T duality of the Zamolodchikov-Zamolodchikov brane

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We examine how nonperturbative effects in string theory are transformed under the T-duality in its nonperturbative framework by analyzing the $c = 1/2$ noncritical string theory as a simplest example. We show that in the T-dual theory they also take the form of $\exp(-S_0/g_s)$ in the leading order and that the instanton actions S_0 of the dual Zamolodchikov-Zamolodchikov (ZZ)-branes are exactly the same as those in the original $c = 1/2$ string theory. Furthermore we present formulas for coefficients of $\exp(-S_0/g_s)$ in the dual theory.

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I. INTRODUCTION

T-duality is a relation between perturbative vacua in string theory and is known to hold at each order in perturbation theory for the critical string theory [1]. Since it is a characteristic feature of string theory, it is believed to play an important role even in constructing its nonperturbative formulation. From this point of view, it is quite intriguing to examine how nonperturbative effects are transformed under the T-duality in a possible nonperturbative framework of string theory. As such a framework, the noncritical string theory provides a useful toy model, because it can be formulated nonperturbatively by matrix models and their nonperturbative effects are identified as a kind of instanton effects [2–7]. Moreover, we can explicitly formulate the T-duality in the noncritical string theory, for example, in the $c = 1/2$ string theory as discussed in [8].

However, as for the $c = 1/2$ noncritical string theory, which does not have a continuous target space, it has been shown explicitly in [8] that the theory is not invariant under the T-duality for lack of global winding modes associated to string world sheets of higher genus topology. This viewpoint for the T-duality still stays at the perturbative world sheet picture, although the corresponding string field theory has been explicitly constructed. Therefore, as a first step to understanding the T-duality truly at the nonperturbative level, it is important and interesting to identify nonperturbative effects in the dual $c = 1/2$ string theory and to clarify how they correspond to those in the original $c = 1/2$ string theory. This is the aim of this paper.

Since as shown in Refs. [8,9] the dual theory is defined by the $O(n)$ model on a random surface [10] with $n = 1$, we analyze this model and show that the leading nonperturbative effects in the dual theory also take the same form of $\exp(-S_0/g_s)$ as in the original theory, where g_s is the string coupling constant. This form of the nonperturbative effects in general reflects the large-order behavior of perturbation series in string theory [11]. Thus, we find that the perturbation series in the dual theory also shows the same large-order behavior. Moreover, for the standard $c < 1$ noncritical string theory, the values of S_0 are deduced from the string equations [2,12,13], or directly from matrix models as their instanton effects mentioned above. Other techniques deriving S_0 are also found in Refs. [14,15]. For the $O(n)$ model with general n on a random surface, in contrast to the matrix models above, the string equation has not been derived (to the best of our knowledge), from which the nonperturbative effects in the dual theory can be seen. However, since the total free energies of the original theory and the dual theory are equivalent by definition for the case $n = 1$,[1] the free energy of the dual theory should satisfy the same string equation as that of the original theory. Therefore, we expect that the dual theory has exactly the same values of S_0 as those of the original theory. We will see that this is indeed the case. Since it is known in the standard noncritical string theories that S_0 can be identified [16–20] as the classical actions of the Zamolodchikov-Zamolodchikov (ZZ)-branes [21], we can conclude that the classical actions of the dual ZZ-branes are the same as those of the original theory, and it gives an important basis in determining the T-duality transformation rule of the ZZ-branes.

The organization of this paper is as follows. In the next section, we formulate the T-duality in the $c = 1/2$ noncritical string theory in terms of the two-matrix model, and we see that the dual theory is defined by the $O(n)$ model on a random surface with $n = 1$. In Sec. III, we make the saddle point analysis in the $O(1)$ model and derive nonperturbative effects in the dual $c = 1/2$ string theory. We proceed to examine the next-to-leading contribution to

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[1]Because of the subtlety of well-definedness for the matrix integrals in the double scaling limit, we can safely say that the equivalence holds in the sense of the $1/N$-expansion.

these nonperturbative effects in Sec. IV. The last section is devoted to the conclusions and discussions. In Appendix A, we explain the double scaling limits of the original and dual two-matrix models, and present a derivation of resolvents in the dual model. Some other computational details are discussed in Appendix B.

II. *T*-DUALITY IN THE $c = 1/2$ NONCRITICAL STRING THEORY

The original $c = 1/2$ string theory is defined by the double scaling limit [22] of a two-matrix model

$$\begin{aligned} Z &= \int d^{N^2}A d^{N^2}B \exp[-N \operatorname{tr} S(A, B)], \\ S(A, B) &= \tfrac{1}{2}A^2 + \tfrac{1}{2}B^2 - cAB - \frac{g}{3}A^3 - \frac{g}{3}B^3, \end{aligned} \tag{2.1}$$

where A, B are the $N \times N$ Hermitian matrices. This model describes the Ising model on a random surface [23] with the inverse temperature given by

$$\beta = -\tfrac{1}{2}\log c. \tag{2.2}$$

Various amplitudes in the $c = 1/2$ string theory can be computed based on (2.1), for example by solving the Schwinger-Dyson equations as discussed in Refs. [24,25].

In order to perform the T-duality transformation to this model, we first note that the familiar T-duality transformation in closed string theories compactified on a circle is basically the same as the Kramers-Wannier dual transformation to the XY-model defined on the string world sheet (see, e.g. [26] or Appendix A in [8]). From this point of view, the T-duality transformation for this model will be naturally formulated as the Kramers-Wannier transformation of the Ising model on a random surface. Since the matrices A and B can be regarded as the up and down spins on a random surface, respectively, the Boltzmann factors of the original Ising model on a random surface are related to the bare propagators for A and B as

$$\begin{aligned} \langle A_{ij}A_{kl}\rangle_{\rm bare} &= \langle B_{ij}B_{kl}\rangle_{\rm bare} = \frac{1}{N}\delta_{il}\delta_{jk}Le^{\beta}, \\ \langle A_{ij}B_{kl}\rangle_{\rm bare} &= \frac{1}{N}\delta_{il}\delta_{jk}Le^{-\beta}, \end{aligned} \tag{2.3}$$

with $L = \sqrt{c}/(1-c^2)$. Therefore, the Boltzmann factors in the dual model should be proportional to $e^{\pm\tilde{\beta}}$, where $\tilde{\beta}$ is obtained by the $\mathbf{Z}_2$ Fourier transformation

$$e^{\beta} = K(e^{\tilde{\beta}} + e^{-\tilde{\beta}}), \qquad e^{-\beta} = K(e^{\tilde{\beta}} - e^{-\tilde{\beta}}). \tag{2.4}$$

K is an overall normalization constant given by $K = (e^{2\tilde{\beta}} - e^{-2\tilde{\beta}})^{1/2}$. It is easy to introduce matrices realizing such bare propagators. If we define new matrix variables

$$X = \frac{1}{\sqrt{2}}(A + B), \qquad Y = \frac{1}{\sqrt{2}}(A - B), \tag{2.5}$$

then their bare propagators exactly give the Boltzmann factors of the dual model:

$$\begin{aligned} \langle X_{ij}X_{kl}\rangle_{\rm bare} &= \frac{1}{N}\delta_{il}\delta_{jk}\frac{1}{\sqrt{1-c^2}}e^{\tilde{\beta}}, \\ \langle Y_{ij}Y_{kl}\rangle_{\rm bare} &= \frac{1}{N}\delta_{il}\delta_{jk}\frac{1}{\sqrt{1-c^2}}e^{-\tilde{\beta}}. \end{aligned} \tag{2.6}$$

In this sense, these represent "stick" or "flip" of the dual spin, respectively. Thus, the T-duality transformation amounts to making a field redefinition (2.5) and the dual $c = 1/2$ string theory is defined by the double scaling limit of a dual two-matrix model [8,9]

$$\begin{aligned} Z &= \int d^{N^2}X d^{N^2}Y \exp[-N \operatorname{tr} \tilde{S}_D(X, Y)], \\ \tilde{S}_D(X, Y) &= \frac{1-c}{2}X^2 + \frac{1+c}{2}Y^2 - \frac{\hat{g}}{3}(X^3 + 3XY^2), \end{aligned} \tag{2.7}$$

where $\hat{g} = g/\sqrt{2}$. This model is also known as the $n = 1$ case of the $O(n)$ loop gas model on a random surface [10].

The T-duality transformation (2.5) is a trivial change of the integration variables, and the total free energies given by

$$F = -\frac{1}{N^2}\log Z, \tag{2.8}$$

take the same value at least order-by-order in the $1/N$-expansions for both models. However, the coincidence of the free energies does not always lead to the T-duality. Indeed, we can see that the T-dual relation is broken in correlation functions on a surface of higher genus topology. For example, it is shown explicitly in [8] that disk amplitudes in both theories have the same functional form, while a disk amplitude with one handle in the dual theory have different functional form from a corresponding amplitude in the original theory. More precisely, for the universal part of the disk and cylinder amplitudes, the following identification between the original and dual models holds:

$$\frac{1}{\sqrt{2}}\left(\operatorname{tr}\frac{1}{\zeta - A} + \operatorname{tr}\frac{1}{\zeta - B}\right) \Leftrightarrow \operatorname{tr}\frac{1}{\xi - X}, \tag{2.9}$$

in the sense that they take the same functional form as functions of ζ and ξ respectively with certain identifications of parameters. This is expected because both operators in (2.9) are interpreted as the Dirichlet type boundary conditions for the original and dual spins, respectively, under the T-dual relation. Recall that $\langle X_{ij}X_{kl}\rangle_{\mathrm{bare}}$ represents stick of the dual spin. On the other hand, the disk amplitude with one handle

$$\left\langle \frac{1}{N}\,\mathrm{tr}\frac{1}{\xi - X}\right\rangle_1 \tag{2.10}$$

no longer has the same form as

$$\left\langle \mathrm{tr}\left(\frac{1}{\zeta - A} + \frac{1}{\zeta - B}\right)\right\rangle_1 \tag{2.11}$$

even for the universal part. (The subscript "1" put to the expectation values represents the random surface having one handle on which the expectation values are evaluated.) This difference originates from the excitations of odd number of Y-loops along a handle. Note that, if they are present, such a configuration cannot be interpreted as a dual spin configuration because $\langle Y_{ij}Y_{kl}\rangle_{\mathrm{bare}}$ represents the flip of the dual spin. Of course, from Eqs. (2.5) we have

$$\left\langle \frac{1}{N}\,\mathrm{tr}\frac{1}{\xi - X}\right\rangle_1 = \left\langle \frac{1}{N}\,\mathrm{tr}\frac{\sqrt{2}}{\sqrt{2}\xi - A - B}\right\rangle_1. \tag{2.12}$$

However, it does not imply the T-dual relation between the amplitudes, because the operator appearing in the right-hand side is not consistent with the interpretation as stick of the dual spin in the left-hand side. Thus, we should remark that, in order to show the T-duality, it is necessary to give a consistent interpretation of X and Y as dual spins, not only the transformation of variables (2.5). It is the same situation to the case of the Ising model on the regular lattice. The high temperature expansion of the Ising partition function on the plane has an one-to-one correspondence to the low temperature expansion. (For example, see [27].) It is a manifestation of the Kramers-Wannier duality (the T-duality). All the terms in the high temperature expansion can be written as configurations of loop gas. In the case of the surface with higher genus, some of them contain the loop gas configurations where odd number of the loops surround topologically nontrivial cycles. They cannot be interpreted as dual spin configurations and violate the T-dual relation. Therefore, we can conclude that, although the total free energies are same between the original and dual theories, the T-duality does not hold in their higher genus parts due to the excitations of odd number of Y-loops along the handles.

In string theory with continuous target space, the T-dual symmetry arises when the target space is compactified. It is a symmetry under the interchange between momentum modes and winding modes in the compactified directions. From this viewpoint, the Ising model, whose target space is discrete (consists of two points), contains counterparts of the momentum modes, but not those of the winding modes for either case of random or regular lattice. We can understand that this asymmetry between the momentum and winding modes is the origin of the breaking of the T-duality.[2]

In the original model given by (2.1), we can apply the method of the orthogonal polynomial [29] and derive the string equation in the double scaling limit as

$$f^3 - \tfrac{3}{4}g_s^2 f f'' - \tfrac{3}{8}g_s^2(f')^2 + \tfrac{1}{24}g_s^4 f^{(4)} = t, \tag{2.13}$$

where t is the cosmological constant, g_s is the string coupling constant, and $f(t)$ is the second derivative of the free energy $f(t) = g_s^2\ddot{F}(t)$ as a function of t [30,31].[3] From this equation we can deduce the asymptotic expansion of F as

$$F(t) = \frac{9}{28}\frac{t^{7/3}}{g_s^2} + \frac{1}{24}\log t + \cdots, \tag{2.14}$$

which is nothing but the genus expansion. Since as shown in Appendix A [Eqs. (A4) and (A6)] the double scaling limit is taken for the dual model (2.7) in the same way as in the original model [8], the free energy of the dual model should also satisfy the same string equation as in (2.13) in the double scaling limit[4] and should have the same genus expansion as given in (2.14). However, even in this case, it is possible that they have different nonperturbative effects. Namely, suppose f_1 and f_2 satisfy the same string equation (2.13), then the semiclassical treatment of (2.13) with $g_s \sim \hbar$ leads to the difference $\Delta f = f_1 - f_2$ of the form:

$$\begin{aligned}\Delta f = C_1\frac{g_s^{1/2}}{t^{1/4}}\exp\left(-\frac{6\sqrt{6}}{7g_s}t^{7/6}\right), \quad \text{or}\\ C_2\frac{g_s^{1/2}}{t^{1/4}}\exp\left(-\frac{12\sqrt{3}}{7g_s}t^{7/6}\right),\end{aligned} \tag{2.15}$$

which yields a nonperturbative ambiguity of the free energy $F(t)$ in the following form:

[2] It is possible to introduce the counterparts of the winding modes to matrix models with discrete target space to have the exact T-dual symmetry as discussed in [28].

[3] Note that we have changed the sign of the free energy compared to that in [30] and multiplied it by 2 because the potential in (2.1) is not even.

[4] In the dual model (2.7), we have the expression of the integrals over the eigenvalues of X and Y (3.4) after integrating out the angle variables. It contains the factor $\prod_{i<j}1/(\mu_i + \mu_j)$, in addition to the Vandermonde determinants usually appearing in the case of the original model. Because of the additional factor, the orthogonal polynomial method does not work well, and it makes directly deriving the string equation difficult.

$$\Delta F = \frac{C_1}{6}\frac{g_s^{1/2}}{t^{7/12}}\exp\left(-\frac{6\sqrt{6}}{7g_s}t^{7/6}\right), \quad \text{or} \quad \frac{C_2}{12}\frac{g_s^{1/2}}{t^{7/12}}\exp\left(-\frac{12\sqrt{3}}{7g_s}t^{7/6}\right). \tag{2.16}$$

C_1 and C_2 are numerical constants which cannot be determined from the string equation alone. Therefore, if we interpret one of f_1 and f_2 as a quantity of the original model and the other as that of the dual model, as long as C_1 or C_2 is nonzero, the dual theory has a different nonperturbative effect from that in the original theory. It is known that the nonperturbative effects in the original theory itself take the form (2.16) [3,7], so the dual theory must also have nonperturbative effects of the same form (up to the overall constants). We will see that this is indeed the case by computing the nonperturbative effects in the dual theory directly from the matrix model (2.7). It is well known that the exponents of the nonperturbative effects are identified as the actions of the ZZ-branes and are provided by the disk amplitudes in the presence of them. Therefore, the fact that the dual theory also has the nonperturbative effects as in (2.16) implies that the actions of the ZZ-branes in the dual theory (the dual ZZ-branes) are the same as those in the original theory. It is worth noticing that the string equation can fix not only the exponents in the nonperturbative effects but the power of t in the factors in front of them.

On the other hand, the coefficients C_1 and C_2 in the nonperturbative effects cannot be fixed from the string equation. However, it is shown in [4,7] that, if we compute these coefficients in the $c < 1$ noncritical string theory directly from a matrix model, they turn out to be unique and universal in the sense that they do not depend on details of a potential in the matrix model. Thus, it is interesting to examine whether this is also the case with the nonperturbative effects in the dual $c = 1/2$ string theory and whether they agree with those in the original theory. Moreover, since the nonperturbative effects are computed from certain disk and cylinder amplitudes [4,7], it will be possible to find out the T-duality at the nonperturbative level from the knowledge of the T-dual relation of the disk and cylinder amplitudes. It is expected that the analysis reveals the existence of large universality including T-duality for nonperturbative effects in string theory, or for string theory itself.

III. NONPERTURBATIVE EFFECTS IN THE DUAL THEORY

In this section, we derive the leading part of the nonperturbative effects in the dual $c = 1/2$ noncritical string theory directly by evaluating instanton contributions in the matrix model (2.7).

A. Chemical potential of instanton

As a preparation for instanton calculus in the dilute gas approximation, here we formulate the chemical potential of an instanton in the dual two-matrix model (2.7).

By rescaling the matrices, the partition function in (2.7) becomes

$$Z_N(h) = \int dXdY \exp\left[-\frac{N}{h}\mathrm{tr}S_D(X,Y)\right], \quad S_D(X,Y) = \frac{1-c}{2}X^2 + \frac{1+c}{2}Y^2 - \frac{1}{3}(X^3 + 3XY^2), \tag{3.1}$$

where $h = \hat{g}^2$ and the measures dX, dY are defined with the normalization:

$$\int dX \exp\left[-\frac{N}{h}\mathrm{tr}\left(\frac{1-c}{2}X^2\right)\right] = 1, \quad \int dY \exp\left[-\frac{N}{h}\mathrm{tr}\left(\frac{1+c}{2}Y^2\right)\right] = 1, \tag{3.2}$$

so that the free energy has the standard $1/N$-expansion

$$F = -\frac{1}{N^2}\log Z_N(h) = F_0(h) + \frac{1}{N^2}F_1(h) + \frac{1}{N^4}F_2(h) + \cdots, \tag{3.3}$$

where $F_k(h)$ represents the contribution from random surfaces with k-handles.

The formula given in [29] enables us to rewrite the partition function in terms of the eigenvalues of X and Y as

$$Z_N(h) = D_N^{-1}\int\left(\prod_i d\lambda_i d\mu_i\right)\frac{\Delta^{(N)}(\lambda)\Delta^{(N)}(\mu)}{\prod_{i<j}(\mu_i+\mu_j)} \times \exp\left[-\frac{N}{h}\sum_i\left(V(\lambda_i) + \frac{1+c}{2}\mu_i^2 - \lambda_i\mu_i^2\right)\right], \tag{3.4}$$

where λ_i, μ_i $(i = 1, \cdots N)$ are the eigenvalues of X and Y respectively, and $\Delta^{(N)}(\lambda) = \prod_{N\geq i>j\geq 1}(\lambda_i - \lambda_j)$ is the Vandermonde determinant. $V(\lambda)$ is defined as

$$V(\lambda) = \frac{1-c}{2}\lambda^2 - \frac{1}{3}\lambda^3. \tag{3.5}$$

The proportional constant D_N is explicitly computed in Appendix B [See Eq. (B31)].

Hereafter, as a configuration of one instanton, we consider a situation where one pair of the eigenvalues (say (λ_N, μ_N)) is separated from the other pairs (λ_i, μ_i) $(i = 1, \cdots, N-1)$ as a point on the (λ, μ)-plane. Setting $x = \lambda_N$, $y = \mu_N$, the partition function is expressed as

$$\begin{aligned}
Z_N(h) &= D_N^{-1}\int dxdy\int\Bigg(\prod_{i=1}^{N-1} d\lambda_i d\mu_i\Bigg)\frac{\Delta^{(N-1)}(\lambda)\Delta^{(N-1)}(\mu)}{\prod_{i<j\le N-1}(\mu_i+\mu_j)}\prod_{i=1}^{N-1}\frac{(x-\lambda_i)(y-\mu_i)}{(y+\mu_i)} \\
&\quad\times e^{-(N/h)\sum_{i=1}^{N-1}(V(\lambda_i)+((1+c)/(2))\mu_i^2-\lambda_i\mu_i^2)-(N/h)(V(x)+((1+c)/2)y^2-xy^2)} \\
&= D_N^{-1}\int dxdyD_{N-1}\int dX'dY'\frac{\det(x-X')\det(y-Y')}{\det(y+Y')}e^{-(N/h)\,\mathrm{tr}S_D(X',Y')}e^{-(N/h)S_D(x,y)} \\
&= \frac{D_{N-1}}{D_N}Z_{N-1}(h')\int dxdy\Bigg\langle\frac{\det(x-X')\det(y-Y')}{\det(y+Y')}\Bigg\rangle' e^{-((N-1)/h')S_D(x,y)} \\
&= \frac{D_{N-1}}{D_N}Z_{N-1}(h')\int dxdye^{-V_{\mathrm{eff}}(x,y)},
\end{aligned} \tag{3.6}$$

where X', Y' are $(N-1)\times(N-1)$ Hermitian matrices, and

$$\begin{aligned}
Z_{N-1}(h') &= \int dX'dY'e^{-((N-1)/h')\,\mathrm{tr}S_D(X',Y')}, \\
\langle\mathcal{O}\rangle' &= \frac{1}{Z_{N-1}(h')}\int dX'dY'\mathcal{O}e^{-((N-1)/h')\,\mathrm{tr}S_D(X',Y')}.
\end{aligned} \tag{3.7}$$

In the above equation, we have defined h' as

$$\frac{N}{h}=\frac{N-1}{h'}, \tag{3.8}$$

and $V_{\mathrm{eff}}(x,y)$ as

$$e^{-V_{\mathrm{eff}}(x,y)}\equiv\Bigg\langle\frac{\det(x-X')\det(y-Y')}{\det(y+Y')}\Bigg\rangle' e^{-((N-1)/h')S_D(x,y)}. \tag{3.9}$$

As in the one-matrix model case considered in [4], the total partition function is divided into multi-instanton sectors as

$$Z_N(h)=Z_N^{(0-\mathrm{inst})}(h)+Z_N^{(1-\mathrm{inst})}(h)+\cdots, \tag{3.10}$$

where the k-instanton sector is characterized by k-pairs of the eigenvalues separated from the other pairs on the (λ,μ)-plane. More precisely, the partition functions in the 1-instanton sector and 0-instanton sector are given by

$$Z_N^{(1-\mathrm{inst})}(h)=N\frac{D_{N-1}}{D_N}Z_{N-1}(h')\int_{(x,y)\notin\mathcal{S}}dxdye^{-V_{\mathrm{eff}}(x,y)}, \tag{3.11}$$

$$Z_N^{(0-\mathrm{inst})}(h)=\frac{D_{N-1}}{D_N}Z_{N-1}(h')\int_{(x,y)\in\mathcal{S}}dxdye^{-V_{\mathrm{eff}}(x,y)}, \tag{3.12}$$

respectively, where the factor N in (3.11) means the number of ways to specify a separated pair of the eigenvalues, and $\mathcal{S}$ is the support of the eigenvalue distribution on the (λ,μ)-plane in the large-N limit. Taking the ratio between them, we obtain

$$\mu\equiv\frac{Z_N^{(1-\mathrm{inst})}(h)}{Z_N^{(0-\mathrm{inst})}(h)}=N\frac{\int_{(x,y)\notin\mathcal{S}}dxdye^{-V_{\mathrm{eff}}(x,y)}}{\int_{(x,y)\in\mathcal{S}}dxdye^{-V_{\mathrm{eff}}(x,y)}}. \tag{3.13}$$

Following the argument given in [4], it is easy to see that μ defined in this equation is in fact the chemical potential of the instanton, namely, a statistical weight of the instanton, in the dilute gas approximation for the computation of the free energy of the dual two-matrix model.

B. Saddle point analysis

In this subsection, we apply the saddle point method to the numerator in (3.13), which is valid in the large-N limit. Since $V_{\mathrm{eff}}(x,y)$ can be rewritten as

$$e^{-V_{\mathrm{eff}}(x,y)}=\langle e^{\mathrm{tr}\log(x-X')+\mathrm{tr}\log(y-Y')-\mathrm{tr}\log(y+Y')}\rangle' e^{-((N-1)/h')S_D(x,y)}, \tag{3.14}$$

one may expect that $V_{\mathrm{eff}}(x,y)$ in the large-N limit can be expanded in terms of connected Green functions as

$$\begin{aligned}
e^{-V_{\mathrm{eff}}(x,y)} &= \exp\Bigg[-\frac{N-1}{h'}S_D(x,y)+\langle\mathrm{tr}\log(x-X')\rangle_d'+\langle\mathrm{tr}\log(y-Y')\rangle_d'-\langle\mathrm{tr}\log(y+Y')\rangle_d'+\frac{1}{2}\langle(\mathrm{tr}\log(x-X'))^2\rangle_c' \\
&\quad+\frac{1}{2}\langle(\mathrm{tr}\log(y-Y'))^2\rangle_c'+\frac{1}{2}\langle(\mathrm{tr}\log(y+Y'))^2\rangle_c'+\langle\mathrm{tr}\log(x-X')\mathrm{tr}\log(y-Y')\rangle_c' \\
&\quad-\langle\mathrm{tr}\log(y-Y')\,\mathrm{tr}\log(y+Y')\rangle_c'-\langle\mathrm{tr}\log(x-X')\,\mathrm{tr}\log(y+Y')\rangle_c'+\mathcal{O}\Big(\frac{1}{N}\Big)\Bigg],
\end{aligned} \tag{3.15}$$

where the subscripts "d" and "c" represent amplitudes of the disk and cylinder topologies, respectively. However, if x or y

is inside the support of the eigenvalue distribution of X' or Y' respectively, the operator $\mathrm{tr}\log(x - X')$ or $\mathrm{tr}\log(y - Y')$ becomes large by itself to make the expansion (3.15) not valid [32]. This motivates us to divide the numerator in (3.13) as

$$N\int_{(x,y)\notin\mathcal{S}} dxdye^{-V_{\mathrm{eff}}(x,y)} = N\int_{x\notin I_X, y\notin I_Y} dxdye^{-V_{\mathrm{eff}}(x,y)} + N\int_{x\notin I_X, y\in I_Y} dxdye^{-V_{\mathrm{eff}}(x,y)} + N\int_{x\in I_X, y\notin I_Y} dxdye^{-V_{\mathrm{eff}}(x,y)}, \tag{3.16}$$

where $\mathcal{S} = I_X \times I_Y$, and I_X (I_Y) is the support of the eigenvalue distribution of X' (Y'). In our choice of the potential, I_X and I_Y should be connected intervals, and their explicit forms are given in Appendix A as Eqs. (A42) and (A44).

1. The first term in (3.16)

First we consider the case where both x and y are outside the supports of the eigenvalue distributions of X' and Y'. Then, the expansion (3.15) is justified and the leading part of $V_{\mathrm{eff}}(x, y)$ in the large-N limit, denoted as $V^{(0)}_{\mathrm{eff}}(x, y)$, is given by

$$\begin{aligned} e^{-V^{(0)}_{\mathrm{eff}}(x,y)} &= \exp\bigg[-\frac{N-1}{h'}S_D(x, y) + \langle \mathrm{tr}\log(x - X')\rangle'_d \\ &\quad + \langle \mathrm{tr}\log(y - Y')\rangle'_d - \langle \mathrm{tr}\log(y + Y')\rangle'_d\bigg] \\ &= \exp\bigg[-\frac{N-1}{h'}S_D(x, y) + \langle \mathrm{tr}\log(x - X')\rangle'_d\bigg], \end{aligned} \tag{3.17}$$

where in the last line we have used the $\mathbf{Z}_2$ symmetry under $Y \to -Y$ of the action (3.1). Therefore, we have

$$V^{(0)}_{\mathrm{eff}}(x, y) = \frac{N-1}{h'}\left(V(x) + \frac{1+c}{2}y^2 - xy^2\right) - \langle \mathrm{tr}\log(x - X')\rangle'_d. \tag{3.18}$$

Since $V^{(0)}_{\mathrm{eff}}(x, y)$ is proportional to $N - 1$, we can apply the saddle point method to evaluate a leading contribution to the integrals of the first term in (3.16) in the large-N limit. The saddle point equations read

$$\begin{aligned} 0 &= \frac{\partial V^{(0)}_{\mathrm{eff}}(x, y)}{\partial x} = \frac{N-1}{h'}(V'(x) - y^2 - h'R_{X'}(x)), \\ 0 &= \frac{\partial V^{(0)}_{\mathrm{eff}}(x, y)}{\partial y} = \frac{N-1}{h'}(1 + c - 2x)y, \end{aligned} \tag{3.19}$$

where $R_X(x)$ and $R_{X'}(x)$ are the resolvents for X and X', respectively:

$$\begin{aligned} R_X(x) &= \left\langle \frac{1}{N}\mathrm{tr}\frac{1}{x - X}\right\rangle_d, \\ R_{X'}(x) &= \left\langle \frac{1}{N-1}\mathrm{tr}\frac{1}{x - X'}\right\rangle'_d. \end{aligned} \tag{3.20}$$

In the large-N limit, these two become coincident, and the explicit form is given by Eqs. (A40) and (A41) in Appendix A. Since it is seen from (A44) that the origin $y = 0$ belongs to I_Y (which is also suggested by the $\mathbf{Z}_2$ symmetry), the second equation in (3.19) leads to the solution $x_0 = (1 + c)/2 \equiv \hat{P}_*$ as a saddle point of x. This value coincides with the critical point of x, at which a cubic equation satisfied by the universal part of the resolvent $\hat{R}_X(x)$ becomes triply degenerate as $\hat{R}_X(x_0)^3 = 0$ when $\hat{g} = \hat{g}_*$, $c = c_*$. (For details, see Ref. [8] or Appendix A).[5] The nonuniversal part of $R_X(x)$ is given by

$$\begin{aligned} R^{\mathrm{non}}_X(x) &= \frac{1}{3h}(2V'(x) - V'(1 + c - x)), \\ R_X(x) &= R^{\mathrm{non}}_X(x) + \hat{R}_X(x). \end{aligned} \tag{3.21}$$

Making use of (3.21) to the first equation in (3.19) determines saddle points of y as

$$y_0 = \pm 2c \equiv \pm\hat{Q}_*, \tag{3.22}$$

which respects the $\mathbf{Z}_2$ symmetry. From Eqs. (A42) and (A44) in Appendix A, we recognize these saddle points to be outside the supports of the eigenvalue distributions in the double scaling limit:[6]

$$x_0 = \hat{P}_* \notin I_X, \qquad y_0 = \pm\hat{Q}_* \notin I_Y. \tag{3.23}$$

At the saddle points, $V^{(0)}_{\mathrm{eff}}(x, y)$ in (3.18) takes the form

$$V_{\mathrm{eff}}(\hat{P}_*, \pm\hat{Q}_*) = \frac{N-1}{h'}V(\hat{P}_*) - \langle \mathrm{tr}\log(\hat{P}_* - X')\rangle'_d. \tag{3.24}$$

[5] Hereafter, in taking the large-N limit, c is fixed at the critical value $c_* = \frac{-1+2\sqrt{7}}{27}$.

[6] It turns out that the $o(a)$ term of γ in (A44) is positive, by considering next-to-leading contributions to the solution of (A34). Thus, strictly speaking, y_0 is slightly inside the cut I_Y by the order of $o(a)$, and coincides with the right or left edge of I_Y in the double scaling limit. Hence there is some subtlety in justification of $V^{(0)}_{\mathrm{eff}}(x, y)$ given in (3.18). However, since the quantity $o(a)$ is in fact negligible compared to contributions of $\mathcal{O}(a)$ usually playing relevant roles in the double scaling limit, we may assume that the eigenvalue distribution at y_0 is almost zero, and that the operator $\mathrm{tr}\log(y_0 - Y)$ is not so singular that it can invalidate (3.18).

2. The second term in (3.16)

For the case $y \in I_Y$, we have the solution $y_0 = 0 \in I_Y$ in the second equation of (3.19). The solution is also expected from the $\mathbf{Z}_2$ symmetry. However, as noticed above, in this case we cannot trust the form of $V_{\rm eff}^{(0)}(x, y)$ given in (3.18). This kind of saddle point first appears in the dual model, not seen in the case of the original model [3,7]. In order to resolve this problem, we consider performing the Y-integration first in the partition function and then calculating instanton effects from isolated eigenvalues in the resulting one-matrix model for X. In evaluating the second term of (3.16), we replace the integration over the interval I_Y with that over the whole real axis. Then, saddle points in the $y \notin I_Y$ region, which are nothing but the ones considered in (3.23), would give rise to an error in this replacement. However, as shown in Sec. III C 1, they only give contributions exponentially small by the factor (3.40) compared to those from the integration over $y \in I_Y$. Thus, concerning the leading contribution of the second term in (3.16), we can neglect such an error and justify the replacement of the integration region.

We go back to the expression (3.6) to rewrite the second term in (3.16) as

$$N\int_{x\notin I_X, y\in I_Y} dxdye^{-V_{\rm eff}(x,y)} = \frac{D_{N-1}^{-1}}{Z_{N-1}(h')}\int_{x\notin I_X} dx \int\bigg(\prod_{i=1}^N d\lambda_i d\mu_i\bigg)\sum_{i=1}^N \delta(x-\lambda_i)\frac{\Delta^{(N)}(\lambda)\Delta^{(N)}(\mu)}{\prod_{i<j\le N}(\mu_i+\mu_j)}e^{-((N-1)/h')\sum_{i=1}^N S_D(\lambda_i,\mu_i)}$$
$$= \frac{D_{N-1}^{-1}}{Z_{N-1}(h')}\int_{x\notin I_X} dxD_N\int dXdY\,{\rm tr}\delta(x-X)e^{-(N/h)\,{\rm tr}S_D(X,Y)}, \tag{3.25}$$

where we have inserted $1 = \int d\lambda_N \delta(x-\lambda_N)$ and used the fact that the integrand is symmetric under the interchange among $\{\lambda_i, x\}$ $(i = 1, \cdots, N-1)$. The measures dX, dY are normalized as (3.2). As the result of the Y-integration, we obtain

$$N\int_{x\notin I_X, y\in I_Y} dxdye^{-V_{\rm eff}(x,y)} = \frac{D_{N-1}^{-1}D_N}{Z_{N-1}(h')}\int_{x\notin I_X} dx\int dX\,{\rm tr}\delta(x-X)$$
$$\times \exp\bigg[-\frac{N}{h}{\rm tr}V(X) - \frac{1}{2}{\rm tr}\log\bigg(\mathbf{1}\otimes\mathbf{1} - \frac{1}{1+c}(X\otimes\mathbf{1}+\mathbf{1}\otimes X)\bigg)\bigg]. \tag{3.26}$$

Next, integration over the angular variables of X yields

$$N\int_{x\notin I_X, y\in I_Y} dxdye^{-V_{\rm eff}(x,y)} = \frac{D_{N-1}^{-1}D_N}{Z_{N-1}(h')}\int_{x\notin I_X} dx(J_N^X)^{-1}\int\bigg(\prod_{i=1}^N d\lambda_i\bigg)\Delta^{(N)}(\lambda)^2\sum_{i=1}^N\delta(x-\lambda_i)$$
$$\times\exp\bigg[-\frac{N}{h}\sum_{i=1}^N V(\lambda_i) + \frac{1}{2}\sum_{n=1}^\infty\frac{1}{n}\bigg(\frac{1}{1+c}\bigg)^n\sum_{k=0}^n {}_nC_k\sum_{i=1}^N\lambda_i^k\sum_{j=1}^N\lambda_j^{n-k}\bigg], \tag{3.27}$$

where the normalization constant J_N^X arises upon the angular integration and satisfies for an arbitrary $U(N)$-invariant function of X: $f({\rm tr}X, {\rm tr}X^2, \cdots)$

$$J_N^X\int dXf({\rm tr}X, {\rm tr}X^2, \cdots) = \int\bigg(\prod_{i=1}^N d\lambda_i\bigg)\Delta^{(N)}(\lambda)^2 f\bigg(\sum_i\lambda_i, \sum_i\lambda_i^2, \cdots\bigg). \tag{3.28}$$

J_N^X is calculated in Appendix B as (B24). We rename one of λ_i's as x by integrating the delta functions $\sum_{i=1}^N\delta(x-\lambda_i)$ and introduce the $(N-1)\times(N-1)$ Hermitian matrix X' again to obtain the effective potential $V_{\rm eff}(x)$:

$$
\begin{aligned}
N\int_{x\notin I_X, y\in I_Y} dxdy e^{-V_{\rm eff}(x,y)} &= N\frac{D_{N-1}^{-1}D_N(J_N^X)^{-1}}{Z_{N-1}(h')}\int_{x\notin I_X} dx J_{N-1}^X \int dX' \det(x-X')^2 \\
&\quad \times \exp\biggl[-\frac{N-1}{h'}\,{\rm tr}V(X') - \frac{N-1}{h'}V(x) - \frac{1}{2}\log\Bigl(1-\frac{2x}{1+c}\Bigr) - {\rm tr}\log\Bigl(1-\frac{x+X'}{1+c}\Bigr) \\
&\quad - \frac{1}{2}\,{\rm tr}\log\Bigl(\mathbf{1}\otimes\mathbf{1} - \frac{1}{1+c}(X'\otimes\mathbf{1}+\mathbf{1}\otimes X')\Bigr)\biggr] \\
&= N\frac{D_{N-1}^{-1}D_N(J_N^X)^{-1}J_{N-1}^X}{Z_{N-1}(h')}\int_{x\notin I_X} dx \int dX'dY' \det(x-X')^2(c+1)^{N-(1/2)} \\
&\quad \times \exp\biggl[-\frac{N-1}{h'}V(x) - \frac{1}{2}\log(c+1-2x) - {\rm tr}\log(c+1-x-X')\biggr] \\
&\quad \times \exp\biggl[-\frac{N-1}{h'}\,{\rm tr}\Bigl(V(X') + \frac{1+c}{2}Y'^2 - X'Y'^2\Bigr)\biggr] \\
&= N\frac{D_N J_{N-1}^X}{D_{N-1}J_N^X}(c+1)^{N-(1/2)}\int_{x\notin I_X} dx \Bigl\langle \frac{\det(x-X')^2}{\det(c+1-x-X')}\Bigr\rangle' e^{-((N-1)/h')V(x)-(1/2)\log(c+1-2x)} \\
&\equiv N\frac{D_N J_{N-1}^X}{D_{N-1}J_N^X}(c+1)^{N-(1/2)}\int_{x\notin I_X} dx e^{-V_{\rm eff}(x)}. \qquad (3.29)
\end{aligned}
$$

Therefore, when $y \in I_Y$, the effective potential for $x \notin I_X$ is expressed as

$$
e^{-V_{\rm eff}(x)} = \Bigl\langle \frac{\det(x-X')^2}{\det(c+1-x-X')}\Bigr\rangle' e^{-((N-1)/h')V(x)-(1/2)\log(c+1-2x)}. \qquad (3.30)
$$

In the large-N limit, the leading term of $V_{\rm eff}(x)$ is reduced to

$$
V_{\rm eff}^{(0)}(x) = \frac{N-1}{h'}V(x) - 2\,{\rm Re}\langle {\rm tr}\log(x-X')\rangle_d' + \langle {\rm tr}\log(c+1-x-X')\rangle_d'. \qquad (3.31)
$$

Note that for large $x \notin I_X$, the third term may yield the imaginary part which prevents us from interpreting $V_{\rm eff}^{(0)}(x)$ as the effective potential. However, at least near the saddle point for x to be determined in Sec. III C 2, the point $c+1-x$ is also outside the cut and the third term stays real. Since $V_{\rm eff}^{(0)}(x)$ is again proportional to $N-1$, we can apply the saddle point method to compute the second term of (3.16) in the large-N limit.

The saddle point equation from $V_{\rm eff}^{(0)}(x)$ reads

$$
0 = \frac{\partial V_{\rm eff}^{(0)}(x)}{\partial x} = -(N-1)(2\,{\rm Re}\hat{R}_{X'}(x) + \hat{R}_{X'}(c+1-x)), \qquad (3.32)
$$

where we have used (3.21). The saddle point equation (3.32) is solved in the double scaling limit in the next subsection.

Finally we comment on the third term in (3.16). The solutions (3.23) for the Eqs. (3.19) are unique in the case $y \notin I_Y$, and thus (3.18) has no saddle point in the region $x \in I_X$, $y \notin I_Y$. Although strictly speaking the expression of the effective potential in the large-N limit given in (3.18) cannot be sufficiently trusted when $x \in I_X$, $y \notin I_Y$, here we assume that the third term in (3.16) does not contribute to the numerator in (3.13).

C. Nonperturbative effects

The leading term of the denominator in (3.13) is computed as

$$
\begin{aligned}
\int_{(x,y)\in S} dxdy e^{-V_{\rm eff}(x,y)}\Big|_{\rm leading} &= \exp\biggl[\langle {\rm tr}\log(\beta - X')\rangle_d' + (N-1)\int_{c+1-\beta}^{\beta} R_{X'}(x')dx' \\
&\quad - \frac{N-1}{h'}V(\beta)\biggr], \qquad (3.33)
\end{aligned}
$$

where β is the right edge of the eigenvalue distribution of X in the one-matrix model appearing in (3.26):

$$
\begin{aligned}
Z_N(h) = \int dX \exp\biggl[&-\frac{N}{h}\,{\rm tr}V(X) \\
&- \frac{1}{2}\,{\rm tr}\log\Bigl(\mathbf{1}\otimes\mathbf{1} - \frac{1}{1+c}(X\otimes\mathbf{1}+\mathbf{1}\otimes X)\Bigr)\biggr]. \qquad (3.34)
\end{aligned}
$$

Since this model is obtained simply by performing the Gaussian integration over Y in the dual model (3.1), the partition function is exactly the same as in (3.1). Details in the derivation of (3.33) are given in Appendix B [See Eq. (B19)].

1. The case $x_0 \notin I_X$, $y_0 \notin I_Y$

At the saddle points (3.23), the leading term of the numerator in (3.13) is given by the effective potential (3.24). Putting it together with the contribution from the denominator (3.33), the leading term of the chemical potential μ takes the form as

$$(\mu \text{ leading}) = \exp\bigg[-\frac{N-1}{h'}(V(\hat{P}_*) - V(\beta)) + (N-1) \times \bigg(\int_\beta^{\hat{P}_*} R_{X'}(x')dx' - \int_{c+1-\beta}^{\beta} R_{X'}(x')dx'\bigg)\bigg]. \tag{3.35}$$

Note that the potential terms above exactly cancel the nonuniversal parts of the resolvents, by using the expression (3.21), to give

$$(\mu \text{ leading}) = \exp\bigg[(N-1)\bigg(\int_\beta^{\hat{P}_*} \hat{R}_{X'}(x')dx' - \int_{c+1-\beta}^{\beta} \hat{R}_{X'}(x')dx'\bigg)\bigg]. \tag{3.36}$$

It can be written solely in terms of the universal part of the resolvent, and the nonuniversal part does not contribute. This observation would be relevant if we consider proving the universality of nonperturbative effects in the dual theory as done for the standard $c < 1$ noncritical string theories [4,5,7].

In order to take the double scaling limit, let us introduce the scaling variable ζ as $x' = \hat{P}_*(1 + a\zeta)$ with the lattice constant a. Then, the resolvent is expressed as

$$\hat{R}_X(x') = a^{4/3}\hat{\alpha}w(\zeta) + \mathcal{O}(a^{5/3}), \quad w(\zeta) = \Big(\zeta + \sqrt{\zeta^2 - \hat{T}}\Big)^{4/3} + \Big(\zeta - \sqrt{\zeta^2 - \hat{T}}\Big)^{4/3}, \tag{3.37}$$

where $\hat{\alpha}$ is a numerical constant and $\hat{T}$ is proportional to the cosmological constant t defined by

$$\hat{\alpha} = \frac{\hat{s}^{4/3}}{2^{2/3}\cdot 5c}, \qquad \hat{g} = \hat{g}_*(1 - a^2 t) = \hat{g}_*\Big(1 - a^2\frac{\hat{s}^2}{10}\hat{T}\Big). \tag{3.38}$$

$\hat{s}$ is an irrational number $\hat{s} = 1 + \sqrt{7}$, and $\hat{g}_* = \sqrt{5c^3}$ is the critical point of $\hat{g}$. (For more details in the double scaling limit of the dual model, see Appendix A.) We carry out the integrations in (3.36) by moving to the variable ζ as

$$x' = \hat{P}_* \Leftrightarrow \zeta = 0, \quad x' = \beta = \hat{P}_*\Big(1 - a\sqrt{\hat{T}}\Big) \Leftrightarrow \zeta = -\sqrt{\hat{T}}, \quad x' = c + 1 - \beta = 2\hat{P}_* - \beta \Leftrightarrow \zeta = \sqrt{\hat{T}}, \tag{3.39}$$

and end up with the result

$$(\mu \text{ leading}) = \exp\Big(-5^{1/6}\frac{12\sqrt{6}}{7}(N-1)a^{7/3}t^{7/6}\Big). \tag{3.40}$$

Since the sphere free energy is expressed as

$$N^2 F_0 = \frac{9}{7}5^{1/3}N^2 a^{14/3}t^{7/3} \tag{3.41}$$

in the parametrization used here,[7] we take the double scaling limit in such a way that (3.41) agrees with the first term in (2.14). Explicitly,

$$N \to \infty \quad \text{and} \quad a \to 0 \quad \text{with} \quad \frac{1}{g_s} \equiv 2\cdot 5^{1/6}Na^{7/3} \text{ fixed.} \tag{3.43}$$

Then, (3.40) becomes

$$(\mu \text{ leading}) = \exp\Big(-\frac{6\sqrt{6}}{7g_s}t^{7/6}\Big), \tag{3.44}$$

which exactly reproduces one of the leading terms in the nonperturbative effects given in (2.16). Moreover, we have found two degenerate saddle points (3.23) which give this nonperturbative effect. Notice that the original $c = 1/2$ theory also has two degenerate nonperturbative effects exactly given by (3.44) [3,7]. Thus, the leading nonperturbative effects (3.44) precisely match between the original theory and the dual one including their multiplicity.

2. The case $x_0 \notin I_X$, $y_0 \in I_Y$

From Eqs. (3.31) and (3.33), the leading contribution to μ, which comes from a saddle point $x = x_0 \notin I_X$ satisfying (3.32), takes the form

$$(\mu \text{ leading}) = \exp\bigg[-\frac{N-1}{h'}(V(x_0) - V(\beta)) + (N-1)\bigg(2\int_\beta^{x_0} R_{X'}(x')dx' + \int_\beta^{x_0} R_{X'}(c+1-x')dx'\bigg)\bigg]. \tag{3.45}$$

It is again expressed only by the universal part:

$$(\mu \text{ leading}) = \exp\bigg[(N-1)\int_\beta^{x_0}(2\hat{R}_{X'}(x') + \hat{R}_{X'}(c+1-x'))dx'\bigg], \tag{3.46}$$

as anticipated from (3.32). Note that both of $R_{X'}(x')$ and $R_{X'}(c+1-x')$ do not have the imaginary parts for $\beta \leq$

[7] Equation (3.41) is obtained, for example, by integrating the leading term of $\hat{W}_3$ in (A15) with respect to t, since

$$W_3 = \Big\langle\frac{1}{N}\text{tr}A^3\Big\rangle_d = -\frac{3}{2}\frac{\partial F_0}{\partial g} \tag{3.42}$$

in the original model, and the spherical free energy takes the same form for both of the original and dual models.

$x' \le x_0$. In terms of the scaling variable ζ, the saddle point equation (3.32) is written as

$$2w(\zeta) + w(-\zeta) = 0, \tag{3.47}$$

and it is easy to see that a solution to this equation on the first (physical) sheet is given by $\zeta_0 = \sqrt{\hat{T}/2}$. Substituting this value into (3.46), we find another nonperturbative effect

$$(\mu \text{ leading}) = \exp\left(-5^{1/6}\frac{24\sqrt{3}}{7}(N-1)t^{7/6}a^{7/3}\right). \tag{3.48}$$

In the double scaling limit (3.43), this becomes

$$(\mu \text{ leading}) = \exp\left(-\frac{12\sqrt{3}}{7g_s}t^{7/6}\right), \tag{3.49}$$

to reproduce the leading term of the remaining nonperturbative effect in (2.16). It is also exactly the same as the remaining one of three nonperturbative effects in the original $c = 1/2$ string theory. Combining the results in (3.44) and (3.49), we conclude that the T-duality transformation does not change the number of the ZZ-branes and their actions (weights).

Before closing this section, we make the following one comment. The disk amplitude $\langle \mathrm{tr}\log(y - Y')\rangle'_d$ [or $R_{Y'}(y)$] represents a quite singular dual spin configuration along the boundary, and its counterpart in the original model does not appear in the instanton calculus [7]. It is worthy of noticing that such amplitudes cancel out due to the $\mathbf{Z}_2$ symmetry in the process of the computation of $V^{(0)}_{\mathrm{eff}}(x, y)$ in (3.17) and they do not affect the interpretation of the dual ZZ-branes.

IV. COEFFICIENTS OF THE NONPERTURBATIVE EFFECTS

In this section, we consider next-to-leading terms in the chemical potential of the instanton, namely, coefficients in front of the nonperturbative effects (3.44) and (3.49).

The coefficient coming from the denominator in (3.13) is evaluated in Appendix B. In Eq. (B40), the first term in the exponential $2(N-1)R$ represents nothing but the leading term (3.33), and the remaining parts

$$(2\pi)^{3/2}\sqrt{(N-1)h'}\exp\left(R + h'\frac{\partial R}{\partial h'}\right) \tag{4.1}$$

contribute to the coefficient. R is defined by Eq. (B38), and $R + h'\frac{\partial R}{\partial h'}$ can be written in terms of several disk and cylinder amplitudes as (B43).

In the following, let us consider contributions to the coefficient from the numerator in each of the cases $x_0 \notin I_X$, $y_0 \notin I_Y$ and $x_0 \notin I_X$, $y_0 \in I_Y$.

In the first case $x_0 \notin I_X$, $y_0 \notin I_Y$, the next-to-leading contributions are provided by the $\mathcal{O}(N^0)$ part in the exponent in (3.15) evaluated at the saddle point and also by the Gaussian integration of (3.17) around the saddle point. The latter is easy to calculate to yield for both of $(x_0, y_0) = (\hat{P}_*, \pm\hat{Q}_*) = (\frac{1+c}{2}, \pm 2c)$

$$N\int_{x\notin I_X, y\notin I_Y} dxdy e^{-V_{\mathrm{eff}}(x,y)} = i\frac{\pi h}{2c}e^{-V^{(1)}_{\mathrm{eff}}(x_0,y_0)}e^{-V^{(0)}_{\mathrm{eff}}(x_0,y_0)} \times\left(1 + \mathcal{O}\left(\frac{1}{N}\right)\right), \tag{4.2}$$

where

$$\begin{aligned} V^{(0)}_{\mathrm{eff}}(x, y) &= \frac{N-1}{h'}\left(V(x) + \frac{1+c}{2}y^2 - xy^2\right) \\ &\quad - \langle \mathrm{tr}\log(x - X')\rangle'_d, \\ V^{(1)}_{\mathrm{eff}}(x, y) &= -\frac{1}{2}\langle(\mathrm{tr}\log(x - X'))^2\rangle'_c \\ &\quad - \langle(\mathrm{tr}\log(y - Y'))^2\rangle'_c \\ &\quad + \langle \mathrm{tr}\log(y - Y')\mathrm{tr}\log(y + Y')\rangle'_c \end{aligned} \tag{4.3}$$

are obtained from (3.15) by using the $\mathbf{Z}_2$ symmetry. For both of the two saddle points (4.2) takes the same value, which should be expected from the $\mathbf{Z}_2$ symmetry. Together with the contribution from the denominator, $e^{-V^{(0)}_{\mathrm{eff}}(x_0,y_0)-2(N-1)R}$ gives the leading part (3.44), and $V^{(1)}_{\mathrm{eff}}(x, y)$ represents the next-to-leading term in $V_{\mathrm{eff}}(x, y)$. The cylinder amplitudes appearing in $V^{(1)}_{\mathrm{eff}}(x, y)$ will be found by integrating with respect to z and z'

$$\left\langle \mathrm{tr}\frac{1}{z - X'}\,\mathrm{tr}\frac{1}{z' - X'}\right\rangle'_c, \qquad \left\langle \mathrm{tr}\frac{1}{z - Y'}\,\mathrm{tr}\frac{1}{z' - Y'}\right\rangle'_c, \qquad \left\langle \mathrm{tr}\frac{1}{z - Y'}\,\mathrm{tr}\frac{1}{-z' - Y'}\right\rangle'_c. \tag{4.4}$$

For this purpose, we will need the explicit form of these cylinder amplitudes without taking the double scaling limit. In fact, as long as the first one in (4.4) is concerned, such an amplitude is calculated in [33]. However, the result has a complicated form to makes it difficult to perform the z, z'-integration explicitly. Moreover, to the best of our knowledge, the cylinder amplitudes for Y' like the remaining two in (4.4) have not been found in the literature.

Similarly, in the second case $x_0 \notin I_X$, $y_0 \in I_Y$, from (3.29) the numerator takes the form of

$$N\int_{x\notin I_X, y\in I_Y} dx e^{-V_{\mathrm{eff}}(x)} = N\frac{D_N J^X_{N-1}}{D_{N-1}J^X_N}(c+1)^{N-(1/2)}\frac{1}{\sqrt{c+1-2x_0}}\sqrt{\frac{2\pi}{V^{(0)\prime\prime}_{\mathrm{eff}}(x_0)}}e^{-V^{(1)}_{\mathrm{eff}}(x_0)}e^{-V^{(0)}_{\mathrm{eff}}(x_0)}\left(1 + \mathcal{O}\left(\frac{1}{N}\right)\right), \tag{4.5}$$

where $x_0 = \hat{P}_*(1 + a\zeta_0) = \hat{P}_*(1 + a\sqrt{\hat{T}/2})$,

$$V^{(0)}_{\rm eff}(x) = \frac{N-1}{h'}V(x) - 2\,{\rm Re}\langle {\rm tr}\log(x-X')\rangle'_d + \langle {\rm tr}\log(c+1-x-X')\rangle'_d,$$
$$V^{(1)}_{\rm eff}(x) = -\frac{1}{2}\langle({\rm tr}\log(x-X')^2)^2\rangle'_c - \frac{1}{2}\langle({\rm tr}\log(c+1-x-X'))^2\rangle'_c + \langle {\rm tr}\log(x-X')^2\,{\rm tr}\log(c+1-x-X')\rangle'_c, \tag{4.6}$$

and $e^{-V^{(0)}_{\rm eff}(x_0)-2(N-1)R}$ provides the leading contribution (3.49). The factor coming from the Gaussian integration can be evaluated easily in the double scaling limit. Then we find

$$N\int_{x\notin I_X, y\in I_Y} dxdy e^{-V_{\rm eff}(x,y)} = \sqrt{\frac{\sqrt{3}\pi^2 h}{4\hat{\alpha}a^{4/3}\hat{T}^{2/3}}}e^{-V^{(1)}_{\rm eff}(x_0)}e^{-V^{(0)}_{\rm eff}(x_0)}. \tag{4.7}$$

Note that the factor has the a-dependence of $a^{-(2/3)}$ and it is real. Therefore, it is important to check that $e^{-V^{(1)}_{\rm eff}(x_0)}$ exactly cancels the a-dependence in the prefactor so that the nonperturbative effect will be finite in the double scaling limit. Also, it is interesting to see whether $e^{-V^{(1)}_{\rm eff}(x_0)}$ provides the factor i. In examining $V^{(1)}_{\rm eff}(x_0)$, however we encounter again the technical problem in the z, z'-integration of the first amplitude in (4.4).

V. CONCLUSIONS AND DISCUSSIONS

Based on the dual two-matrix model, we formulated the chemical potential of the instanton and showed that the instantons in the dual model give the same nonperturbative effects as those in the original model in the leading order. Since the dual model defines the T-dual theory of the standard $c = 1/2$ noncritical string theory, it implies that we have identified the degrees of freedom of the dual ZZ-branes, which provide nonperturbative effects in the dual theory, and shown that the number of species of the dual ZZ-branes and their tensions coincide with the original ones. Thus our result gives further support on the identification of the ZZ-brane or nonperturbative effect in string theory as the instanton in the matrix model.

As mentioned in Sec. II, this result is expected because the free energy of the dual model is the same as the original one (rigorously speaking, in the sense of the genus expansion) and hence it should satisfy the same string equation in the double scaling limit as the free energy of the original theory does, which then implies that both theory have the same form of the leading nonperturbative effects. Another argument which supports our result is that the leading part of the nonperturbative effect is given by the disk amplitude in the ZZ-brane background, but at the level of the disk topology the T-dual relation (2.9) holds. In this interpretation, it is crucial that the disk amplitudes $\langle {\rm tr}\log(y \pm Y')\rangle'_d$ cancel out in the process of the computation and do not contribute to the instanton effects as shown in Sec. III.

However, regarding the next-to-leading part in the chemical potential of the instanton, the string equation does not guarantee that the dual theory has the same coefficients in the nonperturbative effects as those in the original theory. In the direct calculation from the matrix model in Sec. IV, several cylinder amplitudes appear to contribute to them. In particular, the amplitudes $\langle({\rm tr}\log(y-Y'))^2\rangle'_c$ and $\langle {\rm tr}\log(y-Y')\,{\rm tr}\log(y+Y')\rangle'_c$ represent quite singular dual spin configurations and their counterparts in the original model do not appear in the calculation of the nonperturbative effects [7]. Thus, if the above amplitudes persist in contributing after taking the double scaling limit, there is a possibility that the T-duality does not hold at the level of the next-to-leading order. From this point of view, the comparison of the coefficients in the nonperturbative effects in both theories is quite interesting because it may possibly reflect the violation of the T-duality. On the other hand, if the nonperturbative effects computed from the original and dual matrix models coincide even including their coefficients, this may imply that the matrix model is more fundamental than the string equation and that the nonperturbative effect in string theory has large universality which contains the T-duality. In fact, it is shown in [7] that the coefficient in the nonperturbative effect is universal within the original $c < 1$ string theory in the sense that it does not depend on details of the potential of the matrix model. Therefore, it is an interesting problem to investigate whether this universality involves even the T-duality. We hope that this kind of study would give some insight into universality of string theory itself. For this purpose, it is desirable to calculate the cylinder amplitudes of trace-log operators like (4.3) and (4.6) which contribute to the coefficients. We hope we will be able to report on it in the near future.

Apart from the coefficients of the nonperturbative effects, there are lots of quantities of interest to be computed. For example, in order to reinforce the identification between the dual ZZ-brane and the instanton in the dual matrix model, it is preferable to calculate the loop amplitudes under the corresponding backgrounds for both of the continuum theory and the matrix model and then to compare the obtained results. In particular, it is quite interesting how the T-duality and the dual ZZ-branes are realized in the continuum Liouville theory. In the case of the standard $c = 1/2$ conformal field theory, we know how the T-duality maps the relevant operators into themselves. If we clarify it for the gravitationally dressed case ($c = 1/2$ noncritical string theory), we will be able to make more explicit the correspondence between relevant operators and associated ZZ-branes in the dual theory and to construct the T-duality transformation law of the ZZ-branes in the noncritical string theory. The result in this paper strongly suggests that the T-duality transformation rule of the relevant operators is the same as in the nongravitational case and that the T-duality does not change the tension of the ZZ-brane associated with each relevant operator.

Finally, in Refs. [25,8], $c = 1/2$ noncritical string field theories corresponding to the original and dual two-matrix models have been constructed, respectively, and it is clarified how string fields in the two theories are related each other under the T-duality. It is quite interesting to express the original and dual ZZ-branes in terms of the string fields[8] and to determine the T-duality transformation rule between the ZZ-branes by utilizing the established transformation rule of the string fields. It will be also intriguing from the viewpoint of finding the relations between different descriptions of nonperturbative objects in string theory—(ZZ-)branes and string fields.

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APPENDIX A: DOUBLE SCALING LIMIT AND RESOLVENTS IN THE DUAL MODEL

In the paper [8], various amplitudes in the dual two-matrix model (2.7) are calculated and their expressions in the continuum limit are obtained. Here, for convenience, we present the result of the disk amplitude for the resolvent operator of X:

$$R_X(x) = \left\langle \frac{1}{N} \mathrm{tr} \frac{1}{x - X} \right\rangle_d \tag{A1}$$

derived in [8]. Also, we compute the disk amplitude for the resolvent of Y:

$$R_Y(y) = \left\langle \frac{1}{N} \mathrm{tr} \frac{1}{y - Y} \right\rangle_d, \tag{A2}$$

which has not appeared in the literature.

Since the dual model (2.7) is obtained from the original one (2.1) by changing the integration variables as (2.5), the partition functions for both models coincide and have the same critical point. The original model has the critical point

$$c_* = \frac{-1 + 2\sqrt{7}}{27}, \qquad g_* = \sqrt{10 c_*^3}, \tag{A3}$$

and the double scaling limit to the $c = 1/2$ string theory is taken as $N \to \infty$ and the lattice spacing $a \to 0$ with fixing

$$c = c_*, \qquad t = \frac{g_* - g}{g_* a^2} \quad \text{and} \quad \frac{1}{g_s} = (\mathrm{const}) N a^{7/3}. \tag{A4}$$

t and g_s represent the cosmological constant and the string coupling constant in the continuum theory. Correspondingly, the critical point of the dual model is

$$(c, \hat{g}) = (c_*, \hat{g}_*) = \left(\frac{-1 + 2\sqrt{7}}{27}, \sqrt{5 c_*^3} \right), \tag{A5}$$

and we obtain the dual $c = 1/2$ string theory by taking the double scaling limit $N \to \infty$, $a \to 0$ with keeping

$$c = c_*, \qquad t = \frac{\hat{g}_* - \hat{g}}{\hat{g}_* a^2} \quad \text{and} \quad \frac{1}{g_s} = (\mathrm{const}) N a^{7/3} \tag{A6}$$

fixed. In what follows, the coupling c is understood to be fixed at the critical point c_*.

1. $R_X(x)$

As in [8,9], the following closed equation for $R_X(x)$ is derived by combining five Schwinger-Dyson equations obtained in the planar limit:

$$\hat{g} R_X(x)^3 + f_2 R_X(x)^2 + f_1 R_X(x) + f_0 = 0, \tag{A7}$$

where

$$\begin{aligned} f_2 &= \hat{g}^2 x^2 + (5c - 1)\hat{g} x - 2c(1 + c), \\ f_1 &= 4c\hat{g}^2 x^3 - (6c - 2c^2)\hat{g} x^2 + 2c(1 - c^2)x - 2\hat{g}^2 V_X \\ &\quad + \hat{g}(1 - 3c), \\ f_0 &= (6c - 2c^2 - 4c\hat{g}x)\hat{g} V_X - 4c\hat{g}^2 V_{X^2} - 4c\hat{g}^2 x^2 \\ &\quad + (6c - 2c^2)\hat{g} x - 2c(1 - c^2) - \hat{g}^2. \end{aligned} \tag{A8}$$

We used the notation: $V_{X^n Y^m} \equiv \langle \frac{1}{N} \mathrm{tr}(X^n Y^m) \rangle_d$. The amplitudes V_{X^n} $(n = 1, 2)$ are related to the amplitudes in the original model $W_n \equiv \langle \frac{1}{N} \mathrm{tr} A^n \rangle_d$ $(n = 1, 3)$ as

$$V_X = \sqrt{2} W_1, \qquad V_{X^2} = \frac{1 - c^2}{cg} W_1 - \frac{g}{c} W_3 - \frac{1}{c}. \tag{A9}$$

After the shift

$$R_X(x) = -\frac{f_2}{3\hat{g}} + \hat{R}_X(x), \tag{A10}$$

(A7) becomes

$$\hat{R}_X(x)^3 - \tfrac{1}{3} F_1 \hat{R}_X(x) - \tfrac{1}{27} F_0 = 0 \tag{A11}$$

with

$$F_1 = \frac{1}{\hat{g}^2} f_2^2 - \frac{3}{\hat{g}} f_1, \qquad F_0 = \frac{9}{\hat{g}^2} f_1 f_2 - \frac{2}{\hat{g}^3} f_2^3 - \frac{27}{\hat{g}} f_0. \tag{A12}$$

At $\hat{g} = \hat{g}_*$, F_1 and F_0 can be written as

[8] For somewhat related works, see Refs. [34].

$$F_1 = \frac{c}{5}(z-\hat{s})^4,$$
$$F_0 = -2\left(\frac{c}{5}\right)^{3/2}(z-\hat{s})^4(z-\hat{s}+3\sqrt{6})(z-\hat{s}-3\sqrt{6}), \tag{A13}$$

where $z \equiv \sqrt{5c}x$, $\hat{s} = 1+\sqrt{7}$, and the fact was used that W_1, W_3 expressed in [25]:

$$W_1 = W_1^{\rm non} + \hat{W}_1,$$
$$W_1^{\rm non} = \frac{-8\rho_*^4 + 3(2g_*)^2}{3(2g_*)^3} + a^2\frac{-136\rho_*^4+27(2g_*)^2}{27(2g_*)^3}t,$$
$$\hat{W}_1 = a^{8/3}\frac{8\rho_*^4}{27(2g_*)^3}(5t)^{4/3} + a^{10/3}\frac{4\rho_*^4}{81(2g_*)^3}(5t)^{5/3} + a^4\frac{-8527\rho_*^4+972(2g_*)^2}{972(2g_*)^3}t^2 + \mathcal{O}(a^{14/3}), \tag{A14}$$

$$W_3 = W_3^{\rm non} + \hat{W}_3,$$
$$W_3^{\rm non} = \frac{32(420-839\rho_*)\rho_*^5}{729(2g_*)^5} + a^2\frac{160(252-611\rho_*)\rho_*^5}{729(2g_*)^5}t,$$
$$\hat{W}_3 = a^{8/3}\frac{320\rho_*^6}{81(2g_*)^5}(5t)^{4/3} + a^{10/3}\frac{160\rho_*^6}{243(2g_*)^5}(5t)^{5/3} + a^4\frac{70(1152-3593\rho_*)\rho_*^5}{729(2g_*)^5}t^2 + \mathcal{O}(a^{14/3}), \tag{A15}$$

with $\rho_* = 3c$ become at the critical point

$$W_{1*} = \left(\frac{9}{40c}\right)^{3/2}\left(-8c+\frac{40}{27}\right),$$
$$W_{3*} = \frac{32}{243}\left(\frac{9}{40c}\right)^{5/2}(140-839c).$$

Equations (A13) determine the critical point of x, denoted by $\hat{P}'_*$, as

$$\hat{P}'_* = \frac{\hat{s}}{\sqrt{5c}}, \tag{A16}$$

where the cubic equation (A11) becomes triply degenerate: $\hat{R}_X(\hat{P}'_*)^3 = 0$. In general, the equation (A11) is solved by

$$\hat{R}_X(x) = \frac{1}{3}\left[\left(\frac{F_0+\sqrt{F_0^2-4F_1^3}}{2}\right)^{1/3} + \left(\frac{F_0-\sqrt{F_0^2-4F_1^3}}{2}\right)^{1/3}\right]. \tag{A17}$$

Introducing the variables in the continuum theory $\hat{T}$ and ζ as

$$\hat{g} = \hat{g}_*\left(1-a^2\frac{\hat{s}^2}{10}\hat{T}\right), \qquad x = \hat{P}'_*(1+a\zeta) \tag{A18}$$

($\hat{T}$ is proportional to the cosmological constant t), the continuum limit of the solution (A17) takes the form

$$\hat{R}_X(x) = a^{4/3}\hat{\alpha}' w(\zeta) + \mathcal{O}(a^{5/3}),$$
$$w(\zeta) = \left(\zeta+\sqrt{\zeta^2-\hat{T}}\right)^{4/3} + \left(\zeta-\sqrt{\zeta^2-\hat{T}}\right)^{4/3}, \tag{A19}$$

with $\hat{\alpha}' = \frac{c^{1/2}\hat{s}^{4/3}}{2^{2/3}\sqrt{5}}$. Also,

$$R_X(x) = R_X^{\rm non}(x) + \hat{R}_X(x), \qquad R_X^{\rm non}(x) = -\frac{f_2}{3\hat{g}}, \tag{A20}$$

where $R_X^{\rm non}$ and $\hat{R}_X$ stand for the nonuniversal and universal parts, respectively.

Eigenvalue distribution of the matrix X is seen as a cut of $\hat{R}_X(x)$ on the x-plane, which is determined in principle from (A17) under the condition of $\hat{R}_X(x)$ having one cut and being analytic for ${\rm Re}x > x_c$ with x_c some finite number. However, practically it is not an easy task in general, since $F_0^2 - 4F_1^3$ is a complicated polynomial of x with the degree ten. $w(\zeta)$ in (A19) has one cut $\zeta \in (-\infty, -\sqrt{\hat{T}}]$, which means that the right edge of the cut for $\hat{R}_X(x)$ is $x = \beta' \equiv \hat{P}'_*(1-a\sqrt{\hat{T}})$ when $\hat{g}$ is near the critical point $\hat{g}_*$. The leading value of the left edge also can be determined from the expression of (A13). Plugging it into (A17), we find the form of $\hat{R}_X(x)$, with c and $\hat{g}$ set to the critical values but x left generic, as

$$\hat{R}_X(x) = \frac{1}{3}\sqrt{\frac{c}{5}}(z-\hat{s})^{4/3}(g_+(z)^{1/3} + g_-(z)^{1/3}),$$
$$g_\pm(z) \equiv -(z-\hat{s}+3\sqrt{6})(z-\hat{s}-3\sqrt{6}) \pm 6\sqrt{3}\sqrt{-(z-\hat{s}+3\sqrt{3})(z-\hat{s}-3\sqrt{3})}. \tag{A21}$$

Here, it is easy to see that $g_+(z)$ has no zero and $g_-(z)$ has the unique zero at $z = \hat{s}$. For $z \sim \hat{s}$,

$$g_+(z) = 108 + \mathcal{O}((z-\hat{s})^2),$$
$$g_-(z) = \frac{1}{108}(z-\hat{s})^4 + \mathcal{O}((z-\hat{s})^5). \tag{A22}$$

From these properties of $g_\pm(z)$, it can be shown that (A21) has one cut of the interval $z \in [\hat{s}-3\sqrt{3}, \hat{s}]$. Thus, the left edge of the cut for $\hat{R}_X(x)$ is given by $x = \alpha' \equiv \hat{P}'_* - \frac{3\sqrt{3}}{\sqrt{5c}} + \mathcal{O}(a)$ in the case of $\hat{g}$ near the critical point.

2. $R_Y(y)$

To obtain a closed equation for $R_Y(y)$, we combine the six Schwinger-Dyson equations derived from the following identities in the planar limit:

$$
\begin{aligned}
0 &= \int d^{N^2}X d^{N^2}Y \sum_\alpha \frac{\partial}{\partial Y_\alpha} \operatorname{tr}\left(\frac{1}{y-Y}t^\alpha\right)e^{-N\operatorname{tr}\tilde{S}_D(X,Y)},\\
0 &= \int d^{N^2}X d^{N^2}Y \sum_\alpha \frac{\partial}{\partial X_\alpha} \operatorname{tr}\left(\frac{1}{y-Y}t^\alpha\right)e^{-N\operatorname{tr}\tilde{S}_D(X,Y)},\\
0 &= \int d^{N^2}X d^{N^2}Y \sum_\alpha \frac{\partial}{\partial Y_\alpha} \operatorname{tr}\left(\frac{1}{y-Y}Xt^\alpha\right)e^{-N\operatorname{tr}\tilde{S}_D(X,Y)},\\
0 &= \int d^{N^2}X d^{N^2}Y \sum_\alpha \frac{\partial}{\partial Y_\alpha} \operatorname{tr}\left(\frac{1}{y-Y}Xt^\alpha X\right)e^{-N\operatorname{tr}\tilde{S}_D(X,Y)},\\
0 &= \int d^{N^2}X d^{N^2}Y \sum_\alpha \frac{\partial}{\partial X_\alpha} \operatorname{tr}\left(\frac{1}{y-Y}XYt^\alpha\right)e^{-N\operatorname{tr}\tilde{S}_D(X,Y)},\\
0 &= \int d^{N^2}X d^{N^2}Y \sum_\alpha \frac{\partial}{\partial X_\alpha} \operatorname{tr}\left(\frac{1}{y-Y}t^\alpha YX\right)e^{-N\operatorname{tr}\tilde{S}_D(X,Y)}.
\end{aligned}
\tag{A23}
$$

Here, t^α ($\alpha = 1, \cdots, N^2$) are a basis of $N \times N$ Hermitian matrices satisfying

$$
\begin{aligned}
\sum_\alpha \operatorname{tr}(Wt^\alpha Z t^\alpha) &= \operatorname{tr}W \operatorname{tr}Z,\\
\sum_\alpha \operatorname{tr}(Wt^\alpha)\operatorname{tr}(Zt^\alpha) &= \operatorname{tr}(WZ)
\end{aligned}
\tag{A24}
$$

for arbitrary matrices W, Z. The result is

$$
R_Y(y)^4 + b_1 R_Y(y)^3 + b_2 R_Y(y)^2 + b_3 R_Y(y) + b_4 = 0,
\tag{A25}
$$

where

$$
\begin{aligned}
b_1 &= -8cy,\\
b_2 &= 4\hat{g}^2y^4 + (-1 + 2c + 19c^2)y^2 + 2(1 + c - 2\hat{g}V_X),\\
b_3 &= -16c\hat{g}^2y^5 + 4c(1+c)(1-3c)y^3\\
&\quad - 8c(1 + c - 2\hat{g}V_X)y,\\
b_4 &= 16c\hat{g}^2y^4 + \{-4c(1+c)(1-3c)\\
&\quad - 4(1-c)(1-3c)\hat{g}V_X + 4(1-3c)\hat{g}^2V_{X^2}\\
&\quad + 8\hat{g}^3V_{XY^2}\}y^2 + (1 + c - 2\hat{g}V_X)^2.
\end{aligned}
\tag{A26}
$$

(Note that $V_Y = V_{XY} = 0$ from the $\mathbf{Z}_2$ symmetry under $Y \to -Y$.) Also, V_{XY^2} can be expressed in terms of the amplitudes W_1 and W_3 in the original model, similarly to V_X and V_{X^2}, as

$$
V_{XY^2} = \frac{1}{\sqrt{2}}\left(-\frac{(1-c)(1-c^2)}{cg^2}W_1 + \frac{1+c}{c}W_3 + \frac{1}{cg}\right).
\tag{A27}
$$

We shift $R_Y(y)$ as

$$
R_Y(y) = -\tfrac{1}{4}b_1 + \hat{R}_Y(y),
\tag{A28}
$$

so that the cubic term of $\hat{R}_Y(y)$ in (A25) vanishes, to obtain

$$
\hat{R}_Y(y)^4 + B_1\hat{R}_Y(y)^2 + B_2\hat{R}_Y(y) + B_3 = 0,
\tag{A29}
$$

where

$$
\begin{aligned}
B_1 &= -\tfrac{3}{8}b_1^2 + b_2, \qquad B_2 = \tfrac{1}{8}b_1^3 - \tfrac{1}{2}b_1b_2 + b_3,\\
B_3 &= -\tfrac{3}{256}b_1^4 + \tfrac{1}{16}b_1^2b_2 - \tfrac{1}{4}b_1b_3 + b_4.
\end{aligned}
\tag{A30}
$$

The critical points of y are determined by

$$
B_1|_* = B_2|_* = B_3|_* = 0,
\tag{A31}
$$

with the symbol "$|_*$" meaning c, $\hat{g}$, W_1 and W_3 set to the critical values. The equation for B_2 is trivially satisfied, and the remaining two lead to

$$
\begin{aligned}
B_1|_* &= 20c^3(y - \hat{Q}'_*)^2(y + \hat{Q}'_*)^2 = 0,\\
B_3|_* &= -80c^5(y - \hat{Q}'_*)^3(y + \hat{Q}'_*)^3 = 0, \qquad \left(\hat{Q}'_* \equiv \frac{2}{\sqrt{5c}}\right),
\end{aligned}
\tag{A32}
$$

to determine the critical points as

$$
y = \pm\hat{Q}'_*.
\tag{A33}
$$

The result reflects the $\mathbf{Z}_2$ symmetry as should be. We introduce the scaling variable η as $y = \hat{Q}'_*(1 + a\eta)$ and take the continuum limit of the quartic equation (A29) by using the expressions of W_1, W_3 in (A14) and (A15). The result is

$$
\hat{R}_Y(y)^4 - a^2\frac{8}{5}c(32\eta^2 - \hat{s}^2\hat{T})\hat{R}_Y(y)^2 - a^3 2\left(\frac{8^2}{5}c\right)^2\eta^3 + \mathcal{O}(a^{11/3}) = 0,
\tag{A34}
$$

which is solved as

$$
\begin{aligned}
R_Y(y) &= R_Y^{\mathrm{non}}(y) + \hat{R}_Y(y), \qquad R_Y^{\mathrm{non}}(y) = -\tfrac{1}{4}b_1 = 2cy,\\
\hat{R}_Y(y) &= a^{3/4}8\cdot 2^{1/4}\sqrt[4]{\frac{c}{5}}\eta^{3/4} + \mathcal{O}(a^{5/4}).
\end{aligned}
\tag{A35}
$$

(We took the branch of $\hat{R}_Y(y)$ to be real positive for y or η large positive.) The scaling of $\hat{R}_Y(y)$ with the power $a^{3/4}$ is clearly different from the scaling of correlators ever computed in the original and dual models [8,25], which is characterized by the power of cubic roots. $R_Y(y)$ is regarded as a disk amplitude with the disorder operators distributed densely over the boundary, and such an amplitude has not been computed in the literature concerning either of the original and dual theories. Also, the leading expression of $\hat{R}_Y(y)$ does not depend on the cosmological constant, meaning that the disk collapses into a surface of zero area although its boundary has a finite length. In the above, we took the continuum limit around the critical point $y = \hat{Q}'_*$. Similarly, the limit around the other critical point setting $y = -\hat{Q}'_*(1 + a\eta)$ leads to the same equation as (A34), which is a consequence from the $\mathbf{Z}_2$ symmetry.

The expression (A35) has one cut $\eta \in (-\infty, 0]$, whose endpoint does not depend on $\hat{T}$. It shows that the right edge of the cut $y = \hat{Q}'_*$ on y-plane receives no $\mathcal{O}(a)$ correction even when $\hat{g}$ moves slightly off the critical point. In order to see the global structure of the cut on y-plane, we solve Eq. (A29) by using (A32) in the case of $(c, \hat{g})$ fixed at the critical point but y left generic. The result is

$$\hat{R}_Y(y) = \sqrt{10c^3}(y - \hat{Q}'_*)^{3/4}(y + \hat{Q}'_*)^{3/4} \times \left[y - \sqrt{(y - \hat{Q}'_*)(y + \hat{Q}'_*)} \right]^{1/2}, \quad \text{(A36)}$$

(We took the same choice of the branch as above.) which has one cut of the interval $y \in [-\hat{Q}'_*, \hat{Q}'_*]$. Therefore, we see that $\hat{R}_Y(y)$ has the $\mathbf{Z}_2$-symmetric one cut of the form $y \in [-\gamma', \gamma']$ with $\gamma' = \hat{Q}'_* + o(a)$ for the case $\hat{g}$ near the critical point.

3. Expressions for the action (3.1)

The calculation of the instanton effects in Sec. III is done by using the rescaled action $S_D(X, Y)$ in (3.1) instead of $\tilde{S}_D(X, Y)$ in (2.7). The former is obtained from the latter by

$$X \to \frac{1}{\hat{g}} X, \qquad Y \to \frac{1}{\hat{g}} Y. \quad \text{(A37)}$$

Here, we consider the effect of the rescaling to the results obtained in Secs. A 1 and A 2 to match them with the expressions in Sec. III.

Let us rename the resolvents calculated using the action $\tilde{S}_D(X, Y)$ as $\tilde{R}_X(x)$ and $\tilde{R}_Y(y)$, which are nothing but those appearing in the Secs. A 1 and A 2. As a result of the rescaling, the resolvents R_X, R_Y considered in Sec. III are related to $\tilde{R}_X$, $\tilde{R}_Y$ as

$$R_X(x) = \frac{1}{\hat{g}} \tilde{R}_X\left(\frac{1}{\hat{g}} x\right), \qquad R_Y(y) = \frac{1}{\hat{g}} \tilde{R}_Y\left(\frac{1}{\hat{g}} y\right). \quad \text{(A38)}$$

Hence, the critical points of x and y for $R_X(x)$ and $R_Y(y)$, denoted by $\hat{P}_*$ and $\hat{Q}_*$ respectively, are given as

$$\hat{P}_* = \hat{g}_* \hat{P}'_* = \hat{s}c = \frac{1+c}{2}, \qquad \hat{Q}_* = \hat{g}_* \hat{Q}'_* = 2c. \quad \text{(A39)}$$

For $R_X(x) = R_X^{\rm non}(x) + \hat{R}_X(x)$, the nonuniversal part becomes

$$R_X^{\rm non}(x) = \frac{1}{\hat{g}} \tilde{R}_X^{\rm non}\left(\frac{1}{\hat{g}} x\right) = -\frac{1}{3\hat{g}^2} f_2 \Big|_{x \to (1/\hat{g})x} = -\frac{1}{3h}(x^2 + (5c-1)x - 2c(1+c)) = \frac{1}{3h}(2V'(x) - V'(1+c-x)), \quad \text{(A40)}$$

where $h = \hat{g}^2$, and $V(x) \equiv \frac{1-c}{2} x^2 - \frac{1}{3} x^3$. The scaling variable ζ is introduced as $x = \hat{P}_*(1 + a\zeta)$, and the universal part takes the form

$$\hat{R}_X(x) = \frac{1}{\hat{g}} \hat{\tilde{R}}_X\left(\frac{1}{\hat{g}} x\right) = a^{4/3} \hat{\alpha} w(\zeta) + \mathcal{O}(a^{5/3}),$$
$$\hat{\alpha} = \frac{1}{\hat{g}_*} \hat{\alpha}' = \frac{\hat{s}^{4/3}}{2^{2/3} \cdot 5c}, \quad \text{(A41)}$$
$$w(\zeta) = \left(\zeta + \sqrt{\zeta^2 - \hat{T}}\right)^{4/3} + \left(\zeta - \sqrt{\zeta^2 - \hat{T}}\right)^{4/3}.$$

When $\hat{g}$ is near the critical point, endpoints of the cut of $\hat{R}_X(x)$: $x \in I_X = [\alpha, \beta]$ are expressed as

$$\beta = \hat{g}_* \beta' = \hat{P}_*\left(1 - a\sqrt{\hat{T}}\right), \ \alpha = \hat{g}_* \alpha' = \hat{P}_* - 3\sqrt{3}c + \mathcal{O}(a). \quad \text{(A42)}$$

Similarly, for $R_Y(y) = R_Y^{\rm non}(y) + \hat{R}_Y(y)$, we define the variable η by $y = \hat{Q}_*(1 + a\eta)$, and obtain

$$R_Y^{\rm non}(y) = \frac{1}{\hat{g}} \tilde{R}_Y^{\rm non}\left(\frac{1}{\hat{g}} y\right) = -\frac{1}{4\hat{g}} b_1 \Big|_{y \to (1/\hat{g})y} = -\frac{2c}{h} y,$$
$$\hat{R}_Y(y) = \frac{1}{\hat{g}} \hat{\tilde{R}}_Y\left(\frac{1}{\hat{g}} y\right) = a^{3/4} \frac{8 \cdot 2^{1/4}}{5c} \eta^{3/4} + \mathcal{O}(a^{5/4}). \quad \text{(A43)}$$

The cut of $\hat{R}_Y(y)$ is given by the $\mathbf{Z}_2$-symmetric interval $y \in I_Y$ as

$$I_Y = [-\gamma, \gamma] \quad \text{with} \quad \gamma = \hat{Q}_* + o(a), \quad \text{(A44)}$$

for the case $\hat{g}$ near the critical point.

Since the one-matrix model (3.34) is obtained after the Gaussian integration over Y in (3.1), the partition functions of (3.1) and (3.34) are identical and have the same double scaling limit (A6). Also, concerning the correlators among operators independent of Y [for example, $R_X(x)$], both models give the same result.

APPENDIX B: COMPUTATION OF DENOMINATOR

In this appendix, we compute the denominator in (3.13) based on the method developed in [32]. Using (3.12), the denominator can be written in terms of the partition functions as

$$\int_{(x,y)\in\mathcal{S}} dx dy e^{-V_{\rm eff}(x,y)} = \frac{D_N Z_N^{(0-{\rm inst})}(h)}{D_{N-1} Z_{N-1}(h')}. \quad \text{(B1)}$$

The difference between the 0-instanton partition function $Z_N^{(0-{\rm inst})}(h)$ and the total partition function $Z_N(h)$ is exponentially small as $\exp(-C/g_s)$ which is nothing but the nonperturbative effect. Since it is negligible in the computation of μ, we can replace $Z_N^{(0-{\rm inst})}(h)$ with $Z_N(h)$ in (B1). Therefore,

$$\int_{(x,y)\in\mathcal{S}} dx dy e^{-V_{\rm eff}(x,y)} = \frac{D_N Z_N(h)}{D_{N-1} Z_{N-1}(h')}. \quad \text{(B2)}$$

Namely, the denominator can be obtained basically as the ratio between the partition functions of the matrix model with rank N and that with rank $N-1$. As mentioned in (3.3), under the measure (3.2) the total free energy has the standard $1/N$-expansion

$$\begin{aligned} Z_N(h) &= \exp\left[-N^2F_0(h) - F_1(h) + \mathcal{O}\left(\frac{1}{N^2}\right)\right], \\ Z_{N-1}(h') &= \exp\left[-(N-1)^2F_0(h') - F_1(h') + \mathcal{O}\left(\frac{1}{(N-1)^2}\right)\right]. \end{aligned} \tag{B3}$$

Hence,

$$\begin{aligned} \int_{(x,y)\in\mathcal{S}} dxdy e^{-V_{\rm eff}(x,y)} &= \frac{D_N}{D_{N-1}}\exp\left[-(N-1)(2F_0(h') + h'F_0'(h')) - \left(F_0(h') + 2h'F_0'(h') + \frac{1}{2}h'^2F_0''(h')\right)\right] \\ &\quad \times\left[1 + \mathcal{O}\left(\frac{1}{N}\right)\right]. \end{aligned} \tag{B4}$$

The computation is essentially reduced to finding the sphere free energy $F_0(h)$.

1. Sphere free energy

In order to compute the sphere free energy, we first perform the integration with respect to Y in (3.1) and express $Z_N(h)$ as the integration over the eigenvalues of X. Then we obtain

$$Z_N(h) = (J_N^X)^{-1}\int\prod_{i=1}^N d\lambda_i \exp\left[\sum_{i<j}\log(\lambda_i - \lambda_j)^2 - \frac{N}{h}\sum_i V(\lambda_i) - \frac{1}{2}\sum_{ij}\log(c + 1 - \lambda_i - \lambda_j) + \frac{N^2}{2}\log(c+1)\right], \tag{B5}$$

where J_N^X is defined in (3.28) whose explicit form is given in Appendix B 4 a. In the large-N limit, by introducing the eigenvalue distribution $\rho(\lambda) = \langle\frac{1}{N}\sum_i\delta(\lambda - \lambda_i)\rangle$, this becomes

$$\begin{aligned} Z_N^{(0)}(h) = (J_N^X)^{-1}\exp\bigg[N^2\bigg\{&\int d\lambda d\mu\rho(\lambda)\rho(\mu)\log|\lambda - \mu| - \frac{1}{h}\int d\lambda\rho(\lambda)V(\lambda) \\ &-\frac{1}{2}\int d\lambda d\mu\rho(\lambda)\rho(\mu)\log(c + 1 - \lambda - \mu) + \frac{1}{2}\log(c+1)\bigg\}\bigg]. \end{aligned} \tag{B6}$$

Therefore, the sphere free energy is expressed in terms of the eigenvalue distribution as

$$\begin{aligned} F_0(h) = &-\int d\lambda d\mu\rho(\lambda)\rho(\mu)\log|\lambda - \mu| + \frac{1}{h}\int d\lambda\rho(\lambda)V(\lambda) + \frac{1}{2}\int d\lambda d\mu\rho(\lambda)\rho(\mu)\log(c + 1 - \lambda - \mu) \\ &-\frac{1}{2}\log(c+1) + \frac{1}{N^2}\log J_N^X. \end{aligned} \tag{B7}$$

2. Identity of the resolvent

Here we derive an identity of the resolvent $R_X(z)$ introduced in (3.20) which plays an important role in the following. In terms of the eigenvalue distribution $\rho(\lambda)$, the resolvent is given by

$$R_X(z) = \int d\lambda\frac{\rho(\lambda)}{z - \lambda}. \tag{B8}$$

Then the support of the eigenvalue distribution is characterized as the cut of $R_X(z)$ on the complex z-plane.

On the other hand, in the large-N limit $\rho(\lambda)$ satisfies a saddle point equation which follows from (B7)

$$\begin{aligned} 0 &= 2P\int d\mu\frac{\rho(\mu)}{\lambda - \mu} - \frac{1}{h}V'(\lambda) + \int d\mu\frac{\rho(\mu)}{c + 1 - (\lambda + \mu)} \\ &= R_X(\lambda + i0) + R_X(\lambda - i0) + R_X(c + 1 - \lambda) - \frac{1}{h}V'(\lambda), \quad \text{for } \lambda \text{ on the cut.} \end{aligned} \tag{B9}$$

Note that, for λ on the cut, $\lambda < \hat{P}_*$ as shown in (A42) and $c+1-\lambda > \hat{P}_*$ is always outside the cut. Hence the third term in the last equation in (B9) does not have the imaginary part. A particular solution to this equation is provided by

$$R_X^{\rm non}(z) = \frac{1}{3h}(2V'(z) - V'(1+c-z)). \tag{B10}$$

Therefore, the resolvent $R_X(z)$ is in general given by

$$R_X(z) = R_X^{\rm non}(z) + \hat{R}_X(z), \tag{B11}$$

where $\hat{R}_X(z)$ satisfies

$$0 = \hat{R}_X(z+i0) + \hat{R}_X(z-i0) + \hat{R}_X(1+c-z), \tag{B12}$$

for z on the cut.

In fact, this form of the resolvent coincides with the result in (A40) and (A41) in Appendix A.

3. $F_0(h)$ in terms of the resolvent

Now let us express each term in (B7) in terms of the resolvent. From (B8), it is easy to see that the following identities hold:

$$\begin{aligned} \int dzdz'\rho(z)\rho(z')\log|z-z'| &= \int_\Lambda^\beta R_X(z')dz' + \log\Lambda + \int_\alpha^\beta dz\rho(z)\int_\beta^z {\rm Re}R_X(z')dz', \\ \int dzdz'\rho(z)\rho(z')\log(c+1-z-z') &= \int_\Lambda^\beta R_X(z')dz' + \log\Lambda + \int_\alpha^\beta dz\rho(z)\int_\beta^{c+1-z} R_X(z')dz', \end{aligned} \tag{B13}$$

where $\Lambda \to \infty$ limit is understood to be taken eventually, and β is the right edge of the cut explicitly given in (A42). Substituting these into (B7) and using the saddle point equation (B9) leads to

$$F_0(h) = -\frac{1}{2}\left\langle\frac{1}{N}\,{\rm tr}\log(\beta - X)\right\rangle_d - \frac{1}{2}\int_{c+1-\beta}^\beta R_X(z')dz' + \frac{1}{2h}V(\beta) + \frac{1}{2h}\int dz\rho(z)V(z) + \mathcal{J}, \tag{B14}$$

where $\mathcal{J}$ represents terms independent of the potential V

$$\mathcal{J} = -\frac{1}{2}\log(c+1) + \frac{1}{N^2}\log J_N^X. \tag{B15}$$

According to (B4), we also need derivatives of $F_0(h)$ in computing the denominator. In order to make explicit h-dependence of the sphere free energy, we rewrite $F_0(h)$ again in terms of the integration with respect to the eigenvalues as

$$F_0(h) - \mathcal{J} = -\frac{1}{N^2}\log\left[\int\prod_{i=1}^N d\lambda_i \Delta^{(N)}(\lambda)^2 e^{-(N/h)\sum_i V(\lambda_i)}\left(\frac{1}{1+c}\right)^{N^2/2}\prod_{i,j=1}^N\left(1 - \frac{1}{1+c}(\lambda_i+\lambda_j)\right)^{-(1/2)}\right], \tag{B16}$$

where we have used (B5). This shows

$$h\frac{\partial}{\partial h}(F_0(h) - \mathcal{J}) = -\frac{1}{N^2}\left\langle\frac{N}{h}\,{\rm tr}V(X)\right\rangle_d = -\frac{1}{h}\int dz\rho(z)V(z). \tag{B17}$$

From (B14) and (B17), for the $(N-1)\times(N-1)$ matrix model with h' defined as (3.8), we obtain

$$2F_0(h') + h'F_0'(h') = -\left\langle\frac{1}{N-1}\,{\rm tr}\log(\beta - X')\right\rangle_d' - \int_{c+1-\beta}^\beta R_{X'}(z')dz' + \frac{1}{h'}V(\beta) + \left(2 + h'\frac{\partial}{\partial h'}\right)\mathcal{J}', \tag{B18}$$

where $\mathcal{J}'$ is obtained by replacing N and h by $N-1$ and h' in (B15). We will see below that the last term in this equation together with the overall factor D_N/D_{N-1} in (B4) gives subleading contributions not affecting the leading term. Therefore, we find that the leading term in the denominator is given as

$$\int_{(x,y)\in\mathcal{S}} dxdye^{-V_{\rm eff}(x,y)}\Bigg|_{\rm leading} = \exp\left[\langle{\rm tr}\log(\beta - X')\rangle_d' + (N-1)\int_{c+1-\beta}^\beta R_{X'}(x')dx' - \frac{N-1}{h'}V(\beta)\right]. \tag{B19}$$

B. Next-to-leading contributions in the denominator

a. Derivation of D_N and J_N^X

In this subsection, we evaluate next-to-leading contributions in the denominator in (3.13).First let us evaluate the measure factors D_N, J_N^X introduced in (3.4) and (3.28).

In order to calculate J_N^X, we make a connection between the measure dX defined in (3.2) and the standard measure

$$\widetilde{dX} \equiv \prod_i dX_{ii} \prod_{i<j} 2(d\,\mathrm{Re}X_{ij})(d\,\mathrm{Im}X_{ij}). \tag{B20}$$

For this purpose, it is sufficient to consider the Gaussian integration as follows. In the case of the standard measure,

$$\int \widetilde{dX} e^{-(1/2)\,\mathrm{tr}X^2} = (2\pi)^{N^2/2}. \tag{B21}$$

Comparing this to (3.2) yields

$$dX = \left(\frac{(1-c)N}{2\pi h}\right)^{N^2/2} \widetilde{dX}. \tag{B22}$$

On the other hand, by using the method of orthogonal polynomials,

$$J_N^X \int dX e^{-(1/2)\,\mathrm{tr}X^2} = \int \prod_i d\lambda_i \Delta^{(N)}(\lambda)^2 e^{-(1/2)\sum_i \lambda_i^2} = N! \prod_{n=0}^{N-1} h_n = (2\pi)^{N/2} \prod_{p=0}^{N} p!, \tag{B23}$$

where we have used the well-known fact that in the Gaussian case the orthogonal polynomial is nothing but the Hermite polynomial with $h_n = \sqrt{2\pi}n!$. Thus we find

$$J_N^X = (2\pi)^{N/2} \left(\prod_{p=0}^{N} p!\right) \left(\frac{h}{(1-c)N}\right)^{N^2/2}. \tag{B24}$$

Similarly to J_N^X in (3.28), if we introduce J_N^Y by

$$J_N^Y \int dY f(\mathrm{tr}Y, \mathrm{tr}Y^2, \cdots) = \int \left(\prod_{i=1}^{N} d\mu_i\right) \Delta^{(N)}(\mu)^2 \times f\left(\sum_i \mu_i, \sum_i \mu_i^2, \cdots\right) \tag{B25}$$

with dY defined in (3.2), we obtain

$$J_N^Y = (2\pi)^{N/2} \left(\prod_{p=0}^{N} p!\right) \left(\frac{h}{(1+c)N}\right)^{N^2/2}. \tag{B26}$$

Using these, (3.1) becomes

$$\int dX dY e^{-(N/h)\,\mathrm{tr}(V(X)+((1+c)/2)Y^2 - XY^2)} = (J_N^X J_N^Y)^{-1} \int \left(\prod_{i=1}^{N} d\lambda_i d\mu_i\right) \Delta^{(N)}(\lambda)^2 \Delta^{(N)}(\mu)^2 e^{-(N/h)\sum_i (V(\lambda_i)+((1+c)/2)\mu_i^2)} I(\lambda, \mu), \tag{B27}$$

where

$$I(\lambda, \mu) = \int dU \exp\left(\frac{N}{h}\,\mathrm{tr}(XUY^2U^\dagger)\right). \tag{B28}$$

It is calculated by the method in [29] as

$$I(\lambda, \mu) = \left(\frac{N}{h}\right)^{-(N(N-1)/2)} \left(\prod_{p=1}^{N-1} p!\right) \frac{\det_{ij} e^{(N/h)\lambda_i \mu_j^2}}{\Delta^{(N)}(\lambda)\Delta^{(N)}(\mu^2)}. \tag{B29}$$

Substituting this into (B27), we find

$$\int dX dY e^{-(N/h)\,\mathrm{tr}(V(X)+((1+c)/2)Y^2 - XY^2)} = (J_N^X J_N^Y)^{-1} \left(\frac{N}{h}\right)^{-(N(N-1)/2)} \left(\prod_{p=1}^{N} p!\right) \times \int \left(\prod_{i=1}^{N} d\lambda_i d\mu_i\right) \times \frac{\Delta^{(N)}(\lambda)\Delta^{(N)}(\mu)}{\prod_{i<j}(\mu_i + \mu_j)} e^{-(N/h)\sum_i (V(\lambda_i)+((1+c)/2)\mu_i^2 - \lambda_i \mu_i^2)}. \tag{B30}$$

Comparing this equation with the definition of D_N (3.4), we finally obtain

$$D_N = J_N^X J_N^Y \left(\frac{N}{h}\right)^{(N(N-1)/2)} \left(\prod_{p=1}^{N} p!\right)^{-1} = \left(\prod_{p=0}^{N} p!\right)\left(\frac{4\pi^2 h}{N}\right)^{N/2}\left(\frac{h}{(1-c^2)N}\right)^{N^2/2}. \tag{B31}$$

Thus the overall factor in (B4) becomes

$$\frac{D_N}{D_{N-1}} = (2\pi)^{3/2}\sqrt{N-1}e^{-(N-1)}\frac{h'^N}{(1-c^2)^{N-(1/2)}} \times \left(1 + \mathcal{O}\left(\frac{1}{N}\right)\right), \tag{B32}$$

which agrees with the result in [7].

b. Other contributions in (B4)

Using the Euler-Maclaurin formula, we evaluate the following quantity

$$\log\left(\prod_{p=0}^{N} p!\right) = \log\left(\prod_{k=1}^{N} k^{N+1-k}\right) = \frac{(N+1)^2}{2}\log(N+1) - \frac{3}{4}N^2 - \frac{N}{2} + \mathcal{O}(1), \tag{B33}$$

which is also derived in [35]. From (B24), we see that $\mathcal{J}$ given in (B15) becomes

$$\begin{aligned}\mathcal{J} &= -\frac{1}{2}\log(c+1) + \frac{1}{N^2}\log\left(\prod_{p=0}^{N} p!\right) + \frac{1}{2}\log\left(\frac{h}{(1-c)N}\right) + \mathcal{O}\left(\frac{1}{N}\log N\right)\\ &= -\frac{1}{2}\log\left(\frac{(1-c^2)N}{h}\right) + \frac{1}{N^2}\left(\frac{(N+1)^2}{2}\log(N+1) - \frac{3}{4}N^2\right) + \mathcal{O}\left(\frac{1}{N}\log N\right)\\ &= -\frac{1}{2}\log\left(\frac{(1-c^2)}{h}\right) - \frac{3}{4} + \mathcal{O}\left(\frac{1}{N}\log N\right).\end{aligned} \tag{B34}$$

Therefore, the last term in (B18) gives

$$\left(2 + h'\frac{\partial}{\partial h'}\right)\mathcal{J}' = -\log\left(\frac{1-c^2}{h'}\right) - 1. \tag{B35}$$

Next, let us examine the $\mathcal{O}(N^0)$ contribution in the exponent in (B4). From (B17) we have

$$h'F_0'(h') = -\frac{1}{h'}\int dz\rho(z)V(z) + \frac{1}{2}. \tag{B36}$$

Combining this equation with (B18) and (B35), we find

$$F_0(h') + 2h'F_0'(h') + \frac{1}{2}h'^2F_0''(h') = -R - h'\frac{\partial}{\partial h'}R - \frac{1}{2}\log\left(\frac{1-c^2}{h'}\right), \tag{B37}$$

where R is given as

$$R \equiv \frac{1}{2}\left\langle\frac{1}{N-1}\operatorname{tr}\log(\beta - X')\right\rangle_d' + \frac{1}{2}\int_{c+1-\beta}^{\beta} R_{X'}(z')dz' - \frac{1}{2h'}V(\beta). \tag{B38}$$

Also, (B18) is written as

$$2F_0(h') + h'F_0'(h') = -2R - \log\left(\frac{1-c^2}{h'}\right) - 1. \tag{B39}$$

We substitute (B32), (B37), and (B39) into (B4), to obtain a simple expression of the denominator in terms of R:

$$\int_{(x,y)\in\mathcal{S}} dxdye^{-V_{\rm eff}(x,y)} = (2\pi)^{3/2}\sqrt{(N-1)h'} \times \exp\left(2(N-1)R + R + h'\frac{\partial R}{\partial h'}\right). \tag{B40}$$

The first term in the exponential $2(N-1)R$ represents the leading part (B19), and the remaining gives the next-to-leading contributions. A similar expression is obtained for the denominator in the definition of the chemical potential of an instanton in the standard $c<1$ noncritical string theory [7].

Finally, we have a comment on the fact that $\mathcal{O}(N^0)$ part in the exponent in the denominator (B40) can be written in terms of a cylinder amplitude. Noting that

$$h'\frac{\partial}{\partial h'}\langle\mathcal{O}\rangle' = \frac{N-1}{h'}\langle\mathcal{O}\operatorname{tr}V(X')\rangle'_{\rm conn}, \tag{B41}$$

where the subscript "conn" represents taking the connected part of the correlator, we find that

$$h'\frac{\partial R}{\partial h'} = \frac{1}{h'}\langle \mathrm{tr}\log(\beta - X')\,\mathrm{tr}V(X')\rangle'_c - \frac{1}{2h'}\langle \mathrm{tr}\log(c+1-\beta-X')\,\mathrm{tr}V(X')\rangle'_c + \frac{1}{2h'}V(\beta) + \left(\left\langle\frac{1}{N-1}\,\mathrm{tr}\frac{1}{\beta - X'}\right\rangle'_d + \frac{1}{2}\left\langle\frac{1}{N-1}\,\mathrm{tr}\frac{1}{c+1-\beta-X'}\right\rangle'_d - \frac{1}{2h'}V'(\beta)\right)h'\frac{\partial\beta}{\partial h'}, \tag{B42}$$

where the last term vanishes due to (B9). Thus

$$\begin{aligned} R + h'\frac{\partial R}{\partial h'} &= \left\langle\frac{1}{N-1}\,\mathrm{tr}\log(\beta - X')\right\rangle'_d - \frac{1}{2}\left\langle\frac{1}{N-1}\,\mathrm{tr}\log(c+1-\beta-X')\right\rangle'_d + \frac{1}{h'}\langle \mathrm{tr}\log(\beta - X')\,\mathrm{tr}V(X')\rangle'_c \\ &\quad - \frac{1}{2h'}\langle \mathrm{tr}\log(c+1-\beta-X')\,\mathrm{tr}V(X')\rangle'_c \\ &= \int_\Lambda^\beta dz\left(R_{X'}(z) + \oint\frac{dz'}{2\pi i}\frac{1}{h'}V(z')\left\langle \mathrm{tr}\frac{1}{z-X'}\,\mathrm{tr}\frac{1}{z'-X'}\right\rangle'_c\right) \\ &\quad - \frac{1}{2}\int_{\Lambda'}^{c+1-\beta} dz\left(R_{X'}(z) + \oint\frac{dz'}{2\pi i}\frac{1}{h'}V(z')\left\langle \mathrm{tr}\frac{1}{z-X'}\,\mathrm{tr}\frac{1}{z'-X'}\right\rangle'_c\right) + \log\Lambda - \frac{1}{2}\log\Lambda'. \end{aligned} \tag{B43}$$

Namely, once we calculate the cylinder amplitude $\langle \mathrm{tr}\frac{1}{z-X'}\,\mathrm{tr}\frac{1}{z'-X'}\rangle'_c$, we should determine the $\mathcal{O}(N^0)$ coefficient in the denominator. In fact, the explicit form of this cylinder amplitude is given in [33] for the $O(n)$ model with arbitrary n. (For the case $|n| < 2$, see Sec. 3.4 in the first paper of [33].) Using that expression in our case $n = 1$, we find that the $\mathcal{O}(N^0)$ part in the exponent takes rather a simple form

$$\begin{aligned} R + h'\frac{\partial R}{\partial h'} &= \int_{-\Lambda/h'}^{z_0} dz\frac{1}{\sqrt{3}}G(z) + \log\Lambda \\ &\quad - \frac{1}{2}\int_{-\Lambda'/h'}^{-z_0} dz\frac{1}{\sqrt{3}}G(z) - \frac{1}{2}\log\Lambda', \end{aligned} \tag{B44}$$

where $G(z)$ is a function introduced in [33], which is universal in the sense that it is uniquely determined only by the homogeneous saddle point equation (B12) and by specifying its behavior near the edge of the cut and at the infinity. For details in the case $|n| < 2$, see Secs. 3.2 and 3.3 in the first paper of [33]. It is quite interesting that, even in the case of $c = 0$ string theory defined by the one-matrix model, the denominator in the chemical potential of an instanton also has the next-to-leading term given by $G(z)$ for the $O(0)$ model as

$$R + h'\frac{\partial R}{\partial h'} = \int_\Lambda^\beta dz\frac{1}{\sqrt{2}}G(z) + \log\Lambda = \log\frac{\beta - \alpha}{4}. \tag{B45}$$

It would be intriguing to examine whether this property holds for other $O(n)$ models. In our case (the $O(1)$ model), however complex form of $G(z)$ prevents us from performing the z-integration explicitly in (B44) and we have not yet succeeded in obtaining a concrete value of the coefficient in the denominator. It would be natural to expect that it is somehow related to the length of the cut as in the case of the $O(0)$ model (B45).

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