PHYSICAL REVIEW D **75,** 044010 (2007)

Effective action and heat kernel in a toy model of brane-induced gravity

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(Received 29 November 2006; published 7 February 2007)

We apply a recently suggested technique of the Neumann-Dirichlet reduction to a toy model of brane-induced gravity for the calculation of its quantum one-loop effective action. This model is represented by a massive scalar field in the $(d + 1)$-dimensional flat bulk supplied with the d-dimensional kinetic term localized on a flat brane and mimicking the brane Einstein term of the Dvali-Gabadadze-Porrati (DGP) model. We obtain the inverse mass expansion of the effective action and its ultraviolet divergences which turn out to be nonvanishing for both even and odd spacetime dimensionality d. For the massless case, which corresponds to a limit of the toy DGP model, we obtain the Coleman-Weinberg type effective potential of the system. We also obtain the proper-time expansion of the heat kernel in this model associated with the generalized Neumann boundary conditions containing second-order tangential derivatives. We show that in addition to the usual integer and half-integer powers of the proper time this expansion exhibits, depending on the dimension d, either logarithmic terms or powers multiple of one quarter. This property is considered in the context of strong ellipticity of the boundary value problem, which can be violated when the Euclidean action of the theory is not positive definite.

DOI: 10.1103/PhysRevD.75.044010 PACS numbers: 04.60.−m, 04.50.+h, 04.62.+v

I. INTRODUCTION

Modified theories of gravity in the form of braneworld models can in principle account for the phenomenon of dark energy as well as for nontrivial compactifications of multidimensional string models. It becomes increasingly more obvious that one should include in such models the analysis of quantum effects beyond the tree-level approximation [1]. This is the only way to reach an ultimate conclusion on the resolution of such problems as the presence of ghosts [2] and low strong-coupling scale [3]. Quantum effects in brane models are also important for the stabilization of extra dimensions [4], fixing the crossover scale in the Brans-Dicke modification of the DGP model [5] and in the recently suggested mechanism of the cosmological acceleration generated by the four-dimensional conformal anomaly [6].

A general framework for treating quantum effective actions in brane models (or, more generally, models with timelike and spacelike boundaries) was recently suggested in [7–9]. The main peculiarity of these models is that due to quantum field fluctuations on the branes the field propagator is subject to generalized Neumann boundary conditions involving normal and tangential derivatives on the brane/boundary surfaces. This presents both technical and conceptual difficulties, because such boundary conditions are much harder to handle than the simple Dirichlet ones. The method of [9] provides a systematic reduction of the generalized Neumann boundary conditions to Dirichlet conditions. As a byproduct it disentangles from the quantum effective action the contribution of the surface modes mediating the brane-to-brane propagation (that is, within one brane), which play a very important role in the zero-mode localization mechanism of the Randall-Sundrum type [10]. The main purpose of our paper here is to apply this method to a simplified toy model of brane-induced gravity in order to demonstrate how it works for the first nontrivial example of a field system with boundary conditions involving second-order tangential derivatives. As we will see, this model leads to qualitatively new structures in the one-loop effective action, its renormalization, and the associated heat kernel.

Briefly the method of [9] looks as follows. The action of a (free field) brane model generally contains the bulk and the brane parts,

$$S[\phi] = \frac{1}{2}\int_{\mathbf{B}} d^{d+1}X\,\phi(X)\overleftrightarrow{F}(\nabla)\phi(X) + \frac{1}{2}\int_{\mathbf{b}} d^d x\,\varphi(x)\kappa(\partial)\varphi(x), \tag{1.1}$$

where the $(d + 1)$-dimensional bulk and the d-dimensional brane coordinates are labeled, respectively, by $X = X^A$ and $x = x^\mu$, and the boundary values of bulk fields $\phi(X)$ on the brane/boundary $\mathbf{b} = \partial\mathbf{B}$ are denoted by $\varphi(x)$,

$$\phi(X)|_{\mathbf{b}} = \varphi(x). \tag{1.2}$$

The kernel of the bulk Lagrangian is given by the second-order differential operator $F(\nabla)$, whose derivatives $\nabla \equiv \partial_X$ are integrated by parts in such a way that they form bilinear combinations of first-order derivatives acting on two different fields (this is denoted in (1.1) by the double-headed arrow). Integration by parts in the bulk gives nontrivial surface terms on the brane/boundary. In particular, this operation results in the Wronskian relation for generic test functions $\phi_{1,2}(X)$,

$$\int_{\mathbf{B}} d^{d+1}X(\phi_1\vec{F}(\nabla)\phi_2 - \phi_1\overleftarrow{F}(\nabla)\phi_2) = -\int_{\partial\mathbf{B}} d^d x(\phi_1\vec{W}\phi_2 - \phi_1\overleftarrow{W}\phi_2). \tag{1.3}$$

Arrows everywhere here indicate the direction of action of derivatives either on ϕ_1 or ϕ_2.

The brane part of the action contains as a kernel some local operator $\kappa(\partial)$, $\partial = \partial_x$. Its order in derivatives depends on the model in question. In the Randall-Sundrum model [10], for example, it is for certain gauges just an ultralocal multiplication operator generated by the tension term on the brane. In the Dvali-Gabadadze-Porrati (DGP) model [11] this is a second-order operator induced by the brane Einstein term on the brane, $\kappa(\partial) \sim \partial\partial/m$, where m is the DGP scale which is of the order of magnitude of the horizon scale,being responsible for the cosmological acceleration [12]. In the context of the Born-Infeld action in D-brane string theory with vector gauge fields, $\kappa(\partial)$ is a first-order operator [13].

In all these cases the variational procedure for the action (1.1) with dynamical (not fixed) fields on the boundary $\varphi(x)$ naturally leads to generalized Neumann boundary conditions of the form

$$(\vec{W}(\nabla) + \kappa(\partial))\phi|_{\mathbf{b}} = 0, \tag{1.4}$$

which uniquely specify the propagator of quantum fields and, therefore, a complete Feynman diagrammatic technique for the system in question. The method of [9] allows one to systematically reduce this diagrammatic technique to the one subject to the Dirichlet boundary conditions $\phi|_{\mathbf{b}} = 0$. The main additional ingredient of this reduction procedure is the brane operator $\boldsymbol{F}^{\text{brane}}(x, x')$ which is constructed from the Dirichlet Green's function $G_D(X, X')$ of the operator $F(\nabla)$ in the bulk,

$$\boldsymbol{F}^{\text{brane}}(x, x') = -\vec{W}(\nabla_X)G_D(X, X')\overleftarrow{W}(\nabla_{X'})|_{X=e(x),X'=e(x')} + \kappa(\partial)\delta(x, x'). \tag{1.5}$$

This expression expresses the fact that the kernel of the Dirichlet Green's function is being acted upon both arguments by the Wronskian operators with a subsequent restriction to the brane, with $X = e(x)$ denoting the brane embedding function.

As shown in [9], this operator determines the brane-to-brane propagation of the physical modes in the system with the classical action (1.1) (its inverse is the brane-to-brane propagator) and additively contributes to its full one-loop effective action according to

$$\Gamma_{\text{1-loop}} \equiv \frac{1}{2}\,\mathrm{Tr}_N^{(d+1)} \ln F = \frac{1}{2}\,\mathrm{Tr}_D^{(d+1)} \ln F + \frac{1}{2}\,\mathrm{Tr}^{(d)} \ln \boldsymbol{F}^{\text{brane}}, \tag{1.6}$$

where $\mathrm{Tr}_{D,N}^{(d+1)}$ denotes functional traces of the bulk theory subject to Dirichlet and Neumann boundary conditions, respectively, while $\mathrm{Tr}^{(d)}$ is a functional trace in the boundary d-dimensional theory. The full quantum effective action of this model is obviously given by the functional determinant of the operator $F(\nabla)$ subject to the generalized Neumann boundary conditions (1.5), and the above equation reduces its calculation to that of the Dirichlet boundary conditions plus the contribution of the brane-to-brane propagation.

Here we apply (1.6) to a simple model of a scalar field which mimics, in particular, the properties of the brane-induced gravity models and the DGP model [11]. This is the $(d+1)$-dimensional massive scalar field $\phi(X) = \phi(x, y)$ with mass M living in the half-space $y \geq 0$ with the additional d-dimensional kinetic term for $\varphi(x) \equiv \phi(x, 0)$ localized at the brane (boundary) at $y = 0$,

$$S[\phi] = \frac{1}{2}\int_{y\geq 0} d^{d+1}X((\nabla\phi(X))^2 + M^2\phi^2(X)) + \frac{1}{2m}\int d^d x(\partial\varphi(x))^2. \tag{1.7}$$

Here and in what follows we work in a flat Euclidean (positive-signature) spacetime. Therefore, this action corresponds to the following choice of $F(\nabla)$ in terms of $(d+1)$-dimensional and d-dimensional D'Alembertians (Laplacians)

$$F(\nabla) = M^2 - \Box^{(d+1)} = M^2 - \Box - \partial_y^2, \quad \Box = \Box^{(d)} \equiv \partial_\mu^2. \tag{1.8}$$

Its Wronskian operator is given by the normal derivative with respect to outward-pointing normals to the brane, $W = -\partial_y$, and the boundary operator $\kappa(\partial)$ equals

$$\kappa(\partial) = -\frac{1}{m}\Box, \tag{1.9}$$

where the dimensional parameter m mimics the role of the DGP scale [11]. Thus, the generalized Neumann boundary conditions in this model involve second-order derivatives tangential to the brane,

$$\left(\partial_y + \frac{1}{m}\Box\right)\phi|_{\mathbf{b}} = 0, \tag{1.10}$$

cf. (1.4) with $W = -\partial_y$ and $\kappa = -\Box/m$.

As we show below, the brane-to-brane operator for such a model has the form of the following pseudodifferential operator,

$$\boldsymbol{F}^{\text{brane}}(\partial) = \frac{1}{m}(-\Box + m\sqrt{M^2 - \Box}). \tag{1.11}$$

In the massless case of the DGP model [11], $M = 0$, this operator is known to mediate the gravitational interaction on the brane, interpolating between the four-dimensional Newtonian law at intermediate distances and the five-dimensional law at the horizon scale $\sim 1/m$ [3]. We calculate the effective action (1.6) for this model both in the form of the $1/M$-expansion and exactly in terms of a special hypergeometric function representation (as a function of M and the dimensionless ratio m/M) and find its ultraviolet divergences.

As a byproduct of the Neumann to Dirichlet reduction (1.6), the technique of [9] also yields a method for obtaining the proper-time expansion for the heat kernel associated with the boundary conditions (1.4). For simple Neumann (Robin) boundary conditions containing at most first-order derivatives tangential to the boundary, this expansion has the form

$$\mathrm{Tr}^{(d+1)} e^{-sF(\nabla)} = \frac{1}{(4\pi s)^{(d+1)/2}} \sum_{n=0}^{\infty} (s^n A_n + s^{n/2} B_{n/2}). \tag{1.12}$$

In addition to the well-known bulk integrals A_n of the local Schwinger-DeWitt coefficients of integer powers of the proper time [14–16], it contains surface integrals $B_{n/2}$ as coefficients of both integer and half-integer powers of s [17–21]. They are sufficiently easy to calculate for the Dirichlet boundary conditions [17,21], but become much harder to obtain for the Robin and generalized Neumann case with a growing number of tangential derivatives [19,22,23]. As shown in [9], the Neumann-Dirichlet reduction method simplifies their calculation essentially. For second-order derivatives they are not known at all, and a toy DGP model of the above type seems to be the first application of the heat kernel method subject to the boundary conditions (1.10).

It turns out that in the case of (1.10) even the very structure (1.12) is incorrect, because for even d it contains also terms logarithmic in s and for odd d it has also powers of s which are multiples of a quarter. We calculate these additional terms in the heat kernel expansion, discuss their relation to a nontrivial analytic structure of the brane part of the effective action (1.6) and also to the problem of strong ellipticity [24] of the boundary value problem (1.10). In particular, the latter is shown to be determined by the positivity of the action (1.7) or the positive-definiteness of the brane operator (1.11).

The paper is organized as follows. In Sec. II we derive the Dirichlet and brane parts of the effective action (1.6). Sections III and IV present its inverse mass expansion and ultraviolet divergences in various dimensions. In Sec. V we obtain a new type of the proper-time expansion for the heat kernel associated with the boundary conditions involving second-order tangential derivatives. In Sec. VI we present the hypergeometric function representation of the effective action and analyze the limit of a simple Neumann boundary condition corresponding to $m \to \infty$; we also consider the massless limit $M = 0$ which gives the effective potential in the toy DGP model. In the Conclusions we discuss these results in the context of a possible violation of strong ellipticity for the boundary conditions (1.10), their potential applications in braneworld models including gravitation and the use of the proper-time method in brane models. Three appendices contain the derivation of the inverse mass expansion of the effective action, its exact hypergeometric function representations, and the presentation of the status of the strong ellipticity problem in a toy DGP model.

II. DIRICHLET AND BRANE-TO-BRANE CONTRIBUTIONS

We begin by constructing the Dirichlet part of the effective action. The basic Dirichlet Green's function of the model can be obtained by the proper-time integration of the corresponding heat equation kernel,

$$K_D(s|X, X') = e^{s\Box_{(d+1)}} \delta(X, X'). \tag{2.1}$$

It follows by the method of images from its well-known expression in the flat $(d+1)$-dimensional spacetime without boundaries

$$K_D(s|X, X') = \frac{1}{(4\pi s)^{(d+1/2)}} \times \left\{\exp\left(-\frac{(x-x')^2 + (y-y')^2}{4s}\right) - \exp\left(-\frac{(x-x')^2 + (y+y')^2}{4s}\right)\right\}. \tag{2.2}$$

The functional trace of this heat kernel contains two terms—bulk and boundary integrals of the only two nonvanishing Schwinger-DeWitt coefficients,

$$\mathrm{Tr}_D^{(d+1)} e^{s_{(d+1)}} = \int_{y \geq 0} d^{d+1}X K_D(s|X, X) = \frac{1}{(4\pi s)^{(d+1/2)}} \left(\int_{y\geq 0} d^{d+1}X - \frac{\sqrt{\pi}}{2} s^{1/2} \int d^d x\right). \tag{2.3}$$

The corresponding Dirichlet-type effective action for the model with the mass M in the bulk can be obtained by the following proper-time integration,

$$\frac{1}{2}\,\mathrm{Tr}_D \ln[-\Box_{(d+1)} + M^2] = -\frac{1}{2}\,\mathrm{Tr}_D \int_0^\infty \frac{ds}{s}\, e^{s\Box_{(d+1)} - sM^2} = -\frac{1}{2}\left(\frac{M^2}{4\pi}\right)^{(d+1/2)} \Gamma\left(-\frac{d+1}{2}\right) \times \int_{y\geq 0} d^{d+1}X + \frac{1}{8} \times \left(\frac{M^2}{4\pi}\right)^{d/2} \Gamma\left(-\frac{d}{2}\right) \int d^d x. \tag{2.4}$$

The dimensionality d will be treated as a parameter of the dimensional regularization. Therefore, this expression contains ultraviolet divergences as the poles of Gamma functions at negative integer values of their arguments. These divergences are represented here either by the bulk or boundary surface integrals, depending on whether the total spacetime dimensionality $(d+1)$ is even or odd.

For the brane part of the action (1.6) we need the brane-to-brane operator (1.5) which is based on the Dirichlet Green's function of the model. The latter is also exactly calculable in elementary functions because the corresponding proper-time integral can be expressed in terms of the modified Bessel function of half-integer order,

$$G_D(X, X') = \int_0^\infty ds K_D(s|X, X') e^{-sM^2} = \int_0^\infty \frac{ds}{\sqrt{4\pi s}} (e^{-(y-y')^2/4s} - e^{-(y+y')^2/4s}) e^{s(\Box - M^2)} \delta(x, x') = \frac{1}{2\sqrt{M^2 - \Box}} (e^{-|y-y'|\sqrt{M^2-\Box}} - e^{-(y+y')\sqrt{M^2-\Box}}) \delta(x, x'). \tag{2.5}$$

Therefore the first term of (1.5) takes the form of a square-root operator [9],

$$\vec{\partial}_y G_D(X, X') \overleftarrow{\partial}_y |_{X=(x,0), Y=(x',0)} = \sqrt{M^2 - \Box}\, \delta(x, x'), \tag{2.6}$$

and the full brane-to-brane operator (1.5) is given by (1.11).

The operator (1.11) is of a nonlocal pseudodifferential nature, and no conventional proper-time representation is known for its functional determinant (see Sec. VII, though). Therefore we will calculate the latter in the basis of Fourier modes—the eigenmodes of $\boldsymbol{F}^{\text{brane}}$. By resolving the d-dimensional delta-function in the Fourier integral we have

$$\frac{1}{2}\,\mathrm{Tr}^{(d)} \ln \boldsymbol{F}^{\text{brane}} = \frac{1}{2} \int d^d x \ln(-\Box + m\sqrt{M^2 - \Box}) \delta(x, x')|_{x'=x} = \frac{1}{2} \int d^d x \frac{1}{(2\pi)^d} \int d^d p \ln(p^2 + m\sqrt{M^2 + p^2}) = \frac{1}{(4\pi)^{d/2}\Gamma(d/2)} \int d^d x \int_0^\infty dp\, p^{d-1} \ln(p^2 + m\sqrt{M^2 + p^2}), \tag{2.7}$$

where p is the radial integration variable in the momentum-space,

$$p = \sqrt{p_\mu p^\mu}. \tag{2.8}$$

As we see, the mass parameter M^2 enters the logarithmic function here in a very nontrivial way, so a typical $1/M^2$-expansion of the local Schwinger-DeWitt expansion is far from being straightforward. In the next section we derive this expansion by converting the expression (2.7) into the form of a so-called integral with a weak singularity to which a known asymptotic expansion technique can be directly applied.

III. INVERSE MASS EXPANSION

By integrating in (2.7) by parts and using the rules of the dimensional regularization, which annihilates purely power-divergent integrals, we have

$$\int_0^\infty dp\, p^{d-1} \ln(p^2 + m\sqrt{M^2 + p^2}) = -\frac{1}{d} \int_0^\infty dp\, p^{d+1} \frac{2 + m/\sqrt{M^2 + p^2}}{p^2 + m\sqrt{M^2 + p^2}}. \tag{3.1}$$

Then, with the change of the integration variable from p to t,

$$t = \frac{p^2}{2M\sqrt{M^2 + p^2}}, \qquad p = M[2t(\sqrt{1 + t^2} + t)]^{1/2}, \tag{3.2}$$

the integral takes the form

$$\int_0^\infty dp\, p^{d-1} \ln(p^2 + m\sqrt{M^2 + p^2}) = \frac{(2M^2)^{d/2}}{d} \varepsilon I, \tag{3.3}$$

where

$$I = \int_0^\infty dt\, t^{d/2-1} (t + \varepsilon)^{-1} \varphi(t), \tag{3.4}$$

$$\varphi(t) = (\sqrt{1+t^2}+t)^{d/2}, \tag{3.5}$$

and

$$\varepsilon = \frac{m}{2M}. \tag{3.6}$$

Obviously the $1/M$ asymptotic expansion corresponds to the asymptotic expansion in $\varepsilon \to 0$—the limit in which the power-law singularity of the integrand occurs at the lower integration limit $t = 0$. Here the integrand is not analytic because of the factor $t^{d/2-1}$ having a branch point at $t = 0$. Remember that the integral should be calculated for a generic dimensionality d that should be analytically continued to the complex plane in order to regularize the ultraviolet divergences appearing at the upper integration limit $t \to \infty$. Therefore, one should expect that the expansion of the integral will also have a part nonanalytic in $\varepsilon \to 0$.

As shown in Appendix A by the asymptotic expansion method for integrals with a weak singularity [25], this expansion indeed has the form

$$I = \frac{\pi\varepsilon^{d/2-1}}{\sin(\pi d/2)}\varphi(-\varepsilon) + \sum_{j=0}^{\infty} a_{2j}\varepsilon^{2j}, \tag{3.7}$$

where the first term—the nonanalytic part—has a branch-point singularity. Interestingly, the coefficient of this non-analytic factor is exactly expressed through the function (3.5) itself but with the flipped sign of the argument. Its expansion in higher powers of ε, given by Eq. (A9) in Appendix A, is determined by the derivatives $\varphi^{(n)}(0) \equiv d^n\varphi/dt^n(0)$ which explicitly equal

$$\begin{aligned}\varphi^{(n)}(0) &= 2^{n-2}(-1)^{n+1}\frac{d\Gamma(\frac{n}{2}-\frac{d}{4})}{\Gamma(1-\frac{n}{2}-\frac{d}{4})}\\ &= \frac{d}{2}\left(\frac{d}{2}-(n-2)\right)\left(\frac{d}{2}-(n-4)\right)\times\cdots\\ &\quad\times\left(\frac{d}{2}-(2-n)\right).\end{aligned} \tag{3.8}$$

The analytic part of (3.7) contains only even powers of ε, with the coefficients

$$a_{2j} = -2^{2j-1-d/2}d\frac{\Gamma(j-\frac{d-1}{2})\Gamma(\frac{d}{2}-1-2j)}{\Gamma(-j+\frac{1}{2})}. \tag{3.9}$$

Thus, finally, on account of (2.7) and (3.3) the inverse mass expansion for the brane part of the action takes the form of two terms having qualitatively different analytic behavior in the mass parameters M and m,

$$\frac{1}{2}\mathrm{Tr}^{(d)}\ln\boldsymbol{F}^{\mathrm{brane}} = \frac{1}{d\Gamma(d/2)}\int d^dx\left[\frac{\pi}{\sin\frac{\pi d}{2}}\left(\frac{Mm}{4\pi}\right)^{d/2}\varphi(-\varepsilon) + \left(\frac{M^2}{2\pi}\right)^{d/2}\sum_{j=0}^{\infty}a_{2j}\varepsilon^{2j+1}\right]. \tag{3.10}$$

IV. ULTRAVIOLET DIVERGENCES VERSUS SPURIOUS INFRARED POLES

The integral (3.4) has a power divergence of order $d-1$ at the upper integration limit. Its differentiation with respect to ε improves the convergence of the integral which becomes ultraviolet finite after d differentiations. This means that the ultraviolet divergence of (3.4) is a polynomial in ε of order $d-1$. The ultraviolet divergence of the brane action is a polynomial of order d in m and M, respectively, cf. Equation (3.3). In dimensional regularization these divergences manifest themselves as poles in the dimensionality d analytically continued to the complex plane. From (3.10) it follows, however, that for even d the nonanalytic and analytic parts of the inverse mass expansion separately have poles to all orders in ε—the poles of $\pi/\sin(\pi d/2)$ and the poles of one of the Gamma functions in the numerator of (3.9) for all $2j \geq d/2-1$. This implies an intrinsic cancellation between the infinite sequence of poles in the nonanalytic and analytic parts of (3.10). This cancellation of spurious poles, which have the nature of infrared divergences, can be directly observed by calculating separately these two contributions.

The real ultraviolet divergences of (3.4) can be independently obtained by means of integration by parts. In terms of the new integration variable $x = 1/t$ the integral becomes divergent at the lower integration limit,

$$I = \int_0^\infty dx x^{-d} f(x), \tag{4.1}$$

$$f(x) \equiv (1+\varepsilon x)^{-1}(\sqrt{1+x^2}+1)^{d/2}. \tag{4.2}$$

We call in the following the physical dimension of spacetime N in order to distinguish it from the formal dimension used in the integrals. Analytically continuing the spacetime dimensionality from its physical value N to the domain of convergence, where integrations by parts are possible without introducing extra surface terms, we have

$$I = \frac{1}{(d-1)(d-2)\cdots(d-N)}\int_0^\infty dx x^{N-d} f^{(N)}(x). \tag{4.3}$$

Then taking the limit to the real physical dimensionality,

$$d = N - \delta, \qquad \delta \to +0, \tag{4.4}$$

we have the logarithmic divergence of this integral as a residue of the pole in $\delta \to 0$,

$$I^{\mathrm{div}} = \frac{1}{\delta}\frac{f^{(N-1)}(0)}{\Gamma(N)}. \tag{4.5}$$

By applying this formula we obtain the ultraviolet divergences which for even and odd dimensionalities look as follows. For even d being a multiple of 4, $d = 4k - \delta$, $k = 0, 1, \cdots$, the divergences read

$$\frac{1}{2}\,\mathrm{Tr}^{(d)}\ln \boldsymbol{F}^{\rm brane}|^{\rm div} = -\frac{1}{\delta}\frac{1}{2(4\pi)^{d/2}\Gamma(\frac{d}{2})}\int d^{4k}x\sum_{j=k-1}^{2k-1}\frac{\Gamma(j+1)M^{4k-2j-2}m^{2j+2}}{\Gamma(2k-j)(2j+2-2k)!}. \tag{4.6}$$

For even d being a multiple of 2, $d = 4k + 2 - \delta$, $k = 0, 1, \cdots$, they have the form

$$\frac{1}{2}\,\mathrm{Tr}^{(d)}\ln \boldsymbol{F}^{\rm brane}|^{\rm div} = -\frac{1}{\delta}\times\frac{1}{2(4\pi)^{d/2}\Gamma(\frac{d}{2})}\int d^{4k+2}x\sum_{j=k}^{2k}\frac{\Gamma(j+1)M^{4k-2j}m^{2j+2}}{\Gamma(2k+1-j)(2j+1-2k)!}. \tag{4.7}$$

Similarly, for odd $d = 2k + 1 - \delta$ the ultraviolet divergences of the brane effective action read

$$\frac{1}{2}\,\mathrm{Tr}^{(d)}\ln \boldsymbol{F}^{\rm brane}|^{\rm div} = \frac{1}{\delta}\times\frac{1}{2(4\pi)^{d/2}\Gamma(\frac{d}{2})}\int d^{2k+1}x\sum_{j=0}^{k}\frac{(-1)^{k+j+1}\Gamma(k-2j-\frac{1}{2})}{\Gamma(\frac{1}{2}-j)(k-j)!}M^{2k-2j}m^{2j+1}. \tag{4.8}$$

These results can be directly checked by disentangling the poles in spacetime dimensionality in the general inverse mass expansion (3.10). This calculation confirms the cancellation of the fictitious infrared divergences in even d mentioned above. Although these divergences have a spurious nature and only arise at intermediate calculational stages, their presence reflects a nontrivial analytic structure of the asymptotic mass expansion (3.10). As we will see below, they entail a nontrivial form of the heat kernel expansion corresponding to the generalized Neumann boundary conditions with second-order derivatives tangential to the boundary.

V. HEAT KERNEL EXPANSION

Because of the nontrivial pseudodifferential nature of the brane-to-brane operator (1.11) and the inverse mass expansion of its action (3.10) the heat kernel for the generalized Neumann boundary conditions (1.10) does not have a typical expansion in integer and half-integer powers of the proper time (1.12). Thus we assume a more general structure of this expansion in the form

$$\mathrm{Tr}_N^{(d+1)} e^{s\Box_{(d+1)}} = \frac{1}{(4\pi s)^{(d+1/2)}}\bigg(\int_{y\geq 0} d^{d+1}X + \sum_{\{p\}} s^p\int d^dx b_{\{p\}}\bigg), \tag{5.1}$$

where the summation runs over some unknown set of powers $\{p\}$ of the proper time with some unknown coefficients of the surface integrals $b_{\{p\}}$. The bulk integral here involves only one term corresponding to the only nonvanishing bulk Schwinger-DeWitt coefficient $a_0(X, X) = 1$ (which is independent of the boundary conditions and coincides with the one in spacetime without boundaries).

This expansion generates the inverse mass expansion for the effective action of the problem (1.1),

$$\frac{1}{2}\mathrm{Tr}_N\ln[M^2 - \Box_{(d+1)}] = -\frac{1}{2}\mathrm{Tr}_N\int_0^\infty\frac{ds}{s}e^{s\Box_{(d+1)}-sM^2}$$
$$= -\frac{1}{2}\bigg(\frac{M^2}{4\pi}\bigg)^{(d+1/2)}\bigg\{\Gamma\bigg(-\frac{d+1}{2}\bigg)\int_{y\geq 0}d^{d+1}X + \sum_{\{p\}}\frac{\Gamma(p-\frac{d+1}{2})}{M^{2p}}\int d^dx b_{\{p\}}\bigg\}. \tag{5.2}$$

Our goal now will be to determine the range of summation $\{p\}$ and the coefficients $b_{\{p\}}$ by comparing this expression with the inverse mass expansion of (1.6) known from (2.4) and (3.10).

We begin by considering the most interesting case of even dimensionality which under dimensional regularization reads as $d = 2k - \delta$, $\delta \to 0$. Even though the heat kernel is an ultraviolet finite object (which generates UV divergences in the action due to the divergence of the proper-time integration), we need this regularization to regulate intermediate infrared divergences which cancel out in the final answer. Thus, assembling together (2.4) and (3.10) we get

$$\frac{1}{2}\mathrm{Tr}_N\ln[M^2 - \Box_{(d+1)}] = -\frac{1}{2}\bigg(\frac{M^2}{4\pi}\bigg)^{(d+1/2)}\Gamma\bigg(-\frac{d+1}{2}\bigg)\int_{y\geq 0}d^{d+1}X + \bigg(\frac{M^2}{4\pi}\bigg)^{d/2}\int d^dx\bigg(\frac{1}{8}\Gamma(-d/2) + \sum_{j=0}^{\infty}\frac{2^{d/2-2j-1}}{d\Gamma(\frac{d}{2})}a_{2j}\frac{m^{2j+1}}{M^{2j+1}}$$
$$-\frac{1}{2}\Gamma(-d/2)\sum_{n=k}^{\infty}\bigg(-\frac{1}{2}\bigg)^{n-k}\frac{\varphi^{(n-k)}(0)}{(n-k)!}\frac{m^{n-\delta/2}}{M^{n-\delta/2}}\bigg), \qquad d = 2k - \delta. \tag{5.3}$$

Comparing it with (5.2) we immediately get the range of summation over the proper-time powers $\{p\}$ which we will label by the integer numbers j and n,

$$\{p\} = \begin{cases} 1/2, & \\ j, & j \geq 1, \\ n/2 - \delta/4, & n \geq k+1, \end{cases} \tag{5.4}$$

and the corresponding coefficients

$$b_{\{p\}} = \begin{cases} b_{1/2}, & \\ b_j, & j \geq 1, \\ b_{n/2-\delta/4} \equiv \tilde{b}_{n/2}, & n \geq k+1. \end{cases} \tag{5.5}$$

In terms of $b_{1/2}$, b_j and $\tilde{b}_{n/2}$ the heat kernel trace takes the form

$$\mathrm{Tr}_N^{(d+1)} e^{s\Box_{(d+1)}} = \frac{1}{(4\pi s)^{(d+1/2)}} \int_{y\geq 0} d^{d+1}X + \frac{1}{(4\pi s)^{(d+1/2)}} \times \int d^d x \bigg\{ b_{1/2} s^{1/2} + \sum_{j=1}^{\infty} b_j s^j + \sum_{n=k+1}^{\infty} \tilde{b}_{n/2} s^{(n-k)/2+d/4} \bigg\}. \tag{5.6}$$

The lowest order surface coefficient coincides with the Dirichlet one,

$$b_{1/2} = b_{1/2}^D \equiv -\frac{\sqrt{\pi}}{2}, \tag{5.7}$$

whereas the higher order ones take the following form in view of the known expressions (3.8) and (3.9) for the coefficients of the nonanalytic and analytic parts of the action,

$$b_j = \sqrt{\pi} \frac{\Gamma(k - 2j + 1 - \frac{\delta}{2})}{\Gamma(\frac{d}{2})\Gamma(\frac{3}{2} - j)} m^{2j-1}, \qquad j \geq 1, \tag{5.8}$$

$$\tilde{b}_{n/2} = \sqrt{\pi} \frac{\Gamma(1 - k + \frac{\delta}{2})}{\Gamma(\frac{3-n}{2} + \frac{\delta}{4})} \frac{1}{(n-1-k)!} m^{n-1-\delta/2}, \qquad n \geq k+1. \tag{5.9}$$

The first few coefficients b_j are ultraviolet finite,

$$b_j \to \sqrt{\pi} \frac{\Gamma(k - 2j + 1)}{\Gamma(k)\Gamma(\frac{3}{2} - j)} m^{2j-1}, \qquad \delta \to 0, \qquad 2 \leq 2j \leq k. \tag{5.10}$$

However, for $2j > k$ they are divergent,

$$b_j = -2^{4-2j}(-1)^{j+k} \frac{\Gamma(2j-2)}{\Gamma(j-1)\Gamma(2j-k)\Gamma(k)} m^{2j-1} \frac{1}{\delta} + \ldots, \qquad \delta \to 0, \tag{5.11}$$

but the complimentary coefficients $\tilde{b}_j$ have poles with residues that are exactly opposite in sign, $(b_j + \tilde{b}_j)^{\text{pole}} = 0$. This is certainly a manifestation of the cancellation of infrared divergences between the analytic part of (3.10) related to b_j and the nonanalytic part related to $\tilde{b}_j$. As a result the heat kernel stays well defined but acquires logarithmic terms in s because

$$b_j s^j + \tilde{b}_j s^{j-\delta/4} = s^j (\beta_j \ln(sm^2) + \gamma_j), \tag{5.12}$$

$$\beta_j = -2^{2-2j}(-1)^{j+k} \frac{\Gamma(2j-2)}{\Gamma(j-1)\Gamma(2j-k)\Gamma(k)} m^{2j-1}. \tag{5.13}$$

The rest of the coefficients with half-integer numbers $\tilde{b}_{j+1/2}$ are finite,

$$\tilde{b}_{j+1/2} = \sqrt{\pi}(-1)^{k+j} \frac{\Gamma(j)}{2\Gamma(2j+1-k)\Gamma(k)} m^{2j}. \tag{5.14}$$

Thus finally the heat kernel trace in even integer (unregulated) dimensionality of a brane $d = 2k$ takes the form

$$\mathrm{Tr}_N^{(d+1)} e^{s\Box_{(d+1)}} = \frac{1}{(4\pi s)^{(d+1/2)}} \int_{y\geq 0} d^{d+1}X + \frac{1}{(4\pi s)^{(d+1/2)}} \int d^d x \bigg\{ b_{1/2} s^{1/2} + \sum_{j=1}^{[k/2]} b_j s^j + \sum_{2j\geq k}^{\infty} \tilde{b}_{j+1/2} s^{j+1/2} + \sum_{2j\geq k+1}^{\infty} (\beta_j \ln(sm^2) + \gamma_j) s^j \bigg\}. \tag{5.15}$$

In odd dimensions, $d = 2k + 1$, the poles of infrared nature are absent, so that both b_j and $\tilde{b}_{n/2}$ are finite, and no logarithmic terms arise in the heat kernel. One can check that its trace has explicitly the form (5.6) with $d = 2k + 1$. Thus it has not only half-integer powers of the proper time, but also powers multiple of a quarter.

Note that $b_{1/2}$ for all *finite* m has the value (5.7) characteristic of the Dirichlet problem. At the same time, for $m \to \infty$ our problem reduces to the case of Neumann conditions, cf. Equation (1.10), corresponding to $b_{1/2}^N = +\sqrt{\pi}/2$ [17]. However, this does not present any contradiction, because the asymptotic expansion in $1/M$ is obviously not homogeneous in $m \to \infty$, as it involves growing positive powers of m. To analyze the limit of large m or $\varepsilon \to \infty$, which includes the case of the massless DGP model, $M = 0$, we have to consider another representation of the effective action. This is discussed in the next section.

VI. MASSLESS LIMIT: EFFECTIVE POTENTIAL IN A TOY DGP MODEL

As shown in Appendix B, the brane part of the effective action (2.7) can be exactly represented in terms of the hypergeometric function. This representation has the form

$$\frac{1}{2}\,\mathrm{Tr}^{(d)}\ln\boldsymbol{F}^{\mathrm{brane}}=\frac{1}{d\Gamma(d/2)}\int d^dx\,\frac{\pi}{\sin\frac{\pi d}{2}}\bigg(\frac{Mm}{4\pi}\bigg)^{d/2}\varphi(-\varepsilon)-\bigg(\frac{M^2}{4\pi}\bigg)^{d/2}\frac{\Gamma(\frac{1-d}{2})}{2d\sqrt{\pi}}\int d^dx\sum_{v=v_\pm}vF\bigg(1,\frac{1-d}{2};1-\frac{d}{2};1-v^2\bigg),\tag{6.1}$$

where $F(a, b; c; u)$ is a hypergeometric function given by (B4), and its argument u is defined in terms of the following two functions of $\varepsilon = m/2M$,

$$v_\pm\equiv\pm\sqrt{1+\varepsilon^2}-\varepsilon=-1/v_\mp.\tag{6.2}$$

This representation exactly recovers the nonanalytic part of the $1/M$-expansion (3.10), whereas the analytic part of the latter follows from the known hypergeometric series $F(a, b; c; u) = 1 + O(u)$ in powers of the argument $u = 1 - v_\pm^2 \sim \varepsilon$.

In the opposite limit of large ε (corresponding to large m or small M) the brane action can be equivalently represented in a form useful for the expansion in $v_+ \sim 1/2\varepsilon \ll 1$. As shown in Appendix B by using the transformation formulas for $F(a, b; c; u)$ it reads

$$\frac{1}{2}\,\mathrm{Tr}^{(d)}\ln\boldsymbol{F}^{\mathrm{brane}}=\bigg(\frac{M^2}{4\pi}\bigg)^{d/2}\int d^dx\bigg\{-\frac{1}{4}\Gamma(-d/2)(1-v_+^2)^{d/2}+\frac{\pi}{2\Gamma(1+\frac{d}{2})\sin(\pi d)}\frac{(1-v_+^2)^{d/2}}{v_+^d}+\frac{\Gamma(-\frac{1+d}{2})}{4\sqrt{\pi}}v_+\bigg[(d+1)F\bigg(1,\frac{1-d}{2};\frac{3}{2};v_+^2\bigg)-F\bigg(1,\frac{1}{2};\frac{3+d}{2};v_+^2\bigg)\bigg]\bigg\}.\tag{6.3}$$

Using this representation one can consider the limit of exactly Neumann boundary conditions corresponding to $m \to \infty$ and a finite value of the mass M in the bulk. In this limit $v_+ \to 0$ and the last two terms of (6.3) vanish. Naively, the second term $\sim\varepsilon^d$ is growing to infinity, but in the domain of ultraviolet convergence $d < 0$, so that with the appropriate order of taking the limits (first in $m \to \infty$ and second in the dimensionality) it vanishes, too. Thus, only the first term remains,

$$\frac{1}{2}\,\mathrm{Tr}^{(d)}\ln\boldsymbol{F}^{\mathrm{brane}}=-\frac{1}{4}\Gamma(-d/2)\bigg(\frac{M^2}{4\pi}\bigg)^{d/2}\int d^dx,\tag{6.4}$$

which is obviously one-half of the contribution of the d-dimensional massive particle corresponding to the brane-to-brane mode propagating with the square-root operator $\boldsymbol{F}^{\mathrm{brane}} = \sqrt{M^2 - \Box}$,

$$\frac{1}{2}\,\mathrm{Tr}^{(d)}\ln\boldsymbol{F}^{\mathrm{brane}}=\frac{1}{4}\,\mathrm{Tr}^{(d)}\ln(M^2-\Box).\tag{6.5}$$

When added to the Dirichlet effective action (2.4) it alters the sign of the brane (d-dimensional integral) term, which corresponds to the transition from the Dirichlet value of the $b_{1/2}^D$ surface coefficient (5.7) to the Neumann value,

$$b_{1/2}^N=\frac{\sqrt{\pi}}{2}.\tag{6.6}$$

Another interesting limit of $\varepsilon \to \infty$ corresponds to the massless case of the DGP model with $M = 0$. In this case the only nonvanishing term is contained in the second term of (6.3), and it yields

$$\frac{1}{2}\,\mathrm{Tr}^{(d)}\ln\boldsymbol{F}^{\mathrm{brane}}=\frac{1}{2}\bigg(\frac{m^2}{4\pi}\bigg)^{d/2}\frac{\pi}{\Gamma(1+\frac{d}{2})\sin\pi d}\int d^dx,\tag{6.7}$$

because $M/v_+ \to 2\varepsilon M = m$. This result can be confirmed by a direct calculation of the effective potential of the brane mode, circumventing the operation of taking the limit $M \to 0$ in the general answer (6.3).

Indeed, the effective potential $V_{\mathrm{eff}}(m)$,

$$\frac{1}{2}\,\mathrm{Tr}^{(d)}\ln\boldsymbol{F}^{\mathrm{brane}}=\int d^dxV_{\mathrm{eff}}(m),\tag{6.8}$$

for the brane operator in the toy DGP model, $\boldsymbol{F}^{\mathrm{brane}} = (-\Box + m\sqrt{-\Box})/m$, can be written down in the form of a momentum-space integral similar to (2.7),

$$V_{\rm eff}(m) = \frac{1}{(4\pi)^{d/2}\Gamma(\frac{d}{2})}\int_0^\infty dp\, p^{d-1}\ln(p^2+mp)$$
$$= \left(\frac{m^2}{4\pi}\right)^{d/2}\frac{1}{\Gamma(\frac{d}{2})}\int_0^\infty dt\, t^{d-1}[\ln(t+1)+\ln t+\ln m^2]. \tag{6.9}$$

Within the dimensional regularization only the first logarithmic term gives a nonvanishing contribution which we transform by using the proper-time representation of the logarithm and changing the order of integrations,

$$V_{\rm eff}(m) = -\left(\frac{m^2}{4\pi}\right)^{d/2}\frac{1}{\Gamma(\frac{d}{2})}\int_0^\infty dt\, t^{d-1}\int_0^\infty \frac{ds}{s}e^{-st-s}$$
$$= \frac{1}{2}\left(\frac{m^2}{4\pi}\right)^{d/2}\frac{1}{\Gamma(\frac{d}{2}+1)}\frac{\pi}{\sin\pi d}. \tag{6.10}$$

After subtracting the ultraviolet divergence in the limit of the physical dimensionality, $d\to N$, this gives the renormalized effective potential of the usual Coleman-Weinberg structure,

$$V_{\rm eff}(m) = \frac{1}{2}\frac{(-1)^N}{\Gamma(\frac{N}{2}+1)}\left(\frac{m^2}{4\pi}\right)^{N/2}\ln\frac{m}{\mu}. \tag{6.11}$$

Here μ is the parameter reflecting the renormalization ambiguity, and the role of the field (argument of the potential) is played by the scale m of the brane term in the classical action (1.7)—the analogue of the DGP scale. This result confirms the boundary effective action calculation of [5].

VII. CONCLUSIONS

In conclusion, we have derived the one-loop effective action in a simplified model of brane-induced gravity which gives rise to special boundary conditions involving second-order tangential derivatives. The main peculiarity of this action is the presence of logarithmic ultraviolet divergences for both even and odd dimensionalities of the spacetime. This is different from analogous one-loop calculations in spacetimes without boundaries leading to divergences only for even spacetime dimensionalities. The action has a nontrivial analytic structure in the mass parameter—for generic M and m it is given by two representations in terms of hypergeometric functions (6.1) and (6.3) and simplifies for the case of the massless field in the bulk, $M=0$, to the form (6.10). After ultraviolet renormalization this gives rise to the familiar logarithmic Coleman-Weinberg effective potential $\sim\varphi^d\ln(\varphi^2/\mu^2)$ with the field φ played by the parameter m in the boundary conditions (1.10)—the result used in [5] for the stabilization of the DGP crossover scale in the Brans-Dicke modification of the DGP model [26].

We also derived the proper-time expansion for the functional trace of the heat kernel subject to these generalized Neumann boundary conditions. This turns out to be nontrivial, because for even dimensionalities of the boundary it involves together with the well-known half-integer powers of the proper time s also the logarithmic terms $\sim\ln s$, cf. Equation (5.15), and for odd dimensionalities contains powers of s which are multiples of one quarter, see Eq. (5.6). Such peculiarities of the proper-time expansion are usually associated with the lack of strong ellipticity of the boundary value problem [24] when a naively positive elliptic operator acquires due to the presence of the boundary an infinite set of negative modes. These modes make the heat kernel operator unbounded and violate usual assumptions underlying its proper-time expansion. But, as we show in Appendix C, strong ellipticity of our problem gets violated only for negative m in (1.10)—when the relevant classical action (1.7) and the brane-to-brane operator $\boldsymbol{F}^{\rm brane}$, Eq. (1.11), are both not positive definite. However, the heat kernel expansions become exotic also for $m>0$ with no violation of strong ellipticity, which implies deeper reasons of these peculiarities.

To summarize, we conclude that the technique for quantum effects in brane models is more complicated than in systems without boundaries. Moreover, it does not reduce to a simple bookkeeping of surface terms in the heat kernel expansion of [17,19,21], and so on, because of the complicated square-root structure of the brane propagator (1.11) mediating the effect of the generalized Neumann boundary conditions (1.10). The proper-time method that was fundamentally efficient in models without boundaries [14,16] in our calculations above became a derivative of an alternative calculation. Namely, the surface terms in the heat kernel expansion were recovered from the $1/M$-expansion of the action obtained by a different method of a Fourier decomposition.

Nevertheless, the proper-time method still does not loose its power and can be used in realistic brane models including gravity. In these models the effective action should be expanded in powers of the bulk spacetime curvature and the extrinsic curvature of the brane, starting with the approximation considered above. The momentum-space decomposition is not very efficient for sake of such an expansion—the difficulty usually circumvented in background field formalism by the use of the Schwinger-DeWitt proper-time method [14–16]. Here we present without derivation (that will be given in forthcoming publications) such a representation for the Green's function of the DGP brane-to-brane operator and its functional determinant. They have a form of the weighted proper-time integrals

$$\frac{1}{\Box - +m\sqrt{-\Box}} = \int_0^\infty ds\, e^{s\Box}w(s), \tag{7.1}$$

$$\mathrm{Tr}\,\ln(-\Box+m\sqrt{-\Box}) = -\mathrm{Tr}\int_0^\infty\frac{ds}{s}e^{s\Box}\frac{1+w(s)}{2}, \tag{7.2}$$

with the weight function $w(s)$ given in terms of the error function $\Phi(x) = \frac{2}{\sqrt{\pi}} \int_0^x dy \exp(-y^2)$ and having the following ultraviolet and infrared asymptotics

$$w(s) = e^{sm^2}(1 - \Phi(m\sqrt{s})) \to \begin{cases} 1, & m\sqrt{s} \to 0, \\ 1/m\sqrt{s\pi}, & m\sqrt{s} \to \infty. \end{cases} \tag{7.3}$$

The advantage of this representation[1] is that it applies also to the case of the curved-space d'Alembertian $\Box$, so that the generally covariant expansion of (7.1) and (7.2) in curvatures can be directly obtained by using a well-known Schwinger-DeWitt expansion for $e^{s\Box}$. Thus the lowest order approximation for the exact brane-to-brane operator (1.5) in models with a curved bulk and curved branes can be considered by means of the manifestly covariant technique which can be systematically extended to higher orders. Combined with the method of fixing the background covariant gauge for diffeomorphism invariance in brane models, developed in [27], this will ultimately give the universal background field method of the Schwinger-DeWitt type in gravitational brane systems.

ACKNOWLEDGMENTS

A. B. thanks D. Vassilevich for helpful thought provoking discussions. A. B. and A. Yu. K. are grateful for the hospitality of the Institute for Theoretical Physics at the University of Cologne where a major part of this work has been done under the grant No. 436 RUS 17/8/06 of the German Science Foundation (DFG). A. B. was partially supported by the RFBR grant No. 05-01-00996 and the LSS grant No 4401.2006.2. A. K. was partially supported by the RFBR grant No. 05-02-17450 and the LSS grant No. 1757.2006.2. D. N. was supported by the RFBR grant No. 05-02-17661 and the LSS grant No. 4401.2006.2 and also thanks the Center of Science and Education of the Lebedev Institute and the target funding program of the Presidium of Russian Academy of Sciences for support. At the completing stage of this work A. B. was also supported by the SFB 375 grant at the Physics Department of the Ludwig-Maximilians University in Munich.

APPENDIX A: ASYMPTOTIC EXPANSION FOR INTEGRALS WITH A WEAK SINGULARITY

The analytic and nonanalytic parts of the asymptotic expansion for the integral (3.4) (a so-called integral with a weak singularity) can be found by the method of [25]. For an integral of a slightly more general form this expansion reads

$$\int_0^\infty dt t^{\beta-1}(t+\varepsilon)^\alpha \varphi(t) = \varepsilon^{\alpha+\beta} \sum_{n=0}^\infty \frac{\Gamma(n+\beta)\Gamma(-\alpha-\beta-n)}{\Gamma(-\alpha)} \frac{\varphi^{(n)}(0)}{n!} \varepsilon^n + \sum_{n=0}^\infty a_n \varepsilon^n, \qquad \varepsilon \to 0. \tag{A1}$$

Here $\varphi(t)$ is a function which is analytic at $t = 0$ and has a Taylor series with coefficients $\varphi^{(n)}(0) \equiv d^n\varphi/dt^n(0)$. The parameter β is positive in order to guarantee the convergence of the integral at $t = 0$. The first sum gives a nonanalytic part of the expansion determined entirely by the derivatives of the function φ, $\varphi^{(n)}(0)$, whereas the second sum determines the analytic part with coefficients involving a (nonlocal) dependence of the function $\varphi(t)$ at all t. These coefficients are given by the following expression:

$$a_n = \frac{\alpha(\alpha-1)\ldots(\alpha-n+1)}{n!} \left\{ \int_0^\infty dt \varphi_S(t) t^{\alpha+\beta-n-1} + \sum_{m=0}^{[n-\alpha-\beta]} \frac{T^{\alpha+\beta+m-n}}{\alpha+\beta+m-n} \frac{\varphi^{(m)}(0)}{m!} \right\}, \tag{A2}$$

where $\varphi_S(t)$ is a piecewise smooth function obtained from $\varphi(t)$ by subtracting its first few terms of the Taylor expansion at $t = 0$ on a finite segment of the t-axes, $0 \le t < T$,

$$\varphi_S(t) = \varphi(t) - \sum_{m=0}^{[n-\alpha-\beta]} \frac{\varphi^{(m)}(0)}{m!} t^m, \qquad t < T, \qquad \varphi_S(t) = \varphi(t), \qquad t \ge T. \tag{A3}$$

Here the number of subtracted terms is given by $[n - \alpha - \beta]$—the integer part of $n - \alpha - \beta$, and T is arbitrary positive. The value of the latter is immaterial, because it is easy to check that $\partial a_n/\partial T = 0$. These subtractions are necessary to guarantee the convergence of the integrals in (A2) at $t = 0$. For the first few a_n these subtractions are absent,

$$a_n = \frac{\alpha(\alpha-1)\ldots(\alpha-n+1)}{n!} \int_0^\infty dt \varphi(t) t^{\alpha+\beta-n-1}, \qquad n < \alpha + \beta, \tag{A4}$$

while for $n > \alpha + \beta$ their effect can be explicitly circumvented by multiple integrations by parts in (A2). After $(N_n + 1)$ integrations by parts, all nonintegral terms of (A2) cancel out and the expansion coefficients take the following alternative form,

[1]Note, in passing, that the interpretation of this weight and its asymptotics is very transparent. In the ultraviolet domain of small proper time $m\sqrt{s} \ll 1$ (or big $\sqrt{-\Box} \gg m$) it approximates the brane operator by $-\Box$, whereas in the infrared domain $m\sqrt{s} \gg 1$ (or $\sqrt{-\Box} \ll m$) it corresponds to its low-energy behavior $m\sqrt{-\Box}$. All the results above could be obtained with the aid of this representation generalized to the case of a nonzero M. We did not use it, however, because this generalization has a complicated weight function.

$$a_n = \frac{\alpha(\alpha-1)\ldots(\alpha-n+1)}{n!}\frac{\Gamma(n-\alpha-\beta-N_n+1)}{\Gamma(n-\alpha-\beta+1)} \times \frac{1}{n-\alpha-\beta-N_n}\int_0^\infty dt\varphi^{(N_n+1)}(t)t^{\alpha+\beta-n+N_n}, \quad n>\alpha+\beta, \tag{A5}$$

where

$$N_n = [n-\alpha-\beta]. \tag{A6}$$

Finally, when $\alpha+\beta$ is a positive integer N the two sums of (A1) formally become analytic, but their coefficients develop pole singularities in $\alpha+\beta-N\to 0$. These singularities come from these two sums with opposite signs and cancel. The finite remnant of this cancellation is a term logarithmic in ε.

If we apply now this asymptotic expansion to the case of our integral (3.3) and (3.4) with

$$\varphi(t) = (\sqrt{1+t^2}+t)^{d/2}, \qquad \alpha=-1, \qquad \beta=\frac{d}{2}, \tag{A7}$$

then in view of

$$\frac{\Gamma(n+\beta)\Gamma(-\alpha-\beta-n)}{\Gamma(-\alpha)} = (-1)^n\frac{\pi}{\sin(\pi d/2)}, \tag{A8}$$

the nonanalytic part is explicitly expressed in terms of the function $\varphi(-\varepsilon)$ with the flipped sign of its argument,

$$\frac{\pi\varepsilon^{d/2-1}}{\sin(\pi d/2)}\sum_{n=0}^{\infty}\frac{\varphi^{(n)}(0)}{n!}(-\varepsilon)^n = \frac{\pi\varepsilon^{d/2-1}\varphi(-\varepsilon)}{\sin(\pi d/2)}. \tag{A9}$$

The coefficients of the analytic part can be explicitly calculated by taking the integrals (A4) in the domain $n<d/2-1$ and extended beyond this domain by analytic continuation (which is equivalent to using (A5)). The result is

$$a_n = 2^{n-1-d/2}(-1)^{n+1}d\frac{\Gamma(\frac{n-d+1}{2})\Gamma(\frac{d}{2}-1-n)}{\Gamma(-\frac{n-1}{2})}. \tag{A10}$$

Therefore for odd $n=2j+1$ they vanish due to the unregulated (by the dimensionality d) infinity in the denominator, whereas for even $n=2j$ they are given by (3.9), and the final form of the expansion for I is given by (3.7).

APPENDIX B: HYPERGEOMETRIC FUNCTION REPRESENTATION

The integral (2.7) can be rewritten in terms of the integration variable $x=\sqrt{p^2+M^2}/M$ as

$$\int_0^\infty dpp^{d-1}\ln(p^2+m\sqrt{M^2+p^2}) = M^d\int_1^\infty dxx(x^2-1)^{d/2-1}\ln(x^2+2\varepsilon x-1). \tag{B1}$$

By factorizing the argument of the logarithm and integrating the result by parts we convert it to the sum of two terms,

$$\int_1^\infty dxx(x^2-1)^{d/2-1}\ln(x^2+2\varepsilon x-1) = \sum_{v=v_\pm}\int_1^\infty dxx(x^2-1)^{d/2-1}\ln(x-v) = -\frac{1}{d}\sum_{v=v_\pm}\int_1^\infty dx(x^2-1)^{d/2}(x-v)^{-1}, \tag{B2}$$

where $v_\pm$ are the roots (6.2) of the quadratic polynomial $x^2+2\varepsilon x-1=(x-v_+)(x-v_-)$.

With the change of the integration variables $x=1/\sqrt{t}$ we have

$$\int_1^\infty dx(x^2-1)^{d/2}(x-v)^{-1} = \frac{1}{2}\int_0^1 dtt^{-1-d/2}(1-t)^{d/2}(1-tv^2)^{-1} + \frac{v}{2}\int_0^1 dtt^{-1/2-d/2}(1-t)^{d/2}(1-tv^2)^{-1}, \tag{B3}$$

so that finally in terms of the hypergeometric function

$$F(a,b;c;u) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 dtt^{b-1}(1-t)^{c-b-1}(1-tu)^{-a} \tag{B4}$$

the basic integral reads as

$$\int_1^\infty dx(x^2-1)^{d/2}(x-v)^{-1} = \frac{\Gamma(-\frac{d}{2})\Gamma(1+\frac{d}{2})}{2\Gamma(1)}F\left(1,-\frac{d}{2};1;v^2\right) + \frac{v}{2}\frac{\Gamma(1+\frac{d}{2})\Gamma(\frac{1-d}{2})}{\Gamma(\frac{3}{2})}F\left(1,\frac{1-d}{2};\frac{3}{2};v^2\right). \tag{B5}$$

We will need its ε-expansion. At $\varepsilon\to 0$ the parameter $v=v_\pm\to\pm 1$, so we have to transform the hypergeometric functions to the series in $1-v^2$. Because of the known relation $F(a,b;a;u)=F(b,a;a;u)=(1-u)^{-b}$ we have

$$F\left(1,-\frac{d}{2};1;v^2\right) = (1-v^2)^{d/2} \tag{B6}$$

and in view of Eq. 9.131.2 of [28],

$$F\left(1,\frac{1-d}{2};\frac{3}{2};v^2\right)=\frac{\Gamma(\frac{3}{2})\Gamma(\frac{d}{2})}{\Gamma(\frac{1}{2})\Gamma(1+\frac{d}{2})}F\left(1,\frac{1-d}{2};1-\frac{d}{2};1-v^2\right)+(1-v^2)^{d/2}\frac{\Gamma(\frac{3}{2})\Gamma(-\frac{d}{2})}{\Gamma(1)\Gamma(\frac{1-d}{2})}F\left(\frac{1}{2},1+\frac{d}{2};1+\frac{d}{2};1-v^2\right), \tag{B7}$$

where again $F(\frac{1}{2},1+\frac{d}{2};1+\frac{d}{2};1-v^2)=[1-(1-v^2)]^{-1/2}=|v|^{-1}$. Therefore

$$F\left(1,\frac{1-d}{2};\frac{3}{2};v^2\right)=-\frac{\sqrt{\pi}}{d}\frac{\Gamma(1-\frac{d}{2})}{\Gamma(\frac{1-d}{2})}\frac{(1-v^2)^{d/2}}{|v|}+\frac{1}{d}F\left(1,\frac{1-d}{2};1-\frac{d}{2};1-v^2\right) \tag{B8}$$

Substituting (B6) and (B8) into (B5) we have

$$\int_1^\infty dx(x^2-1)^{d/2}(x-v)^{-1}=-\frac{\pi}{\sin(\pi d/2)}\theta(v)(1-v^2)^{d/2}+\frac{1}{2\sqrt{\pi}}\Gamma\left(\frac{d}{2}\right)\Gamma\left(\frac{1-d}{2}\right)vF\left(1,\frac{1-d}{2};1-\frac{d}{2};1-v^2\right), \tag{B9}$$

where a step function $\theta(v)$ arose as the result of summation of two terms,

$$\theta(v)=\frac{1}{2}\left(1+\frac{v}{|v|}\right). \tag{B10}$$

The first term in (B9) exists only for positive v and is nonanalytic at $v\to 1$, whereas the second term is analytic. Obviously for negative v this integral is an analytic function because the argument of the logarithm nowhere tends to zero in the integration domain. This explains the absence of the first term for $v<0$.

Substituting (B9) to (B1) and (B2) we finally get

$$\int_0^\infty dp p^{d-1}\ln(p^2+m\sqrt{M^2+p^2})=\frac{M^d}{d}\frac{\pi}{\sin(\pi d/2)}(1-x_+^2)^{d/2}-\frac{M^d}{d}\frac{\Gamma(\frac{d}{2})\Gamma(\frac{1-d}{2})}{2\sqrt{\pi}}\sum_{v=v_\pm}vF\left(1,\frac{1-d}{2};1-\frac{d}{2};1-v^2\right). \tag{B11}$$

Bearing in mind that $(1-v_+^2)^{d/2}=(2\varepsilon v_+)^{d/2}=(2\varepsilon)^{d/2}\varphi(-\varepsilon)$ we finally have the representation useful for small ε,

$$\int_0^\infty dp p^{d-1}\ln(p^2+m\sqrt{M^2+p^2})=\frac{M^d}{d}\frac{\pi}{\sin(\pi d/2)}(2\varepsilon)^{d/2}\varphi(-\varepsilon)-\frac{M^d}{d}\frac{\Gamma(\frac{d}{2})\Gamma(\frac{1-d}{2})}{2\sqrt{\pi}}\sum_{v=v_\pm}vF\left(1,\frac{1-d}{2};1-\frac{d}{2};1-v^2\right), \tag{B12}$$

which gives rise to the representation (6.1).

To consider the limit of $\varepsilon\to\infty$ we need another representation, because in this limit $v_+\to 1/2\varepsilon$ and $v_-\sim -2\varepsilon\to -\infty$, so that the contribution of $v=v_-$ in (B2) should be expandable in $1/v$. This can be achieved by the transformation formula 9.132.2 of [28] which in our case reads as

$$F\left(1,\frac{1-d}{2};\frac{3}{2};v^2\right)=\frac{1}{1+d}\frac{1}{v^2}F\left(1,\frac{1}{2};\frac{3+d}{2};\frac{1}{v^2}\right)+\frac{\sqrt{\pi}}{2}\frac{\Gamma(\frac{1+d}{2})}{\Gamma(1+\frac{d}{2})}\frac{(1-v^2)^{d/2}}{(-v^2)^{1/2}}. \tag{B13}$$

Using this in the representation (1.6) of the $v=v_-$ term of (B2) and taking into account that $1/v_-=-v_+$ we finally arrive at the equation underlying the representation (6.3):

$$\int_0^\infty dp p^{d-1}\ln(p^2+m\sqrt{M^2+p^2})=\frac{\pi M^d}{2d\sin(\pi d/2)}(1-v_+^2)^{d/2}+\frac{\pi M^d}{d\sin(\pi d)}\frac{(1-v_+^2)^{d/2}}{v_+^d}+M^d\frac{\Gamma(\frac{d}{2})\Gamma(-\frac{1+d}{2})}{4\sqrt{\pi}}v_+\left[(d+1)F\left(1,\frac{1-d}{2};\frac{3}{2};v_+^2\right)-F\left(1,\frac{1}{2};\frac{3+d}{2};v_+^2\right)\right]. \tag{B14}$$

APPENDIX C: STRONG ELLIPTICITY PROBLEM

The strong ellipticity problem for the operator (1.8) with generalized Neumann boundary condition (1.10) consists in the existence of an infinite set of normalizable eigenmodes with a spectrum which is unbounded from below [24]. This implies that the relevant heat kernel is an unbounded operator which cannot be rendered bounded by the elimination of a finite number of states from its functional space, and therefore it has an unusual proper-time asymptotics different from (1.12). This situation occurs when the parameter m in the operator (1.9) of the boundary condition is negative.[2] Here we show that these negative modes correspond to the negative modes of the brane-to-brane operator (1.11), localized near the brane/boundary and responsible for brane-to-brane propagation.

Indeed, in this case there is a set of eigenmodes localized near the boundary $y = 0$ of the form

$$\Phi_p(x, y) = e^{ipx-\lambda_p y} \tag{C1}$$

in which λ_p is given on account of the boundary condition by the expression

$$\lambda_p = -\frac{p^2}{m} > 0, \qquad m < 0. \tag{C2}$$

Negative m guarantees the normalizability of these eigenmodes which, therefore, cannot be excluded from the functional space of the operator. Their eigenvalues Λ_p,

$$(M^2 - \Box_{(d+1)})\Phi_p(X) = \Lambda_p \Phi_p(X), \tag{C3}$$

$$\Lambda_p = \left(M^2 + p^2 - \frac{(p^2)^2}{m^2}\right), \tag{C4}$$

are negative for sufficiently high Fourier momenta p,

$$\Lambda_p < 0, \qquad p^2 > \frac{m^2}{2}(1 + \sqrt{1 + 4M^2/m^2}), \tag{C5}$$

and tend to $-\infty$ for $p^2 \to \infty$. But the momentum-space domain where they are negative exactly coincides with the domain in which the brane operator is negative definite for $m < 0$,

$$\boldsymbol{F}^{\text{brane}}\varphi_p(x) = \left(\frac{p^2}{m} + \sqrt{M^2 + p^2}\right)\varphi_p(x),\ \varphi_p(x) = e^{ipx}. \tag{C6}$$

Thus, the lack of strong ellipticity of the generalized Neumann boundary value problem is in fact the lack of positivity of the action (1.7) with $m < 0$, from which this problem originates by the variational procedure.

[2] We are grateful to D. Vassilevich for pointing out this example of strong ellipticity violation.

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