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Toroidal halos in a nontopological soliton model of dark matter

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Soliton type solutions of an axionlike scalar model with self-interaction are analyzed further as a toy model of dark matter halos. For a "nonlinear superposition" of round and flattened configurations we found ringlike substructures in the density profile similarly as has been inferred for our Galaxy from the observed excess of the diffuse component of cosmic gamma rays.

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I. INTRODUCTION

The dominant nonvisible "dark" fraction of the total energy of the Universe is known to exist from its gravitational effects. Since the *dark matter* (DM) part is distributed over rather large distances, its interaction, including a possible self-interaction [1], must be rather weak [2]. The main candidates for such weakly interacting particles (WIMPs) are the (universal) *axion* or the lightest supersymmetric particle, as the neutralino in most models, cf. Ref. [3].

Recently, an observed excess of diffuse gamma rays has been attributed [4] to the annihilation of DM in our Galaxy. The flux of the presumed neutralino annihilation allows a reconstruction of the distribution of DM in our Galaxy. Most probable is a pseudoisothermal profile but with a substructure of two doughnut-shape *rings* in the galactic plane. It is believed that these possibly *transient* substructures have their origin in the hierarchical clustering of DM into galaxies, and recent optical observations [5] seem to provide independent indications of a giant ringlike substructure in the galactic plane at 15 to 20 kpc distance from the center. However, there are reservations concerning the particular nature of the DM particle: Given the same normalization for its cross section, it appears unlikely [6] that DM annihilation is the main source of the extragalactic gamma-ray background (EGB). Moreover, for internal consistency the large accompanied antiproton flux needs to meet several constraints [7].

On the other hand, heterotic string theory provides a very light universal *axion* which may avoid [8] the strong *CP* problem in QCD [9]. Given the existence of such almost massless (pseudo-)scalars, it has been speculated that a coherent *nontopological soliton* (NTS) type solution of a nonlinear Klein-Gordon equation can account for the observed halo structure, simulating a *Bose-Einstein condensate* (BEC) of astronomical size [10–12]. In particular, a Φ^6 toy model [13] yields *exact* Emden type solutions in flat spacetime, including a flattening [14] of halos with ellipticity $e < 1$ as observed via microlensing [15,16]. In this paper, we are going to probe if a superposition of NTS halos can induce a *toroidal substructure* similar to that suggested by the de Boer *et al.* [4] interpretation of the recent EGRET observations.

Our paper is organized as follows: In Sec. II we reconsider exact NTS solutions [17] with *angular momentum l* and provide analytic expressions for their energy density and pressure profiles in Sec. III. A "nonlinear superposition" of a round halo and one flattened by rotation is tentatively considered for achieving a substructure. In fact, Sec. IV provides intriguing 3D profiles with two toriodal rings in the density. Their relative locations turn out to depend on an initial value of the NTS. As a consequence, the DM part of the galactic rotation curve exhibits a deviation from the universal Burkert profile with a wiggle at the second maxima, similar but less profound as has been inferred from observations, cf. Sec. V. Ramifications of the idea of modeling DM by scalar fields, branches of (meta-)stability of the NTS halos, as well as possible unifications with dark energy or even inflation are briefly discussed in Sec. VI.

II. NONTOPOLOGICAL SOLITONS WITH ANGULAR MOMENTUM

Nontopological soliton solutions of a *Lane-Emden type equation* familiar in astrophysics were considered already in 1978, cf. Ref. [17,18]. Here, we continue our previous studies [13,14] of the solvable *toy model* of a scalar field with the self-interaction

$$U(|\Phi|) = m^2|\Phi|^2(1 - \chi|\Phi|^4), \qquad \chi|\Phi|^4 \leq 1, \tag{1}$$

where m is the "bare" mass of the boson and χ a coupling constant in natural units ($\hbar = c = 1$).

In flat spacetime, stationary solutions $\Phi = P(r) \times \exp(-imt)$ of the corresponding nonlinear Klein-Gordon equation obey the radial equation

$$P'' + \frac{2}{x}P' - \frac{l(l+1)}{x^2}P + 3\chi P^5 = 0, \tag{2}$$

where $' = d/dx$ is the derivative with respect to *dimensionless* radial coordinate $x := mr$, cf. Eq. (2.6) of Ref. [17]. Equation (2) has the completely *regular* exact

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solution

$$P(r) = \pm\chi^{-1/4}(Ax)^l\sqrt{\frac{(2l+1)A}{1+(Ax)^{2(2l+1)}}} \tag{3}$$

without nodes, where $A = \sqrt{\chi}P(0)^2$ is a free integration constant in the range $0 < A < \sqrt{2}$ for round halos with $l = 0$. For nonvanishing angular momentum the constant $A = l^{-l/(2l+1)}(l+1)^{-(l+1)/(2l+1)}\sqrt{\chi}P^2(x_{\max})$ is related to the maximum at $Ax_{\max} = [l/(l+1)]^{1/2(2l+1)} \leq 1$.

III. DENSITY AND PRESSURE PROFILES OF NTSS

Although the canonical energy-momentum tensor $T_\mu{}^\nu(\Phi) = \mathrm{diag}(\rho, -p, -p_\theta, -p_\varphi)$ of a relativistic spherically symmetric configuration is diagonal, the radial and tangential pressures generated by the scalar field are in general different, i.e. the stresses are *anisotropic*, cf. Ref. [19] for details. Since the tangential pressures p_θ and p_φ do not play a role for the rotation curves, they will be disregarded in the following.

In flat spacetime the energy-density of the soliton (3) is given by

$$\rho^l = \frac{1}{2}\left(m^2P^2 + \left(\frac{dP}{dr}\right)^2 + U\right) = (2l+1)Am^2(Ax)^{2l}\frac{l^2 + 2x^2 + (Ax)^{4(2l+1)}((l+1)^2 + 2x^2) + (Ax)^{2(2l+1)}(4x^2 - 6l(l+1) - 1)}{2x^2(1+(Ax)^{2(2l+1)})^3\sqrt{\chi}}. \tag{4}$$

For spherically symmetric configurations with $l = 0$ the central density $\rho_0 = A(2 - A^2)m^2/2\sqrt{\chi}$ remains positive in the range $0 < A < \sqrt{2}$.

In addition, this nonlinearly coupled scalar field exerts the radial pressure:

$$p^l = \rho - U = (2l+1)Am^2(Ax)^{2l}\frac{(l+1)^2(Ax)^{2(2l+1)} + l^2}{2x^2(1+(Ax)^{2(2l+1)})^2\sqrt{\chi}} \tag{5}$$

on the configuration with a positive pressure $p_0 = A^3m^2/2\sqrt{\chi}$ at the center. At spatial infinity, the density and the radial pressure behave like

$$\rho^l \to \frac{(2l+1)m^2}{A^{2l+1}\sqrt{\chi}}\frac{1}{x^{2l+2}}, \qquad p^l \to \frac{(l+1)^2}{2x^2}\rho^l. \tag{6}$$

Thus the energy density of a round ($l = 0$) NTS halo has a $1/r^2$ decay exactly as the density profile of a *pseudoisothermal sphere* (IS), whereas a rotating NTS halo decrease for $l = 1$ similarly as the truncated isothermal sphere (TIS), cf. Ref. [20]. Halos with angular momentum are exhibiting interesting features of nonlinearly coupled *standing density waves* with two wiggles which possibly could have effects on the galactic spiral structure [21].

Let us study here in some more detail a kind of superposition of a spherically symmetric halo and those flattened by rotation.

IV. HALO SUBSTRUCTURE FROM NONLINEAR SUPERPOSITION

Motivated by the recent mathematical progress for the two-dimensional nonlinear Korteweg-de Vries equation [22], we have considered in Ref. [14] the speculative possibility of a "nonlinear superposition" of a round ($l = 0$) and K flattened NTSs with angular momentum $l > 0$ such that

$$\rho_{\sup} \simeq \frac{1}{1+K}\sum_{l=0}^{K}\frac{\rho^l}{2l+1}, \qquad p_{\sup} \simeq \frac{1}{1+K}\sum_{l=0}^{K}\frac{p^l}{2l+1} \tag{7}$$

hold. This superposition has been "tailored" such that the $(2l+1)$-degeneracy of the (quantum-mechanical) rotator is factored out, both, in the density (4) and the pressure (5). As a consequence, the *scaling law*

$$\rho_0 \propto r_c^{-1} \tag{8}$$

between the observed central density and the core radius emerges due to an exact cancellation of higher order terms such as r_c^{-3}. Since the initial value A, which may vary from halo to halo, also *cancels out* [14], the relation (8) could successfully be tested [23] on a quite large number of galaxies of different types.

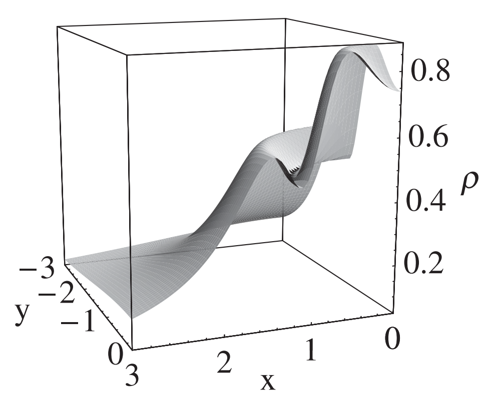

FIG. 1. Normalized energy density $\rho := \rho_{\sup}/\rho_0$ for a superposition of a spherical ($l = 0$) and a rotating ($l = 1$) NTS halo where $\rho_0 = A(2 - A^2)m^2/2\sqrt{\chi}$ and $A = 0.805$.

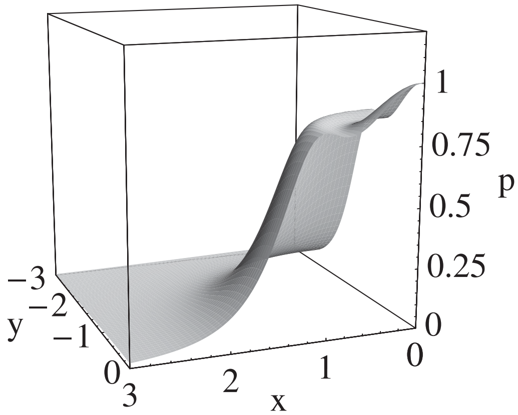

FIG. 2. Normalized pressure distribution p_{sup}/p_0 for a superposition of a spherical ($l = 0$) and a rotating ($l = 1$) NTS halo where $p_0 = A^3 m^2 / 2\sqrt{\chi}$ and $A = 0.805$.

The resulting density distribution (7) of superposed configurations exhibits an intriguing substructure with two rings of relative density maxima, see Fig. 1, located at $x_{1\,\mathrm{max}} = 0.60$ and $x_{2\,\mathrm{max}} = 1.46$ in dimensionless radial coordinates for $A = 0.805$. Moreover, the inner slope before the first maximum at x_{max} simulates a "cusp" familiar from cold DM simulations which, eventually, is alleviated by a smaller nonzero central density ρ_0. Viewed the other way round, the two local maxima of the pressure at the origin and $x_{2\,\mathrm{max}} = 1.03$ of Fig. 2 cause the corresponding depressions into two local minima in the density, with one located at the center.

Variation of the maxima of the rings

Since the location of the rings depends on the initial value A of our configuration, we have monitored in Fig. 3 the dimensionless quotient $Q := (x_{2\,\mathrm{max}}/x_{1\,\mathrm{max}})$ of the first two local maxima in order to facilitate a comparison with observations. Our result qualitatively complies with the reconstructed DM density of our Galaxy, cf. the right top row in Fig. 5 of Ref. [4], where a double ring structure on top of a (probably) triaxial halo has been adopted.

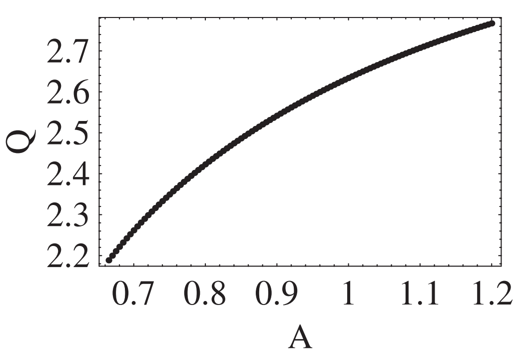

FIG. 3. Quotient $Q := (x_{2\,\mathrm{max}}/x_{1\,\mathrm{max}})$ of the location of the first two relative maxima of the "nonlinearly superposed" NTS density with $l = 0$ and $l = 1$, as a function of the initial value A.

The diffuse galactic gamma rays observed by EGRET indicates [4] the existence of two rings, on top of a possible triaxial form, at about 4.15 and 12.9 kpc from the galactic center according to the updated values, cf. Ref. [7]. If confirmed, this observation corresponds to an $Q_{\mathrm{obs}} = 3.1$, albeit rather large error bars, which is marginally consistent with our NTS model. The observed ellipticity is quite strong with an axis ratio of 0.65 ± 0.15 and 0.85 ± 0.1, respectively, largely due to the adiabatic compression of the bar. On the other hand the ellipticity of our superposed NTS configuration is $e_{\mathrm{NTS}} := 1 - c/a = 1/2 - 2/3\pi$, cf. Ref. [14].

V. WIGGLES IN THE ROTATION CURVE

For revealing the shape of DM halos one commonly uses the circular velocities of stars or HI gas in galaxies which are bounded by $v_\varphi/c \leq 10^{-3}$. Although they are *nonrelativistic*, we nevertheless depart from the general relativistic formula [24,25]

$$v_\varphi^2 := \frac{r}{2}\frac{d\nu}{dr} = \frac{\kappa}{2}\left[\frac{M(r) + 4\pi p_{\mathrm{sup}} r^3}{4\pi r - \kappa M(r)}\right] \simeq \frac{\kappa}{2}\left[\frac{M(r)}{4\pi r} + p_{\mathrm{sup}} r^2\right] \tag{9}$$

for the tangential velocity squared, in order to account for the effect of pressure. (In Schwarzschild type radial coordinates, $N = e^{\nu/2}$ is known as the lapse function.) For the superposed configuration (7), the total mass is given by

$$M(r) \simeq \frac{1}{1+K}\sum_{l=0}^{K}\frac{M(r)^l}{2l+1}, \tag{10}$$

where the resulting Newtonian *mass function*

$$\begin{aligned} M(r)^l &:= 4\pi\int_0^r \rho^l \tilde{r}^2 d\tilde{r} \\ &= \frac{\pi}{2m\sqrt{\chi}}\Bigg[(4l^2 - 1 - (2l+1)(2l+3)(Ax)^{2(2l+1)}) \\ &\quad \times \frac{(Ax)^{2l+1}}{(1+(Ax)^{2(2l+1)})^2} + \arctan((Ax)^{2l+1}) \\ &\quad + 4x^2(Ax)^{2l+1}\Gamma\left(\frac{2l+3}{2(2l+1)}\right){}_2F_1\left(1, \frac{2l+3}{2(2l+1)}; \right. \\ &\quad \left. \frac{6l+5}{2(2l+1)}; -(Ax)^{2(2l+1)}\right)\Bigg]. \end{aligned} \tag{11}$$

of the l-dependent NTS profile (4) is obtained by integration with the aid of MATHEMATICA 5.2 and further simplification [26] in terms of the regularized hypergeometric function ${}_2F_1(a,b;c;z) := \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\sum_{n=0}^{\infty}\frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)}\frac{z^n}{n!}$.

For a round halo with $l = 0$, we recover Eq. (3.10) of Ref. [13]. Our Newtonian approximation reveals that also the radial pressure $p \neq 0$ of an anisotropic scalar "fluid" contributes to the shape of the configuration as well as to the rotation curve. Although for a round halo ($l = 0$) the "Keplerian" $M(r)/r$ term is dominating over the pressure term pr^2, this is not the case for $l \neq 0$ in which both terms, due to a relativistic "conspiracy", have the same $1/x^{2l}$ fall-off at infinity. Obviously, without radial pressure and for a weak gravitational field, Eq. (9) would reduce to the Kepler law $v^2_{\varphi,\mathrm{Newt}} \simeq GM(r)/r$.

For a spherical NTS halo ($l = 0$), we recover from the mass function (11) and its radial pressure (5) the rotation velocity (4.3) of Ref. [13]. The bound $v_\infty := \sqrt{(2l+1)\kappa/2\chi P^2(0)} \leq 10^{-3}c$ on the observed rotation velocities can be used to constrain the coupling constant χ of our NTS model. The *superposed rotation velocity*, drawn Fig. 4 in the approximation (9), turns out to be the quadratic mean of the two individual velocities arising from a round ($l = 0$) and a flattened ($l = 1$) NTS halo. It has a local maximum around the location of the outer ring, qualitatively resembling the rotation curve in Fig. 8 of Ref. [4] as inferred from DM annihilation, except for a less profound effect of the inner ring which is observationally anyhow more problematic. For the superposed NTS halo this can be traced back to the already mentioned conspiracy of mass density and pressure in Eq. (9), in contradistinction to standard cold DM models where pressure is negligible.

Our results for the superposed NTS halo can also be compared with the *universal* fitting formula

$$v^2_{\varphi\mathrm{B}}/v^2_0 = \frac{1}{2x}\{\ln[(1+x)^2(1+x^2)] - 2\arctan(x)\}, \quad (12)$$

proposed by Burkert [27,28] in 1995. It continues to serve as a useful *empirical* description of DM-dominated galactic halos. After a maximum at $x = 3.3$ in dimensionless units $x = r/r_c$, it amounts to a *logarithmic modification* $v^2 \longrightarrow \ln x/x$ of the Kepler law at spatial infinity. Except for the wiggle, the concordance in Fig. 4 is still rather good for the choice of $A = 0.805$, which yields the overall best fit [24] for a round halo.

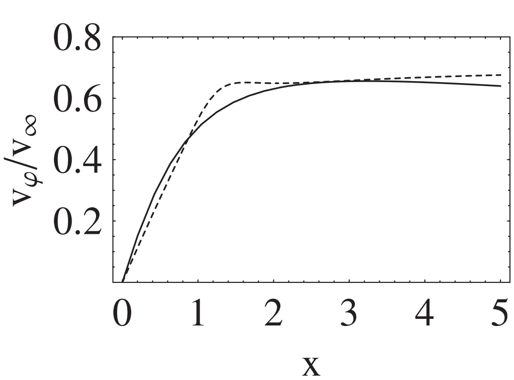

FIG. 4. Rotation curve of the superposed NTS halo (dashed line) normalized by $v_\infty := \sqrt{\kappa/2A\sqrt{\chi}}$ including angular momentum up to $l = 1$ for $A = 0.805$, in comparison to the empirical Burkert fit (solid line).

VI. DISCUSSION: BRANCHES OF STABILITY

For several models exhibiting symmetry breaking, the stability of a configuration is assured by a topological charge, such as the winding number, associated with the first (or higher) homotopy group $\pi_n(G/H)$ of the coset space G/H of the ground state [29].

For *nontopological* soliton stars as devised by T. D. Lee [30], the situation is quite different, usually conserved Noether charges such as the particle number or the total mass

$$N := 4\pi \int_0^\infty j^0 r^2 dr, \qquad M := 4\pi \int_0^\infty \rho r^2 dr, \quad (13)$$

respectively, arising from a global symmetry, provide stability of the configuration up to a certain limit. Typically the binding energy $M - mN$ for solitons in the lower branch is negative, indicating stability, whereas the situation gets reversed for the higher branches which correspond to pulsating or even collapsing solitons. These results from linear perturbation theory [31] have been confirmed and extended by global methods borrowed from *catastrophe theory* [32]. In fact, for self-gravitating solitons a behavior similar to that of neutron stars has been corroborated [33].

In our toy model, the exact solution of the nonlinear radial Emden type Eq. (2) with an angular momentum barrier have already been analyzed in 1977 for their stability with respect to linear perturbations, cf. Eq. (6) of Ref. [34]. Accordingly, their energy grows as $E = m^2[1 + 4l(l+1)]/\sqrt{\chi}$ with increasing angular momentum l and the exact NTS type solutions (3) decay exponentially with a decay time of $\tau = \lambda_m/2\pi A\gamma_0(l)c$, where $\lambda_m = 2\pi\hbar/mc$ is the Compton wave length and $\gamma_0(0) = 1.92$, $\gamma_0(1) = 4.46$, and $\gamma_0(2) = 7.23$, etc. are eigenvalues determined numerically from the linearized perturbation equation. If we assume a Compton wave length $\lambda_m = 100000$ Ly of the extension of our Galaxy, corresponding to an ultralight boson mass $m = 10^{-22}$ eV of the so-called fuzzy cold DM models [12,35], our spherical NTS type halo (with $l = 0$) would already decay in 10^4 years, which is much shorter than the average life time of 10^{10} yr for galaxies. Since $\gamma_0(l)$ increases with l, transient DM halo substructures would decay even more rapidly.

Therefore, as in the vortex rings of a BEC [36], one needs another bosonic "confining" component for stabilization, which is here, however, automatically provided by the tensor field of gravity. Moreover, the self-generated positive pressure (5) of our soliton also needs to be compensated by *self-gravity*, in order to stabilize a NTS halo. As is well-known from boson or soliton stars, up to the limit $M_{\mathrm{crit}} = 0.633 M^2_{\mathrm{PL}}/m_{\mathrm{resc}}$ first found numerically by Kaup, there exist a branch of configurations which are *absolutely* stable against radial perturbations, cf. Ref. [19] for a review. Higher angular momentum solitons appear to be less stable due to the increasing rotational energy. However, an isolated configuration can get rid of excess angular momentum only via $\Delta l = 2$ transitions, due to the quadrupole nature of gravitational waves, cf. Ref. [37]. This new feature of self-gravitating systems implies that our ''superposition'' of a round NTS with a $l = 1$ ''vortex'' should be meta-stable. Details need to be confirmed by numerical simulations, which are beyond the scope of this paper.

The doughnut type rings in the ''nonlinear'' superposition of a spherically symmetric NTS with one of (quantized) angular momentum $l = 1$ resemble rotating vortices [38] of Bose-Einstein condensates, in which anisotropic pressure sustains the substructure. As a toy model for DM halos, it indeed could induce a toruslike substructure, in contrast to the supposition of Ref. [7], but it needs to be seen, if the effective angular momentum matches observations [39].

Our choice of the self-interaction (1) is dictated by simplicity, in order to dispose on analytic expressions for the density and pressure in the Newtonian approximation. Self-interactions with a degenerated vacuum would require another order six potential like $U_{\rm NTS} = m^2|\Phi|^2(1 - \sqrt{\chi}|\Phi|^2)^2$, leading to the NTS stars [30] already mentioned. A more realistic WIMP, like the axion as a pseudoscalar, would rather feel a periodic potential with interesting repercussions on cosmology, cf. Ref. [40]. Despite some preliminary analysis, cf. Ref. [41], its utility as a model of DM halos needs to be seen.

In the context of scalar field models, there exist phases which account for both, dark energy and dark matter, on different scales, cf. Ref. [42]. On the other hand, the real part $\varphi := (\Phi + \Phi^*)/2$ of a complex scalar Φ can be conformally mapped into a nonlinear $L(R)$ gravity which exhibits an interesting *bifurcation* [43] into different branches which likewise may correspond to various initial patches of the early Universe or to different scales in the present epoch. They are distinguished by zero or nonvanishing induced cosmological constant and thereby may pave the way to a solution of the *coincidence problem*. Such scalar fields may even incorporate the inflaton, cf. Refs. [44,45] in the very early Universe.

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