

Cosmology of a universe with spontaneously broken Lorentz symmetryP. G. Ferreira,^{1,2} B. M. Gripcios,¹ R. Saffari,^{1,3} and T. G. Zlosnik¹¹*Astrophysics and Theoretical Physics, University of Oxford, Denys Wilkinson Building, Keble Road, Oxford OX1 3RH, United Kingdom*²*African Institute for Mathematical Sciences, 6-8 Melrose Road, Muizenberg 7945, South Africa*³*Institute for Advanced Studies in Basic Sciences, Gava Zang, P.O.Box 45195-1159 Zanjan, Iran*

(Received 13 November 2006; published 12 February 2007)

A self-consistent effective field theory of modified gravity has recently been proposed with spontaneous breaking of local Lorentz invariance. The symmetry is broken by a vector field with the wrong-sign mass term and it has been shown to have additional graviton modes and modified dispersion relations. In this paper we study the evolution of a homogeneous and isotropic universe in the presence of such a vector field with a minimum lying along the timelike direction. A plethora of different regimes is identified, such as accelerated expansion, loitering, collapse, and tracking.

DOI: [10.1103/PhysRevD.75.044014](https://doi.org/10.1103/PhysRevD.75.044014)

PACS numbers: 04.50.+h

I. INTRODUCTION

The evolution and current state of the Universe has become amenable to a number of astronomical and experimental probes. In the past few years, it has become possible to measure its geometry, expansion rate, and constituents with unprecedented precision. The recent measurement of the anisotropy and polarization of the cosmic microwave background with the Wilkinson Anisotropy Microwave Probe satellite has further improved constraints on cosmological parameters [1]. We now believe that we have identified an accurate and consistent model of the Universe based on the general theory of relativity.

However our current model relies on the existence of extremely exotic forms of energy: dark matter which clumps gravitationally but does not interact with light and dark energy which is repulsive under gravity. Furthermore, we must posit that 95% of the energy density of the Universe is taken up by these exotic components. There are proposals for their fundamental origin but as yet no compelling explanation.

An alternative possibility is that gravity is not what it seems and that general relativity (GR) should be modified on certain scales. A notable example of this is the relativistic theory of modified Newtonian dynamics [2,3]. In this theory, dark matter is replaced by a modification of the gravitational interaction, through the inclusion of a scalar and vector field. These extra fields can compensate for the absence of cold dark matter on galactic and galactic-cluster scales, and at the cosmological horizon. Other proposals involving additional metrics (e.g. [4]), additional functions of the Ricci scalar in the gravitational action (e.g. [5]), and combinations of new scalar, tensor, and vector fields (e.g. [6]) have been proposed. Typically these models are constructed with the goal of reproducing observation but with little or no basis on fundamental principles. Mostly they are plagued with inconsistencies at the quantum level (see for example [7]).

Recently, one of us has proposed an action for Einstein gravity coupled with a vector field that leads to modifications to conventional gravitational interactions [8]. The action is constructed strictly according to the rules for an effective field theory with a mass scale cutoff, M : all Lorentz-invariant terms containing the vector field and metric up to a predefined order in E/M (where E is the energy scale at which the theory is being considered) are included. The theory is manifestly self-consistent at energy scales below the cutoff M , and the usual problems of theories of modified gravity, such as ghosts [9], strong-coupling [10], and discontinuities [11] are absent. The vector field has a Lorentz-invariant mass term with the wrong sign, which leads to a vacuum expectation value for the vector field and spontaneous breaking of Lorentz invariance. As a result one finds that there are additional graviton modes and modified dispersion relations. Moreover, the new terms in the action lead to additional terms in the equations of motion which affect the dynamics of the Universe on very large scales. It is this latter aspect of the theory that we wish to explore in this paper.

Our work in this paper complements that presented in [12] where it has been shown that Lorentz-violating vector fields may slow down the expansion rate of the Universe. In their analysis, the authors consider a fixed-norm vector field, pinned down through a Lagrange multiplier term in the action. Furthermore, they examine a simplification of the theory presented in [13] by restricting themselves to quadratic terms in the vector field. In this paper we include higher order terms and allow the norm of the vector field to vary. A fixed-norm case will be considered as a limiting case where a coupling constant in the potential for the vector field becomes infinite.

The layout of this paper is as follows. In Sec. II we present the action and briefly discuss other attempts at studying Lorentz violation from vector fields. In Sec. III we write down the general equations of motion. We then specialize to the case of a homogeneous and isotropic

spacetime. As will become apparent, the parameter space of this theory is immense. There are effectively 6 coupling constants and 3 energy scales that play a role. We briefly discuss the range of parameters that need to be explored in IV. In the following three sections we look at the dynamics in more detail. In Sec. V we focus on the case in which the norm of the vector field is fixed. This corresponds to the case where a coupling constant of the potential is infinite and the theory becomes an extension of the theory studied in [12,13]. In Sec. VI we set the potential energy to zero and allow the vector field to roam freely. This allows us to study the impact of the multiple derivative couplings be-

tween the vector field and the metric. We then look at the more general case in Sec. VII and conclude in Sec. VIII.

II. AN EFFECTIVE THEORY OF SPONTANEOUSLY BROKEN LORENTZ INVARIANCE: A RECAP

The effective action for gravity coupled to a vector field \mathbf{A} can be written as

$$S = S_G[\mathbf{g}, \mathbf{A}] + S_m, \quad (1)$$

$$\begin{aligned} S_G[\mathbf{g}, \mathbf{A}] = \int d^4x (-g)^{1/2} & \left(\frac{M_{PL}^2}{2} R - \frac{1}{4g^2} F^{ab} F_{ab} - \frac{\alpha_1}{2} R_{ab} A^a A^b - \frac{\alpha_2}{2} (\nabla_a A^a)^2 - \frac{\beta_1}{2M^2} F_{ab} F^{ac} A_c A^b \right. \\ & - \frac{\beta_2}{2M^2} \nabla_a A^a \nabla_b A^c A^b A_c - \frac{\beta_3}{2M^2} \nabla_a A^b \nabla_c A_b A^a A^c - \frac{\beta_4}{2M^2} \nabla_a A^b \nabla_c A^d A^a A^c A_b A_d \\ & \left. - \frac{\gamma}{2} (A_a A^a - M^2 n_a n^a)^2 + \dots \right), \end{aligned} \quad (2)$$

$$S_m = \int d^4x (-g)^{1/2} L_m. \quad (3)$$

Where the $(-, +, +, +)$ metric signature is chosen and Latin indices denote general spacetime components. The gauge coupling constant is small ($g \ll 1$) and the dimensionless coupling constants ($\alpha_1, \alpha_2, \beta_1, \beta_2, \beta_3, \beta_4, \gamma$) and $n_a n^a$ are of order unity. Note that $n_a n^a$ is to be fixed during variation. The ellipsis denotes terms suppressed by powers of M or M_{PL} ; note that such a theory leads to corrections to general relativity of order M^2/M_{PL}^2 in the weak-field limit. The S_m contains all other fields that contribute to the total action, coupled to gravity in a generally covariant fashion. Note that we do not include couplings between the vector field and fields contained in S_m at this order in the effective field theory expansion [14].

We can rewrite the action S_G in a form akin to that presented in [13] as

$$\begin{aligned} S_G[\mathbf{g}, \mathbf{A}] = \frac{1}{16\pi G} \int d^4x (-g)^{1/2} & [R + K^{ab}{}_{mn} \nabla_a A^m \nabla_b A^n \\ & - \gamma 8\pi G (A^a A_a - M^2 n^a n_a)^2], \end{aligned} \quad (4)$$

where the kinetic kernel is defined through

$$\begin{aligned} K^{ab}{}_{mn} &= K(1)^{ab}{}_{mn} + K(2)^{ab}{}_{mn}, \\ K(1)^{ab}{}_{mn} &= c_1 g^{ab} g_{mn} + c_2 \delta^a{}_m \delta^b{}_n + c_3 \delta^a{}_n \delta^b{}_m \\ &\quad + c_4 A^a A^b g_{mn}, \\ K(2)^{ab}{}_{mn} &= c_5 \delta^a{}_n A^b A_m + c_6 g^{ab} A_m A_n + c_7 \delta^a{}_m A^b A_n \\ &\quad + c_8 A^a A^b A_m A_n \end{aligned}$$

and the dimensionful coupling constants c_i are related to the dimensionless ones through

$$\begin{aligned} c_1 &= -\frac{8\pi G}{g^2}, & c_2 &= -8\pi G(\alpha_2 - \alpha_1), \\ c_3 &= -8\pi G\left(\alpha_1 - \frac{1}{g^2}\right), & c_4 &= -\frac{8\pi G(\beta_1 + \beta_3)}{M^2}, \\ c_5 &= +\frac{16\pi G\beta_1}{M^2}, & c_6 &= -\frac{8\pi G\beta_1}{M^2}, \\ c_7 &= -\frac{8\pi G\beta_2}{M^2}, & c_8 &= -\frac{8\pi G\beta_4}{M^4}. \end{aligned}$$

As compared to the theories studied in [12,13] we are not restricting the vector field to have fixed norm, nor are we restricting the vector field to be exactly timelike.

III. COSMOLOGICAL EVOLUTION EQUATIONS

We can find the equations of motion for the effective theory in a general form using the formulation of (4). The Einstein equations are

$$G_{ab} = \tilde{T}_{ab} + 8\pi G T_{ab}^m, \quad (5)$$

where G_{ab} is the Einstein tensor and T_{ab}^m is the energy-momentum tensor of the fields contained in S_m . \tilde{T}_{ab} is defined to be

$$\begin{aligned}
 \tilde{T}_{ab} \equiv & \frac{1}{2}\nabla_m[I_{(a}^m A_{b)} - I_{(ab)}^m A^m] + \frac{1}{2}\nabla_m[J_{(a}^m A_{b)} - J_{(ab)}^m A^m] + c_1[(\nabla_m A_a)(\nabla^m A_b) - (\nabla_a A_m)(\nabla_b A^m)] \\
 & + c_4 A^m \nabla_m A_a A^n \nabla_n A_b + c_5 \delta_p^m A^n \nabla_n A^p A_{(b)} \nabla_m A_{|a)} + c_6 [2\nabla^m A_{(a|} A_n A_{|b)} \nabla_m A^n - A_m A_n \nabla_a A^m \nabla_b A^n] \\
 & + c_7 \delta_n^m A^p \nabla_p A_{(a|} \nabla_m A^n A_{|b)} + c_8 2\nabla_m A_{(a|} \nabla_n A^p A_{|b)} A_p A^m A^n + \frac{1}{2}g_{ab}K - 4\pi G\gamma[g_{ab}V + 4A_a A_b \sqrt{V}], \quad (6)
 \end{aligned}$$

in which

$$I_n^b = K^{ab}{}_{mn} \nabla_a A^m, \quad J_m^a = K^{ab}{}_{mn} \nabla_b A^n, \quad K = K^{ab}{}_{mn} \nabla_a A^m \nabla_b A^n, \quad \sqrt{V} = (g_{ab}A^a A^b - M^2 n^a n_a). \quad (7)$$

The vector field equation of motion is given by

$$\begin{aligned}
 0 = & \nabla_a I_m^a + \nabla_a J_m^a - 2c_4 A^e \nabla_e A_a \nabla_m A^a - c_5 \delta_n^a A_e \nabla_a A^e \nabla_m A^n - c_5 \delta_n^a A^b g_{em} \nabla_a A^m \nabla_b A^n - 2c_6 g^{ab} A_n g_{em} \nabla_a A^e \nabla_b A^n \\
 & - c_7 \delta^a_e A_n \nabla_a A^e \nabla_m A^n - c_7 g_{nm} \delta^a_e A^b \nabla_a A^e \nabla_b A^n - 2c_8 A^b A_a A_n \nabla_m A^a \nabla_b A^n - 2c_8 g_{em} A^a A^b A_n \nabla_a A^e \nabla_b A^n \\
 & + 32\pi G\gamma \sqrt{V} g_{ma} A^a. \quad (8)
 \end{aligned}$$

We now wish to restrict ourselves to a homogeneous and isotropic universe, i.e. one in which the metric is of the form

$$\mathbf{g} = -dt^2 + a(t)^2 \delta_{ij} d\mathbf{x}^i \otimes d\mathbf{x}^j \quad (9)$$

(where we have restricted ourselves to Euclidean spatial sections). To do so we must pick $n_a n^a = -1$, i.e. the minimum of the vector field must be timelike (a spacelike minimum would require the vector field to have spatial components and would violate isotropy). The vector field then has the isotropic and homogeneous form

$$\mathbf{A} = (A(t), 0, 0, 0). \quad (10)$$

The contribution from S_m is taken to be:

$$\mathbf{T}^m = \rho \mathbf{U} \otimes \mathbf{U} + P(\mathbf{g} + \mathbf{U} \otimes \mathbf{U}), \quad (11)$$

where $g_{ab} U^a U^b = -1$.

The equations of motion now simplify dramatically. The nonvanishing Christoffel symbols are

$$\begin{aligned}
 \Gamma_{11}^0 = \Gamma_{22}^0 = \Gamma_{33}^0 &= a\dot{a}, \\
 \Gamma_{01}^1 = \Gamma_{10}^1 = \Gamma_{02}^2 = \Gamma_{20}^2 = \Gamma_{03}^3 = \Gamma_{30}^3 &= \frac{\dot{a}}{a},
 \end{aligned}$$

which can be used to calculate various contributions to Eqs. (7) and (8), for example

$$\nabla_e A^e = \dot{A} + 3\frac{\dot{A}}{A}, \quad \sqrt{V} = -(A^2 - M^2 n^0 n_0),$$

$$\begin{aligned}
 K = & (c_1 + c_3)\left(\dot{A}^2 + 3\frac{\dot{A}^2}{A^2} A^2\right) + c_2 (\nabla_e A^e)^2 - (c_4 + c_5 \\
 & + c_6 - c_8 A^2) A^2 \dot{A}^2 - c_7 A^2 \dot{A} \nabla_e A^e,
 \end{aligned}$$

and the dot denotes differentiation with respect to t .

The Friedmann equation is now modified. From the 00th component of the Einstein equations we have

$$\frac{3\dot{a}^2}{a^2} = 8\pi G\rho + \sum c_i \Lambda_i + 4\pi G\gamma(V - 4\sqrt{V}A^2), \quad (12)$$

where

$$\begin{aligned}
 \Lambda_1 &= -\frac{3}{2}\frac{\dot{a}^2}{a^2} A^2 + 3\frac{\dot{a}}{a} A\dot{A} + \frac{\dot{A}^2}{2} + A\ddot{A}, \\
 \Lambda_2 &= +\frac{3}{2}\frac{\dot{a}^2}{a^2} A^2 + 6\frac{\dot{a}}{a} \dot{A}A + 3\frac{\ddot{a}}{a} A^2 + A\ddot{A} + \frac{\dot{A}^2}{2}, \\
 \Lambda_3 &= -\frac{3}{2}\frac{\dot{a}^2}{a^2} A^2 + 3\frac{\dot{a}}{a} A\dot{A} + \frac{\dot{A}^2}{2} + A\ddot{A}, \\
 \Lambda_4 &= -3\frac{\dot{a}}{a} A^3 \dot{A} - \frac{3}{2} A^2 \dot{A}^2 - A^3 \ddot{A}, \\
 \Lambda_5 &= -3\frac{\dot{a}}{a} A^3 \dot{A} - \frac{3}{2} A^2 \dot{A}^2 - A^3 \ddot{A}, \\
 \Lambda_6 &= -3\frac{\dot{a}}{a} A^3 \dot{A} - \frac{3}{2} A^2 \dot{A}^2 - A^3 \ddot{A}, \\
 \Lambda_7 &= -3\frac{\dot{a}^2}{a^2} A^4 - \frac{9}{2}\frac{\dot{a}}{a} A^3 \dot{A} - \frac{3}{2}\frac{\ddot{a}}{a} A^4 - \frac{3}{2} A^2 \dot{A}^2 - A^3 \ddot{A}, \\
 \Lambda_8 &= 3\frac{\dot{a}}{a} A^5 \dot{A} + \frac{5}{2} A^4 \dot{A}^2 + A^5 \ddot{A}.
 \end{aligned}$$

From the trace of the spatial part of the Einstein equations we find

$$-2\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} = 8\pi GP + \sum c_i Y_i - 4\pi G\gamma V, \quad (13)$$

where

$$\begin{aligned}
 Y_1 &= -\frac{1}{2}\frac{\dot{a}^2}{a^2}A^2 - \frac{\ddot{a}}{a}A^2 - 2\frac{\dot{a}}{a}A\dot{A} + \frac{1}{2}\dot{A}^2, \\
 Y_2 &= -3\frac{\ddot{a}}{a}A^2 - \frac{3}{2}\frac{\dot{a}^2}{a^2}A^2 - 6\frac{\dot{a}}{a}A\dot{A} - \frac{1}{2}\dot{A}^2 - A\ddot{A}, \\
 Y_3 &= -\frac{1}{2}\frac{\dot{a}^2}{a^2}A^2 - \frac{\ddot{a}}{a}A^2 - 2\frac{\dot{a}}{a}A\dot{A} + \frac{1}{2}\dot{A}^2, \\
 Y_4 &= -\frac{1}{2}A^2\dot{A}^2, & Y_5 &= -\frac{1}{2}A^2\dot{A}^2, \\
 Y_6 &= -\frac{1}{2}A^2\dot{A}^2, & Y_7 &= \frac{1}{2}A^2\dot{A}^2 + \frac{1}{2}A^3\ddot{A}, \\
 Y_8 &= +\frac{1}{2}A^4\dot{A}^2.
 \end{aligned}$$

Similarly to the matter fields, one can imagine the vector field as a perfect fluid of energy density ρ_A and P_A given by $8\pi G\rho_A = \sum c_i \Lambda_i + 4\pi G\gamma(V - 4\sqrt{V}A^2)$ and pressure $8\pi GP_A = \sum c_i Y_i - 4\pi G\gamma V$.

The equation of motion for the vector field becomes

$$0 = \sum c_i \Psi_i - 16\pi G\gamma\sqrt{V}A, \quad (14)$$

where

$$\begin{aligned}
 \Psi_1 &= \ddot{A} + 3\frac{\dot{a}}{a}\dot{A} - 3\frac{\dot{a}^2}{a^2}A, \\
 \Psi_2 &= 3\frac{\ddot{a}}{a}A - 3\frac{\dot{a}^2}{a^2}A + 3\frac{\dot{a}}{a}\dot{A} + \ddot{A}, \\
 \Psi_3 &= \ddot{A} + 3\frac{\dot{a}}{a}\dot{A} - 3\frac{\dot{a}^2}{a^2}A, \\
 \Psi_4 &= -3\frac{\dot{a}}{a}A^2\dot{A} - A\dot{A}^2 - A^2\ddot{A}, \\
 \Psi_5 &= -3\frac{\dot{a}}{a}A^2\dot{A} - A\dot{A}^2 - A^2\ddot{A}, \\
 \Psi_6 &= -3\frac{\dot{a}}{a}A^2\dot{A} - A\dot{A}^2 - A^2\ddot{A}, \\
 \Psi_7 &= +\frac{3}{2}\frac{\ddot{a}}{a}A^3 + 3A^3\frac{\dot{a}^2}{a^2} + 3\frac{\dot{a}}{a}A^2\dot{A} - A\dot{A}^2 - A^2\ddot{A}, \\
 \Psi_8 &= 3\frac{\dot{a}}{a}A^4\dot{A} + 4A^3\dot{A}^2 + A^3\ddot{A}.
 \end{aligned}$$

The additional terms significantly complicate the equations of motion. In particular, we now find that the 00 equation (a constraint equation in general relativity) contains second derivatives of the scale factor which appear in \tilde{T}_{ab} . This situation is similar to the situation one encounters in higher-derivative theories of gravity. It is not unexpected: in [8], two additional graviton modes are found, a clear indication that we should find more degrees of freedom in this theory than are present in conventional Einstein gravity.

IV. THE PARAMETER SPACE

Because of the large number of independent, Lorentz-invariant, two-derivative terms in the effective action, the parameter space for this theory is large. Even though we have restricted ourselves to a homogeneous and isotropic universe, we must still contend with eight coefficients c_i and the coupling constant γ . Inspecting the equations of motion, we find that there are degeneracies. c_1 and c_3 multiply equivalent terms as do c_4 , c_5 , and c_6 . Putting them all together and reverting back to dimensionless parameters we find that we are left with six independent parameters. They can be organized into various groups

Kinetic terms, quadratic in \mathbf{A} : α_1 and α_2

Kinetic terms, quartic in \mathbf{A} : β_2 and β_3

Kinetic terms, sextic in \mathbf{A} : β_4

Potential energy terms: γ

Unless an additional symmetry principle is proposed, none of these terms can be discarded. In order to illustrate the possible dynamics, we shall restrict ourselves to subspaces of the full parameter space in what follows; the full dynamics are simply too complex. But we emphasize that the true spirit of effective field theories does not allow us to selectively discard terms.

There are also three relevant dimensionful scales, namely, the Planck scale M_P , the cutoff M , and the observation scale M_0 . From these, we define the two dimensionless ratios $r = M/M_P$ and $r_0 = M_0/M_P$. Corrections to general relativity in the weak-field limit are of order r^2 . The observation scale can be defined in terms of the energy density, ρ by fixing the scale factor, a to be unity at the time of observation. We then have

$$\rho \equiv \frac{M_0^4}{a^n},$$

where n depends on the equation of state of the matter component. So, in addition to the six dimensionless parameters we have two dimensionless energy scales.

We clearly have a very large space of parameters to explore. To do so, and with the aim of illustrating qualitatively the possible cosmological dynamics, we will restrict ourselves to considering subspaces of the full parameter space in what follows. To this end, in the rest of this paper, we will try to identify what kind of behavior each of the terms in the effective action, pinned to a given parameter, will generate.

V. STRONG-COUPPLING LIMIT: $\gamma \rightarrow \infty$

In the limit of γ tending to infinity, we expect the vector field to be fixed at the minimum of the potential. This can also be realized by including a fixed-norm constraint to the action, of the form

$$\frac{1}{16\pi G} \int d^4x (-g)^{1/2} \lambda (A^a A_a - M^2 n^a n_a), \quad (15)$$

where λ is a Lagrange multiplier, which yields the constraint upon variation. Now, the time derivatives of \mathbf{A} vanish. If $n^a n_a = -1$, as if the norm of a timelike unit vector, then we have $\mathbf{A} = (M, 0, 0, 0)$. The equations obtained by varying the action with respect to the vector field and the 00th component of the metric then reduce to

$$0 = 3 \frac{\dot{a}^2}{a^2} (c_1 + c_2 + c_3 + c_7 M^2) - 3 \frac{\ddot{a}}{a} \left(c_2 - \frac{1}{2} c_7 M^2 \right) - \lambda, \quad (16)$$

$$3 \frac{\dot{a}^2}{a^2} = 8\pi G \rho - \frac{3}{2} \frac{\dot{a}^2}{a^2} M^2 (c_1 - c_2 + c_3 + 2c_7 M^2) + 3 \frac{\ddot{a}}{a} M^2 \left(c_2 - \frac{1}{2} c_7 M^2 \right) + \lambda M^2. \quad (17)$$

Eliminating the Lagrange multiplier λ and converting to α and β coefficients yields the following modified Friedmann equation

$$\left(1 + \left(\frac{3}{2} \alpha_2 - \alpha_1 \right) r^2 \right) 3 \frac{\dot{a}^2}{a^2} = 8\pi G \rho. \quad (18)$$

In this limit, we then have an effective rescaling of the gravitational constant G in the cosmological background, as found in [12].

VI. WEAK-COUPPLING LIMIT: $\gamma \rightarrow 0$

Let us now look at the dynamics of the system which result from discarding the potential term. The vector field is now free to vary subject to the couplings between its kinetic terms and the metric.

We first focus on the case in which $\alpha_1 \neq 0$, with the remaining coupling constants vanishing (i.e. only the $R_{ab} A^a A^b$ term in the vector field Lagrangian). Varying with respect to the vector and i th component of the metric yields

$$3A \frac{\ddot{a}}{a} = 0, \\ 2 \frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a} \right)^2 = -8\pi G P + \alpha_1 8\pi G \left[A\ddot{A} + 2A^2 \frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a} \right)^2 A^2 + 4 \frac{\dot{a}}{a} A\dot{A} + \dot{A}^2 \right].$$

The first of these equations has the solution

$$a(t) = \frac{\dot{s}_i a_i}{t_i} \left(t + t_i \frac{1 - \dot{s}_i}{\dot{s}_i} \right),$$

where a_i and \dot{s}_i are the initial conditions at time t_i . Note that we can always rescale $t \rightarrow t + t_i \frac{1 - \dot{s}_i}{\dot{s}_i}$, such that $a \propto t$, and we will do so from now on.

To solve the second equation we define a new variable, $X = \frac{1}{2} a^4 \dot{A}^2$. We then have

$$\dot{X} = 2a^3 \dot{a} \dot{A}^2 + a^4 \ddot{A} \dot{A},$$

$$\ddot{X} = a^4 A \ddot{A} + 2a^3 \ddot{a} \dot{A}^2 + a^4 \dot{A}^2 + 6a^2 \dot{a}^2 \dot{A}^2 + 8a^3 \dot{a} \dot{A} \ddot{A}.$$

We can now rewrite the second equation as

$$\ddot{X} - 4 \frac{\dot{a}}{a} \dot{X} + 6 \left(\frac{\dot{a}}{a} \right)^2 X = \frac{1}{8\pi G \alpha_1} (\dot{a}^2 a^2 + 8\pi G P a^4),$$

whose homogeneous part has the solution

$$X = A t^2 + B t^3.$$

The dominant term at late times comes from the particular integral

$$X = \frac{(\dot{s}_i a_i)^4}{16\pi G \alpha_1 t_i^4} t^4.$$

The particular integrals due to the pressure term of the matter/radiation are

$$X = \frac{P_0}{5\alpha_1} t^2 \ln t, \quad \text{radiation era,}$$

$$X = \frac{P_0}{\alpha_1} t^3 \ln t, \quad \text{matter era.}$$

We can find the vector field from the definition of X . The dominant solution is

$$A^2 \simeq \frac{1}{2\alpha_1} M_{Pl}^2.$$

As we can see, in this reduced space of parameters we have an attractor solution given by $a \propto t$ and $A = \sqrt{1/2\alpha_1} M_{Pl}$.

An altogether different type of behavior emerges if we consider all constants to be zero, except for α_2 (corresponding to the $\nabla_a A^a \nabla_b A^b$ term in the vector field Lagrangian). The two equations to solve now are:

$$\ddot{A} + 3 \frac{\dot{a}}{a} \dot{A} - 3 \left(\frac{\dot{a}}{a} \right)^2 A + 3 \frac{\dot{a}}{a} \dot{A} = 0, \\ 2 \frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a} \right)^2 = -8\pi G P - 8\pi G \alpha_2 \left[A\ddot{A} + 3A^2 \frac{\ddot{a}}{a} + \frac{3}{2} \times \left(\frac{\dot{a}}{a} \right)^2 A^2 + 6 \frac{\dot{a}}{a} A\dot{A} + \frac{1}{2} \dot{A}^2 \right].$$

Defining $Y = a^3 A$, we find that the first equation reduces to

$$\ddot{Y} - 3 \frac{\dot{a}}{a} \dot{Y} = 0,$$

with the solution

$$\dot{Y} = C a^3, \quad Y = C \int dt a^3 + D.$$

We can also rewrite the second equation in terms of Y , which conveniently simplifies to

$$2 \frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a} \right)^2 = -8\pi G P - 4\pi G \alpha_2 \frac{\dot{Y}^2}{a^6},$$

where $\frac{\alpha_2}{2} \frac{\dot{Y}^2}{a^6}$ is the effective pressure of the vector field. Let us consider the case where P is negligible. Then, with $b = \ln a$, we find

$$\ddot{b} + \frac{3}{2}\dot{b}^2 = -2\pi G\alpha_2 C^2 \equiv J,$$

which can be solved as follows. First, we consider separately the cases where J is positive or negative, denoted by J_+ and J_- , respectively. Integrating, we find

$$\dot{b} = \sqrt{\frac{2J_+}{3}} \tanh\left(\sqrt{\frac{3J_+}{2}}(t - t_a)\right)$$

and

$$\dot{b} = -\sqrt{\frac{-2J_-}{3}} \tan\left(\sqrt{\frac{-3J_-}{2}}(t - t_a)\right),$$

respectively, where t_a is a constant of integration. Integrating again yields

$$b = \frac{2}{3} \ln\left(\cosh\left(\sqrt{\frac{3J_+}{2}}(t - t_a)\right)\right) + D$$

and

$$b = \frac{2}{3} \ln\left(\cos\left(\sqrt{\frac{-J_-3}{2}}(t - t_a)\right)\right) + D, \quad (19)$$

where D is an integration constant. We now have a complete solution for $a = \exp(b)$. Note that the solution is singular, with $a \rightarrow 0$ in a finite time, for $J < 0$.

This type of behavior arises because of the novel type of coupling that arises when considering a vector field. The Lorentz structure of the vector field leads to a coupling with derivatives of the metric and hence to second derivatives in the would-be constraint equations and equations of motion. These lead to instabilities in the solutions of the equations of motion, a finite time singularity in this case. These instabilities presumably signal deeper pathologies in the effective field theory description.

If only β_3 (corresponding to the $\nabla_a A^b \nabla_c A_b A^a A^c$ term in the vector field Lagrangian) is nonzero, the 00th and i th equations are, respectively:

$$3\left(\frac{\dot{a}}{a}\right)^2 = 8\pi G\rho - \frac{8\pi G\beta_3}{M^2} \left(-3\frac{\dot{a}}{a}A^3\dot{A} - \frac{3}{2}A^2\dot{A}^2 - A^3\ddot{A}\right),$$

$$-2\frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^2 = 8\pi GP + \frac{8\pi G\beta_3}{M^2} \left(\frac{1}{2}A^2\dot{A}^2\right).$$

The vector field equation is

$$0 = 3\frac{\dot{a}}{a}A^3\dot{A} + 3A^2\dot{A}^2 + A^3\ddot{A}.$$

Multiplying the i th equation by three, and adding to the 00th equation we recover

$$-6\frac{\ddot{a}}{a} = 8\pi G(3P + \rho),$$

where we have used the vector equation. We see that, in this case, the evolution is identical to that of general relativity. For the case of only $\beta_4 \neq 0$ (corresponding to the $A_a A_b A^c A^d \nabla_c A^a \nabla_d A^b$ term in the vector field Lagrangian), the same applies, with the caveat that, in the weak-coupling limit, there is nothing to stop $A(t)$ approaching values beyond the applicability of the effective action.

Finally, we remark that the case in which $\beta_2 \neq 0$ (corresponding to the $\nabla_a A^a \nabla_b A^c A^b A_c$ term in the vector field Lagrangian) is significantly more complicated, given the appearance of a wide variety of terms in each equation. We may, however, combine the vector and i th equations so as to obtain two equations, each containing just one type of second-order time derivative term. In this way, we obtain expressions for $\frac{\ddot{a}}{a}$ and \ddot{A} . It turns out that the \ddot{A} equation is singular at $A = 0$ and $A = (8/3\beta_2)^{1/4} \sqrt{MM_P}$. We will see later that in the corresponding case with a natural value of γ , physically viable solutions are obtained only for positive β_2 . Moreover, for acceptable values of A the second singularity will be real and lie above $A = M$. Solving the equations numerically, we find that for $\dot{A}(0) < 0$ the system generically reaches the singularity at $A = 0$ (see Fig. 1), while for $\dot{A}(0) > 0$ the system reaches the second singularity.

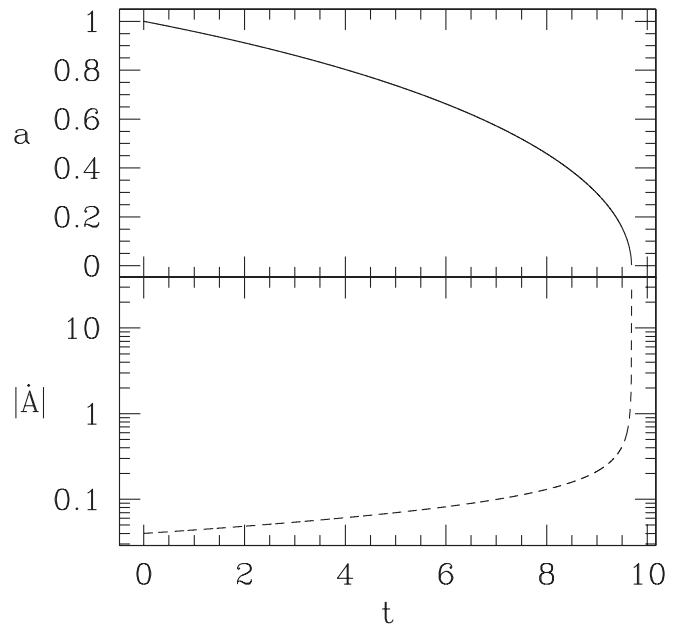


FIG. 1. Weak-coupling limit response when $\beta_2 \neq 0$ and $\dot{A}(0) < 0$ (note that the units are arbitrary).

VII. THE GENERAL CASE

We now consider solutions for natural values of the parameters α_i , β_i , γ (order unity). Furthermore, γ is assumed to be positive. In [8], it was argued that r could be as high as $\sim 10^{-2}$ and as low as $\sim 10^{-31}$ and we will opt to look for behavior between these bounds. Even with these considerations, the parameter space remains vast so we will proceed, as before, by considering the effect of isolated nonvanishing α_i and β_i parameters. Following this, we will consider a few nonvanishing combinations of α_i and β_i which yield novel behavior.

Reasonable initial conditions are

$$\begin{aligned} b(0) &= 0, & \dot{b}(0) &\approx \frac{rM}{3^{1/2}}, \\ A(0) &= M, & |\dot{A}(0)| &\approx rM^2. \end{aligned}$$

We will consider the Universe to be radiation dominated.

Let us first consider $\alpha_1 \neq 0$ and all remaining constants equal to zero. Recall that this is the case where only the $R_{ab}A^aA^b$ term contributes in the action. From (8) see that, peculiarly, second derivatives of the vector field \ddot{A} are absent from the vector field equation and present in the metric field equation whereas \ddot{b} appears in both. Hence the vector equation becomes an evolution equation for b .

We find for all values of r that the field A undergoes damped oscillations, irrespective of the sign of α_1 . Indeed, for $r \ll 1$, \ddot{A} is dominated by

$$\ddot{A} = -\frac{4\gamma M(A^2 - M^2)}{3\alpha_1^2 r^2 A} + \dots,$$

where the ellipsis denotes damping terms, which are small independently of the sign of α_1 . In the limit of $A \rightarrow M$, we have that $\ddot{b} \rightarrow 0$. Thus we generically approach a loitering solution where the scale factor a grows linearly in time.

If we now consider only $\alpha_2 \neq 0$, we note that second derivatives of b and A appear in each field equation, an indication of how the coupling to the vector field modifies the kinetic terms of the metric. Consequently, for instance, terms in the metric equation of order 1 or lower in time derivatives contribute to \ddot{A} to a degree suppressed by r^2 . As in the weak-coupling limit [8], the sign of α_2 has an impact on the evolution. It is found that positive values of α_2 generically lead to unbounded growth in A , while negative values (of order unity) lead to oscillations about M , damped roughly on a timescale $(rM)^{-1}$. Therefore, at times much greater than this, the vector field will lie fixed at $A = M$, and we can read off the resulting contribution to the metric equation as $\tilde{T}_{ii} = (-3\alpha_2 r^2/2)a^{-2}G_{ii}$. The evolution equation for the scale factor now becomes:

$$-2\frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{1 + \frac{3}{2}\alpha_2 r^2} P. \quad (20)$$

We can interpret this as a rescaling of the gravitational constant G . As discussed in [12], this can only be detected

by comparison with regimes where the vector field may rescale G by a differing amount.

It is interesting to note that another consistent solution may emerge for high values of r —that is values of order unity. In this range we see that A may be attracted to a fixed solution at $A = \sqrt{\frac{2}{-3\alpha_2}} M_{Pl} \equiv B$, now close to M (see top panel of Fig. 2). At $A = B$ we have that

$$\ddot{b} = -\frac{\gamma}{4M_{Pl}^2} (3B^2 - M^2)(B^2 - M^2) \equiv C.$$

To understand the nature of this solution, we consider the evolution of the ratio of the scale factor $a(t)$ to the scale factor $a_{GR}(t)$ resulting from the same initial conditions but in the absence of the vector field, i.e. as in pure general relativity. This ratio is plotted in the bottom panel of Fig. 2. The stability of this solution depends on the sign of C : For $C > 0$, \dot{b} grows linearly with time and A is constant, leading to an asymptotic solution at late times of the form

$$a(t) \propto e^{C(t-t_a)^2}, \quad A(t) = B, \quad (21)$$

where t_a is a constant of integration. In this limit the evolution of the vector field and metric are overwhelmingly dominated by terms in \dot{b} . We can consider the stability of this solution by considering the evolution of small perturbations to b and A . The first order perturbation A_1 to A is found to obey

$$\ddot{A}_1 + 3Ct\dot{A}_1 + 9C^2t^2A_1 = 0. \quad (22)$$

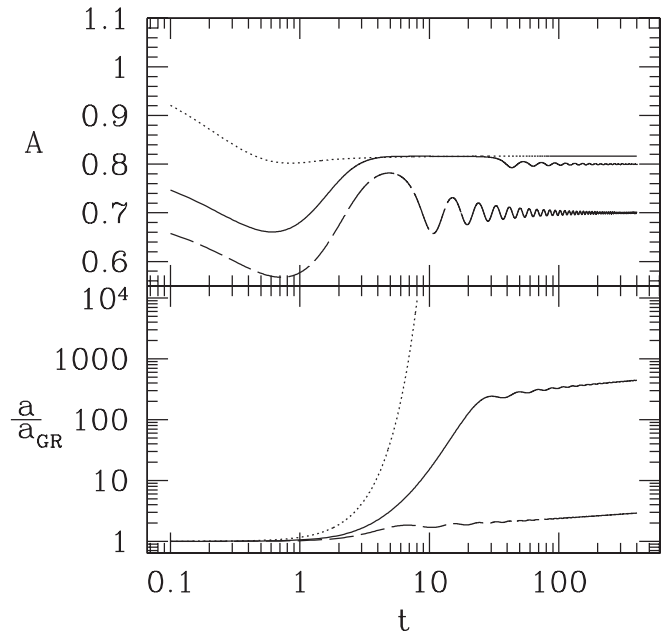


FIG. 2. Response of the system for $r = 1$ (dotted line), $r = 0.8$ (solid line), $r = 0.7$ (dashed line). $\alpha_2 = -1$, $\gamma = 1$, and $M_{Pl} = 1$.

The general solution to this equation is:

$$A_1(t) = k_1 \frac{e^{(-3Ct^2/4)}}{t^{1/2}} W_M\left(\frac{i\sqrt{3}}{4}, \frac{1}{4}, \frac{i\sqrt{3}Ct^2}{2}\right) + k_2 \frac{e^{(-3Ct^2/4)}}{t^{1/2}} W_W\left(\frac{i\sqrt{3}}{4}, \frac{1}{4}, \frac{i\sqrt{3}Ct^2}{2}\right), \quad (23)$$

where k_1 and k_2 are integration constants and W_M and W_W are Whittaker M and W functions. We have checked that $A_1 \rightarrow 0$ as $t \rightarrow \infty$ for the appropriate values of r ; thus the solution may be considered to be stable.

In the case that $C < 0$, \dot{b} will decrease linearly in time while the vector field is in the vicinity of B . Note that with the opposite sign of C , perturbations (as in (22)) are no longer stable. Thus we may expect that if the system can reach a fixed solution at B it will only do so briefly before moving away. This is amply illustrated in Fig. 2. For $r = 1$ ($C > 0$) the system quickly settles to the solution $A = B$ prompting runaway growth in a/a_{GR} . For $r = 0.8$ ($C < 0$) the system settles briefly at $A = B$, during which time a/a_{GR} is enhanced significantly, before departing to the solution $A = M$. For $r = 0.6$ (and all lower values of r) the vector field never reaches B and settles to the $A = M$ tracking solution described by (20).

We now restrict ourselves to $\beta_2 \neq 0$ and discard all other terms. We note that for the ranges of r considered, the values at which \ddot{A} is singular (see VI) arguably lie beyond the regime of validity of the effective action. For negative β_2 , A experiences unbounded growth in the direction of the initial perturbation, as in Sec. VI, and so will generically reach singularities. For positive β_2 , the vector field undergoes oscillations about M . Numerical exploration suggests that the system will evolve so that the oscillations will be of increasing frequency and amplitude up until $t \approx (Mr)^{-1}$. Terms in $\tilde{\mathbf{T}}$, the effective stress-energy of the vector field, then act to halt the expansion of the Universe leading to eventual collapse (see Fig. 3).

If we consider $\beta_3 \neq 0$ or $\beta_4 \neq 0$, we recover familiar behavior. The vector field's stress-energy tensor contributes only first derivatives in time and terms such as $\frac{\dot{a}}{a}^2$ and $\frac{\ddot{a}}{a}$ do not appear in the vector field equation; thus, A contributes and evolves much like a scalar field. It is found that for positive values of β_3 and for negative values of β_4 , the vector field oscillates about A . It undergoes Hubble friction, the oscillations decaying on a timescale $\sim M^{-1}$ (see Fig. 4). Settling towards $A = M$, the potential terms vanish as before and so the effect on the expansion of the Universe is as in the weak-coupling limit i.e. identical to that of general relativity.

An interesting simplification is obtained for particular combinations of coefficients, given by $\alpha_1 = \alpha_2 < 0$. With this combination, \dot{b} drops out from the vector equation. For acceptable values of r , the vector field oscillates about M , the background expansion gradually settling the field to this value. This happens on a timescale of roughly M^{-1} .

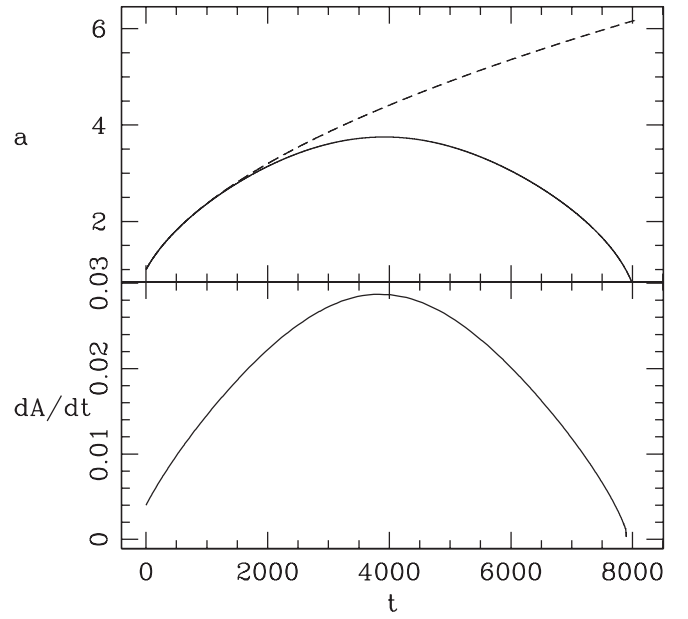


FIG. 3. Top panel: Actual scale factor (solid line) compared to GR scale factor (dashed evolution). Bottom panel: Oscillation amplitude of $\frac{\dot{a}}{M}$. $r = 0.02$, $b_2 = 1$, $\gamma = 1$, and $M_{Pl} = 10$

Thus at times larger than this the metric equation reduces to

$$-2\frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{1 - 2r^2\alpha_1} P.$$

Again this behavior can be interpreted as a rescaling of G or, in this case, as the tensor $\tilde{\mathbf{T}}$ tracking any form of matter field in the Universe.

Finally, we would like to make a general point about the appearance of singularities in this system. The field equations can be written schematically as:

$$\ddot{b} = \frac{1}{-f}(\dots), \quad \ddot{A} = \frac{1}{f}(\dots),$$

where f is of the form

$$f = (-2 + r^2(2\alpha_1 - 3\alpha_2)\bar{A}^2) \times (-\alpha_2 + (\beta_3 + \beta_2)\bar{A}^2 - \beta_4\bar{A}^4) - \left(\alpha_1 - \alpha_2 + \frac{\beta_2\bar{A}^3}{2}\right) \times \left(3\alpha_1 - 3\alpha_2 - \frac{3\beta_2}{2}\bar{A}^2\right) \times r^2\bar{A}^2, \quad (24)$$

and where $\bar{A} = \frac{A}{M}$. When the terms in parentheses are nonvanishing, there may occur singularities in the evolution for particular values of \bar{A} . An example of this was encountered in VI for only $\beta_2 \neq 0$.

We expect r to be of order 10^{-2} or smaller and \bar{A} to be of order unity. Hence, generally, $(-2 + r^2(2\alpha_1 - 3\alpha_2)\bar{A}^2) \approx -2$. Therefore, the second terms in parenthesis may be expected to be suppressed relative to the first by r^2 . It is

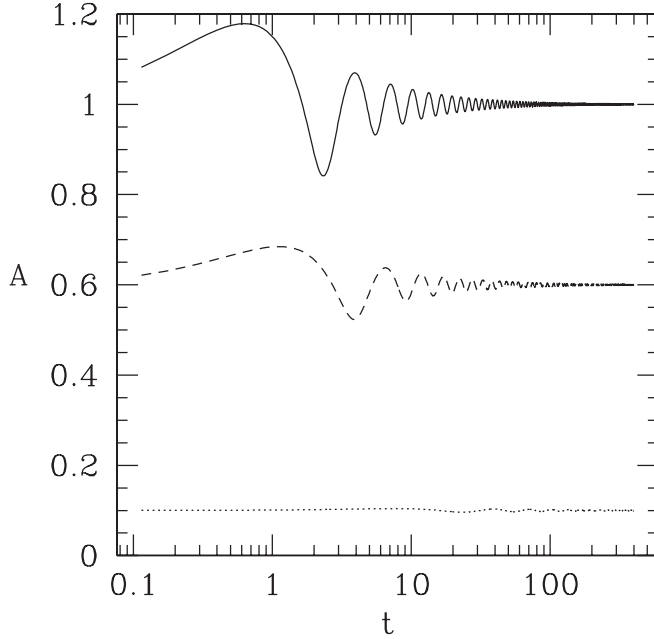


FIG. 4. Comparative evolution of A for r values in ratio 10:6:1 (solid line:dashed line:dotted line) with $r = 1$ being the highest scale. Such a high value is chosen only for ease of illustration. $\beta_4 = -1, \gamma = 1$, and $M_{pl} = 1$.

unlikely then that f may vanish due to equality of the first and second terms in parenthesis. An alternative is for both groups of terms to vanish identically. This would seem to require of the first that $\delta \equiv -\alpha_2 + (\beta_3 + \beta_2)\dot{A}^2 - \beta_4\dot{A}^4$ vanishes along with at least one term from the second group. When considering isolated nonvanishing coefficients, restrictions on their sign were found to be $\alpha_2, \beta_4 \leq 0$ and $\beta_2, \beta_3 \geq 0$, respectively. This would seem to imply that δ is inherently positive. It may only vanish when each of the coefficients are zero, in which case f may only vanish if α_1 is also zero. However, we emphasize that the restrictions on the sign of individual coefficients need not hold when general combinations of α_i, β_i are considered.

VIII. CONCLUSIONS

In this paper, we have studied the cosmology of the model proposed in [8]. As expected, there is a wide range of possible behaviors. We achieve accelerated expansion if we have a term with $\alpha_2 \neq 0$ and a high mass scale M . The system will scale when $\alpha_1 = \alpha_2$, or when $\alpha_2 \neq 0$ with a

low mass scale. If $\beta_2 \neq 0$, the Universe will recollapse, while it will loiter if we only have $\alpha_1 \neq 0$. Clearly we have only looked at a small subset of the parameter space but our analysis has allowed us to probe these different regimes. It also allows us to make some comments about the structure of the theory.

First, we stress again that this is an effective field theory, valid at energies below the cutoff M . In some cases, we have found that this theory is unstable, in the sense that runaway solutions push the theory beyond its regime of validity. To properly understand its behavior in these regimes, we would need to at least consider higher order corrections, and ultimately some ultraviolet completion of the theory would be necessary. Such a completion might conceivably simply involve extra fields [15], or, more likely, more complicated dynamics.

Second, the Lorentz structure of the vector field in this framework introduces a novel phenomenon. For theories with $c_2 \neq 0$ and $c_7 \neq 0$ we find second derivatives of the scale factor appear in what would have been (in standard general relativity) constraint equations, and in the evolution equations for the vector field. We find that this leads to possible instabilities in the cosmology. Clearly, there is a need for a complete perturbative analysis of this theory in a cosmological setting, akin the study of Gauss-Bonnet or higher-derivative modifications to gravity [16].

Third, the results found here can be added to known cosmological consequences of Lorentz-violating fields: de Sitter expansion and dustlike stress-energy tensor in background [17], pervasive tracking solutions in background [12], and generating the instability permitting the growth of large scale structure [18].

Finally, the theory discussed here, though complex in its structure, should be amenable to a detailed comparison with current cosmological observations. We have laid down the framework for looking at constraints on the background evolution. The next step is to construct the evolution equations for linear perturbations. With these in hand, it should be possible to harness the wealth of new, high precision cosmological observations and use them to provide constraints on this theory.

ACKNOWLEDGMENTS

The authors thank J. March-Russell, C. Skordis, and G. Starkman for useful comments and discussions.

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