Light cone quantization and Savvidy instability in dense quark matter

Aiichi Iwazaki

Department of Physics, Nishogakusha University, Ohi Kashiwa Chiba 277-8585, Japan
(Received 9 October 2006; published 28 February 2007)

Solving instability of the Savvidy vacuum in QCD is a longstanding problem. Using light cone quantization, we analyze the problem not in the real confining vacuum but in dense quark matter where gluons interact weakly with each other. We find a stable ferromagnetic ground state of gluons which carry a single longitudinal momentum. Their states are composed as if they are confined in a two-dimensional quantum well. This supports our previous result that gluons form a quantum Hall state in dense quark matter.

PACS numbers: 12.38.Lg, 12.38.Aw, 24.85.+p

About 30 years ago, Savvidy [1] showed that a color magnetic field is generated spontaneously in the Yang-Mills gauge theory. Namely, when one calculates an effective potential of the color magnetic field using the one loop approximation, it is found that the nontrivial color magnetic field is generated spontaneously. Soon after, it was shown [2] that some of the gluons have imaginary energies under the color magnetic field and produce an imaginary part in the effective potential. The existence of the nontrivial imaginary part in the effective potential implies the instability of the vacuum with the color magnetic field. Some of the gluons are unstable in the vacuum. We call it Savvidy instability. This magnetic instability of the vacuum in the Yang-Mills gauge theory was expected by many authors to lead to a confining vacuum. Namely, it was expected that the confining vacuum would be realized by the condensation of the unstable gluons. The subsequent analysis [3] of the gluons has revealed the complication of the color magnetic flux due to the production of an additional magnetic field generated by the unstable gluons. Although the formation of a lattice of the flux tube has been argued [3], any other clear pictures of such complicate states formed by the unstable gluons have not been presented. Eventually, a confining vacuum could not be obtained.

We have recently investigated the Savvidy instability in dense quark matter and shown [4] that the instability is solved by the formation of a stable quantum Hall state [5,6] of the unstable gluons. Since perturbative arguments such as loop expansions are applicable in sufficiently dense quark matter, it is reliable that the spontaneous generation of the color magnetic field arises in the matter. Although the quantum Hall state is realized nonperturbatively due to the effect of the gluon's repulsive self-interactions, this formation mechanism is well established in the physics of quantum Hall states of electrons. This is similar to the formation mechanism of BCS states; BCS states arise due to the effect of attractive forces between electrons on the Fermi surface, even if the forces are fairly weak. Hence, it is also reliable that the quantum Hall states of the gluons arise in the dense quark matter. Consequently, we may

understand that Savvidy instability is solved in the dense quark matter. It is composed of quarks, the color magnetic field, and the colored quantum Hall state of gluons. The phase of the quark matter is called the color ferromagnetic phase. Although quarks occupy Landau levels, they do not form quantum Hall states in general. (We have shown [4] that the color ferromagnetic phase is realized in the quark matter with lower densities than ones with which color superconductivity [7] is realized. Thus, the phase is phenomenologically more important than the color superconducting phase. We have discussed an astrophysical implication of the phase [8] and also have pointed out the similarity [9] between the gluons in the dense quark matter and color glass condensate in nucleons.)

Quantum Hall states arise only in two-dimensional space. For example, quantum Hall states of electrons are realized in quantum wells of semiconductors, which are effectively two dimensional. Excitations with nontrivial momenta perpendicular to the two-dimensional well are forbidden energetically as far as we are concerned with smaller energies (or lower temperature) than a finite gap. Thus, only excitations with smaller energies than the gap are allowed and they are excitations in the two-dimensional well. This is a feature of the two-dimensional quantum well. Then, it is natural to ask how two-dimensional quantum Hall states of gluons are formed in the three-dimensional dense quark matter. In this paper we analyze the problem as well as Savvidy instability, using the light cone quantization [10,11]. Since the QCD Hamiltonian with a color magnetic field can be well defined in the quantization, it is easy to analyze ground states of the gluons with the use of an approximation valid at small couplings. As a result, we find that the gluons in the lowest Landau level form a ground state in which all of the gluons have a single longitudinal momentum. Furthermore, the energies of excitations with the same longitudinal momentum as the momentum of the gluons in the ground state are much smaller than the energies of excitations with nontrivial longitudinal momenta. Such excitations arise in the two-dimensional transverse space. Thus, the gluons are two dimensional since their excitations are only allowed

in the transverse directions as far as we are concerned with much smaller energies. In this way, two-dimensional gluonic states arise effectively in the three-dimensional quark matter. All of them carry the single longitudinal momentum. These gluons may form quantum Hall states, but we do not discuss in this paper how the two-dimensional gluons form a quantum Hall state. Hereafter we analyze SU(2) gauge theory.

First of all, we will give a brief review of our previous result [4,6] obtained with the use of equal-time quantization, that is, how the two-dimensional states of gluons arise in a ground state. We assume the presence of a color magnetic field, $B_i^a \propto \delta_{i,3}\delta^{a,3}$, generated spontaneously; indices of a and i denote a coordinate of color and of space, respectively. Then, some of the gluons under a spatially uniform color magnetic field, $B = \epsilon_{3ij}\partial_i A_j^3$, have imaginary energies such that $E^2 = k^2 - gB$. These gluons ($A_l^1 + iA_l^2 \propto e^{-iEt}$ perpendicular to magnetic field, B, in color space) occupy the lowest Landau level and have color magnetic moments parallel to B. Here g is the gauge coupling constant and k is a momentum parallel to the magnetic field $\vec{B} = (0, 0, B)$. Indices j of A_l^j denote colors of SU(2). The other stable gluons occupy higher Landau levels and their energies are given by $E^2 = k^2 + gBn$ with integer $n \geq 0$. Thus, the unstable gluons with small momentum, such as $k^2 < gB$, have imaginary energies and their amplitudes grow up rapidly in time. Among them, the most unstable gluons are the ones with vanishing momentum $k = 0$. (There are infinitely many degenerate states with $k = 0$ in the lowest Landau level.) Hence, their wave functions are uniform in the direction parallel to the magnetic field. These most unstable gluons with $k = 0$ are expected to form a new stable ground state similar to the case in the Higgs model with a negative mass term $-m^2|\phi|^2$. In the model, unstable Higgs modes with energies such as $E^2 = \vec{k}^2 - m^2 < 0$ are present in a naive vacuum $\langle\phi\rangle = 0$. Among them, the most unstable modes with $\vec{k} = 0$ form a new ground state, $\langle\phi\rangle \neq 0$, with the condensation of the field ϕ. Similarly, we expect that only the gluons with $k = 0$ form a stable ground state. They are two-dimensional objects since they are uniform in the direction of $\vec{B}$. In this way, the two dimensionality of the gluonic ground state arises since only the most unstable gluons with $k = 0$ make a stable ground state. It is reasonable to assume that any gluons with $k \neq 0$ do not contribute to the formation of the ground state. This is because the similar phenomena arise in the Higgs model. This is our previous result derived with the use of the equal-time quantization. (It is well known that bosons as well as fermions in two-dimensional space can form a quantum Hall state if repulsive interaction between them is present. Hence, the two-dimensional gluons may make a quantum Hall state due to the repulsive self-interaction [5,12].)

In such a circumstance, we wish to examine in detail whether the ground states of gluons are two dimensional or not, by using the light cone quantization. In the formulation, the states with imaginary light cone energy never appear. Instead, a naive Fock vacuum is not the lowest energy state. The real ground state with the lowest energy is formed as a composite state of gluons in the lowest Landau level. In this paper we do not investigate the real confining vacuum which is a complicated composite of strongly interacting gluons; the problem of obtaining such a real QCD vacuum is beyond our scope [13]. Instead, we investigate a ground state of gluons in dense quark matter, in which gluons are weakly interacting with each other. We assume the presence of a color magnetic field, which is generated spontaneously. Here, we do not address a question of spontaneous generation of color magnetic field in the light cone quantization.

Now, we show that a ground state of gluons is two dimensional, that is, it depends trivially on a longitudinal momentum p^+. We use the notations of the light cone time coordinate, $x^+ = (x^0 + x^3)/\sqrt{2}$, and longitudinal coordinate, $x^- = (x^0 - x^3)/\sqrt{2}$. Transverse coordinates are denoted by x^i, or $\vec{x}$. We assume a finite length, $-L \leq x^- \leq L$, in the longitudinal space and impose a periodic boundary condition such that $A_j^a(x^- = L) = A_j^a(x^- = -L)$. Then, corresponding momentum becomes discrete denoted by $p_n^+ = n\pi/L$ with integer n. Light cone components of gauge fields, A^+, A^-, A_i, are defined similarly. Then, the Hamiltonian, H, with the light cone gauge, $A^+ = 0$, is given by

$$H = \frac{1}{4}F_{ij}^a F_{ij}^a + \frac{g^2}{2}\rho^a \frac{1}{(-\partial_-^2)}\rho^a, \qquad (1)$$

with field strength, $F_{ij}^a = \partial_i A_j^a - \partial_j A_i^a + g\epsilon_{abc}A_i^b A_j^c$, where color indices a run from 1 to 3 and space indices i, j run from 1 to 2. ρ^a is defined by

$$\rho^a \equiv (D_i A_i)^a = (\partial_i \delta^{ab} + g\epsilon^{a3b}A_i^B)A_i^b, \qquad (2)$$

where A_i^B denotes the gauge potential of color magnetic field, which is assumed to direct into σ_3 in color SU(2); $B = \partial_1 A_2^B - \partial_2 A_1^B$. We have neglected a dynamical gauge potential $A_i^{a=3}$ aside from the classical one A_i^B since it does not couple directly with A_i^B. We have only taken dynamical gauge fields $A_i^{a=1,2}$ perpendicular in color space to the color magnetic field. They form Landau levels under B. We treat only the quantum effects of the gluons, but treat quarks classically for simplicity. Their color charge neutralizes the color charge of the gluons.

We make a comment that our treatment of "zero mode" [14,15] in the light cone quantization is similar to the one used by Thorn [16]: We quantize gauge fields in the finite volume, $-L \leq x^- \leq L$, and neglect zero modes of the fields. Consequently, the Hamiltonian becomes a simple form involving at most quartic terms of creation or annihilation operators in addition to quadratic ones. As has been shown [16] in a two-dimensional model of the scalar

field, the true ground state can be gripped even if we neglect the zero mode of the field, at least in the limit of $L \to \infty$. We assume that it also holds in the gauge theory. We may justify neglecting the zero mode in the analysis of dense quark matter as follows. That is, our concern is not the real vacuum, but a ground state of gluons in dense quark matter. The zero mode may play an important role in the real vacuum of strongly interacting gluons. But, it may not play such a role in a ground state of gluons weakly interacting with each other in the dense quark matter. Typical energy scale of quarks and gluons in the quark matter is given by the chemical potential of the quarks and is much larger than Λ_{QCD}. In such dense quark matter, the zero mode does not play an important role for realizing the ground state. It is similar to the case of QCD at high energy scattering [17] where the typical energy scale is much higher than Λ_{QCD}. Hence, the zero mode does not play an important role for realizing so-called color glass condensation. Therefore, it is reasonable to neglect the zero modes of the gluons in the quark matter.

We assume the spontaneous generation of the color magnetic field, B, in the light cone quantization. The condition of neglecting the zero mode requires the coherent length of B in x^- being shorter than L, but becoming infinite as $L \to \infty$. Thus, our discussion below is limited for the case such that momentum scale p^+ of gluons is larger than the inverse of the coherent length of B.

In the light cone gauge, only dynamical variables are transverse components A_i^a of gauge fields. This can be expressed in terms of creation and annihilation operators:

$$A_i^b = \sqrt{\frac{\pi}{L}} \sum_{p^+>0} \frac{1}{\sqrt{2\pi p^+}}$$
$$\times (a_{i,p}^b(x^+, \vec{x})e^{-ip^+x^-} + a_{i,p}^{b\dagger}(x^+, \vec{x})e^{ip^+x^-}), \quad (3)$$

with $p^+ = \pi n/L$ (integer, $n \geq 1$), where operators $a_{i,p}^l$ satisfy the commutation relations $[a_{i,p}^b(x^+, \vec{x}), a_{j,k}^{\dagger c}(x^+, \vec{y})] = \delta_{ij}\delta^{bc}\delta_{pk}\delta(\vec{x} - \vec{y})$, with other commutation relations being trivial. As we mentioned before, we have neglected the zero modes, $p^+ = 0$, of the gauge fields.

Then, the gauge fields satisfy the equal time, x^+, commutation relation,

$$[\partial_- A_i^a(x^+, x^-, \vec{x}), A_j^b(x^+, y^-, \vec{y})]$$
$$= -i\delta_{ij}\delta^{ab}\delta(\vec{x} - \vec{y})\left(\delta(x^- - y^-) - \frac{1}{2L}\right), \quad (4)$$

where the last factor, $1/2L$, on the right-hand side of the equation comes from neglecting the zero modes of the gauge fields.

We should mention that the second term in H represents a Coulomb interaction. It is derived by solving a constraint equation, $\partial_-^2 A^{-,a} = \rho^a$, that is, Gauss law associated with the light cone gauge condition, $A^+ = 0$. In order to assure that the gauge field A^- is periodic in x^-, the zero mode of

ρ ($\rho \propto \sum_{n=\text{integer}} \rho_n e^{i\pi n x^-/L}$) must vanish; $\rho_{n=0} = 0$. Then, the operation of $1/(-\partial_-^2)$ is well defined. The condition of $\rho_{n=0} = 0$ implies that the total color charge $\int_{-L}^{L} dx^- \rho_{\text{tot}}$ vanishes. Here we should include background classical color charge of quarks in ρ_{tot}. ρ in Eq. (1) should be replaced by $\rho_{\text{tot}} = \rho - \rho_{\text{quark}}$ with $\rho_{\text{tot},n=0} = 0$. This requirement of $\rho_{\text{tot},n=0} = 0$ is consistent with the condition $A_{n=0}^- = 0$.

We now rewrite the Hamiltonian in terms of "charged vector fields," $\Phi_i = (A_i^1 + iA_i^2)/\sqrt{2}$, which may be decomposed into the spin parallel (antiparallel) component, $\Phi_p = (\Phi_1 + \Phi_2)/\sqrt{2}$ ($\Phi_{ap} = (\Phi_1 - \Phi_2)/\sqrt{2}$). These fields transform as Abelian charged fields under the U(1) gauge transformation, $A_i \to U^\dagger A_i U$ with $U = \exp(i\theta\sigma_3)$.

Then, using the fields Φ_p (Φ_{ap}), we obtain the following Hamiltonian:

$$H = \frac{1}{2}B^2 + \Phi_p^\dagger(-\vec{D}^2 - 2gB)\Phi_p + \Phi_{ap}^\dagger(-\vec{D}^2 + 2gB)\Phi_{ap}$$
$$+ \frac{g^2}{2}(|\Phi_p|^2 - |\Phi_{ap}|^2)^2 + \frac{g^2}{2}\rho\frac{1}{(-\partial_-^2)}\rho, \quad (5)$$

with $\vec{D} = \vec{\partial} + ig\vec{A}^B$, where ρ is given by

$$\rho = i(\Phi_p^\dagger\partial_-\Phi_p - \partial_-\Phi_p^\dagger\Phi_p + \Phi_{ap}^\dagger\partial_-\Phi_{ap} - \partial_-\Phi_{ap}^\dagger\Phi_{ap})$$
$$- \rho_{\text{quark}}. \quad (6)$$

The first term in Eq. (5) represents the classical energy of the color magnetic field, and the second (third) term represents the kinetic energy of the charged gluons with spin parallel (antiparallel) under the color magnetic field, B. The fourth term represents the energy of the repulsive self-interactions. The last term represents the Coulomb energy coming from the second term with the $\rho^{a=3}$ component in Eq. (1).

Since eigenstates of the operator $\vec{D}^2$ are classified by Landau levels, the second and the third terms can be rewritten as

$$\sum_{n=0,1,2,...} (\Phi_{p,n}^\dagger(2n-1)gB\Phi_{p,n} + \Phi_{ap,n}^\dagger(2n+3)gB\Phi_{ap,n}),$$
$$(7)$$

where the fields $\Phi_{p,n}$ ($\Phi_{ap,n}$) denote operators in the Landau level specified by integer n. (We have implicitly assumed integration over the transverse directions in the above equation.)

Now, we take only the field $\Phi_{p,n=0}$ in the lowest Landau level $n = 0$, that is, the component having negative kinetic energy. It is most important, among others, for realizing the ground state of the Hamiltonian in the limit of strong magnetic field B. The field corresponds to the unstable gluon in our previous discussions [4] with the use of the timelike quantization. Therefore, we obtain the following reduced Hamiltonian for analyzing the ground state of the system:

$$H_r = -gB|\Phi|^2 + \frac{g^2}{2}|\Phi|^4 + \frac{g^2}{2}\rho_r\frac{1}{(-\partial_-^2)}\rho_r, \quad (8)$$

with $\rho_r \equiv i(\Phi^\dagger\partial_-\Phi - \partial_-\Phi^\dagger\Phi) - \rho_{\text{quark}}$, where we have put $\Phi \equiv \Phi_{p,n=0}$ for simplicity. The field Φ can be expressed by using creation and annihilation operators,

$$\Phi = \sqrt{\frac{\pi}{L}}\sum_{p>0,m=0,1,2,...}\frac{1}{\sqrt{2\pi p}}$$
$$\times (a_{p,m}\phi_m(\vec{x})e^{-ipx} + b_{p,m}^\dagger\phi_m^\dagger(\vec{x})e^{ipx}), \quad (9)$$

where simplified notation such as $x = x^-$ and $p = p^+$ is exploited and will be used below. $\phi_m(\vec{x}) = g_m z^m \exp(-|z|^2/4l^2)$ represents the normalized eigenfunction of $\vec{D}^2$ with angular momentum m, in the lowest Landau level $\int d^2\vec{x}\phi_m^\dagger\phi_n = \delta_{m,n}$; $z = x_1 + ix_2$ and $g_m \equiv \frac{1}{\pi m!(2l^2)^{m+1}}$ with $l^2 \equiv 1/gB$. $a_{p,m}$ and $b_{p,m}$ satisfy the commutation relations; $[a_{p,m}, a_{k,n}^\dagger] = \delta_{p,k}\delta_{m,n}$, $[b_{p,m}, b_{k,n}^\dagger] = \delta_{p,k}\delta_{m,n}$, others $= 0$.

When we express the first term in Eq. (8) in terms of the operators, $a_{p,m}$ and $b_{p,m}$,

$$\int_{-L}^{L} dx d^2\vec{x}: -gB|\Phi|^2$$
$$:= -gB\sum_{p>0,m}\frac{1}{p}(a_{p,m}^\dagger a_{p,m} + b_{p,m}^\dagger b_{p,m}), \quad (10)$$

we find that there exist states with lower energies than a trivial Fock vacuum, $|\text{vac}\rangle$; $a_{p,m}|\text{vac}\rangle = b_{p,m}|\text{vac}\rangle = 0$. Namely, gluons in the lowest Landau level are produced spontaneously to form a state with lower energy than that of the vacuum. The production of the gluons is limited by the second term in Eq. (8) representing repulsion among the gluons. This is similar to the case of the Higgs model. Contrary to the model, the gluons do not condense. Thus, we postulate $\langle\Phi\rangle = 0$. (This is consistent with neglecting zero mode.) In order to find the ground state of the gluons, we will express approximately the energy $\langle H_r\rangle$ of the state in terms of the distribution functions $\langle a_{p,m}^\dagger a_{p,m}\rangle$ and $\langle b_{p,m}^\dagger b_{p,m}\rangle$. Then, by minimizing the energy we will find the momentum p and angular momentum m distribution of gluons. The state of the gluons may carry nonvanishing color charge density, in general. The fact that quarks and gluons are weakly interacting in the dense quark matter certificates that our approximation gives rise to reliable results; any quantum corrections to the results are small.

Before finding the ground state of the Hamiltonian H_r, we should note that there are two conserved quantities such as total color charge and momentum,

$$Q = \int_{-L}^{L} dx d^2\vec{x}\rho_r = \sum_{p>0,m}(a_{p,m}^\dagger a_{p,m} - b_{p,m}^\dagger b_{p,m})$$
$$- \int_{-L}^{L} dx d^2\vec{x}\rho_{\text{quark}} = 0 \quad (11)$$
$$P_{\text{total}} = \sum_{p>0,m} p(a_{p,m}^\dagger a_{p,m} + b_{p,m}^\dagger b_{p,m}),$$

where we have required that the total color charge of gluons and quarks must vanish. The ground state must be found under the condition of the conserved quantities being given. Since our concern is the ground state of gluons in dense quark matter, not real vacuum, the total color charge of the gluons can be nonzero and is neutralized by the color charge of quarks.

In order to evaluate the expectation value of H_r with the ground state $|g\rangle$, we assume the following approximation:

$$\langle a_\alpha^\dagger a_\beta^\dagger a_\gamma a_\delta\rangle \simeq \langle a_\alpha^\dagger a_\gamma\rangle\langle a_\beta^\dagger a_\delta\rangle + \langle a_\alpha^\dagger a_\delta\rangle\langle a_\beta^\dagger a_\gamma\rangle,$$
$$\text{for } \alpha \neq \beta, \qquad \text{similar for } b_\alpha \quad (12)$$

$$\langle a_\alpha^\dagger b_\beta^\dagger a_\gamma b_\delta\rangle \simeq \langle a_\alpha^\dagger a_\gamma\rangle\langle b_\beta^\dagger b_\delta\rangle,$$
$$\langle a^\dagger a^\dagger b^\dagger a\rangle = \langle a^\dagger b^\dagger b^\dagger b\rangle = \langle a^\dagger aab\rangle = \langle b^\dagger bba\rangle = 0, \quad (13)$$

$$\langle a_\alpha^\dagger a_\beta\rangle \propto \delta_{\alpha,\beta}, \qquad \langle b_\alpha^\dagger b_\beta\rangle \propto \delta_{\alpha,\beta}, \quad (14)$$

where indices $\alpha, \beta, \ldots$ denote a set of p and m. Namely, we assume no mixing among states with different p and m in Eq. (12). Furthermore, we assume no pair creations and annihilations in Eq. (13) so that the numbers of particle and antiparticle are conserved, respectively. We also postulate momentum and angular momentum conservation in Eq. (14). These formulas are satisfied by eigenstates of the number operators $a_\alpha^\dagger a_\alpha$ and $b_\alpha^\dagger b_\alpha$. Hence, we minimize the Hamiltonian by using the eigenstates of the number operators in our approximation.

Using the approximation, we evaluate an expectation value $\langle:H_r:\rangle$ of normal ordered Hamiltonian $:H_r:$ and find the ground state minimizing the expectation value,

$$\langle:H_r:\rangle = -gB\sum_{p>0,m}(a(p,m) + b(p,m)) + \frac{g^2}{2L}\sum_{p,q>0,m,n}(a(p,m)a(q,n) + b(p,m)b(q,n) + 2a(p,m)b(q,n))N_{m,n}$$
$$+ \frac{g^2}{2L}\sum_{p\neq q>0}\left(\frac{p+q}{p-q}\right)^2\sum_{m,n}(a(p,m)a(q,n) + b(p,m)b(q,n))N_{m,n} + \frac{g^2}{2L}\sum_{p,q>0}\left(\frac{p-q}{p+q}\right)^2\sum_{m,n}2a(p,m)b(q,n)N_{m,n},$$
$$(15)$$

with $N_{m,n} = (m + n)!(\pi m!n!2^{m+n+2}l^2)^{-1}$, where $a(p, m)$ and $b(p, m)$ are defined by

$$\langle a^\dagger_{p,m} a_{q,n} \rangle = \delta_{p,q} \delta_{m,n} p a(p, m),$$
$$\langle b^\dagger_{p,m} b_{q,n} \rangle = \delta_{p,q} \delta_{m,n} p b(p, m),$$
(16)

where $a(p, m)$ and $b(p, m)$ represent distributions of p and m in the ground state, respectively. We have used the color neutrality condition $\rho_{r,n=0} = 0$ in the derivation of Eq. (15); the condition is imposed as a constraint such as $p \neq q$ in the third term.

It is easy to see that the expectation value of the color charge density becomes uniform in the longitudinal direction $x = x^-$,

$$\langle \rho_r \rangle = \frac{1}{L} \sum_{p>0,m} p(a(p, m) - b(p, m)) |\phi_m(\vec{x})|^2 - \rho_{\text{quark}},$$
(17)

which may depend on transverse coordinate $\vec{x}$.

In Eq. (15) the first term represents the kinetic energy in the Landau level and the second term represents the energy of the repulsion between gluons. These two terms are denoted by E_1. The third and fourth terms represent the Coulomb energy between gluons and are denoted by E_2. Obviously, these terms are non-negative except for the first one. Therefore, we can find a ground state with the lowest energy $\langle :H: \rangle \equiv E_1 + E_2$, by minimizing the first two terms E_1 and the last two terms E_2 of the Coulomb energy, respectively. Simultaneously, we need to take into account the condition that the state carries a given total momentum and vanishing total color charge. We may assume that the color charge of quarks ρ_{quark} is spatially uniform.

It is easy to minimize the first two terms, i.e. E_1, which is given by a set of values $c(m) \equiv \sum_{p>0}(a(p, m) + b(p, m))$. This is because E_1 can be rewritten such that $E_1 = -gB\sum_m c(m) + g^2/(2L)\sum_{m,n} c(m)H_{m,n}c(n)$. Thus, minimizing the energy E_1 does not determine the dependence of $a(p, m)$ or $b(p, m)$ on the longitudinal momentum $p = p^+$. It simply gives the summation over the momentum p, namely, $c(m) \propto gBL/g^2$. The dependence on p is determined only by minimizing the Coulomb energy E_2. The energy $E_2 \geq 0$ can be minimized easily by assuming that the ground state depends only on a single momentum $p = p_0$, that is, $a(p, m) \propto \delta_{p,p_0}$ and $b(q, n) \propto \delta_{q,p_0}$. This distribution of the momentum leads to the minimum $E_2 = 0$. [Any other distributions with the dependence on various momenta give rise to higher energies (> 0).] It follows from the distribution that the color charge density is given such that $\langle \rho_r \rangle = p_0\sum_m(a(p_0, m) - b(p_0, m))|\phi_m(\vec{x})|^2/L - \rho_{\text{quark}}$. For the color charge density to vanish, $a(p_0, m) - b(p_0, m)$ should be independent of m. Then, the color charge density of the gluons becomes uniform in $\vec{x}$, and can cancel that of quarks ρ_{quark} since $\sum_m |\phi_m(\vec{x})|^2 = 1/(2l^2\pi)$. On the other hand, the total momentum is given by $P_{\text{tot}} = \sum_m p_0^2(a(p_0, m) + b(p_0, m)) = p_0^2\sum_m c(m)$.

Therefore, the ground state minimizing the energy in Eq. (15) is characterized by the trivial distribution of the longitudinal momentum; $\langle a^\dagger_{p,m} a_{p,m} \rangle \propto \delta_{p,p_0}$ and $\langle b^\dagger_{p,m} b_{p,m} \rangle \propto \delta_{p,p_0}$. The Coulomb energy E_2 of the state vanishes. All of the gluons in the lowest Landau level occupy the states with a single momentum, e.g. $p = p_0$. It implies that the ground state is composed of two-dimensional gluons occupying the lowest Landau level. It is also important to note that the displacement of a gluon with the momentum p_0 in the ground state to a state with a momentum $k \neq p_0$ gives rise to an energy gap ΔE_k; $\langle k|:H_r:|k \rangle = \langle :H_r: \rangle + \Delta E_k$ ($\Delta E_k > 0$ for any k), where $|k\rangle \equiv a^\dagger_{k,m} a_{p_0,m}|g\rangle$. We note that $\Delta_k \to \infty$ as $k \to p_0$. This simple argument suggests the existence of a finite gap energy needed to excite modes with longitudinal momenta different than p_0. On the other hand, we can show that there exist gapless excitations with the momentum p_0, which have different distributions in m with $a(p_0, m)$ and $b(p_0, m)$ in the ground state. The fact implies that gluons are confined in a two-dimensional quantum well extending in the transverse directions. Only motions are allowed in the well as far as we are concerned with smaller energies than the gap energy. Motions in the longitudinal direction are visible only when we are concerned with higher energies than the gap energy. On this point, more detail analysis is necessary to demonstrate the conclusion.

Here, we wish to mention that, for P_{total} to be finite in the limit of $L \to \infty$, p_0 goes to 0 in the limit because $c(m) \propto L$. On the other hand, the color charge density of gluons $\propto p_0(a(p_0, m) - b(p_0, m))$ can take any finite value even in the limit. Hence, all of the gluons occupy the states with a vanishingly small longitudinal momentum. In the terminology used in high energy scattering of hadrons or nuclei, parton (gluons) distribution behaves such as $\delta(x)$ since $x = p_0/P_{\text{tot}} \to 0$. (The distribution contains only contributions of charged vector field $\Phi_\pm$ or $A_i^{a=1,2}$. If we take into account momentum distribution of the gauge field $A_i^{a=3}$ neglected in our argument, it would give a parton distribution with a finite support in x.)

We have not yet shown that the gluons form quantum Hall states, but have only shown that they form two-dimensional states. It is a necessary condition for the realization of quantum Hall states of the gluons. Whether or not the quantum Hall states of the gluons are made depends on the color charge density of the gluons. If a specific condition of "filling factor" $\nu = 2\pi\rho_r L/gB$ is satisfied, the quantum Hall state can be realized, that is, $\nu = 2\pi p_0\sum_m(a(p_0, m) - b(p_0, m))|\phi_m(\vec{x})|^2/gB = 1/2, 1/4, \ldots$. Otherwise, gluons simply form a two-dimensional compressible state, i.e. gapless state.

As we have shown, the two dimensionality of the ground state of gluons arises from the Coulomb interaction $\rho \frac{1}{-\partial_-^2} \rho$ in the longitudinal direction x^-. In our approximation, the Coulomb energy E_2 is a positive semidefinite. Thus, the ground state is given by the state with $E_2 = 0$. When we apply the similar approximation to quarks coupled with the magnetic field, we obtain a negative semidefinite Coulomb energy. Thus, we do not obtain the two dimensionality in the quarks. In the ground state of the quarks, the distribution of the longitudinal momentum does not take the form of a delta function. The result in the quarks or fermions is natural physically. Electrons in metals never take a form of two-dimensional gases under external magnetic field except for ones confined in two-dimensional space, e.g. quantum wells.

The difference between the case of the gluons and that of quarks comes from their difference in statistics. In order to see this point more explicitly, we take the following simple model of a Coulomb interaction,

$$V_c = \frac{g^2}{2} :\rho \frac{1}{(-\partial_-^2)} \rho: \qquad \rho(x) = \psi^\dagger(x)\psi(x) - \langle \psi^\dagger \psi \rangle \tag{18}$$

with $-L \leq x \leq L$, where the field, ψ represents boson or fermion with an appropriate boundary condition at $x = \pm L$.

Suppose that we have two states with momentum $p > 0$ and $q(\neq p) > 0$, whose wave functions are denoted as $f(p, x)$ and $f(q, x) \propto e^{-iqx}$. We define the field operator such that

$$\psi(x) = a_p f(p, x) + a_q f(q, x), \tag{19}$$

with annihilation operators a_p and a_q satisfying commutation relations $[a_p, a_q^\dagger]_\pm = \delta_{p,q}, \ldots$. As in the previous case, we extract zero modes of ρ for $1/(-\partial_-^2)$ to be well

defined. Thus, $\psi^\dagger \psi = a_p^\dagger a_q f^\dagger(p, x) f(q, x) + $ H.c. When we evaluate the expectation value of V_c with the use of a state $a_p^\dagger a_q^\dagger |0\rangle$,

$$\langle V_c \rangle_\pm = \pm 2 |f(p, x)|^2 |f(q, x)|^2 \frac{1}{(p - q)^2}, \tag{20}$$

we find that the Coulomb energy is positive semidefinite in the bosonic case $\langle V_c \rangle_+ \geq 0$, and negative semidefinite in the fermionic case $\langle V_c \rangle_- \leq 0$. If we allow additional internal states with an identical momentum, i.e. $f_i(p, x)$ with $i = 1, 2$ and evaluate $\langle V_c \rangle$ with the use of a state $a_{p,i=1}^\dagger a_{p,i=2}^\dagger |0\rangle$, we find $\langle V_c \rangle = 0$. Therefore, it turns out that, in the case of bosons, the state of two particles with an identical momentum is more stable than the state of two particles with different momentum. On the other hand, in the case of fermions, the situation is reverse; the state of two particles with different momentum is more stable than the state of two particles with an identical momentum.

To summarize, we have shown using light cone quantization that Savvidy instability is solved in sufficiently dense quark matter where quarks and gluons couple weakly with each other. In the quark matter, the color magnetic field is generated spontaneously and gluons in the lowest Landau level are also produced spontaneously to form a ground state of gluons. All of the gluons in the ground state carry a vanishingly small longitudinal momentum. Thus, they form a nontrivial two-dimensional state with finite color charge. The state arises due to the effect of Coulomb interaction among gluons in the longitudinal direction. We need further analysis to see whether or not these gluons form quantum Hall states.

[1] G. K. Savvidy, Phys. Lett. **71B**, 133 (1977).
[2] N. K. Nielsen and P. Olesen, Nucl. Phys. **B144**, 376 (1978); Phys. Lett. **79B**, 304 (1978).
[3] J. Ambjørn, N. K. Nielsen, and P. Olesen, Nucl. Phys. **B152**, 75 (1979); H. B. Nielsen and M. Ninomiya, Nucl. Phys. **B156**, 1 (1979); H. B. Nielsen and P. Olesen, Nucl. Phys. **B160**, 330 (1979).
[4] A. Iwazaki and O. Morimatsu, Phys. Lett. B **571**, 61 (2003); A. Iwazaki, O. Morimatsu, T. Nishikawa, and M. Ohtani, Phys. Lett. B **579**, 347 (2004); Phys. Rev. D **71**, 034014 (2005).
[5] *The Quantum Hall Effect, 2nd Edition*, edited by R. E. Prange and S. M. Girvan (Springer-Verlag, New York, 1990); *Perspectives in Quantum Hall Effects*, edited by S. Das Sarma and A. Pinczuk (Wiley, New York, 1997).
[6] Z. F. Ezawa, M. Hotta, and A. Iwazaki, Phys. Rev. B **46**, 7765 (1992); Z. F. Ezawa and A. Iwazaki, J. Phys. Soc. Jpn. **61**, 4133 (1992).
[7] K. Rajagopal and F. Wilczek, hep-ph/0011333.
[8] A. Iwazaki, Phys. Rev. D **72**, 114003 (2005).
[9] A. Iwazaki, hep-ph/0604222.
[10] P. A. M. Dirac, Rev. Mod. Phys. **21**, 392 (1949).

[11] For a review, see A. Harindranath, hep-ph/9612244; T. Heintl, Lect. Notes Phys. **572**, 55 (2001).

[12] T. Nakajima and M. Ueda, Phys. Rev. Lett. **91**, 140401 (2003).

[13] K. Kondo, Phys. Lett. B **600**, 287 (2004).

[14] T. Maskawa and K. Yamawaki, Prog. Theor. Phys. **56**, 270 (1976).

[15] S. Tsujimaru and K. Yamawaki, Phys. Rev. D **57**, 4942 (1998).

[16] J. S. Rozowsky and C. B. Thorn, Phys. Rev. Lett. **85**, 1614 (2000).

[17] E. Iancu and R. Venugopalan, hep-ph/0303204.