θ^- , and β . The final result is

$$\begin{aligned}
 \Delta\tilde{\phi} &= \tilde{y} \left\{ \ln(-\sqrt{D_{ds}^2 + (s + b^-)^2} + b^- \cos\gamma + r_s) \right. \\
 &- \ln(b^-(1 + \cos\gamma)) + \ln(b^+(1 - \cos\varphi^+)) \\
 &- \ln(r_s - r_L - b^+ \cos\varphi^+) + \ln(b^- + r_0 \cos\theta^-) \\
 &- \sqrt{b^{-2} + r_0^2} - \ln(r_0(1 + \cos\theta^-)) \\
 &+ \ln(r_0(1 + \cos\theta^+)) \\
 &\left. - \ln(b^+ + r_0 \cos\theta^+ - \sqrt{b^{+2} + r_0^2}) \right\}, \quad (4.10)
 \end{aligned}$$

where $r_s^2 = b^{+2} + r_L^2 + 2b^+ r_L \cos\varphi^+$, $r_L^2 = D_{ds}^2 + (s - b^+)^2$, $\varphi^+ + \alpha^+ + \alpha^- + \gamma - \theta^+ - \theta^- = \pi$ and $\tilde{y} = 2GMk$. This result is exact and independent of the value of \tilde{y} .

A simple quantum mechanical calculation indicates that the probability of finding particles at O contains an oscillating term (two-image interference) that is proportional to $\cos^2 \Delta\tilde{\phi}/2$. In the particular case $b^+ \sim b^- \equiv b$, $\varphi^+ \sim \gamma$, $r_s \sim r_L \gg (b, s)$, $\theta^+ \sim \theta^- \equiv \theta$, $\alpha^+ \sim \alpha^- \equiv \alpha$, we obtain from (4.10), the expression

$$\Delta\tilde{\phi} \sim \tilde{y} \ln \frac{r_L(1 + \sin(\theta - \alpha))}{b \sin(\theta - \alpha)}, \quad (4.11)$$

which is approximate to terms of $O(b/r_L)$ and higher in the argument of the logarithm. The overall probability P_0 of finding particles at O is therefore

$$P_0 \propto \cos^2 \left\{ \frac{\tilde{y}}{2} \ln \left[\frac{r_L}{2b} \frac{(1 + \tan \frac{\theta - \alpha}{2})^2}{\tan \frac{\theta - \alpha}{2}} \right] \right\}, \quad (4.12)$$

which exhibits an oscillating behavior typical of combined interference and diffraction effects. Higher order terms in b/r_L would in general prevent the logarithmic term from diverging when $\theta \rightarrow \alpha$.

V. CONCLUSIONS

We have solved the wave equation (1.1) for massless and massive spin-2 particles propagating in a background gravitational field. The solution is exact to first order in $\gamma_{\mu\nu}$, is covariant and gauge-invariant and is known whenever a solution $\phi_{\mu\nu}$ of the free wave equation is known.

The external gravitational field, represented by the background metric, only appears in the phase of the wave function. It is precisely this phase that provides the general framework for the study of spin-2 particles.

The spin-gravity coupling and Mashhoon's helicity-rotation interaction follow from the gravity-induced phase, (2.5). The origin of (2.5) resides in the skew-symmetric part of the space-time connection terms in (2.1), while, in the case of fermions, it is the spinorial connection [7] that accounts for $S_{\alpha\beta}$. The spin term $S_{\alpha\beta}$ affects the polarization of a gravitational wave, as shown in Sec. III. It also plays a role in the collision of two gravitational waves. If these are represented by $\phi_{22} = -\phi_{33} = \varepsilon_{22} \exp[ik(t-x)]$, $\phi_{23} = \varepsilon_{23} \exp[ik(t-x)]$ and the gravitational background is a wave of the same polarization, but proceeding along the negative direction of the x -axis, then the corresponding metric is

$$ds^2 = 2\Phi'_{02}dx^0dx^2 + 2\Phi'_{03}dx^0dx^3 + 2\Phi'_{12}dx^1dx^2 + 2\Phi'_{13}dx^1dx^3 + 2(\Phi'_{23} + \Phi'_{23})dx^2dx^3, \quad (5.1)$$

where

$$\begin{aligned} \Phi'_{02} &= \frac{-ik}{2}[(\gamma_{22}\phi_{22} + \gamma_{32}\phi_{23})x^2 + (\gamma_{23}\phi_{23} + \gamma_{33}\phi_{23})x^3]; \\ \Phi'_{03} &= \frac{-ik}{2}[(\gamma_{22}\phi_{32} + \gamma_{32}\phi_{23})x^2 + (\gamma_{23}\phi_{32} + \gamma_{33}\phi_{33})x^3]; \\ \Phi'_{12} &= \frac{-ik}{2}[(\gamma_{22}\phi_{22} + \gamma_{32}\phi_{23})x^2 + (\gamma_{23}\phi_{22} + \gamma_{33}\phi_{23})x^3]; \\ \Phi'_{13} &= \frac{-ik}{2}[(\gamma_{22}\phi_{23} + \gamma_{32}\phi_{33})x^2 + (\gamma_{23}\phi_{32} + \gamma_{33}\phi_{33})x^3]; \\ \Phi'_{23} &= \phi_{22}(\gamma_{32} - \gamma_{33}), \end{aligned} \quad (5.2)$$

and the collision takes place at the origin of the coordinates. The metric (5.1) has a singularity at $x^2 = x^3 = 0$. More complete treatments of this problem show that this is a curvature singularity [17–20].

From the phase we have derived the geometrical optics of the particles and verified that their deflection is that predicted by Einstein. The gravitational background behaves as a material medium of index of refraction n given by (3.17).

Wave optics too can be extracted from the phase. We have derived an exact expression for the phase change $\Delta\tilde{\phi}$

given by (4.10) and have shown that (4.10) represents interference and diffraction effects. In gravitational lensing [21,22] and in the gravitational lensing of gravitational waves [12], wave effects for a point source depend on the parameter \tilde{y} which gives an indication of the maximum magnification of the wave flux, or, alternatively, of the number of Fresnel zones contributing to lensing. Different values of \tilde{y} require, in general, different approximations, or different solutions of the wave equation. In particular, diffraction effects are expected to be considerable when $\tilde{y} \approx 1$. In our approach, (4.10) holds true regardless of the value of \tilde{y} . Wave optics problems usually deal with spherical wave solutions of Helmholtz equation in which gravity appears in the form of a potential. The extension of our findings to include spherical wave solutions is, of course, allowed by (2.2), but results in additional terms in (3.2) and in a more cumbersome, but still exact final result. It is left to a future, specific application in which the planar configuration of Fig. 1 will be rescinded. In the present approach, however, gravity makes itself felt in a rather more subtle way than just through a single potential, as evidenced by (1.4).

The framework developed can also be used in the interferometry of atoms and molecules. A laboratory instrument capable of using coherent beams of atoms or molecules would go a long way in probing the interface between gravitational theories and quantum mechanics. For instance, the phase shift of a particle beam in the Lense-Thirring field of the Earth is [23]

$$\Delta\tilde{\phi}_{\text{LT}} = \frac{4G}{R_{\oplus}^3} J_{\oplus} m \ell^2, \quad (5.3)$$

where $J_{\oplus} = 2M_{\oplus}R_{\oplus}^2\Omega/5$ is the angular momentum of Earth (assumed spherical and homogeneous), R_{\oplus} its radius and ℓ the typical dimension of the interferometer. For neutron interferometers with $\ell \sim 10^2$ cm, we find $\Delta\tilde{\phi}_{\text{LT}} \sim 10^{-7}$ rad. The value of the phase difference increases with m and ℓ^2 . This suggests that the development and use of large, heavy particle interferometers would be particularly advantageous in attempts to measure gravitational effects. When (3.5) and (4.10) are used in the case of a square interferometer and extreme relativistic particles, we however obtain $\tilde{\phi} \approx GMk$, irrespective of the size of the interferometer. This is as expected. In fact, for the particular configuration of Fig. 1 (and unlike the problems considered in [6,16]), the gravitational flux of the source is completely contained in the integration path $SLOPS$ and $\Delta\tilde{\phi}$ cannot be made larger by increasing the dimensions of the interferometer.

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