

Perturbative renormalization for static and domain-wall bilinears and four-fermion operators with improved gauge actions

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We calculate one-loop renormalization factors for heavy-light bilinears as well as four-fermion operators relevant for $B^0 - \bar{B}^0$ mixing calculations on the lattice. We use the static approximation for heavy quarks and the domain-wall formulation for light quarks. We present results for different choices of improved gauge action.

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I. INTRODUCTION

Recent experimental observations [1–3] intensified the interest in B -meson physics as the prime area in which the validity of the standard model can be put to test. In order to perform such a test, theoretical predictions for f_B , B_B , and $\Delta m_s/\Delta m_d$ are needed. In principle, the lattice approach to QCD allows one to calculate these and other quantities of interest from the first principles. However, the majority of lattice results to date suffer from substantial systematic errors which make comparison with experimental results inconclusive. One of the largest systematic errors comes from the procedure which relates lattice operators to the corresponding operators in continuum QCD.

There are two main sources of error in this matching procedure. First, lattice actions used in simulations usually have a reduced symmetry group. The lack of symmetry leads to the introduction of additional lattice operators which complicates the matching. Second, the matching is usually performed at the one-loop level in perturbation theory. More often than not, the one-loop contributions turn out to be quite large. Large one-loop contributions undermine the perturbative approach to matching and force the introduction of a large systematic error in order to account for possible higher-order corrections.

In this paper we will demonstrate that both of these issues can be addressed by an appropriate choice of lattice action. Specifically, we will focus on the combination of the static effective field theory [4] for heavy quarks, the domain-wall formulation [5,6] for light quarks and an improved gauge action [7] for gluons. This action provides a good approximation for heavy mesons such as B , and it has already been employed in numerical simulations [8].

We will obtain the renormalization constants connecting bare lattice operators to continuum QCD operators renormalized in the $\overline{\text{MS}}$ (NDR) scheme. In particular, we will consider heavy-light bilinears and four-fermion operators relevant for $B^0 - \bar{B}^0$ mixing. The matching will be done at the one-loop level in perturbation theory. We will split the matching into two parts. First, the continuum QCD operators need to be related to the continuum static theory

operators. This step has already been carried out, and we will quote the results here. Second, the continuum static theory operators need to be matched to their lattice counterparts. This second step has not been performed before for our choice of lattice action, and thus is the main subject of this paper.

This paper is organized as follows. In Sec. II we introduce our choice of lattice action and discuss the corresponding Feynman rules. In Sec. III the symmetries of the action are discussed. In Sec. IV we quote the results of matching the continuum QCD and continuum static theory operators. In Sec. V we will match the static theory bilinears to their lattice counterparts. In Sec. VI we present the matching of four-fermion operators as well as B_B . In Sec. VII we discuss the application of mean-field improvement to our results.

II. LATTICE ACTION AND FEYNMAN RULES

Our lattice action consists of three parts:

$$S = S_{\text{static}} + S_{\text{DW}} + S_{\text{gauge}}, \quad (1)$$

where S_{static} is the static-quark action describing heavy quarks, S_{DW} is the domain-wall fermion action describing light quarks and S_{gauge} is the pure gauge action. In this and all subsequent sections we set the lattice spacing a equal to 1.

A. Static-quark action

The simplest formulation of the static-effective-field-theory action on the lattice is given by [9]

$$S_{\text{static}} = \sum_x \bar{h}(x)[h(x) - U_0^\dagger(x - \hat{0})h(x - \hat{0})], \quad (2)$$

where $U_0(x - \hat{0})$ is the gauge link in temporal direction between sites x and $x - \hat{0}$. The static-quark field h satisfies $\gamma_0 h = h$. The static-quark propagator in momentum space is given by

$$H(k) = \frac{1}{1 - e^{-ik_0} + \epsilon}. \quad (3)$$

The quark-gluon vertices are

$$\tilde{V}_\mu^a(k, k') = -igT^a \delta_{0\mu} e^{-i(k_0 + k'_0)/2}, \quad (4)$$

$$\tilde{V}_{\mu\nu}^{ab}(k, k') = -\frac{1}{2}g^2\{T^a, T^b\}\delta_{0\mu}\delta_{0\nu}e^{-i(k_0 + k'_0)/2}, \quad (5)$$

with T_a being a generator of the $SU(N_c)$ color group. We note that it is not difficult to extend the results of this paper to the case of the modified static-quark action of Ref. [10]. The results of such extension will be reported elsewhere.

B. Domain-wall fermion action

For a quark with mass m the domain-wall fermion action [6,11] is given by

$$S_{\text{DW}} = \sum_{s,s'=1}^N \sum_{x,y} \bar{\psi}_s(x) D_{ss'}^{\text{DW}}(x, y) \psi_{s'}(y) + \sum_x m \bar{q}(x) q(x), \quad (6)$$

$$D_{ss'}^{\text{DW}}(x, y) = D^4(x, y) \delta_{ss'} + D^5(s, s') \delta_{xy} + (M_5 - 5) \delta_{ss'} \delta_{xy}, \quad (7)$$

$$D^4(x, y) = \sum_\mu \frac{1}{2} [(1 + \gamma_\mu) U_\mu(x) \delta_{x+\hat{\mu}, y} + (1 - \gamma_\mu) U_\mu^\dagger(y) \delta_{x-\hat{\mu}, y}], \quad (8)$$

$$D^5(s, s') = \begin{cases} P_R \delta_{2,s'} & (s = 1) \\ P_R \delta_{s+1,s'} + P_L \delta_{s-1,s'} & (1 < s < N) \\ P_L \delta_{N-1,s'} & (s = N) \end{cases} \quad (9)$$

where $\psi_s(x)$ is a $(4 + 1)$ -dimensional Wilson-style fermion field. The fifth dimension extends from 1 to N and is labeled by s . $P_L = (1 - \gamma_5)/2$ and $P_R = (1 + \gamma_5)/2$ are the projectors for the left- and right-handed spinors. The domain-wall height M_5 is a parameter of the theory, which can be set $0 \leq M_5 \leq 2$. Finally, $q(x)$ represents the physical four-dimensional quark field constructed from the five-dimensional field $\psi_s(x)$ at $s = 1$ and N

$$q(x) = P_R \psi_1(x) + P_L \psi_N(x), \quad (10)$$

$$\bar{q}(x) = \bar{\psi}_1(x) P_L + \bar{\psi}_N(x) P_R. \quad (11)$$

In momentum space the free field domain-wall Dirac operator is given by

$$D(p)_{st} = \sum_\mu i\gamma_\mu \sin p_\mu \delta_{st} + (W^+(p)_{st} + m \delta_{s,1} \delta_{N,t}) P_R + (W^-(p)_{st} + m \delta_{s,N} \delta_{1,t}) P_L, \quad (12)$$

where s and t label coordinates in the fifth dimension and

$$W^+(p)_{st} = \begin{pmatrix} -W(p) & 1 & & \\ & \ddots & \ddots & \\ & & -W(p) & 1 \\ & & & -W(p) \end{pmatrix}, \quad (13)$$

$$W^-(p)_{st} = \begin{pmatrix} -W(p) & & & \\ 1 & -W(p) & & \\ & \ddots & \ddots & \\ & & 1 & -W(p) \end{pmatrix}, \quad (14)$$

$$W(p) = 1 - M_5 + \sum_\mu (1 - \cos p_\mu). \quad (15)$$

From now on we will focus on $m = 0$ case and assume that N is large. We will rely on the formalism developed in Refs. [12,13] in our perturbative treatment of domain-wall fermions. By inverting the Dirac equation one can write the tree-level DWF propagator as

$$S(p)_{st} = (-i\gamma_\mu \sin p_\mu \delta_{su} + W_{su}^-) G_{ut}^R(p) P_R + (-i\gamma_\mu \sin p_\mu \delta_{su} + W_{su}^+) G_{ut}^L(p) P_L, \quad (16)$$

where

$$G_{st}^R(p) = -\frac{A}{F} [(1 - W e^{-\alpha}) e^{-\alpha(2N-s-t)} + (1 - W e^\alpha) e^{-\alpha(s+t)}] + A e^{-\alpha|s-t|}, \quad (17)$$

$$G_{st}^L(p) = -\frac{A}{F} [(1 - W e^\alpha) e^{-\alpha(2N-s-t+2)} + (1 - W e^{-\alpha}) e^{-\alpha(s+t-2)}] + A e^{-\alpha|s-t|}. \quad (18)$$

Note that α , A , F , and W in the above formulas all depend on the momentum p as follows:

$$\cosh(\alpha) = \frac{1 + W^2(p) + \sum_\mu \sin^2 p_\mu}{2|W(p)|}, \quad (19)$$

$$A = \frac{1}{2W \sinh(\alpha)}, \quad (20)$$

$$F = 1 - e^\alpha W(p). \quad (21)$$

The form of the propagator in Eq. (16) is valid for positive W . For $0 < M_5 < 1$, W is positive for any p . For $1 < M_5 < 2$, W is negative in the small-momentum region. For that momentum region we need to adjust the formulas using the substitution

$$e^{\pm\alpha} \rightarrow -e^{\pm\alpha}. \quad (22)$$

Using Eqs. (10), (11), and (16) one can obtain the formula for the physical quark propagator

$$S_q(p) = \langle q(-p) \bar{q}(p) \rangle = \frac{i\gamma_\mu \sin p_\mu}{F}. \quad (23)$$

For one-loop calculations we will also need propagators connecting the physical quark field and the domain-wall fermion field

$$\langle q(-p)\bar{\psi}_s(p) \rangle = \frac{i\gamma_\mu \sin p_\mu}{F} (e^{-\alpha(N-s)}P_R + e^{-\alpha(s-1)}P_L) - e^{-\alpha}(e^{-\alpha(N-s)}P_L + e^{-\alpha(s-1)}P_R), \quad (24)$$

$$\langle \psi_s(-p)\bar{q}(p) \rangle = \frac{i\gamma_\mu \sin p_\mu}{F} (e^{-\alpha(N-s)}P_R + e^{-\alpha(s-1)}P_L) - e^{-\alpha}(e^{-\alpha(N-s)}P_R + e^{-\alpha(s-1)}P_L). \quad (25)$$

The relevant quark-gluon interaction vertices are the same as in the N -flavor Wilson case with $r = -1$:

$$V_\mu^a(p, p')_{st} = -igT^a \left(\gamma_\mu \cos \frac{(p+p')_\mu}{2} + i \sin \frac{(p+p')_\mu}{2} \right) \delta_{st}, \quad (26)$$

$$V_{\mu\nu}^{ab}(p, p')_{st} = \frac{1}{2}g^2 \frac{1}{2} \{T^a, T^b\} \left(i\gamma_\mu \sin \frac{(p+p')_\mu}{2} + \cos \frac{(p+p')_\mu}{2} \right) \delta_{\mu\nu} \delta_{st}. \quad (27)$$

C. Gauge action

In this paper we consider the following class of gauge actions:

$$S_{\text{gauge}} = -\frac{2}{g_0^2} \left((1 - 8c_1) \sum_P \text{Re Tr}[U_P] + c_1 \sum_R \text{Re Tr}[U_R] \right), \quad (28)$$

where g_0 denotes the bare lattice coupling, U_P is the path-ordered product of links around the 1×1 plaquette P and U_R is the path-ordered product of links around the 1×2 rectangle R . For $c_1 = 0$, Eq. (28) reduces to the standard Wilson plaquette action. In the case $c_1 = -1.40686$, $c_1 = -0.331$ and $c_1 = -1/12$ we have the doubly-blocked Wilson (DBW2) [14,15], Iwasaki [16], and Symanzik [7,17] actions, respectively.

For the Wilson plaquette action, the gluon propagator in momentum space is straightforward

$$D_{\nu\rho}^0(k) = \frac{\delta_{\nu\rho}}{4\sum_\mu \sin^2(\frac{k_\mu}{2})} = \frac{\delta_{\nu\rho}}{\Delta_1}. \quad (29)$$

For improved gauge action the gluon propagator is given by [17]

$$D_{\nu\rho}^{c_1}(k) = \frac{(1 - A_{\nu\rho}^{c_1}(k))4\sin(\frac{k_\nu}{2})\sin(\frac{k_\rho}{2}) + \delta_{\nu\rho}\sum_\sigma 4\sin^2(\frac{k_\sigma}{2})A_{\nu\sigma}^{c_1}(k)}{\Delta_1^2}, \quad (30)$$

where $A_{\nu\rho}^{c_1}(k)$ is a function symmetric in ν and ρ whose explicit dependence on c_1 and k is given in the appendix. We assumed the Feynman gauge for both propagators.

III. SYMMETRIES OF THE ACTION

We will now discuss the symmetries of our action and their consequences for heavy-light operators. Similar discussions for different choices of action were presented in Refs. [18,19]. We will use the following notation to denote lattice heavy-light bilinears

$$O_\Gamma^{\text{lat}} = \bar{h}\Gamma q = \begin{cases} S & (\Gamma = 1) \\ P & (\Gamma = \gamma_5) \\ V_\mu & (\Gamma = \gamma_\mu) \\ A_\mu & (\Gamma = \gamma_\mu \gamma_5) \\ T_{\mu\nu} & (\Gamma = \sigma_{\mu\nu}) \end{cases}. \quad (31)$$

A. Chiral symmetry

The five-dimensional domain-wall fermion field $\psi_s(x)$ can be subjected to the following chiral rotations:

$$\psi_s(x) \rightarrow e^{i\alpha_L^a t^a} \psi_s(x), \quad 1 \leq s \leq N/2, \quad (32)$$

$$\psi_s(x) \rightarrow e^{i\alpha_R^a t^a} \psi_s(x), \quad N/2 + 1 \leq s \leq N, \quad (33)$$

where $\{t_a\}$ are the generators of the $\text{SU}(N_f)$ flavor group and N is the extent of the fifth dimension. Together with the definition of the physical quark field q in Eq. (10), the transformations in Eqs. (32) and (33) act as standard chiral symmetry transformations on q . For finite N this symmetry is not exact. However, as was shown in Refs. [20,21] the degree of the symmetry breaking decreases rapidly with increasing N . Since we take $N \rightarrow \infty$ limit in this calculation, we can consider the symmetry breaking to be negligible. Thus, under chiral symmetry transformations we have $S \rightarrow iP$, $V_\mu \rightarrow iA_\mu$ and vice versa. This implies that at the level of the static-theory-to-lattice matching we should expect $Z_S = Z_P$ and $Z_V = Z_A$, where Z 's are defined in Eq. (85).

B. Heavy quark symmetries

First, we note that since the static-quark field h satisfies the field equation $\gamma_0 h = h$ we have $V_0 = S$ and $A_0 = P$ as well as $T_{0j} = V_j$ and $T_{ij} = \epsilon_{ijk} A_k$. Second, the action in Eq. (2) as well as the continuum static-effective-field-theory action are invariant under the following $\text{SU}(2)$ transformation

$$h \rightarrow e^{-i\phi_j \epsilon_{jkl} \sigma_{kl}} h, \quad (34)$$

where ϕ_j is a parameter and $\sigma_{kl} = \frac{i}{2}[\gamma_k, \gamma_l]$. This is the heavy-quark spin symmetry [22,23]. As its consequence we have the following transformations $S \rightarrow iA_j$, $P \rightarrow iV_j$ and vice versa. This implies that at the level of the static-theory-to-lattice matching we should expect $Z_S = Z_A$ and $Z_P = Z_V$.

IV. MATCHING FULL-THEORY AND STATIC THEORY OPERATORS

In this section we will briefly review the relation of full QCD operators renormalized in $\overline{\text{MS}}(\text{NDR})$ scheme to the corresponding static theory operators.

A. Heavy-light bilinears

The QCD bilinear $O_\Gamma = \bar{b}\Gamma q$, renormalized in $\overline{\text{MS}}(\text{NDR})$ at the scale μ_b , can be related to the corresponding static theory bilinear $\tilde{O}_\Gamma = \bar{h}\Gamma q$, renormalized at the scale μ , as follows [24,25]:

$$O_\Gamma(\mu_b) = C_{O_\Gamma}(\mu_b, \mu) \tilde{O}_\Gamma(\mu) + \mathcal{O}(\Lambda_{\text{QCD}}/\mu_b), \quad (35)$$

where $C_{O_\Gamma}(\mu_b, \mu)$ is a perturbative coefficient encoding the physics between the scales μ_b and μ . In lattice calculations μ naturally assumes the value of the inverse lattice spacing and μ_b is chosen to match the b quark mass. The coefficients C_{O_Γ} for each O_Γ were calculated perturbatively at one-loop level in Ref. [4] and at two-loop level in Ref. [26]. The coefficients C_{O_Γ} are different for different bilinears.

B. Four-fermion operator

The QCD operator relevant for $B^0 - \bar{B}^0$ mixing, here renormalized according to the $\overline{\text{MS}}(\text{NDR})$ scheme, is

$$O_{(V-A)(V-A)} = [\bar{b}\gamma^\mu(1 - \gamma_5)q][\bar{b}\gamma_\mu(1 - \gamma_5)q]. \quad (36)$$

We need to consider only the parity conserving part of it:

$$O_{VV+AA} = (\bar{b}\gamma^\mu q)(\bar{b}\gamma_\mu q) + (\bar{b}\gamma^\mu \gamma_5 q)(\bar{b}\gamma_\mu \gamma_5 q). \quad (37)$$

The relation to the static theory operators is as follows:

$$\begin{aligned} O_{VV+AA}(\mu_b) &= Z_1(\mu_b, \mu) \tilde{O}_{VV+AA}(\mu) \\ &+ Z_2(\mu_b, \mu) \tilde{O}_{SS+PP}(\mu) + \mathcal{O}(\Lambda_{\text{QCD}}/\mu_b), \end{aligned} \quad (38)$$

where

$$\begin{aligned} \tilde{O}_{VV+AA} &= 2(\bar{h}^{(+)}\gamma^\mu q)(\bar{h}^{(-)}\gamma_\mu q) + 2(\bar{h}^{(+)}\gamma^\mu \gamma_5 q) \\ &\times (\bar{h}^{(-)}\gamma_\mu \gamma_5 q), \end{aligned} \quad (39)$$

$$\begin{aligned} \tilde{O}_{SS+PP} &= 2(\bar{h}^{(+)}q)(\bar{h}^{(-)}q) + 2(\bar{h}^{(+)}\gamma_5 q)(\bar{h}^{(-)}\gamma_5 q), \end{aligned} \quad (40)$$

with

$$h^{(\pm)}(x) = e^{\pm imv \cdot x} \frac{1 \pm \not{v}}{2} b(x). \quad (41)$$

The coefficients $Z_{1,2}$ were calculated at one-loop in perturbation theory in Refs. [27–29]. We note that Z_2 is an order g^2 coefficient.

V. MATCHING STATIC-THEORY AND LATTICE HEAVY-LIGHT BILINEARS

In this section we will match the $\overline{\text{MS}}(\text{NDR})$ renormalized static theory heavy-light bilinears introduced in Sec. IV to their lattice counterparts. We will use the lattice Feynman rules from Sec. II.

The Feynman graphs representing one-loop corrections to the tree-level heavy-light bilinears are depicted in Fig. 1. The light-quark wave function renormalization factor coming from the diagrams (a) and (b) has been calculated in Refs. [13,30]. The vertex correction (c) has not been calculated before for the case of DWF-static quarks. We will present this calculation below. The heavy-quark wave function renormalization factor coming from the diagrams (d) and (e) has been calculated for the Wilson gauge action in Ref. [9]. We will extend the result of Ref. [9] to include the case of improved gauge actions.

A. Vertex correction

At the tree level, the Green's function $\langle(\bar{h}(x)\Gamma q(x))h(y)\bar{q}(z)\rangle$ for small external momenta p is given by

$$\langle(\bar{h}\Gamma q)h\bar{q}\rangle_{\text{tree}} = \frac{1}{ip_0} \Gamma \frac{(1 - w_0^2)}{i\not{p}}, \quad (42)$$

where

$$w_0 = W(0) = 1 - M_5. \quad (43)$$

We need to calculate the one-loop correction to this

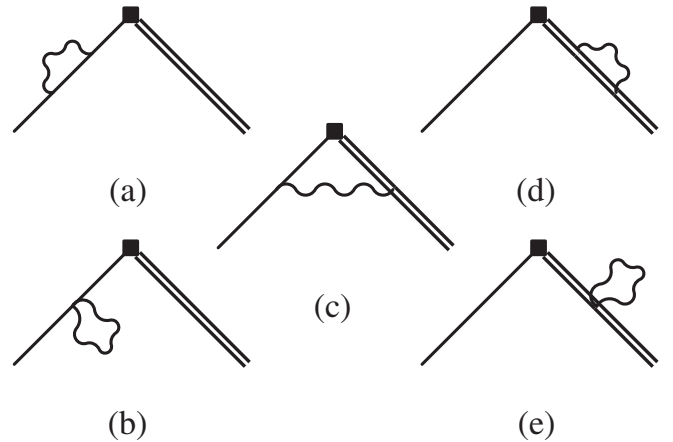


FIG. 1. One-loop corrections to heavy-light bilinears.

Green's function coming from the graph depicted in Fig. 2. As can be seen from Eq. (25), for small external momenta the external line of the light quark propagator is

$$\langle \psi_s(-p \approx 0) \bar{q}(p \approx 0) \rangle = (w_0^{N-s} P_L + w_0^{s-1} P_R) \frac{(1 - w_0^2)}{i \not{p}}. \quad (44)$$

This implies that the one-loop vertex-corrected Green's function will take the form

$$\langle (\bar{h} \Gamma q) h \bar{q} \rangle = \frac{1}{i \not{p}_0} (\Gamma + \delta \Gamma) \frac{(1 - w_0^2)}{i \not{p}}, \quad (45)$$

where

$$\delta \Gamma = \int_k \sum_{\nu, \rho} H(k) \tilde{V}_\nu^a(k) D_{\nu\rho}^{c_1}(k) \Gamma \left\{ \sum_{s=1}^N \langle q(-k) \bar{\psi}_s(k) \rangle V_\rho^a(k) \right. \\ \left. \times (w_0^{N-s} P_L + w_0^{s-1} P_R) \right\}, \quad (46)$$

and

$$\int_k \equiv \int_{-\pi}^{\pi} \frac{d^4 k}{(2\pi)^4}. \quad (47)$$

First, we focus on the factor in curly brackets in the integrand of Eq. (46). Since the light quark-gluon vertex V_ρ^a has γ_ρ and Wilson parts, it is convenient to consider them separately. We denote the part coming from the γ_ρ term as S_χ and the part coming from the Wilson term as S_w . Then

$$S_\chi(k) = \sum_{s=1}^N \langle q(-k) \bar{\psi}_s(k) \rangle (w_0^{N-s} P_R + w_0^{s-1} P_L), \quad (48)$$

$$S_w(k) = \sum_{s=1}^N \langle q(-k) \bar{\psi}_s(k) \rangle (w_0^{N-s} P_L + w_0^{s-1} P_R). \quad (49)$$

Using Eqs. (24) and (25) and performing the sum over s we find

$$I_\chi = 16\pi^2 \int_k \frac{\sin^2(k_0) + 4 \sum_j A_{0j}^{c_1} \sin^2(\frac{k_j}{2}) \cos^2(\frac{k_0}{2}) + \sum_j (1 - A_{0j}^{c_1}) \sin^2(k_j)}{(w_0 - e^\alpha)(W(k) - e^{-\alpha}) \Delta_1^2}, \quad (54)$$

and

$$I_w = 16\pi^2 \int_k \frac{1}{2(w_0 - e^\alpha) \Delta_1}. \quad (55)$$

The integral I_w is finite and can be evaluated numerically. The integral I_χ is infrared divergent. We can write it as follows:

$$I_\chi = I_\chi^{c_1} + I_\chi^w, \quad (56)$$

where the first term is the finite contribution coming from

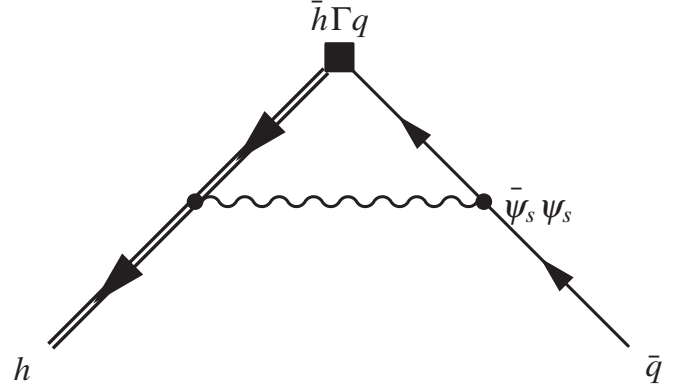


FIG. 2. Heavy-light vertex correction.

$$S_\chi(k) = \frac{1}{w_0 - e^\alpha} \frac{i \sum_\mu \gamma_\mu \sin k_\mu}{W(k) - e^{-\alpha}}, \quad (50)$$

$$S_w(k) = \frac{1}{w_0 - e^\alpha}. \quad (51)$$

Combining this result with the Feynman rules presented in Sec. II, the one-loop correction to the vertex in Fig. 2 can be written as

$$\delta \Gamma = -g^2 C_F \int_k \sum_\rho \frac{e^{-i(k_0/2)}}{1 - e^{-ik_0} + \epsilon} D_{0\rho}^{c_1}(k) \Gamma \left\{ S_\chi(k) \gamma_\rho \cos\left(\frac{k_\rho}{2}\right) \right. \\ \left. + S_w(k) i \sin\left(\frac{k_\rho}{2}\right) \right\}, \quad (52)$$

where the second Casimir $C_F = (N_c^2 - 1)/2N_c$. Performing the sum over ρ and dropping the terms odd in k we can write it as follows:

$$\delta \Gamma = \Gamma \delta V_\Gamma = -\frac{g^2}{16\pi^2} C_F \Gamma (I_\chi + I_w), \quad (53)$$

where

the difference between the improved and the Wilson gauge action

$$I_\chi^{c_1} = 16\pi^2 \int_k \{G(c_1, k) - G(0, k)\}, \quad (57)$$

with $G(c_1, k)$ representing the integrand in Eq. (54). The second term

$$I_\chi^w = 16\pi^2 \int_k G(0, k) \\ = 16\pi^2 \int_k \frac{\cos^2(\frac{k_0}{2})}{(w_0 - e^\alpha)(W(k) - e^{-\alpha})\Delta_1}, \quad (58)$$

is just the Wilson gauge action case added back in. Now, the integral I_χ^w is infrared divergent. We will regularize it by including a gluon mass λ and extract the finite part using the method described in Ref. [31]. We have

$$I_\chi^w = I_\chi^w - \hat{I}_\chi^w - 16\pi^2 \int_k \frac{\theta(1 - k^2)}{k^2(k^2 + \lambda^2)}, \quad (59)$$

where

$$I_\chi^w = 16\pi^2 \int_k \left(\frac{1}{(w_0 - e^\alpha)(W(k) - e^{-\alpha})\Delta_1} + \frac{\theta(1 - k^2)}{k^4} \right), \quad (60)$$

and

$$\hat{I}_\chi^w = 16\pi^2 \int_k \frac{1}{16(w_0 - e^\alpha)(W(k) - e^{-\alpha})}; \quad (61)$$

both are finite and can be calculated numerically.

Then the one-loop correction to the vertex is given by

$$\delta V_\Gamma = \frac{g^2}{16\pi^2} C_F [-\ln(\lambda^2 a^2) - I_\chi^{c_1} - I_\chi^w + \hat{I}_\chi^w - I_w] \\ = \frac{g^2}{16\pi^2} C_F [-\ln(\lambda^2 a^2) + d]. \quad (62)$$

Note that neither the constant d nor the coefficient of the logarithm depend on the matrix Γ at the vertex. The numerical results for I_w , I_χ^w , and $\hat{I}_\chi^w$ are tabulated in Table I. The values of $I_\chi^{c_1}$ for different gauge actions can be found in Table II. The value of the constant d is given in Table III. These and all other numerical integrals in this paper were calculated using the Monte Carlo integration routine VEGAS [32], with 1 000 000 sample points.

B. Heavy quark corrections

Now, let us focus on the heavy quark propagator corrections. The tadpole graph in Fig. 1(e) is given by the following expression

$$\Sigma^{\text{tad}}(p_0) = -\frac{1}{2} g^2 C_F e^{-ip_0} \int_k D_{00}^{c_1}(k) \\ = -\frac{1}{2} g^2 C_F e^{-ip_0} T_{c_1}, \quad (63)$$

where T_{c_1} is the tadpole integral whose numerical values are

TABLE I. Results for vertex integrals defined in Sec. V.

M_5	I_w	I_χ^w	$\hat{I}_\chi^w$
0.05	-4.138 229(319)	-1.909 304(325)	-0.703 771(17)
0.10	-4.034 432(236)	-2.030 270(283)	-0.712 970(18)
0.15	-3.948 298(192)	-2.133 953(315)	-0.722 356(18)
0.20	-3.872 189(163)	-2.228 815(258)	-0.731 938(18)
0.25	-3.802 874(143)	-2.316 957(249)	-0.741 722(18)
0.30	-3.738 464(129)	-2.400 620(227)	-0.751 716(18)
0.35	-3.677 754(117)	-2.480 180(264)	-0.761 929(19)
0.40	-3.619 923(108)	-2.558 862(216)	-0.772 367(19)
0.45	-3.564 378(100)	-2.634 128(243)	-0.783 041(19)
0.50	-3.510 664(93)	-2.709 506(214)	-0.793 959(19)
0.55	-3.458 427(87)	-2.782 736(229)	-0.805 132(20)
0.60	-3.407 373(81)	-2.855 975(230)	-0.816 570(20)
0.65	-3.357 255(75)	-2.928 999(222)	-0.828 285(20)
0.70	-3.307 856(70)	-3.001 320(248)	-0.840 288(20)
0.75	-3.258 987(65)	-3.074 116(255)	-0.852 592(20)
0.80	-3.210 479(60)	-3.147 106(260)	-0.865 211(21)
0.85	-3.162 165(55)	-3.220 438(268)	-0.878 159(21)
0.90	-3.113 899(50)	-3.294 598(273)	-0.891 453(21)
0.95	-3.065 532(44)	-3.369 698(280)	-0.905 107(21)
1.00	-3.016 959(36)	-3.445 750(258)	-0.919 141(22)
1.05	-2.967 904(40)	-3.523 295(291)	-0.933 573(22)
1.10	-2.918 434(44)	-3.602 252(251)	-0.948 424(22)
1.15	-2.868 167(47)	-3.682 678(253)	-0.963 696(24)
1.20	-2.816 869(48)	-3.764 914(309)	-0.979 474(23)
1.25	-2.764 631(52)	-3.849 567(315)	-0.995 722(23)
1.30	-2.711 104(56)	-3.936 509(322)	-1.012 510(29)
1.35	-2.656 052(60)	-4.026 280(329)	-1.029 829(30)
1.40	-2.599 188(65)	-4.119 041(336)	-1.047 733(30)
1.45	-2.540 188(72)	-4.215 449(345)	-1.066 257(31)
1.50	-2.478 673(79)	-4.315 845(352)	-1.085 443(31)
1.55	-2.414 181(87)	-4.421 470(280)	-1.105 334(32)
1.60	-2.346 139(97)	-4.531 634(286)	-1.125 985(35)
1.65	-2.273 804(108)	-4.647 352(381)	-1.147 436(33)
1.70	-2.196 232(120)	-4.770 915(392)	-1.169 762(34)
1.75	-2.112 133(135)	-4.903 247(399)	-1.193 028(34)
1.80	-2.019 661(155)	-5.046 261(405)	-1.217 309(35)
1.85	-1.916 047(182)	-5.202 899(419)	-1.242 693(36)
1.90	-1.796 524(223)	-5.378 317(435)	-1.269 279(37)
1.95	-1.651 230(292)	-5.583 553(420)	-1.297 141(31)

$$T_{c_1} = \begin{cases} 0.154 933(3) & \text{Wilson, } c_1 = 0 \\ 0.094 759(3) & \text{Iwasaki, } c_1 = -0.331 \\ 0.062 426(2) & \text{DBW2, } c_1 = -1.406 86 \\ 0.128 291(3) & \text{Symanzik, } c_1 = -1/12 \end{cases} \quad (64)$$

The formula for the rising sun diagram in Fig. 1(d) is given by

$$\Sigma^{\text{rs}}(p_0) = -g^2 C_F \int_k D_{00}^{c_1}(k - p) \frac{e^{-i(p_0 + k_0)}}{1 - e^{-ik_0} + \epsilon}. \quad (65)$$

The heavy quark wave function renormalization is obtained via

TABLE II. Results for the $I_X^{c_1}$ integral defined in Eq. (57).

M_5	Iwasaki	DBW2	Symanzik
0.05	0.246 331(9)	0.408 168(15)	0.100 822(4)
0.10	0.249 861(9)	0.414 136(16)	0.102 247(4)
0.15	0.253 496(9)	0.420 275(16)	0.103 714(4)
0.20	0.257 237(10)	0.426 590(16)	0.105 225(4)
0.25	0.261 091(10)	0.433 088(16)	0.106 782(4)
0.30	0.265 062(10)	0.439 774(17)	0.108 386(4)
0.35	0.269 153(10)	0.446 657(17)	0.110 041(4)
0.40	0.273 371(10)	0.453 742(17)	0.111 747(4)
0.45	0.277 720(10)	0.461 039(18)	0.113 507(4)
0.50	0.282 207(11)	0.468 556(18)	0.115 323(4)
0.55	0.286 837(11)	0.476 302(18)	0.117 199(5)
0.60	0.291 617(11)	0.484 287(18)	0.119 138(5)
0.65	0.296 554(11)	0.492 522(19)	0.121 141(5)
0.70	0.301 655(11)	0.501 018(19)	0.123 212(5)
0.75	0.306 929(12)	0.509 787(19)	0.125 355(5)
0.80	0.312 383(12)	0.518 844(20)	0.127 573(5)
0.85	0.318 029(12)	0.528 201(20)	0.129 871(5)
0.90	0.323 874(12)	0.537 875(21)	0.132 253(5)
0.95	0.329 931(13)	0.547 883(21)	0.134 722(5)
1.00	0.336 211(13)	0.558 242(21)	0.137 285(5)
1.05	0.342 726(13)	0.568 972(22)	0.139 946(6)
1.10	0.349 490(13)	0.580 095(22)	0.142 712(6)
1.15	0.356 519(14)	0.591 634(23)	0.145 589(6)
1.20	0.363 828(14)	0.603 614(23)	0.148 583(6)
1.25	0.371 435(14)	0.616 064(24)	0.151 703(6)
1.30	0.379 360(15)	0.629 014(24)	0.154 956(6)
1.35	0.387 625(15)	0.642 497(25)	0.158 353(6)
1.40	0.396 252(16)	0.656 551(25)	0.161 902(7)
1.45	0.405 268(16)	0.671 216(26)	0.165 615(7)
1.50	0.414 702(16)	0.686 539(27)	0.169 504(7)
1.55	0.424 586(17)	0.702 570(27)	0.173 584(7)
1.60	0.434 956(17)	0.719 365(28)	0.177 868(8)
1.65	0.445 852(18)	0.736 989(29)	0.182 376(8)
1.70	0.457 321(19)	0.755 513(30)	0.187 125(8)
1.75	0.469 413(19)	0.775 019(31)	0.192 138(8)
1.80	0.482 186(20)	0.795 598(32)	0.197 440(9)
1.85	0.495 709(21)	0.817 357(33)	0.203 059(9)
1.90	0.510 059(22)	0.840 417(35)	0.209 029(10)
1.95	0.525 326(24)	0.864 919(37)	0.215 387(10)

TABLE III. Results for d defined in Eq. (62).

M_5	Wilson	Iwasaki	DBW2	Symanzik
0.05	5.343 762(661)	5.097 431(670)	4.935 594(676)	5.242 940(665)
0.10	5.351 732(537)	5.101 871(546)	4.937 596(553)	5.249 485(541)
0.15	5.359 895(525)	5.106 399(534)	4.939 620(541)	5.256 181(529)
0.20	5.369 066(439)	5.111 829(449)	4.942 476(455)	5.263 841(443)
0.25	5.378 109(410)	5.117 018(420)	4.945 021(426)	5.271 327(414)
0.30	5.387 368(374)	5.122 306(384)	4.947 594(391)	5.278 982(378)
0.35	5.396 005(400)	5.126 852(410)	4.949 348(417)	5.285 964(404)
0.40	5.406 418(343)	5.133 047(353)	4.952 676(360)	5.294 671(347)
0.45	5.415 465(362)	5.137 745(372)	4.954 426(380)	5.301 958(366)
0.50	5.426 211(326)	5.144 004(337)	4.957 655(344)	5.310 888(330)
0.55	5.436 031(336)	5.149 194(347)	4.959 729(354)	5.318 832(341)
0.60	5.446 778(331)	5.155 161(342)	4.962 491(349)	5.327 640(336)
0.65	5.457 969(317)	5.161 415(328)	4.965 447(336)	5.336 828(322)
0.70	5.468 888(338)	5.167 233(349)	4.967 870(357)	5.345 676(343)
0.75	5.480 511(340)	5.173 582(352)	4.970 724(359)	5.355 156(345)
0.80	5.492 374(341)	5.179 991(353)	4.973 530(361)	5.364 801(346)
0.85	5.504 444(344)	5.186 415(356)	4.976 243(364)	5.374 573(349)
0.90	5.517 044(344)	5.193 170(356)	4.979 169(365)	5.384 791(349)
0.95	5.530 123(345)	5.200 192(358)	4.982 240(366)	5.395 401(350)
1.00	5.543 568(316)	5.207 357(329)	4.985 326(337)	5.406 283(321)
1.05	5.557 626(353)	5.214 900(366)	4.988 654(375)	5.417 680(359)
1.10	5.572 262(317)	5.222 772(330)	4.992 167(339)	5.429 550(323)
1.15	5.587 149(324)	5.230 630(338)	4.995 515(347)	5.441 560(330)
1.20	5.602 309(380)	5.238 481(394)	4.998 695(403)	5.453 726(386)
1.25	5.618 476(390)	5.247 041(404)	5.002 412(414)	5.466 773(396)
1.30	5.635 103(407)	5.255 743(422)	5.006 089(431)	5.480 147(413)
1.35	5.652 503(419)	5.264 878(434)	5.010 006(444)	5.494 150(425)
1.40	5.670 496(431)	5.274 244(447)	5.013 945(456)	5.508 594(438)
1.45	5.689 380(448)	5.284 112(464)	5.018 164(474)	5.523 765(455)
1.50	5.709 075(462)	5.294 373(478)	5.022 536(489)	5.539 571(469)
1.55	5.730 317(399)	5.305 731(416)	5.027 747(426)	5.556 733(406)
1.60	5.751 788(418)	5.316 832(435)	5.032 423(446)	5.573 920(426)
1.65	5.773 720(522)	5.327 868(540)	5.036 731(551)	5.591 344(530)
1.70	5.797 385(546)	5.340 064(565)	5.041 872(576)	5.610 260(554)
1.75	5.822 352(568)	5.352 939(587)	5.047 333(599)	5.630 214(576)
1.80	5.848 613(595)	5.366 427(615)	5.053 015(627)	5.651 173(604)
1.85	5.876 253(637)	5.380 544(658)	5.058 896(670)	5.673 194(646)
1.90	5.905 562(695)	5.395 503(717)	5.065 145(730)	5.696 533(705)
1.95	5.937 642(743)	5.412 316(767)	5.072 723(780)	5.722 255(753)

$$\Sigma_{\text{exp}} = g^2 C_F \int_k D_{00}^{c_1}(k) \frac{e^{-ik_0}}{1 - e^{-ik_0} + \epsilon}. \quad (68)$$

Note that

$$\frac{e^{-ik_0}}{1 - e^{-ik_0}} = \frac{1}{2i} \cot\left(\frac{k_0}{2}\right) - \frac{1}{2}. \quad (69)$$

Then Eq. (68) can be simplified to

$$\Sigma_{\text{exp}} = \frac{1}{2} g^2 C_F T_{c_1}^{(3)} - \frac{1}{2} g^2 C_F T_{c_1}, \quad (70)$$

where

$$T_{c_1}^{(3)} = \int_{-\pi}^{\pi} \frac{d^3 k}{(2\pi)^3} D_{00}^{c_1}(0, \vec{k}). \quad (71)$$

The numerical results for $T_{c_1}^{(3)}$ are

$$Z_2^h - 1 = -i \frac{\partial(\Sigma^{\text{tad}} + \Sigma^{\text{rs}})}{\partial p_0} \Big|_{p_0=0}. \quad (66)$$

The calculation of the tadpole contribution to Z_2^h is trivial. In order to calculate the contribution from Σ^{rs} we will use a method similar to the one developed in Ref. [9]. We write

$$-i \frac{\partial \Sigma^{\text{rs}}}{\partial p_0} \Big|_{p_0=0} = \Sigma_{\text{exp}} + \Sigma_D, \quad (67)$$

where Σ_{exp} and Σ_D come from differentiating $\exp[-i(p_0 + k_0)]$ and $D_{00}^{c_1}(k - p)$ in Eq. (65) respectively. We get

$$T_{c_1}^{(3)} = \begin{cases} 0.252\,721(6) & \text{Wilson, } c_1 = 0 \\ 0.164\,361(5) & \text{Iwasaki, } c_1 = -0.331 \\ 0.093\,831(4) & \text{DBW2, } c_1 = -1.406\,86 \\ 0.219\,018(6) & \text{Symanzik, } c_1 = -1/12 \end{cases} \quad (72)$$

For Σ_D in Eq. (67) we obtain

$$\Sigma_D = -ig^2 C_F \int_k D_{00}^{c_1'}(k) 4 \sin\left(\frac{k_0}{2}\right) \cos\left(\frac{k_0}{2}\right) \times \frac{e^{-ik_0}}{1 - e^{-ik_0} + \epsilon}, \quad (73)$$

where

$$D_{00}^{c_1'}(k) = \frac{\partial D_{00}^{c_1}(\hat{k}_0^2, \vec{k})}{\partial \hat{k}_0^2}, \quad (74)$$

with $\hat{k}_0 = 2 \sin(k_0/2)$. We calculated $D_{00}^{c_1'}(k)$ numerically and symbolically using Mathematica. Using Eq. (69) and dropping the term odd in k_0 we can reduce Eq. (73) to

$$\Sigma_D = -2g^2 C_F \int_k D_{00}^{c_1'}(k) \cos^2\left(\frac{k_0}{2}\right). \quad (75)$$

Again, we calculate the difference between the improved and the Wilson gauge action numerically:

$$I_h^{c_1} = 16\pi^2 \int_k \left(D_{00}^{c_1'}(k) + \frac{1}{\Delta_1^2} \right) \cos^2\left(\frac{k_0}{2}\right). \quad (76)$$

The results are

$$I_h^{c_1} = \begin{cases} 1.310\,153(55) & \text{Iwasaki, } c_1 = -0.331 \\ 3.510\,284(162) & \text{DBW2, } c_1 = -1.406\,86 \\ 0.297\,720(15) & \text{Symanzik, } c_1 = -1/12 \end{cases} \quad (77)$$

The contribution of the Wilson part to be added back in is

$$16\pi^2 \int_k \frac{\cos^2(\frac{k_0}{2})}{\Delta_1^2} = 16\pi^2 \int_k \frac{1}{\Delta_1^2} - \pi^2 \int_k \frac{1}{\Delta_1} = 16\pi^2 \int_k \frac{1}{\Delta_1^2} - \pi^2 T_0, \quad (78)$$

where T_0 is the Wilson tadpole integral calculated in Eq. (64). Once again, we subtract numerically the infrared divergent part using the method of Ref. [31]:

$$\Theta = 16\pi^2 \int_k \left(\frac{1}{\Delta_1^2} - \frac{\theta(1-k^2)}{k^4} \right) = 4.791\,861(251). \quad (79)$$

Finally, putting together all one-loop contributions to Z_2^h we have

$$Z_2^h = 1 + 2 \frac{g^2}{16\pi^2} C_F [-\ln(\lambda^2 a^2) - 1 - I_h^{c_1} + \Theta + 4\pi^2 T_{c_1}^{(3)} - \pi^2 T_0], \quad (80)$$

or

$$Z_2^h = 1 + \frac{g^2}{16\pi^2} C_F [-2 \ln(\lambda^2 a^2) + e], \quad (81)$$

where

$$e = \begin{cases} 24.480 & \text{Wilson, } c_1 = 0 \\ 14.883 & \text{Iwasaki, } c_1 = -0.331 \\ 4.914 & \text{DBW2, } c_1 = -1.406\,86 \\ 21.223 & \text{Symanzik, } c_1 = -1/12 \end{cases} \quad (82)$$

with a maximum error of ± 1 in the last digit. Our result for the Wilson case agrees with the result of Ref. [9]. Note that the correction to Z_2^h is significantly reduced in the case of Iwasaki and DBW2 action.

Now, let us discuss the radiative correction to the mass

$$\delta M = -\Sigma^{\text{tad}}(p_0 = 0) - \Sigma^{\text{rs}}(p_0 = 0). \quad (83)$$

We obtain

$$\begin{aligned} \delta M &= \frac{g^2}{16\pi^2} C_F 8\pi^2 T_{c_1}^{(3)} \\ &= \frac{g^2}{16\pi^2} C_F \times \begin{cases} 19.954 & \text{Wilson, } c_1 = 0 \\ 12.977 & \text{Iwasaki, } c_1 = -0.331 \\ 7.409 & \text{DBW2, } c_1 = -1.406\,86 \\ 17.293 & \text{Symanzik, } c_1 = -1/12 \end{cases} \end{aligned} \quad (84)$$

which agrees with the Wilson case result in Refs. [9,33,34].

C. Relation to $\overline{\text{MS}}$ scheme

Now, we can calculate Z_Γ relating the continuum static theory bilinear $\tilde{O}_\Gamma$, renormalized in $\overline{\text{MS}}(\text{NDR})$, to the lattice bilinear O_Γ^{lat} as follows:

$$\tilde{O}_\Gamma(\mu) = (1 - w_0^2)^{-1/2} Z_w^{-1/2} Z_\Gamma(\mu, a) O_\Gamma^{\text{lat}}(a), \quad (85)$$

where $(1 - w_0^2)^{-1/2} Z_w^{-1/2}$ is a DWF-specific factor, whose origin and numerical values are discussed in Refs. [13,30] and

$$\begin{aligned} Z_\Gamma(\mu, a) &= 1 + \frac{g^2}{16\pi^2} C_F \left(-\ln \frac{\lambda^2}{\mu^2} + D + \frac{1}{2} \left(-2 \ln \frac{\lambda^2}{\mu^2} + E \right) \right. \\ &\quad \left. + \frac{1}{2} \left(\ln \frac{\lambda^2}{\mu^2} + F \right) \right) - \frac{g^2}{16\pi^2} C_F \left(-\ln \lambda^2 a^2 + d \right. \\ &\quad \left. + \frac{1}{2} (-2 \ln \lambda^2 a^2 + e) + \frac{1}{2} (\ln \lambda^2 a^2 + f) \right). \end{aligned} \quad (86)$$

The continuum static theory constants $D = 1$, $E = 0$ and $F = 1/2$ were calculated in Ref. [4]. The values for d and e can be found in Table III and Eq. (82). The lattice light quark renormalization factor $f = 1/2 - z_2$, where z_2 was calculated in Refs. [13,30]. Eliminating λ from Eq. (86) we obtain

$$Z_\Gamma(\mu, a) = 1 + \frac{g^2}{16\pi^2} C_F \left(\frac{3}{2} \ln \mu^2 a^2 + \frac{5}{4} - d - \frac{e}{2} - \frac{f}{2} \right). \quad (87)$$

Note that the result for Z_Γ does not depend on Γ since both D and d are Γ independent. This is consistent with the observation in Sec. III that Z_Γ should be the same for all bilinears because of the symmetries of the action.

VI. MATCHING STATIC THEORY AND LATTICE FOUR-FERMION OPERATORS

In Sec. IV we discussed the relationship between the full theory operator O_{VV+AA} and static theory operators $\tilde{O}_{VV+AA}$ and $\tilde{O}_{SS+PP}$. In this section we will match these static theory operators renormalized in $\overline{\text{MS}}(\text{NDR})$ scheme to lattice operators. We will also discuss the renormalization of B_B .

A. Operators matching

First, we note that the coefficient Z_2 in Eq. (38) is already of the order g^2 . Thus, for the purposes of matching at the leading order in g^2 it will suffice to match $\tilde{O}_{SS+PP}$ and O_{SS+PP}^{lat} at tree level.

$$\tilde{O}_{SS+PP}(\mu) = Z_S O_{SS+PP}^{\text{lat}}(a), \quad (88)$$

with

$$Z_S = (1 - w_0^2)^{-1} Z_w^{-1}, \quad (89)$$

being the DWF-specific factor present even at tree level. The matching of $\tilde{O}_{VV+AA}$ to the lattice has been worked out for the Wilson case in Refs. [35,36]. In our case, the mixing is simplified, since the chiral symmetry prohibits mixing with operators of different chirality [18,37,38]. With the exception of the diagrams depicted in Fig. 3, the four-fermions diagrams can be reduced to the diagrams already discussed in Sec. V. Thus, we have

$$\tilde{O}_{VV+AA}(\mu) = Z_L O_{VV+AA}^{\text{lat}}(a), \quad (90)$$

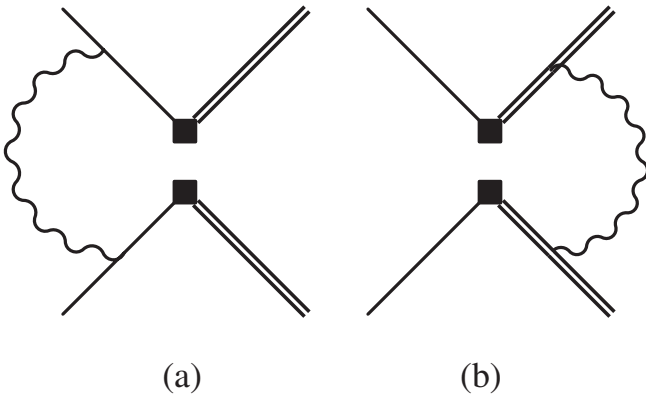


FIG. 3. Four-fermion specific one-loop corrections.

with

$$Z_L = (1 - w_0^2)^{-1} Z_w^{-1} \left(1 + \frac{g^2}{16\pi^2} (4 \ln \mu^2 a^2 + D_L) \right), \quad (91)$$

and

$$D_L = \frac{7}{3} - \frac{1}{3}c - \frac{10}{3}d - \frac{4}{3}e - \frac{4}{3}f - \frac{1}{3}v. \quad (92)$$

The constants d , e and f were discussed in Sec. V. The constant v is the light-light correction depicted in Fig. 3(a), which was calculated in Refs. [13,30,39]. In the notation of Ref. [13] $v = -V_{S,P}$. The constant c is the heavy-heavy correction depicted in Fig. 3(b), which we calculate below.

Omitting the trivial Γ structure, the contribution from the graph in Fig. 3(b) is given by the following expression:

$$\delta V = g^2 C_F \int_k D_{00}^{c_1}(k) \frac{e^{-ik_0}}{(1 - e^{-ik_0} + \epsilon)^2}. \quad (93)$$

Noting that

$$\frac{e^{-ik_0}}{(1 - e^{-ik_0})^2} = -\frac{1}{4\sin^2(k_0/2)}, \quad (94)$$

we integrate Eq. (93) by parts and arrive at

$$\delta V = -2g^2 C_F \int_k D_{00}^{c_1'}(k) \cos^2\left(\frac{k_0}{2}\right), \quad (95)$$

which is identical to Eq. (75) in Sec. V. Using our results from Sec. V we can immediately write

$$\begin{aligned} \delta V &= 2 \frac{g^2}{16\pi^2} C_F [-\ln \lambda^2 a^2 - 1 - I_h^{c_1} + \Theta - \pi^2 T_0] \\ &= \frac{g^2}{16\pi^2} C_F [-2 \ln \lambda^2 a^2 + c], \end{aligned} \quad (96)$$

where

$$c = \begin{cases} 4.525 & \text{Wilson, } c_1 = 0 \\ 1.905 & \text{Iwasaki, } c_1 = -0.331 \\ -2.495 & \text{DBW2, } c_1 = -1.40686 \\ 3.930 & \text{Symanzik, } c_1 = -1/12 \end{cases} \quad (97)$$

with a maximum error of ± 1 in the last digit. Our result for the Wilson case agrees with the result of Ref. [35].

B. B_B renormalization

In QCD, the B -meson parameter B_B is defined as

$$B_B \equiv \frac{\langle \bar{B} | O_{VV+AA}^{\overline{\text{MS}}(\text{NDR})} | B \rangle}{\frac{8}{3} f_B^2 m_B^2}. \quad (98)$$

Combining the results of this section with the results of Sec. IV and V we can relate B_B to lattice as

$$B_B = \frac{Z_1 Z_L}{C_{A_0}^2 Z_A^2} B_{VV+AA}^{\text{lat}} + \frac{Z_2 Z_S}{C_{A_0}^2 Z_A^2} B_{SS+PP}^{\text{lat}}, \quad (99)$$

where

$$B_{O_i}^{\text{lat}} \equiv \frac{\langle \bar{B} | O_i^{\text{lat}}(0) | B \rangle}{\frac{8}{3} |\langle 0 | A_0^{\text{lat}} | B \rangle|^2 m_B}. \quad (100)$$

Note that the B_B renormalization does not depend on $(1 - w_0^2)^{-1} Z_w^{-1}$ since it cancels in the ratios.

VII. MEAN-FIELD IMPROVEMENT

Finally, let us briefly discuss the application of mean-field improvement to our results. We follow the much more detailed discussion in Refs. [13,30,40]. The main reason for the improvement program is the fact that for some values of M_5 , the DWF-specific factor Z_w becomes quite large [12]. This problem can be circumvented by substituting w_0 and Z_w with:

$$w_0^{\text{MF}} = w_0 + 4(1 - u), \quad (101)$$

$$Z_w^{\text{MF}} = Z_w|_{w_0=w_0^{\text{MF}}} + \frac{4w_0^{\text{MF}}}{1 - (w_0^{\text{MF}})^2} 2(1 - u), \quad (102)$$

where $u = P^{1/4}$, with P being the value of the plaquette.

The mean-field improvement will also affect other perturbative constants. The constant $f = 1/2 - z_2$ in Eqs. (87) and (92) should be replaced by $f^{\text{MF}} = 1/2 - z_2^{\text{MF}}$, with z_2^{MF} of Refs. [13,30]. Because of the change in w_0 , the value of the constant d in Eq. (62) should also change. One can obtain the values of d^{MF} from d by observing that $d^{\text{MF}}(M_5) = d(\tilde{M}_5)$, where $\tilde{M}_5 = M_5 - 4(1 - u)$. Similarly, the constant v in Eq. (92) should be replaced by $v^{\text{MF}}(M_5) = v(\tilde{M}_5)$. For the heavy quark, mean-field improvement has no effect on the wave function renormalization as was pointed out in Refs. [40,41]. However, one should be mindful of the subtlety involving different normalization conditions for Z_2^h discussed in Refs. [9,33]. The normalization condition consistent with the mean-field improvement necessitates the use of $e - \delta M$ instead of e in Eqs. (87) and (92). The constant c in Eq. (97) is unaffected by mean-field improvement. Finally, the light-quark field should be multiplied by $\sqrt{u}$, which leads to the multiplication of Z_Γ by $\sqrt{u}$ and of $Z_{L,S}$ by u .

VIII. CONCLUSIONS

In this paper we have calculated one-loop renormalization constants for operators combining static heavy and domain-wall light quarks. We obtained results for Wilson, Iwasaki, DBW2, and Symanzik gauge actions. We have confirmed that all bilinears regularize by a single constant when matched to the static theory. We have shown that the combination of domain-wall fermions and an improved gauge action reduces the size of perturbative corrections

significantly compared to the Wilson case. This suggests that our choice of action is quite suitable for the determination of f_B , B_B , and $\Delta m_s/\Delta m_d$ with increased precision. Our results provide the necessary connection between those phenomenological quantities and lattice calculations.

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APPENDIX

The improved gauge propagator in momentum space is given by [17]

$$D_{\mu\nu}^{c_1}(k) = \frac{1}{(\hat{k}^2)^2} \left([\alpha - A_{\mu\nu}^{c_1}(k)] \hat{k}_\mu \hat{k}_\nu + \delta_{\mu\nu} \sum_\sigma \hat{k}_\sigma^2 A_{\sigma\sigma}^{c_1}(k) \right), \quad (A1)$$

where α is the gauge parameter, $\hat{k}_\mu = 2 \sin(k_\mu/2)$, $\hat{k}^2 = 4 \sum_\mu \sin^2(k_\mu/2)$ and

$$\begin{aligned} A_{\mu\nu}^{c_1}(k) &= A_{\nu\mu}^{c_1}(k) \\ &= \frac{1 - \delta_{\mu\nu}}{\Delta(k)} \left[(\hat{k}^2)^2 - c_1 \hat{k}^2 \left(2 \sum_\rho \hat{k}_\rho^4 + \hat{k}^2 \sum_{\rho \neq \mu, \nu} \hat{k}_\rho^2 \right) \right. \\ &\quad \left. + c_1^2 \left(\left(\sum_\rho \hat{k}_\rho^4 \right)^2 + \hat{k}^2 \sum_\rho \hat{k}_\rho^4 \sum_{\tau \neq \mu, \nu} \hat{k}_\tau^2 \right. \right. \\ &\quad \left. \left. + (\hat{k}^2)^2 \prod_{\rho \neq \mu, \nu} \hat{k}_\rho^2 \right) \right], \end{aligned} \quad (A2)$$

with

$$\begin{aligned} \Delta(k) &= \left(\hat{k}^2 - c_1 \sum_\rho \hat{k}_\rho^4 \right) \left[\hat{k}^2 - c_1 \left((\hat{k}^2)^2 + \sum_\tau \hat{k}_\tau^4 \right) \right. \\ &\quad \left. + \frac{1}{2} c_1^2 \left((\hat{k}^2)^3 + 2 \sum_\tau \hat{k}_\tau^6 - \hat{k}^2 \sum_\tau \hat{k}_\tau^4 \right) \right] \\ &\quad - 4 c_1^3 \sum_\rho \hat{k}_\rho^4 \prod_{\tau \neq \rho} \hat{k}_\tau^2. \end{aligned} \quad (A3)$$

The different choices for the parameter c_1 correspond to different gauge actions discussed in the main text of this paper. In the case $c_1 = 0$, the propagator in Eq. (A1) reduces to the standard Wilson plaquette action propagator.

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