Cosmologies with null singularities and their gauge theory duals

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We investigate backgrounds of Type IIB string theory with null singularities and their duals proposed in S. R. Das, J. Michelson, K. Narayan, S. P. Trivedi, hep-th/0602107. The dual theory is a deformed $\mathcal{N} = 4$ Yang-Mills theory in 3 + 1 dimensions with couplings dependent on a lightlike direction. We concentrate on backgrounds which become $AdS_5 \times S^5$ at early and late times and where the string coupling is bounded, vanishing at the singularity. Our main conclusion is that in these cases the dual gauge theory is nonsingular. We show this by arguing that there exists a complete set of gauge invariant observables in the dual gauge theory whose correlation functions are nonsingular at all times. The two-point correlator for some operators calculated in the gauge theory does not agree with the result from the bulk supergravity solution. However, the bulk calculation is invalid near the singularity where corrections to the supergravity approximation become important. We also obtain pp-waves which are suitable Penrose limits of this general class of solutions, and construct the matrix membrane theory which describes these pp-wave backgrounds.

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I. INTRODUCTION AND SUMMARY

Time-dependent backgrounds, particularly those which contain spacelike or null singularities, are amongst the most poorly understood aspects of string theory. This problem has been attacked from various viewpoints for a long time. These include perturbative string theory [1–10], open and closed string tachyon condensation [11], particularly those which lead to spacelike or null singularities [12–15].

Recently there have been several attempts to attack this problem using holographic duals of various kinds. These include matrix model formulations of noncritical string theory [16–21] (which also involve closed string tachyon condensation), matrix theory duals of backgrounds with null linear dilatons [22–44], and use of the AdS/CFT correspondence [7,45–49]. It has been suspected for a long time that the usual notions of space-time break down near singularities. These holographic approaches attempt to make this idea concrete by replacing usual dynamical space-time by a more fundamental structure provided typically by a gauge theory in lower number of dimensions.

In a previous paper, Ref. [46], we reported on the construction of a family of solutions in Type IIB string theory. The solutions are either time-dependent or depend on a lightlike coordinate,[1] and often have singularities which are spacelike or null, respectively. The solutions can be thought of as deformations of the well-known $AdS_5 \times S^5$ solution. The dilaton and axion are also excited in these solutions, and in some of them the dilaton remains weakly coupled everywhere, including the singularity. Similar solutions were studied in Refs. [45,47].

In this note we continue our study of these backgrounds. In particular, we focus on the null backgrounds, with a *weakly* coupled dilaton. The metric and dilaton in these solutions take the form

$$ds^2 = \left(\frac{r^2}{R^2}\right)\tilde{g}_{\mu\nu}(X^+)dx^\mu dx^\nu + \left(\frac{R^2}{r^2}\right)dr^2 + R^2 d\Omega_5^2,$$
$$\Phi = \Phi(X^+), \tag{1.1}$$

with the four-dimensional metric

$$d\tilde{s}^2 = \tilde{g}_{\mu\nu}dx^\mu dx^\nu = e^{f(X^+)}(-2dX^+dX^- + dx_2^2 + dx_3^2). \tag{1.2}$$

Singularities can arise when the conformal factor e^f vanishes. When this happens, in the solutions of interest, e^Φ vanishes at the singularity. Also, asymptotically as $X^+ \to -\infty$, e^f and the dilaton Φ go to a constant, so that these solutions asymptote to the familiar $AdS_5 \times S^5$ solution. This is in contrast with the kind of backgrounds studied using some other approaches, where the bulk string coupling is typically *large* near the singularity.

In this paper, we argue that these backgrounds have a dual description as the $\mathcal{N} = 4$ Super Yang-Mills theory living in a 3 + 1 dimensional space-time with metric $\tilde{g}_{\mu\nu}$ and with a varying Yang-Mills coupling constant $g_{YM}^2 = e^\Phi$. The Yang-Mills theory starts in the $\mathcal{N} = 4$ vacuum state as $X^+ \to -\infty$ and we want to understand the time

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[1]Below, we will loosely refer to backgrounds which depend on a lightlike coordinate as null backgrounds.

evolution of this system as we approach the singularity at $X^+ = 0$.

Our main conclusion is that the gauge theory is nonsingular. By this we mean that we can find a complete set of gauge invariant operators whose correlation functions are nonsingular even when the bulk geometry has a singularity. This happens because of two features of our null solutions. First, the metric, Eq. (1.2), is conformally flat, and one finds that due to the lightlike dependence of the conformal factor e^f, the conformal anomaly of the gauge theory in this background vanishes. Secondly, the dilaton and hence the Yang-Mills coupling becomes arbitrarily weak near $X^+ = 0$.

In more detail, we carry out the analysis in two parts. First, we neglect the varying dilaton and study the gauge theory in the presence of the nontrivial conformal factor e^f. Using the fact that the conformal anomaly vanishes, we then argue that if the operators, $\mathcal{O}_i(x)$, have conformal dimensions Δ_i, their correlation functions satisfy the relation

$$\langle e^{(f(x_1)\Delta_1)/2}\mathcal{O}(x_1)e^{(f(x_2)\Delta_2)/2}\mathcal{O}(x_2)\cdots e^{(f(x_n)\Delta_n)/2}\mathcal{O}(x_n)\rangle_{[e^f\eta_{\mu\nu}]} = \langle\mathcal{O}(x_1)\mathcal{O}(x_2)\cdots\mathcal{O}(x_n)\rangle_{[\eta_{\mu\nu}]}. \tag{1.3}$$

Here the left-hand side is calculated in the theory with the metric, $\tilde{g}_{\mu\nu} = e^f\eta_{\mu\nu}$, while the right-hand side is calculated with the flat metric. Both correlation functions are calculated in the vacuum of the $\mathcal{N} = 4$ theory, which is conformally invariant.[2]

In the second part of the analysis, we include the effects of the varying coupling constant due to the time-dependent bulk dilaton. The important point here is that for the supergravity solutions of interest, e^Φ is small everywhere and *vanishes* at the singularity. Thus, its effects can be controlled. We show that the varying dilaton, due to its null dependence, does not give rise to any particle production in the interaction picture. Near the singularity, since e^Φ becomes small and vanishes, the gauge field, A_μ, with quadratic terms

$$S_{\rm GF} = -\frac{1}{4}\int d^4x e^{-\Phi}\,{\rm Tr}[F_{\mu\nu}F^{\mu\nu}], \tag{1.4}$$

has singular kinetic terms and is not a well-defined variable. Working in light-cone gauge, $A_- = 0$, a well-defined variable which has canonical quadratic terms is given by

$$\tilde{A}_\mu = e^{-(\Phi/2)}A_\mu. \tag{1.5}$$

For these null backgrounds, we argue that gauge invariant operators made from the $\tilde{A}$ variables are nonsingular at the singularity.[3] This is because the gauge coupling vanishes at $X^+ = 0$ so that coupling effects do not destroy the nonsingular nature of the propagator and also render higher point functions nonsingular. Generically, these operators have supergravity duals which are not local excitations in the bulk.

At the singularity, e^Φ vanishes and the 't Hooft coupling $g_{\rm YM}^2 N$ in the gauge theory goes to zero. This suggests that α' corrections become important near the singularity, since for the $AdS_5 \times S^5$ duality we have $\alpha' \sim \frac{1}{g_{\rm YM}^2 N}$. To understand this issue, we consider the bosonic part of the worldsheet action for a fundamental string in this class of background and fix the light-cone gauge. The gauge fixed action clearly shows that near the singularity all the oscillator modes of the string become very light. We have not performed a detailed analysis of the worldsheet theory to examine whether perturbative string theory can be consistently defined in this background.

Our results strongly suggest that using the nonsingular description of the gauge theory we can extend the bulk space-time past the singularity. In some sense, the singularity therefore appears to be a problem caused by a wrong choice of dynamical variables. In particular, variables which are natural and local from the bulk ten-dimensional supergravity point of view are *not* well behaved at the singularity. However, a correct choice of variables, which is transparent in the dual gauge theory, "resolves" this singularity. We find that the correct description is obtained by appropriately continuing the metric and the dilaton. The resulting space-time is asymptotically $AdS_5 \times S^5$ in the far future, as $X^+ \rightarrow +\infty$, with a constant dilaton.

Finally we perform the Penrose limit of our class of solutions by zooming in on a suitable null geodesic and obtain the resulting pp-wave solutions. The maximally supersymmetric type IIB pp-wave with one compact null direction and an additional compact spacelike direction is a unique background for which two kinds of holographic duality are understood: the holographic duality to a certain sector of Yang-Mills theory in 3 + 1 dimensions [51–53] along the lines of Ref. [54], and the duality to DLCQ matrix theory [55,56] along the lines of Refs. [57–59]. In Ref. [37] type IIB pp-waves with a dilaton which is linear in a null time was considered as a model of a null big bang. It would be of interest to relate the insight gained from a AdS/CFT perspective to that obtained from a DLCQ ma-

[2]Note that we are using the phrase "conformal invariance" in the sense of Weyl invariance, i.e. a position dependent rescaling of the metric without any coordinate transformation. Therefore, the conformal dimensions Δ which appear in Eq. (1.3) can be distinct from usual dimensions of operators under e.g. rescaling of coordinates.

[3]The conformal dimension of A_μ is zero. For example, the action of the gauge theory is classically Weyl invariant, provided A_μ has vanishing conformal weight; see e.g. Ref. [50], p. 448. This means the conformal weight of $\tilde{A}_\mu$ also vanishes and so the $\tilde{A}_\mu$ variables do not require any dressing.

trix perspective and possibly establish a precise relationship between the two. Using the methods outlined in Ref. [37] we write down the $2+1$ dimensional Yang-Mills theory which is dual to string theory in the pp-waves which arise from the solutions considered in this paper. A more detailed analysis of this matrix membrane theory near the singularity is left for the future.

Some of our analysis and conclusions contain substantial overlap with Refs. [45,47]. The supergravity solutions were found and their conjectured gauge theory duals identified in Refs. [45–47]. Furthermore, Ref. [45] calculated the bulk two-point function for the null backgrounds (which agrees with our calculation in Sec. V). Some discussion of dressed correlators and the suggestion that the gauge theory is in fact nonsingular is also contained in Ref. [45].

This paper is structured as follows. The supergravity solutions are reviewed in section II. The gauge theory duals are identified in Sec. III, and extensively analyzed in Sec. IV. Some two-point functions are calculated in the bulk in Sec. V. Section VI concerns the bosonic part of the worldsheet action for a string moving in this class of backgrounds. In Sec. VII we perform a Penrose limit and write down the matrix membrane theory action.

II. REVIEW OF SUPERGRAVITY SOLUTIONS

The solutions we are interested in are discussed in sec. 2 of Ref. [46].[4] The ten-dimensional Einstein frame metric and the dilaton are

$$ds^2 = \left(\frac{r^2}{R^2}\right)\tilde{g}_{\mu\nu}dx^\mu dx^\nu + \left(\frac{R^2}{r^2}\right)dr^2 + R^2 d\Omega_5^2, \tag{2.1a}$$

$$F_{(5)} = R^4(\omega_5 + *_{10}\omega_5), \tag{2.1b}$$

$$\Phi = \Phi(x^\mu). \tag{2.1c}$$

This is a solution of the equations of motion as long as the four-dimensional metric, $\tilde{g}_{\mu\nu}$, and the dilaton, Φ, are only dependent on the four coordinates, x^μ, $\mu = 0, 1, 2, 3$, and satisfy the conditions

$$\tilde{R}_{\mu\nu} = \tfrac{1}{2}\partial_\mu\Phi\partial_\nu\Phi, \tag{2.2a}$$

$$\partial_\mu(\sqrt{-\det(\tilde{g})}\tilde{g}^{\mu\nu}\partial_\nu\Phi) = 0, \tag{2.2b}$$

where $\tilde{R}_{\mu\nu}$ is the Ricci curvature of the metric $\tilde{g}_{\mu\nu}$. In Eq. (2.1), $d\Omega_5^2$ is the volume element and ω_5 is the volume form of the unit five sphere.

Of particular interest in this paper is the case where $\tilde{g}_{\mu\nu}$ is conformally flat and where both $\tilde{g}_{\mu\nu}$ and Φ only depend on one lightlike coordinate which we take to be X^+. In this case, the four-dimensional space-time background takes the form

$$d\tilde{s}^2 = \tilde{g}_{\mu\nu}dx^\mu dx^\nu = e^{f(X^+)}(-2dX^+dX^- + dx_2^2 + dx_3^2), \tag{2.3a}$$

$$\Phi = \Phi(X^+). \tag{2.3b}$$

The conditions, Eq. (2.2), then require

$$\tfrac{1}{2}(f')^2 - f'' = \tfrac{1}{2}(\partial_+\Phi)^2. \tag{2.4}$$

Generically a nonflat metric $\tilde{g}_{\mu\nu}$ as in Eq. (2.1a) introduces curvature singularities at the Poincare horizon $r = 0$. It can be easily checked, however, that these singularities are absent for such null backgrounds. The only possible singularities appear at values of X^+ where timelike or null geodesics entirely lying in the X^μ subspace end or begin at finite affine parameters.

An important case, prototypical of the kind of example we have in mind throughout this paper, is obtained by taking

$$e^f = \tanh^2 X^+. \tag{2.5}$$

Then

$$d\tilde{s}^2 = \tanh^2 X^+(-2dX^+dX^- + dx_2^2 + dx_3^2),$$
$$e^\Phi = g_s\left|\tanh\frac{X^+}{2}\right|^{\sqrt{8}}. \tag{2.6}$$

In this example,[5] in the far past and future as $X^+ \to \pm\infty$, the four-dimensional space-time becomes asymptotically flat and the dilaton goes to a constant. By choosing g_s to be small, the string coupling e^Φ can be made small everywhere in the space-time.[6] In the example, Eq. (2.6), e^Φ is not analytic at $X^+ = 0$. It satisfies the equation, Eq. (2.4), for $X^+ > 0$ and $X^+ < 0$, and is continuous at $X^+ = 0$.

For the metric, Eq. (2.1a) obeying Eq. (2.3a), the affine parameter λ, for a null geodesic moving along a trajectory with X^-, x^2, x^3 constant, is given by

$$\lambda = \int e^{f(X^+)}dX^+. \tag{2.7}$$

The tangent vector along this geodesic is

[4] See also Refs. [7,31,42,60] for time-dependent supergravity solutions in an M-theory context.

[5] Note that a solution with $e^{f(X^+)} = \tanh^2(QX^+)$ is equivalent to Eq. (2.6), by setting to unity the dimensionful scale Q using the symmetry $X^+ \to \lambda X^+$, $Q \to \frac{Q}{\lambda}$, $X^- \to \frac{X^-}{\lambda}$, which is special to these null solutions.

[6] Up to a shift or rescaling (footnote 5) of X^+, the only nontrivial ($e^f \neq$ const) solution with an everywhere constant dilaton is $e^f = \frac{1}{(X^+)^2}$, which can be seen to be flat space by the coordinate transformation

$$x_2 = X^+Y_2, \qquad x_3 = X^+Y_3,$$
$$X^- = Y^- - X^+(Y_2^2 + Y_3^2), \qquad X^+ = -\frac{1}{Y^+}.$$

$$\xi = \xi^\mu \partial_\mu \equiv \frac{\partial}{\partial\lambda} = \left(\frac{d\lambda}{dX^+}\right)^{-1} \frac{\partial}{\partial X^+} = e^{-f} \frac{\partial}{\partial X^+},$$
$$\xi^\mu \xi_\mu = 0. \tag{2.8}$$

For the ten-dimensional metric, Eq. (2.1a), the Ricci scalar, $R^A{}_A$ and the invariants, $R_{AB}R^{AB}$, $R_{ABCD}R^{ABCD}$, are all constant and independent[7] of r, X^+. In contrast, the curvature invariant, for the metric, Eq. (2.1a), $R_{ab}\xi^a\xi^b$, [ξ^a is the component of the tangent vector, Eq. (2.8)] is

$$R_{\lambda\lambda} = R_{++}\left(\frac{dX^+}{d\lambda}\right)^2 = \left(\frac{1}{2}(f')^2 - f''\right)e^{-2f}. \tag{2.9}$$

For a suitably chosen f this can blow up when $e^f \to 0$, and the resulting value of λ, Eq. (2.7), can be finite. This results in a singularity which occurs at finite affine time.

For example, in the case, Eq. (2.6), $R_{++} = \frac{4}{\sinh^2 X^+}$, so that $R_{\lambda\lambda} = \frac{4}{\sinh^2 X^+ \tanh^4 X^+}$, showing a curvature singularity at $X^+ \to 0$, with e^Φ becoming arbitrarily small there.[8] One can see in this example that the singularity occurs at finite affine time.

It is worth understanding the resulting singularity better. Consider null geodesics with X^+ varying and all other coordinates fixed. Take two such nearby geodesics displaced along the x^i, $i = 2, 3$, directions. Following Ref. [50] p. 47, the relative acceleration of these geodesics is

$$a^i = -R_{+i+}{}^i(\xi^+)^2 = -\tfrac{1}{2}(\tfrac{1}{2}(f')^2 - f'')e^{-2f}, \tag{2.10}$$

where ξ is defined in Eq. (2.8). Up to a sign, this is exactly $R_{\lambda\lambda}$, Eq. (2.9). We see that there is a diverging compressional tidal force as the singularity is approached. Another way to see this is to calculate the physical distance between two nearby null geodesics of this type. For two geodesics displaced along x^i, the physical distance is

$$\Delta = \frac{e^{f/2}}{z}\sqrt{(x_i)^2}, \tag{2.11}$$

satisfying the equation

$$\frac{d^2\Delta}{d\lambda^2} = -\frac{1}{2}\left(\frac{1}{2}(f')^2 - f''\right)\Delta. \tag{2.12}$$

The first term on the right-hand side (rhs) agrees with a^i, Eq. (2.10), and $R_{\lambda\lambda}$, Eq. (2.9), showing once again that the compressional tidal force diverges at the singularity.

[7]Here the indices $A, B \cdots$, take values in the tangent space of x^μ, r, Eq. (2.1a). The statement is also true if they range over the full ten-dimensional tangent space including the S^5.

[8]The string frame curvature, $R_{\lambda\lambda}$ blows up at the singularity in this case as well.

It is also worth mentioning that the example, Eq. (2.6), can be regarded as a limiting case of a one-parameter family of solutions, with the conformal factor

$$e^f = (|\tanh X^+| + \epsilon)^2, \tag{2.13}$$

and the dilaton given by solving Eq. (2.4). For small ϵ, the dilaton deviates from its value, Eq. (2.6), only close to $X^+ = 0$. At $X^+ = 0$, $e^\Phi \sim g_s(\epsilon)^{\sqrt{8}}$, so that the string coupling is nonvanishing but small. Note that in Eq. (2.13) the metric is continuous but nonanalytic at $X^+ = 0$ and the first derivative of e^f has a finite discontinuity there. However, the curvature component, R_{++}, is continuous and nonsingular at $X^+ = 0$, and the affine parameter, λ, Eq. (2.7), is also continuous there.

We close by noting that the null backgrounds, Eqs. (2.1) and (2.3), preserve eight supersymmetries.

III. THE DUAL FIELD THEORY

In this section we argue that the backgrounds discussed above are dual to the $\mathcal{N} = 4$ Super Yang-Mills theory in a 3 + 1 dimensional space-time with a varying background metric and a varying gauge coupling. The background metric is $\tilde{g}_{\mu\nu}$, Eq. (2.1a), and the gauge coupling is given in terms of the dilaton, Eq. (2.1c), by, $g_{\rm YM}^2 = e^\Phi$.

We now review the evidence in support of this claim. Notice first that $AdS_5 \times S^5$ is a special case of the solution, Eq. (2.1). When $\tilde{g}_{\mu\nu}$ and Φ are small perturbations about the $AdS_5 \times S^5$ solution,

$$\tilde{g}_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \tag{3.1a}$$
$$\Phi(x) = \ln g_s + \delta\Phi(x), \tag{3.1b}$$

we can use the standard AdS/CFT dictionary and it is easy to see that there is a dual field theory description for these backgrounds. Each supergravity perturbation has a normalizable and a non-normalisable mode [61,62] associated with it; these, respectively, determine the expectation value of the corresponding operator, and the source coupling to the operator in the dual theory. For the solution, Eq. (2.1), we see that only the non-normalizable modes are turned on in these backgrounds. Thus the dual theory in the linearized perturbation case, is the $\mathcal{N} = 4$ theory, in the $\mathcal{N} = 4$ vacuum, with the additional source terms,

$$S_{\rm source} = \int d^4x\left[\frac{\delta\Phi(x^\mu)}{g_{\rm YM}^2}\,{\rm Tr}F^2 + h_{\mu\nu}T^{\mu\nu}\right]. \tag{3.2}$$

For a background which is not a small perturbation about $AdS_5 \times S^5$ the above argument does not directly apply. In this case it is useful to look at things from the perspective of the gauge theory. We are interested in the $\mathcal{N} = 4$ gauge theory subjected to sources, $\tilde{g}_{\mu\nu}$, e^Φ. Suppose we are also interested in a situation where the gauge theory is in the $\mathcal{N} = 4$ vacuum in the far past, as $X^+ \to -\infty$. This is a reasonable initial state, since the sources, $\tilde{g}_{\mu\nu}$, Φ, become the flat metric and a constant dilaton, respectively, as $X^+ \to -\infty$. The initial conditions and the sources specify the gauge theory completely. Now it is reasonable to believe that there is a supergravity dual for this gauge theory. Since the data discussed above specifies the gauge theory completely, the supergravity solution should be unique. The solutions we have given in Sec. II meet all the required conditions. Equations (2.1a) and (2.1c) have the correct asymptotic behavior as $r \to \infty$ to turn on the source terms $\tilde{g}_{\mu\nu}$ and e^{Φ}; the solutions (2.1) also reduce to the $AdS_5 \times S^5$ solution as $X^+ \to -\infty$. Thus they must be the supergravity dual to the gauge theory.

It is also worthwhile examining this issue from the bulk viewpoint. We can try to devise an argument along the lines of the one used to motivate the AdS/CFT correspondence. One can show [46] that a background of the form

$$\begin{aligned} ds^2 &= Z^{-1/2}(x)\tilde{g}_{\mu\nu}dx^\mu dx^\nu + Z^{1/2}(x)dx^m dx_m, \\ \Phi &= \Phi(x^\mu), \end{aligned} \tag{3.3}$$

with the self-dual fiveform, $F = *_{10}F$, with nonzero components

$$F_{0123m} = \frac{1}{4\kappa}\frac{1}{Z\sqrt{\det(-\tilde{g}_{mn})}}\partial_m \log Z, \tag{3.4}$$

and its dual satisfies the equations of motion, as long as $\tilde{g}_{mn}$ and $\Phi(x)$ satisfy the conditions Eq. (2.2), and Z is a harmonic function on the flat six-dimensional space with coordinates, x^m, $m = 1 \cdots 6$. The solution, Eq. (2.1a), corresponds to taking, $Z = R^2/r^2$, $r^2 = x^m x_m$. Another choice is to take

$$Z = 1 + \frac{R^2}{r^2}. \tag{3.5}$$

The solution, Eq. (2.1a), can then be obtained by the "near-horizon limit," $r \to 0$, from this case.

In more detail, for the choice, Eq. (3.5), the asymptotic form of the metric and dilaton, as $r \to \infty$, are

$$ds^2 = \tilde{g}_{\mu\nu}dx^\mu dx^\nu + dx^m dx_m, \qquad \Phi = \Phi(x^\mu). \tag{3.6}$$

One can verify that this background (without any fiveform flux) solves the equations of motion. Now we can add D3-branes to this background. In the probe approximation it is easy to see that each D3-brane will see a four-dimensional metric along its world volume of the form, $\tilde{g}_{\mu\nu}$, and a gauge coupling $g_{YM}^2 = e^{\Phi}$. Adding N D3-branes and taking a low-energy limit, as in the usual AdS/CFT case, would then lead to the field theory dual for the solution, Eq. (2.1), mentioned at the beginning of this section. As an additional final check we note that a probe D3 brane added to the background, Eq. (2.1), also sees a four-dimensional metric along its world volume and a gauge coupling given by $\tilde{g}_{\mu\nu}$, $g_{YM}^2 = e^{\Phi}$, respectively.[9]

In the rest of the paper we will explore the consequences assuming that this conjectured duality with the $\mathcal{N} = 4$ field theory is correct. Before moving on, though, it is worth emphasizing that the arguments given above in support of the dual field theory description while plausible are not airtight. In particular, they are not on as firm footing as for the original AdS/CFT correspondence and might especially fail close to the singularity in the bulk, where the low-energy limit is more subtle. An important check in the AdS/CFT case was that the absorption cross sections [63–65] vanished at low energies in agreement with the required decoupling of the near and far-horizon regions. We have not attempted to devise analogous checks in the case at hand. It is probably useful for this purpose to regard the singular solution, Eq. (2.6) as the limiting case of the one-parameter family, Eq. (2.13). Such checks might reveal subtleties close to the space-time singularity.

It is worth noting that in principle one could study the $\mathcal{N} = 4$ SYM gauge theory with more general time-dependent e^{Φ}, $\tilde{g}_{\mu\nu}$, deformations, but identifying their supergravity duals might be difficult.

We end this section with one comment that will be important in the subsequent discussion. In the null metric, Eq. (2.3a), the conformal anomaly for the $\mathcal{N} = 4$ theory vanishes. The conformal anomaly is given by Refs. [66–70] (see also e.g. Refs. [71–74] in the holographic context of $AdS_5 \times S^5$),

$$\begin{aligned} T_\mu{}^\mu &= \frac{c}{16\pi^2}(C_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta}) \\ &\quad - \frac{a}{16\pi^2}(R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} - 4R_{\alpha\beta}R^{\alpha\beta} + R^2) \\ &\propto -R_{\alpha\beta}R^{\alpha\beta} + \frac{1}{3}R^2, \end{aligned} \tag{3.7}$$

where in the last expression we have used $a = c = \frac{N^2-1}{4}$ for the $SU(N)$ $\mathcal{N} = 4$ super Yang-Mills theory. In the null solutions Eq. (2.3a), since R_{++} is the only nonvanishing component of the stress tensor, the conformal anomaly vanishes.

The null solutions Eq. (2.3), also have a varying dilaton. In general, this could give rise to an additional contribution

[9]The DBI action for a probe D3-brane gives

$$\begin{aligned} \int d^4x e^{-\Phi}\sqrt{-\det(G^{str}_{ab} + F_{ab})} &= \int d^4x\sqrt{-\det(\tilde{g}_{\mu\nu})} \\ &\quad \times \Big(\frac{1}{2}\tilde{g}^{\mu\nu}\partial_\mu x^m \partial_\nu x^m \\ &\quad - \frac{1}{4}e^{-\Phi}\tilde{g}^{\mu\rho}\tilde{g}^{\nu\sigma}F_{\mu\nu}F_{\rho\sigma} \\ &\quad + \ldots\Big), \end{aligned}$$

where $G^{str}_{\mu\nu} = e^{\Phi/2}\tilde{g}_{\mu\nu}$ is the string metric.

in the trace anomaly. However, such a contribution can be ruled out for these solutions by the following argument. The additional term must be generally covariant involving derivatives of the dilaton, the metric and tensors made out of the metric, like the Ricci curvature. But any such term evaluated on the solutions Eq. (2.3) vanishes because the only derivative of the dilaton that is nonvanishing is $\partial_+\Phi$, while the metric component $\tilde{g}^{++}$ and similarly the component of any tensor made out of the metric with two upper $+$ indices vanishes.[10]

IV. ANALYSIS OF THE SINGULARITY IN THE DUAL FIELD THEORY

In this section we analyze the $\mathcal{N}=4$ Super Yang-Mills theory in a space-time with metric $\tilde{g}_{\mu\nu}$ and gauge coupling $g^2_{\rm YM}=e^{\Phi}$ given in Eqs. (2.1a) and (2.1c). We are interested in the case of conformally flat null backgrounds, Eq. (2.3a), which asymptotically become $\mathrm{AdS}_5\times S^5$, as $X^+\to-\infty$, and in which the dilaton remains weakly coupled for all times. In the examples of interest a singularity arises when the conformal factor $e^f\to 0$, and this happens at finite affine time. We will choose coordinates so that the singularity occurs at $X^+=0$. A prototypical example is Eq. (2.6).

Our main conclusion is that appropriately defined correlation functions in the gauge theory are nonsingular at the singularity, i.e. at $X^+=0$. Physically the singularity arises because the metric shrinks to zero. The essential reason why the field theory is nonsingular is that the metric is conformally flat and since the $\mathcal{N}=4$ Super Yang-Mills theory is conformally invariant, appropriately defined correlation functions do not depend on the conformal factor, even when it vanishes. Now the Yang-Mills theory we are interested in also has a varying gauge coupling. This leads to a nontrivial dependence of correlators on the background, but we will argue that the resulting theory is still nonsingular.

The analysis is divided into two parts. In the next subsection we neglect the variation of the gauge coupling and analyze the field theory in a conformally flat background. Here, our discussion is for a general conformally invariant theory. The following section then includes the effects of the varying gauge coupling in the $\mathcal{N}=4$ super Yang-Mills theory.

A. Conformal field theory in a conformally flat background

Consider a conformally invariant field theory in a conformally flat background

$$ds^2=e^f\eta_{\mu\nu}dx^\mu dx^\nu, \tag{4.1}$$

and consider operators $\mathcal{O}_i$ with conformal dimensions Δ_i, in this theory. Then it is straightforward to show that correlation functions of these operators, dressed by powers of the conformal factor determined by their conformal dimensions, in the background Eq. (4.1), are the same as in flat space. That is

$$\langle e^{(f(x_1)\Delta_1)/2}\mathcal{O}(x_1)e^{(f(x_2)\Delta_2)/2}\mathcal{O}(x_2)\cdots e^{(f(x_n)\Delta_n)/2}\mathcal{O}(x_n)\rangle_{[e^f\eta_{\mu\nu}]}=\langle\mathcal{O}(x_1)\mathcal{O}(x_2)\cdots\mathcal{O}(x_n)\rangle_{[\eta_{\mu\nu}]}. \tag{4.2}$$

We are using notation where the background metric is denoted within square brackets suffixed to the correlator. Thus the left-hand side (lhs) is evaluated with the metric, $e^f\eta_{\mu\nu}$, while the rhs is in flat space. One important condition must be met by the metric, Eq. (4.1), for the result, Eq. (4.2), to be true. The trace of the energy-momentum tensor, i.e. the conformal anomaly $T^\mu{}_\mu$ in the metric, Eq. (4.1), must vanish.

To establish Eq. (4.2) we first start with the partition function of the conformal field theory in a general background metric, $g_{\mu\nu}$. We denote the fields in the theory over which the path integral is defined schematically by φ, and write

$$Z[g_{\mu\nu}]=\int[D\varphi]_{[g_{\mu\nu}]}e^{iS[g_{\mu\nu},\varphi]}. \tag{4.3}$$

Here, S is the action of the CFT which depends on the fields φ and the metric $g_{\mu\nu}$, and we have explicitly indicated the dependence of the measure on the background metric. Under an infinitesimal conformal transformation

$$g_{\mu\nu}\to e^{\delta\psi}g_{\mu\nu}, \tag{4.4}$$

the change in the partition function is proportional to the trace of the energy-momentum tensor, $T^\mu{}_\mu$:

$$\delta\log Z=i\left\langle\int d^4x\sqrt{-g}T^\mu{}_\mu\delta\psi\right\rangle. \tag{4.5}$$

Let us be more explicit in how Eq. (4.5) is derived. From Eq. (4.3),

$$Z[e^{\delta\psi}g_{\mu\nu}]=\int[D\varphi]_{[e^{\delta\psi}g_{\mu\nu}]}e^{iS[e^{\delta\psi}g_{\mu\nu},\varphi]}. \tag{4.6}$$

If the fields φ have conformal dimensions Δ, we now change variables in the path integral from φ to $\tilde{\varphi}$ given by

$$\tilde{\varphi}=e^{(\Delta\delta\psi)/2}\varphi, \tag{4.7}$$

and then write

[10] We thank members of the TIFR string theory group and especially Shiraz Minwalla for discussions in this regards and for providing this argument. A concrete realization of this argument in this context can be found in Refs. [75,76]; see, for example, Eq. (25) of Ref. [75] and Eq. (24) of Ref. [76].

$$Z[e^{\delta\psi}g_{\mu\nu}] = \int [D\tilde{\varphi}]_{[g_{\mu\nu}]} e^{iS[g_{\mu\nu},\tilde{\varphi}]} \times \left(1 + i\int d^4x\sqrt{-g}T^{\mu}{}_{\mu}\delta\psi\right). \quad (4.8)$$

In general the term proportional to $T^{\mu}{}_{\mu}$ on the rhs arises both due to the change in the action and the change in the measure. For a conformal field theory the contribution arises entirely due to the change in the measure. Noting that $\tilde{\varphi}$ is a dummy variable in the path integral we can then subtract Eq. (4.3) from Eq. (4.8) giving Eq. (4.5).

Now consider a one-parameter family of metrics:

$$g_{\mu\nu}(x) = e^{\alpha f(x)}\eta_{\mu\nu}, \qquad \alpha \in [0, 1]. \quad (4.9)$$

If $T^{\mu}{}_{\mu}$ vanishes for all values of $\alpha \in [0, 1]$, we learn from Eq. (4.5) that $\partial_\alpha Z = 0$, so that

$$Z[e^f\eta_{\mu\nu}] = Z[\eta_{\mu\nu}]. \quad (4.10)$$

This argument can be easily extended to correlation functions. Consider the two-point function of the dressed fields, $e^{(f(x)\Delta)/2}\varphi(x)$, evaluated in the metric, Eq. (4.1):

$$\langle e^{(f(x)\Delta)/2}\varphi(x)e^{(f(y)\Delta)/2}\varphi(y)\rangle_{[e^f\eta_{\mu\nu}]} = \frac{1}{Z[e^f\eta_{\mu\nu}]}\int [D\varphi]_{[e^f\eta_{\mu\nu}]} e^{iS[e^f\eta_{\mu\nu},\varphi]} e^{(f(x)\Delta)/2}e^{(f(y)\Delta)/2}\varphi(x)\varphi(y) \quad (4.11)$$

One can show that this correlator is the same as the two-point function of $\varphi(x)$ in flat space. That is,

$$\langle e^{(f(x)\Delta)/2}\varphi(x)e^{(f(y)\Delta)/2}\varphi(y)\rangle_{[e^f\eta_{\mu\nu}]} = \langle\varphi(x)\varphi(y)\rangle_{[\eta_{\mu\nu}]} \quad (4.12)$$

The idea is to again "build up" the metric, Eq. (4.1), starting from the flat one, by considering the one-parameter family, Eq. (4.9), and increasing α from 0 to unity. We start with

$$e^{((\alpha+\delta\alpha)f(x)\Delta)/2}e^{((\alpha+\delta\alpha)f(y)\Delta)/2}\langle\varphi(x)\varphi(y)\rangle_{[e^{(\alpha+\delta\alpha)f}\eta_{\mu\nu}]} = \frac{1}{Z}\int [D\varphi]_{[e^{(\alpha+\delta\alpha)f}\eta_{\mu\nu}]} eiS[e^{(\alpha+\delta\alpha)f}\eta_{\mu\nu,\varphi}] e^{((\alpha+\delta\alpha)f(x)\Delta)/2}e^{((\alpha+\delta\alpha)f(y)\Delta)/2} \times \varphi(x)\varphi(y). \quad (4.13)$$

We will again assume that $T^{\mu}{}_{\mu}$ vanishes for the one-parameter family of metrics, Eq. (4.9). Since we have already shown that the partition function Z is independent of α, any change in the two-point function as α changes can only arise from the path integral in the numerator. Now changing the conformal factor of the background metric on the rhs from $e^{(\alpha+\delta\alpha)f}$ to $e^{\alpha f}$ and changing variables in the path integral to $\tilde{\varphi} = e^{(\delta\alpha f\Delta)/2}\varphi$ gives

$$\begin{aligned} e^{((\alpha+\delta\alpha)f(x)\Delta)/2}e^{((\alpha+\delta\alpha)f(y)\Delta)/2}\langle\varphi(x)\varphi(y)\rangle_{[e^{(\alpha+\delta\alpha)f}\eta_{\mu\nu}]} &= \frac{\int D\tilde{\varphi}_{[e^{\alpha f}\eta_{\mu\nu}]} e^{iS[e^{\alpha f}\eta_{\mu\nu},\tilde{\varphi}]} e^{(\alpha f(x)\Delta)/2}e^{(\alpha f(y)\Delta)/2}\tilde{\varphi}(x)\tilde{\varphi}(y)}{Z} \\ &= e^{(\alpha f(x)\Delta)/2}e^{(\alpha f(y)\Delta)/2}\langle\varphi(x)\varphi(y)\rangle_{[e^{\alpha f}\eta_{\mu\nu}]}. \end{aligned} \quad (4.14)$$

The right-hand side of the line above arises by noting that $T^{\mu}{}_{\mu}$ vanishes for the particular background metric under consideration. The rhs of the second line arises from our definition of the two-point function, Eq. (4.11), after noting that $\tilde{\varphi}$ is a dummy variable in the path integral. From Eq. (4.14), we see that the two-point function of the dressed operator does not change under an infinitesimal change in α. It then follows that the two-point function is independent of α leading to Eq. (4.12).

It is now clear that this argument generalizes for n-point correlators of any set of operators $\mathcal{O}_i$, leading to the result, Eq. (4.2).

We have worked in Minkowski space above. This is the relevant setting in the present investigation where we have a time dependent or null background. In Minkowski space we have to specify the state of the system in which the correlator is being computed. The more precise version of Eq. (4.2) is that the correlator on the left-hand side is evaluated in the conformal vacuum appropriate to the metric, Eq. (4.1), while the correlator on the right-hand side is evaluated in the Minkowski vacuum. Our definition of the conformal vacuum is the standard one in the discussion of quantum field theory in curved space [77]. To avoid ambiguities we will discuss Eq. (4.12) in more detail for the concrete case of a conformally coupled scalar field in 4 dimensions in Appendix A.

The above discussion has been for a general conformal field theory. It also applies to the $\mathcal{N} = 4$ super Yang-Mills theory. The only additional condition that needs to be met is that the trace anomaly vanishes. This is true for all backgrounds of the type, Eq. (2.3a), as was discussed in Eq. (3.7) and in the subsequent paragraph.[11]

It is also worth mentioning that in our discussion above, conformal invariance really means Weyl invariance, i.e. position dependent rescalings of the metric without coor-

[11]The one-parameter family, Eq. (4.9), can be obtained by taking the conformal factor in Eq. (2.3a) to be $e^{\alpha f(X^+)}$. Then it is easy to see that the conformal anomaly vanishes for all values of α.

dinate transformations. The conformal dimension of an operator, Δ, which appears in the dressing factor above, is therefore determined by how the operator transforms under a Weyl transformation. Thus in the $\mathcal{N}=4$ theory, the gauge potential A_μ, which does not transform under a Weyl transformation, has $\Delta=0$, while, $F^2 \equiv F_{\mu\nu}F_{\rho\sigma}g^{\mu\rho}g^{\nu\sigma}$, has $\Delta=4$. In particular we see that Δ can be different from the dimension of an operator under a rescaling of the coordinates. The gauge potential A_μ, for example, has dimension 1 under such rescalings.

We are especially interested here in the case where the conformal factor e^f shrinks to zero resulting in a singularity. One might be worried that our conclusions above fail at the singularity. It is useful to think of the singular case as the limiting situation in a one-parameter family. For the case, Eq. (2.6), this family is described in Eq. (2.13) and is labeled by the parameter ϵ. For all values of $\epsilon > 0$, our discussion above does apply, since the conformal anomaly vanishes for all values of the parameter ϵ, and Eq. (4.2) is valid.[12] We can define the correlation functions in the theory with $\epsilon = 0$ as the limiting case obtained by taking $\epsilon \to 0$. We then conclude that the dressed correlators in the case with the vanishing conformal factor equal those in Minkowski space and, in particular, are nonsingular.[13]

B. Effects of the varying dilaton

We now turn to incorporating the effects of the varying dilaton in the Super Yang-Mills theory.

The supergravity backgrounds we are interested in are asymptotically, as $X^+ \to -\infty$, $AdS_5 \times S^5$ with a constant dilaton. For example, in Eq. (2.6), as $X^+ \to -\infty$, we find that

$$\frac{de^\Phi/dX^+}{e^\Phi} = -4\sqrt{2}e^{-|X^+|} \to 0. \tag{4.15}$$

Thus asymptotically the Yang-Mills theory is in a spacetime with flat metric and constant dilaton. Near the singularity, as $X^+ \to 0$, e^Φ goes to zero and the Yang-Mills theory becomes free so once again effects of the dilaton are not serious. However, for intermediate values, $X^+ \sim O(1)$, the variation of the dilaton is of order unity. To obtain a well-defined supergravity description in the far past we need to take $g_sN \gg 1$. So in the intermediate region, when $X^+ \sim O(1)$, the Yang-Mills theory has a large and varying 't Hooft coupling.

There are two concerns about such a varying coupling constant that we address below. First, this variation of the dilaton could potentially also lead to particle production. If such particle production occurs, even if we started in the vacuum of the $\mathcal{N}=4$ theory in the far past we would not in general remain in the conformal vacuum at later times. For example consider the situation when the big crunch at $X^+=0$ is approached; our arguments above, which assumed that the theory was in the conformal vacuum, will now not apply. Second, such X^+ variation might destroy the conformal invariance of the theory and thus render our analysis of the previous section invalid.

It is useful to first consider a toy model of a conformally coupled scalar field in 3 + 1 dimensions subject to a null-dependent perturbation, $\int d^4x\sqrt{-g}J(X^+)\varphi^3$, to analyze both issues. We do so below and then return to the Super Yang-Mills case.

Our conclusions are that due to the variation of the dilaton being null, i.e. along a lightlike direction, there is no particle production.[14] And while the variation of the dilaton destroys conformal invariance, suitably defined correlation functions are expected to be nonsingular in a background of the form, Eq. (2.3a) and (2.6).

1. The conformally coupled scalar

We consider a conformally coupled scalar with Lagrangian

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{2}(\partial\varphi)^2 + \frac{1}{6}R\varphi^2 + J(X^+)\varphi^3\right], \tag{4.16}$$

in a metric

$$ds^2 = e^{f(X^+)}(-2dX^+dX^- + dx_i^2). \tag{4.17}$$

The light-cone quantization of this theory without the $J(X^+)\varphi^3$ term is discussed in Appendix A.

Particle production.—Let us now consider the effects of the perturbation

$$S_{\rm pert} = \int d^4x\sqrt{-g}J(X^+)\varphi^3. \tag{4.18}$$

We take the operator φ^3 to be normal ordered with respect to the creation and anihilation operators, a, $a^\dagger$ defined in Eq. (A2).

In the interaction picture the resulting state of the system is given by

$$|s\rangle = T_+e^{-i\int d^4xe^{2f(X^+)}J(X^+)\varphi^3}|0\rangle. \tag{4.19}$$

The T_+ symbol refers to time ordering with respect to the X^+ direction. At first order we get

$$\delta_1|s\rangle = -i\int d^4xe^{2f(X^+)}J(X^+)\varphi^3|0\rangle. \tag{4.20}$$

Now from Eq. (A2) we see that if φ^3 is normal ordered only the $(a^\dagger)^3$ term in φ^3 will survive. But each of these $a^\dagger$

[12]Since the metric, Eq. (2.13), is nonanalytic at $X^+=0$, some derivatives of the metric have a finite discontinuity there. However, for both $X^+ \to 0^+$ and $X^+ \to 0^-$, the arguments for the vanishing of the conformal anomaly apply.

[13]The variation of the dilaton is not being included here and will be analyzed in the subsection below. Let us note for now that in the family, Eq. (2.13), at $X^+=0$, one has $e^\Phi \sim g_s(\epsilon)^{\sqrt{8}}$. So for small g_s, the string coupling is nonvanishing but small.

[14]A more precise statement will be made below.

terms carries a positive momentum in the X^- direction (momentum in the X^- direction is the value of k_-, so this means that each factor of $a^\dagger$ comes with a factor $e^{ik_-X^-}$ where $k_- > 0$). Since J is independent of X^- the integral over dX^- (which has range $[-\infty, \infty]$) leads to a delta function which means the sum of the three momenta along the k_- directions coming from each of the $a^\dagger$ terms must vanish. Since each of these terms has a positive k_- momentum this constraint cannot be met and thus the first order term Eq. (4.20) vanishes.

In fact it is easy to see that this argument generalises to all orders in perturbation theory and applies even beyond perturbation theory.

To understand what is happening physically let us first consider an analogous situation where the perturbation is not time dependent but depends on a spatial coordinate. In this case, if we start with the vacuum state we cannot produce any particles and the state must remain in the vacuum. Producing particles means adding positive energy, but time translational symmetry prevents that from happening. Here, we have a similar situation except since we are dealing with a null perturbation it is a little less obvious. It is in fact the momentum along the X^- direction that plays the role of the energy. Starting with the vacuum and adding particles means adding positive momentum along the X^- direction. But since the source term $J(X^+)$ does not break translational invariance along X^- this is not allowed and thus the vacuum stays the vacuum.

The summary of the above analysis is that a source term which depends on a null direction does not lead to particle production, and is more analogous, as far as the question of particle production is concerned, to a source which is spatially varying.[15]

Two comments are now in order.

First, there can be other situations of course where there is particle production. Consider, for example, a free scalar field theory whose kinetic term is

$$S = \int d^4x[F(\partial\varphi)^2]. \tag{4.21}$$

If F is a function of time, rather than X^+, there would be particle production in this model in the sense that the out vacuum is related to the in vacuum by a nontrivial Bogoliubov transformation. In such a situation the out vacuum would be a squeezed state of in particles. However if F is a function of X^+, arguments identical to those given in the previous paragraphs ensure that there is no particle production. The out vacuum, whatever it might be, must contain a superposition of arbitrary number of *pairs* of in particles with a vanishing net k_-, and this cannot happen since there are no particles with negative k_-.

Second, the above argument could have subtleties for the $k_- = 0$ modes. We have not examined this issue very carefully: such complications are of course well-known in light-cone quantisation (for example, see Ref. [78,79]). Ultimately we use the fact of no particle production to make some statements about correlators. As long as the correlators are not at zero k_- momentum, we do not expect to be very sensitive to this subtlety.

The source, $J(X^+)$, does have effects of course; it affects correlation functions in the ground state. We turn to examining these next.

Correlation functions.—The field φ in the conformal theory, Eq. (4.16), has conformal dimension 1. We consider the dressed Feynman two-point function in the presence of the source $J(X^+)$. This is given by

$$G_F(x_1, x_2) = \langle 0|T_+ e^{(f(x_1))/2}\varphi(x_1)e^{(f(x_2))/2}\varphi(x_2) \times e^{-i\int d^4x\sqrt{-g}J(X^+)\varphi^3(X)}|0\rangle, \tag{4.22}$$

where the time ordering refers to X^+ ordering. Now we see that when X^+ lies in between x_1^+ and x_2^+ then terms proportional to the source term $J(X^+)$ will not vanish. And thus the Feynman correlation function will depend on $J(X^+)$.

Some comments are now worth making. First, in the example Eq. (4.22), it follows from our discussion in the previous section that if $e^{(f(X^+))/2}J(X^+)$ vanishes as $X^+ \to 0$ then correlation functions where both x_1^+, x_2^+ are close to the origin will to good approximation be the same as in the case when $J(X^+)$ vanishes identically. To see this, we write the interaction Hamiltonian in Eq. (4.22), as,

$$\int d^4X\sqrt{-g}J(X^+)\varphi^3(X) = \int d^4Xe^{f/2}J(X^+)e^{3f/2}\varphi^3(X). \tag{4.23}$$

Now, we know that the dressed operator $e^{3f/2}\varphi^3(x)$ has nonsingular correlation functions, in the background Eq. (4.17) (since the conformal dimension of $\varphi^3(x)$ is 3). Thus if $e^{(f(X^+))/2}J(X^+)$ vanishes, as $X^+ \to 0$, the interaction Hamiltonian becomes negligible.

Second, in perturbation theory in $J(X^+)$, the correlation functions with the source term can be related to those in the theory without the source term. And the latter, we have seen in the previous section can be related after appropriate dressing to correlation functions of the CFT in flat space. For example, to first order we see that

[15]Of course, there is no invariant notion of a particle in general—it is observer and vacuum dependent. This also means that particle production is observer dependent. In our analysis above the vacuum is the vacuum of the free noninteracting theory and is kept unchanged. And we then find that in the interaction picture this state will not change during time development, if the source term is null dependent. This is the more qualified sense in which there is no particle production due to a null dependent source.

$$G_F(x_1, x_2) = -i\langle 0|T_+ e^{(f(x_1^+))/2}\varphi(x_1)e^{(f(x_2^+))/2}\varphi(x_2)\int d^4X\sqrt{-g}J(X^+)\varphi(X)^3|0\rangle$$
$$= -i\int d^4XJ(X^+)e^{(f(X^+))/2}\langle 0|T_+ e^{(f(x_1^+))/2}\varphi(x_1)e^{(f(x_2^+))/2}\varphi(x_2)e^{(3f(X^+))/2}\varphi^3(X)|0\rangle, \quad (4.24)$$

where in the second equation on the right-hand side we have dressed all the operators by powers of e^f proportional to their conformal dimensions. Thus, in perturbation theory possible singular behavior could arise as $X^+ \to 0$ only if $J(X^+)e^{f(X^+)/2}$ diverges as $X^+ \to 0$. The dressing factor, $e^{f/2}$, for the source can be understood easily as it is simply determined by the conformal dimension of the operator φ^3. More generally for a source $J(X^+)$ coupling to an operator of conformal dimension Δ the dressing factor would be $e^{((4-\Delta)/2)f}$. As long as $Je^{((4-\Delta)/2)f}$ is well behaved as $X^+ \to 0$, no singular behavior will arise.

In the SYM theory which we turn to next, the source term is the dilaton which couples to a dimension four operator. Since e^Φ vanishes at the singularity no singular behavior is expected in the correlation functions. The argument above was in perturbation theory. In the SYM theory perturbation theory is not a good approximation when the 't Hooft coupling, $e^\Phi N \geq 1$. However any singular behavior is expected to arise only at the singularity. Since e^Φ vanishes there, Eq. (2.6), we expect that the conclusion that the correlation functions are nonsingular should be more generally true.

We turn to a more detailed discussion of the SYM theory next.

2. The SYM theory in the presence of a varying dilaton

We now show that theories in such null backgrounds admit descriptions in terms of new tilde variables. Before discussing the gauge theory, we first discuss the basic point in the context of a scalar φ with kinetic term containing a nontrivial null-dependent factor $e^{-\Phi(X^+)}$.

$$S = -\int d^4xe^{-\Phi(X^+)}(\partial\varphi)^2. \quad (4.25)$$

If the function $e^{\Phi(X^+)}$ vanishes at some point, say $X^+ = 0$, the propagator for φ would vanish there as well and it might appear that the theory is singular. However we can define new variables $\varphi = \epsilon(x)\tilde{\varphi}$, so that the action becomes

$$S = -\int d^4xe^{-\Phi}\eta^{\mu\nu}(\epsilon^2\partial_\mu\tilde{\varphi}\partial_\nu\tilde{\varphi} + \epsilon\partial_\mu\epsilon\partial_\nu(\tilde{\varphi}^2) + (\partial_\mu\epsilon\partial_\nu\epsilon)\tilde{\varphi}^2). \quad (4.26)$$

Choosing $\epsilon(x) = e^{\Phi(X^+)/2}$, we see that the third term, akin to a "mass-term," vanishes. Also given the null dependence of $\epsilon(x)$, the second (cross-)term becomes a total derivative $\partial_+(\epsilon^2)\partial_-(\tilde{\varphi}^2) = \partial_-[\partial_+(\epsilon^2)\tilde{\varphi}^2]$, which can be dropped. Thus we see that such a theory with a null-dependent kinetic term can be described in terms of new variables $\tilde{\varphi}$ which have canonical kinetic terms. The essential point is that the Fock space of this theory is defined in terms of creation and annihilation operators coming from the usual mode expansion of $\tilde{\varphi}$.

If we had started instead with an interacting theory with action

$$S = -\int d^4xe^{-\Phi(X^+)}[(\partial\varphi)^2 - \lambda\varphi^4], \quad (4.27)$$

then after changing to $\tilde{\varphi}$ variables and dropping a surface term, the resulting action is

$$S = -\int d^4x[(\partial\tilde{\varphi})^2 - \lambda e^{\Phi(X^+)}\tilde{\varphi}^4]. \quad (4.28)$$

We see that the $\tilde{\varphi}^4$ coupling is X^+ dependent. For a dilaton, e^Φ, as in Eq. (2.6), which is bounded and vanishing at $X^+ \to 0$, we see that the interaction term is also bounded and vanishing at $X^+ = 0$. It is clear that this theory has a nonsingular S-matrix in perturbation theory and is therefore well defined. This is transparent in terms of the $\tilde{\varphi}$ variables which, as we noted, are the ones relevant for defining asymptotic states.

In what follows, we will similarly find new nonsingular variables in our case of the SYM theory with null-varying dilaton.

The gauge coupling in the SYM theory is given by $g_{\rm YM}^2 = e^\Phi$. Thus a varying dilaton results in a varying gauge coupling. In more detail the dilaton couples to the terms in the Lagrangian which are quadratic in the gauge field strength:

$$S_{\rm GF} = -\frac{1}{4}\int d^4x\frac{1}{e^\Phi}{\rm Tr}[F_{\mu\nu}F^{\mu\nu}]. \quad (4.29)$$

For now we will work with a flat metric. The effects of the nontrivial conformal factor $f(X^+)$ will be included in the discussion a little later. Since the e^Φ vanishes at the singularity, $X^+ = 0$, the quadratic terms for the gauge field A_μ become singular there. It is therefore useful to carry out a field redefinition to new variables which have well behaved quadratic terms. For simplicity we work in the gauge[16]

[16]The condition, $A_- = 0$, leaves a residual freedom to do X^- independent gauge transformations. This only affects modes with momentum $k_- = 0$. As was mentioned before, there are well-known subtleties associated with quantizing the $k_- = 0$ modes in light-cone gauge quantization [78,79], and we mainly consider modes with $k_- \neq 0$ in our analysis. For such modes the condition, $A_- = 0$ fixes the gauge completely.

$$A_- = 0. \tag{4.30}$$

Then define

$$\tilde{A}_i = e^{-\Phi/2} A_i, \qquad \tilde{A}_+ = e^{-\Phi/2} A_+, \tag{4.31}$$

where the index i in the first equation above takes values in the directions transverse to X^+, X^-. This gives

$$\begin{aligned} S_{\rm GF} = -\frac{1}{4}\int d^4x[&\mathrm{Tr}(\partial_\mu \tilde{A}_\nu - \partial_\nu \tilde{A}_\mu)^2 \\ &- 2ie^{\Phi/2}\,\mathrm{Tr}\{(\partial_\mu \tilde{A}_\nu - \partial_\nu \tilde{A}_\mu)[\tilde{A}^\mu, \tilde{A}^\nu]\} \\ &- e^\Phi\,\mathrm{Tr}([\tilde{A}_\mu, \tilde{A}_\nu])^2 - \partial_{X^-}\{(\partial_+\Phi)\tilde{A}_i\tilde{A}_i\}]. \end{aligned} \tag{4.32}$$

In the last term the index i takes two values transverse to both X^+, X^-. We see that the last term is a total derivative in X^-; it will not affect the equations of motion and can be dropped.[17] The dilaton also appears in other couplings in the Lagrangian when we work in terms of the $\tilde{A}$ fields. These are

$$\begin{aligned} S = \int d^4x[&e^{\Phi/2}J^{\mu a}\tilde{A}_{\mu a} + e^\Phi\,\mathrm{Tr}([\tilde{A}_\mu, \phi^\alpha][\tilde{A}^\mu, \phi^\alpha]) \\ &+ e^\Phi\,\mathrm{Tr}([\phi^\alpha, \phi^\beta][\phi^\alpha, \phi^\beta])], \end{aligned} \tag{4.33}$$

where $J^{\mu a}$ is the $SU(N)$ gauge current arising from the scalars and the fermions, and ϕ^α, $\alpha = 1\cdots 6$ are the six scalars.

Before proceeding let us make two comments. First, in the gauge $A_- = 0$, the variable, A_+ and therefore also $\tilde{A}_+$ is nondynamical. The equation of motion for A_- needs to be imposed as a constraint, and this determines A_+ in terms of the transverse components A_i. Similarly, $\tilde{A}_+$ can be determined in terms of $\tilde{A}_i$. The action in Eq. (4.32) contains both $\tilde{A}_+$ and $\tilde{A}_i$. The correct equations of motion for $\tilde{A}_i$ can be obtained from it by treating all of them as independent variables, deriving the equations of motion for $\tilde{A}_i$ and then substituting for $\tilde{A}_+$ in terms of $\tilde{A}_i$ in them. Alternatively, the same equations of motion for $\tilde{A}_i$ can also be obtained by first substituting in the action, Eq. (4.32), for $\tilde{A}_+$ in terms of $\tilde{A}_i$ and then varying with respect to $\tilde{A}_i$. Second, the $\tilde{A}$ variables can be also be defined in terms of the gauge potential, A_μ, in a general gauge, as

$$\tilde{A}_\mu = e^{-\Phi}(A_\mu + \partial_\mu \chi). \tag{4.34}$$

Where χ is given by

$$\chi = -\partial_-^{-1} A_-. \tag{4.35}$$

Note that Eq. (4.35) uniquely defines χ as long as the momentum component, k_- is nonvanishing (subtleties related to the $k_- = 0$ mode were discussed in footnote 16).

We see that in the terms remaining in Eq. (4.32) after dropping the total derivative in X^-, and in all the terms in Eq. (4.33), e^Φ couples with positive powers. Thus these couplings all vanish when e^Φ vanishes.[18] In fact these couplings are similar to the source terms considered in the discussion above of the conformally coupled scalar field. Therefore the conclusions we reached in the conformally coupled case can also be applied to the dilaton coupling in the SYM theory. First, the lightlike variation of the dilaton will not give rise to any particle production and will leave the $\mathcal{N} = 4$ vacuum unchanged. Second, it will have effects on the correlation functions. So far, we have not included the nontrivial conformal factor in the metric. Once this is included, it is important to work with appropriately dressed operators, as discussed in the previous section. Since the dilaton is a source coupling to dimension four operators, we need to examine its behavior without any dressing factor of e^f. And since e^Φ, Eq. (2.6), vanishes at the singularity $X^+ = 0$, we see that no singularities will arise in the dressed correlation functions.[19] For instance, the scalar kinetic and quartic interaction terms have factors of e^f and e^{2f} respectively, so that redefining the scalars to have canonical kinetic terms removes the e^f dressing factors from both these dimension four operators.

It is worth emphasizing that our arguments go through for the dressed correlation functions constructed out of the $\tilde{A}_\mu$ fields and their field strengths. As was mentioned above it is these fields which have well-defined quadratic terms at the singularity, where e^Φ vanishes. Other fields which are related by singular field redefinitions to $\tilde{A}$ will not have nonsingular correlators at the singularity in general. For example, the field redefinition used to go from the $\tilde{A}$ variables to the A variables is singular when e^Φ vanishes, Eq. (4.31). If we had used the field strength made out of the original A_μ variables, Eq. (4.29), then the correlation functions would not be finite near the singularity. Keeping only the quadratic terms in the field strength and using the relation, Eq. (4.31), we have that

[17]The null-dependence of the dilaton was crucial for this to happen. If the dilaton were time-dependent instead, there would be additional terms in Eq. (4.32), after the redefinition, Eq. (4.31), which are not total derivatives.

[18]In the example, Eq. (2.6), e^Φ is not an analytic function of X^+ at the singularity and sufficiently high order derivatives with respect to X^+ diverge. However the couplings in the YM theory only involve positive powers of the dilaton and not its derivatives. Thus these couplings all vanish at the singularity, and the diverging higher X^+-derivatives of the dilaton do not lead to any singular behavior of the correlation functions.

[19]It is important to note that the varying dilaton does not change the conformal dimensions of operators, assuming that any such contribution to the conformal dimension must come from a term in the effective action. Since the effective action is a local coordinate invariant function of the dilaton, the metric and the tensors made from the metric, any additional terms vanish since the background is null. A similar argument was discussed in the case of the trace anomaly at the end of Sec. III.

$$e^{-\Phi}\,\mathrm{Tr}F^2 = \mathrm{Tr}\tilde{F}^2 - \tfrac{1}{2}\partial_- G_+, \tag{4.36}$$

where $G_+ \equiv (\partial_+\Phi)\tilde{A}_i\tilde{A}^i$. This means

$$\begin{aligned}\langle e^{-\Phi}\,\mathrm{Tr}F^2(x)e^{-\Phi}\,\mathrm{Tr}F^2(y)\rangle &= \langle \mathrm{Tr}\tilde{F}^2(x)\,\mathrm{Tr}\tilde{F}^2(y)\rangle \\ &\quad - \tfrac{1}{2}\partial_{x^-}\langle G_+(x)\,\mathrm{Tr}\tilde{F}^2(y)\rangle \\ &\quad - \tfrac{1}{2}\partial_{y^-}\langle G_+(y)\,\mathrm{Tr}\tilde{F}^2(x)\rangle \\ &\quad + \tfrac{1}{4}\partial_{y^-}\partial_{x^-}\langle G_+(y)G_+(x)\rangle.\end{aligned} \tag{4.37}$$

Now for the solution, Eq. (2.6), close to the singularity, $\Phi \sim \sqrt{8}\log X^+$. Thus $\partial_+\Phi \sim 1/X^+$, and diverges near the singularity. This means if one or both points in the correlator approach the singularity the G_+ dependent terms in Eq. (4.37) blows up. Thus the correlation function of $e^{-\Phi}F^2$ diverges at the singularity.[20] This observation will be relevant for the discussion in the next section where we compare the gauge theory results with those in supergravity. The dilaton in the bulk couples to the operator, $e^{-\Phi}F^2$, and thus the bulk two-point function for the dilaton should be compared with the two-point correlator of this operator in the SYM theory.

Before proceeding let us also note that the two-point function, $\langle \mathrm{Tr}F^2(x)\,\mathrm{Tr}F^2(y)\rangle$, and all higher point functions of $\mathrm{Tr}F^2$, vanish at the singularity for the example, Eq. (2.6), since e^{Φ} vanishes more rapidly at the singularity than $\partial_- G_+$. More generally, depending on the behavior of e^{Φ}, these correlators could diverge at the singularity.

It would be worthwhile to try and identify supergravity duals to the operators made out of the tilde variables. Some operators made out of $\tilde{A}$ fields can be easily related to local operators made out of the field strength, $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]$. For example, let us define

$$\tilde{F}_{\mu\nu} = \partial_\mu\tilde{A}_\nu - \partial_\nu\tilde{A}_\mu - ie^{\Phi/2}[\tilde{A}_\mu, \tilde{A}_\nu]. \tag{4.38}$$

Then as long as μ, $\nu \neq X^+$, we have a local relation,

$$\tilde{F}_{\mu\nu} = e^{-\Phi/2}F_{\mu\nu}. \tag{4.39}$$

Some gauge invariant operators can be built from $F_{\mu\nu}$, μ, $\nu \neq X^+$, by considering quadratic and higher powers and taking a trace over color space. One example is $\mathrm{Tr}(F_{\mu\nu}F_{\rho\sigma})$. Such operators are dual to supergravity modes which are local excitations in the bulk. Using Eq. (4.39) these can be described in terms of the tilde variables. So for these operators, made out of the tilde variables, there is a dual description in terms of local excitations in the bulk.

[20] We have not explicitly put in any dressing factors proportional to e^f in the correlation function, Eq. (4.36). We are actually interested in the operator $\sqrt{-g}F_{\mu\nu}F_{\rho\sigma}g^{\mu\rho}g^{\nu\sigma}$. For the conformally flat metric, Eq. (2.6) factors of e^f drop out. In the expression, F^2, in Eq. (4.36), the indices are contracted using the flat space metric; similarly for $\tilde{F}^2$.

But there are also operators, of a second type, made out of the tilde variables, which are not local functions of the field strength $F_{\mu\nu}$. The supergravity duals of these are not local excitations in the bulk, since the supergravity modes which are local excitations in the bulk, are dual to gauge invariant operators made out of the field strength $F_{\mu\nu}$ and its powers. Example of such operators are $\tilde{A}_\mu$, or $(\partial_\mu\tilde{A}_\nu - \partial_\nu\tilde{A}_\mu)$. Since the tilde variables are defined in the $A_- = 0$ gauge, Eq. (4.31), these are by definition gauge invariant. But since the components of the gauge potential, A_μ cannot be expressed locally in terms of the field strength, neither can the fields $\tilde{A}_\mu$. As another example, take $\tilde{F}_{\mu\nu}$, when either index, μ or ν is X^+. We get

$$\tilde{F}_{\mu\nu} = e^{-\Phi/2}F_{\mu\nu} + \tfrac{1}{2}e^{-\Phi/2}(A_\mu\partial_\nu\Phi - A_\nu\partial_\mu\Phi). \tag{4.40}$$

We see that the extra terms appearing on the right-hand side involve derivatives of Φ, and the gauge potential A. A complete set of operators made out of the tilde variables must include operators of this second type whose supergravity duals are not local excitations in the bulk.

It might be useful here to recall that a familiar example of an operator which is gauge invariant but which is not local in terms of the field strength is the Wilson loop, $\mathrm{Tr}Pe^{i\int A}$. We know this is dual to a string in the bulk and this is not a local excitation. In some very rough sense, the duals of these second type of operators should be similar.

In summary, in the last three sections we have analyzed the dual gauge theory in some detail. Our conclusion is that nonsingular field variables (the $\tilde{A}$ variables) can be found such that correlation functions of these fields, when suitably dressed by appropriate powers of the conformal factor, are nonsingular. This leads to the conclusion that the gauge theory dual is nonsingular.

V. THE BULK TWO-POINT FUNCTION

So far our discussion has been mainly in the gauge theory. In this section we calculate the two-point function of a scalar in the bulk. Some comparisons with the gauge theory are discussed at the end of the section.

It is convenient to work in coordinates in which the 5-dimensional space-time transverse to the S^5 in Eq. (2.1a) has metric

$$ds^2 = \frac{1}{z^2}\left[e^f\left(-2dX^+dX^- + \sum_{i=1,2}(dx^i)^2\right) + dz^2\right]. \tag{5.1}$$

The 5 dimensional space-time, Eq. (5.1) has a boundary at $z = \epsilon$. This serves as an infrared regulator of the bulk theory. The dual gauge theory lives on the boundary as in the standard AdS/CFT correspondence.

A scalar of mass m has the action

$$S = -\int d^5x\sqrt{-g}(g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi + m^2\varphi^2), \tag{5.2}$$

An orthonormal complete set of modes solving the resulting wave equation is given by

$$u_{(k_i,k_-,\omega^2)}(z, x^\mu) = e^{-f(X^+)/2} e^{i(k_i^2 X^+ - \omega^2 \int e^f dX^+)/2k_-} e^{ik_- X^- + ik_i x^i} \zeta_\omega, \quad (5.3)$$

where

$$\zeta_\omega = A\omega^2 z^2 K_\nu(\omega z) + B\omega^2 z^2 I_\nu(\omega z), \qquad \omega^2 > 0,$$
$$\zeta_\omega = A\omega^2 z^2 H_\nu(i\omega z) + B\omega^2 z^2 H_\nu(-i\omega z), \qquad \omega^2 < 0. \quad (5.4)$$

Here $\omega \equiv \sqrt{\omega^2}$, $\nu = \sqrt{4 + m^2}$, and A, B are integration constants.

For the $AdS_5 \times S^5$ background, if we are interested in calculating the bulk two-point function dual to the Feynman propagator in the $\mathcal{N} = 4$ vacuum of the gauge theory, then, for $\omega^2 > 0$, the correct combination of the normalisable and non-normalisable modes is obtained by setting $B = 0$ in the first equation in Eq. (5.4), and only keeping the $K_\nu(\omega z)$ solution

$$\zeta_\omega = (\omega z)^2 K_\nu(\omega z). \quad (5.5)$$

For $\omega^2 < 0$ the correct combination is obtained by analytically continuing this solution

$$\zeta_\omega = A\omega^2 z^2 H_\nu(i\omega z), \quad (5.6)$$

with $|A| = \frac{\pi}{2}$. (Note the analytic continuation is not unique; one could have instead obtained $\zeta = A\omega^2 z^2 H_\nu(-i\omega z)$. This ambiguity does not affect the final answer for the bulk two-point function.)

The choice of these modes is further justified by a standard light front quantization of the field, which is explained in Appendix B.

We are interested here in backgrounds like Eq. (2.6), which become $AdS_5 \times S^5$ asymptotically as $X^+ \to -\infty$. We have argued above that the dual SYM theory starts in the $\mathcal{N} = 4$ vacuum in the far past, as $X^+ \to -\infty$. To calculate the bulk two-point function which is dual to the Feynman propagator of the gauge theory in this case, we once again choose the same combinations of normalizable and non-normalizable modes, as in the $AdS_5 \times S^5$ case. This is clearly correct for correlation functions in the limit, $X^+ \to -\infty$, and since the z dependent part is independent of X^+ it is must then true for all X^+.

A general solution can be expressed in terms of these modes:

$$\varphi(x^\mu, z) = \epsilon^{\Delta_-} \int_{-\infty}^{\infty} d^2k_i \int_{-\infty}^{\infty} dk_- \times \int_{-\infty}^{\infty} \frac{d\omega^2}{2|k_-|} \varphi(k_i, k_-, \omega^2) \frac{u_{(k_i,k_-,\omega^2)}(z, x^\mu)}{\zeta_\omega(\epsilon)}. \quad (5.7)$$

$\varphi(k_i, k_-, \omega^2)$ are Fourier coefficients. $z = \epsilon$ is the boundary of space-time, as discussed after Eq. (5.1). We denote $\Delta_\pm = 2 \pm \nu$.

The action evaluated on this classical solution becomes

$$S = \int d^2k_i dk_- dk_+ C(\nu)\varphi(k_i, k_-, \omega^2)\varphi(-k_i, -k_-, \omega^2)\omega^{2\nu}, \quad (5.8)$$

where the integrals over all four variables, k_i, $i = 1, 2$, k_-, k_+ go from $[-\infty, \infty]$, and $\omega^2 = -2k_-k_+ + k_i^2$. An assumption made in deriving Eq. (B7) is that the affine parameter, λ of Eq. (2.7), has range $\lambda \in [-\infty, \infty]$. This will be true if the conformal factor obeys $e^f \geq 0$ for $X^+ \in [-\infty, \infty]$, since $d\lambda/dX^+ = e^f \geq 0$. In position space, Eq. (5.8) evaluates to

$$S = C \int d^4x d^4x' e^{3f(X^+)/2} e^{3f(X'^+)/2} \varphi(\vec{x})\varphi(\vec{x}')\left(\frac{\Delta\lambda}{\Delta X^+}\right)^{1-\Delta} \times \frac{1}{[(\Delta\vec{x})^2]^\Delta}. \quad (5.9)$$

Here, C is a constant, and

$$\Delta = 2 + \nu. \quad (5.10)$$

Let us relate this to a correlation function in the boundary theory. The mode φ couples to an operator O in the boundary theory with coupling

$$S_{\text{Boundary}} = \int d^4x \sqrt{-\tilde{g}} O(x)\varphi(x), \quad (5.11)$$

Here, $\tilde{g}_{\mu\nu} = e^f \eta_{\mu\nu}$, is the metric on the boundary and $\varphi(x)$ is given by Eq. (B9).

Equating the bulk action with the action of the boundary theory up to second order in the source $\varphi(x)$ gives

$$\sqrt{-\tilde{g}(x)}\sqrt{-\tilde{g}(x')}\langle O(x)O(x')\rangle = \frac{\delta}{\delta\varphi(\vec{x})}\frac{\delta}{\delta\varphi(\vec{x}')} \times \langle e^{\int d^4x\sqrt{-\tilde{g}}O(x)\varphi(x)}\rangle_{\text{CFT}} = \frac{\delta}{\delta\varphi(\vec{x})}\frac{\delta}{\delta\varphi(\vec{x}')} e^{-S_{\text{Sugra}}[\varphi(\vec{x})]}. \quad (5.12)$$

From, Eq. (5.9), we then get

$$\langle O(x)O(x')\rangle = Ce^{-f(x)/2}e^{-f(x')/2}\left(\frac{\Delta\lambda}{\Delta X^+}\right)^{1-\Delta}\frac{1}{[(\Delta\vec{x})^2]^\Delta}. \quad (5.13)$$

In the discussion on correlators of the gauge theory in Sec. IV we emphasized the importance of considering dressed correlators. The operator $O(x)$ has conformal dimension Δ in the SYM theory. Then the dressed correlation function is

$$\langle e^{(f(x)\Delta)/2}O(x)e^{(f(x')\Delta)/2}O(x')\rangle = Ce^{(f(x)(\Delta-1))/2}e^{(f(x')(\Delta-1))/2}\left(\frac{\Delta\lambda}{\Delta X^+}\right)^{1-\Delta}\frac{1}{[(\Delta\vec{x})^2]^\Delta}. \tag{5.14}$$

The behavior of this correlator close to the singularity is worth analyzing. Using the definition of the affine parameter λ, Eq. (2.7), we see that when the two points are close to each other

$$\langle e^{(f(x)\Delta)/2}O(x)e^{(f(x')\Delta)/2}O(x')\rangle = C\frac{1}{[(\Delta\vec{x})^2]^\Delta}, \tag{5.15}$$

which is the two-point function in the $\mathcal{N} = 4$ SYM theory without any sources. This is true when the two points are close to each other in general and, in particular, when they are also close to the singularity. When one of the points, $\vec{x}$, is at the singularity, $X^+ = 0$, we see that the dressing factor e^f leads to the correlation function vanishing. This is true for all values of $(X')^+ \neq 0$. In the limit when both points approach the singularity, X^+, $(X')^+ \to 0$, the correlation function depends on how the limit is taken. For instance, for our prototypical example, Eq. (2.6), we get, upon using $\lambda \sim (X^+)^3$ as $X^+ \to 0$,

$$\langle e^{(f(x)\Delta)/2}O(x)e^{(f(x')\Delta)/2}O(x')\rangle = C\left(\frac{X^+}{(X')^+}\right)^{\Delta-1}\left[\left(\frac{X^+}{(X')^+}\right)^2 + \frac{X^+}{(X')^+} + 1\right]^{1-\Delta}\frac{1}{[(\Delta\vec{x})^2]^\Delta}. \tag{5.16}$$

The correlation function scales with distance $(\Delta\vec{x})^2 = (\Delta x_i)^2$ as in the $\mathcal{N} = 4$ theory, but the coefficient depends on the limiting value of the ratio, $\frac{X^+}{(X')^+}$.

We argued in Sec. IV B 2 that close to the singularity the two-point function in the gauge theory reduces to that in free SYM theory, since the dilaton vanishes. We now see that the correlation function calculated from the bulk does not have this property. The fall off with distance, $\frac{1}{[(\Delta\vec{x})^2]^\Delta}$, is as in the free theory,[21] but as seen above, as the two points approach the singularity, the value of the correlation function depends on the how the limit is approached and is not unique.

This disagreement between the supergravity calculation and the gauge theory does not disprove that the descriptions are dual. The bulk calculation fails close to the singularity where the higher derivative corrections become important. Once these are incorporated presumably the bulk answer will agree with the gauge theory.

Finally, the discussion above, in particular, applies to the dilaton, which is a massless scalar in the five-dimensional theory. The operator it couples to in the gauge theory is $O = e^{-\Phi}\mathrm{Tr}F^2$. We discussed in section IV B 2 near Eq. (4.36) that from the gauge theory point of view the two-point function of this operator should be singular when one or both points approach the singularity. This is very different from the bulk result, which shows a result that is finite but limit dependent. Once again presumably higher derivative corrections are responsible for this disagreement. The gauge theory analysis also tells us that good variables in the gauge theory are the $\tilde{A}$ variables, Eq. (4.31), and gauge invariant field strengths constructed out of these variables. It would be interesting to try and carry out a similar analysis for bulk modes dual to these variables.

VI. WORLDSHEET ACTIONS

In this section we consider the bosonic part of the worldsheet action of a string in the class of backgrounds given in (1.1) and (1.2). For a given string frame metric $g_{\mu\nu}(X)$ and dilaton $\Phi(X)$ the covariant Polyakov Lagrangian density is given by

$$\mathcal{L}_{\text{pol}} = -\tfrac{1}{2}\sqrt{-h}h^{ab}\partial_a X^\mu\partial_b X^\nu g_{\mu\nu}(X) + \sqrt{-h}R\Phi(X), \tag{6.1}$$

where h_{ab} is the worldsheet metric with signature $(-1, 1)$. The string frame metric which follows from Eqs. (1.1) and (1.2) may be rewritten as

$$ds^2 = e^{\Phi/2}\left[\frac{e^{f(x^+)}}{Y^2}[2dX^+dX^- + d\vec{X}^2] + \frac{1}{Y^2}d\vec{Y}^2\right], \tag{6.2}$$

where $\vec{X} = X^1$, X^2, and $\vec{Y} = Y^1\cdots Y^6$, and $Y = |\vec{Y}|$ is related to r of Eq. (1.1) by $Y = R^2/r$.

We will fix a light-cone gauge following [80–82]

$$h_{01} = 0, \qquad X^+ = \tau. \tag{6.3}$$

Let us first ignore the term containing the dilaton. Then the gauge fixed action becomes

$$\mathcal{L}_{\text{pol}} = -\frac{1}{2}\Big[E(\tau,\sigma)g_{ij}\partial_\tau X^i\partial_\tau X^j - \frac{1}{E(\tau,\sigma)}g_{ij}\partial_\sigma X^i\partial_\sigma X^j + 2E(\tau,\sigma)g_{+-}\partial_\tau X^-\Big], \tag{6.4}$$

where we have defined $X^{3\ldots8} = Y^{1\ldots6}$, and

$$E(\sigma,\tau) = \sqrt{-\frac{h_{\sigma\sigma}(\tau,\sigma)}{h_{\tau\tau}(\tau,\sigma)}}. \tag{6.5}$$

The momentum density conjugate to X^- is

$$\mathcal{P}_- = E(\sigma,\tau)g_{+-}(\tau), \tag{6.6}$$

and this is independent of τ by the equations of motion. Since the gauge choice (6.3) still leaves reparametrizations

[21] Bulk modes correspond to operators whose conformal dimensions are unrenormalized and thus remain the same in the free limit.

which are independent of τ, we can choose a σ such that

$$E(\sigma, \tau)g_{+-}(\tau) = 1. \qquad (6.7)$$

and use this to write E in terms of $g_{+-} = e^{\Phi/2+f}/Y^2$. The final form of the action is then

$$S = \frac{1}{2}\int d\sigma d\tau\bigg[(\partial_\tau \vec{X})^2 + e^{-f(\tau)}(\partial_\tau \vec{Y})^2 - \frac{1}{Y^4}e^{2f(\tau)}e^{\Phi(\tau)}(\partial_\sigma \vec{X})^2 - \frac{1}{Y^4}e^{f(\tau)}e^{\Phi(\tau)}(\partial_\sigma \vec{Y})^2\bigg], \qquad (6.8)$$

where we have to remember that $\tau = X^+$. This expression indicates that stringy modes become important when $e^{f(\tau)}$ and/or $e^{\Phi(\tau)}$ become small. Consider, for example, the energy for a piece of a string along the X^1, X^2 directions, at some constant value of $\vec{Y} = \vec{Y}_0$. This corresponds to a classical solution, e.g. $X^1 = \lambda\sigma$. The typical energy, in string units, of this piece at null time $X^+ = \tau = \tau_0$ is $E \sim \frac{1}{Y_0^4}e^{2f(\tau_0)}e^{\Phi(\tau_0)}$ which becomes small when the overall factor is small. The factor of $\frac{1}{Y_0^4}$ is simply the redshift factor of the underlying geometry.

In the example Eq. (2.6) this happens at the singularity $X^+ = 0$. Our analysis then suggests that near the singularity stringy effects become important, in agreement with the conclusions of the previous section.

In the above analysis we have not considered the effect of the dilaton coupling to the worldsheet curvature, the second term in Eq. (6.1). However in the gauge we have chosen the intrinsic worldsheet quantities are functions of τ and the dilaton is a function of τ alone. Therefore this term does not affect the dynamics of the transverse fields $\vec{X}$, $\vec{Y}$ and modify the above conclusion, though the *value* of the energy will be affected.

While it is reasonably clear that stringy effects become important near the singularity, we have no definitive conclusion about whether perturbative string theory is well defined in this background. This would require a much more detailed study incorporating the RR background in the fermionic part of the action and a calculation of correlation functions of vertex operators. We defer this for future work. However since the string coupling is small and stringy effects are large near the singularity there is a distinct possibility that perturbative string theory is well defined, in which case the singularity is really resolved by α' corrections.

VII. PENROSE LIMITS AND MATRIX THEORY

In this section we perform the Penrose limit of our class of solutions, Eqs. (1.1) and (1.2). The main motivation behind this is to write down a matrix theory type action which describes the DLCQ quantization in the resulting pp-wave background.

A. Penrose limit

For this purpose it is convenient to rewrite Eq. (1.1) as

$$ds^2 = r^2[-dt^2 + dq^2 + e^{F(z^+)}(dx_2^2 + dx_3^2)] + \frac{dr^2}{r^2} + d\psi^2 + \sin^2\psi d\Omega_4^2, \qquad (7.1)$$

where we have used the affine parameter z^+ along a null geodesic instead of the original coordinate $X^+ = \frac{1}{\sqrt{2}}(t + q)$

$$z^+ = \int^{x^+} dx e^{f(x)}, \qquad (7.2)$$

and defined a function $F(z^+)$ by

$$F(z^+) = f(X^+(z^+)). \qquad (7.3)$$

The coordinates q, t are defined by

$$z^+ = \frac{1}{\sqrt{2}}(q + t), \qquad X^- = \frac{1}{\sqrt{2}}(t - q). \qquad (7.4)$$

In terms of these coordinates the Eq. (2.4) determining the dilaton becomes

$$\frac{1}{2}\left(\frac{d\Phi}{dz^+}\right)^2 + \frac{d^2F}{dz^{+2}} + \frac{1}{2}\left(\frac{dF}{dz^+}\right)^2 = 0. \qquad (7.5)$$

Now make the following coordinate transformation

$$r = \sin u, \qquad t = -\cot u - \frac{v}{R^2} + \frac{\phi}{R}, \qquad \psi = \frac{\phi}{R} + u, \qquad (7.6)$$

as well as rescale

$$x^2, x^3 \to \frac{x^2, x^3}{R}, \qquad q \to \frac{q}{R}, \qquad \Omega_4^2 \to \frac{\Omega_4^2}{R^2}. \qquad (7.7)$$

Finally we perform the limit

$$R \to \infty, \quad \text{and} \quad u, v, X^i, \Omega_4 = \text{fixed}. \qquad (7.8)$$

In this limit the S^4 decompactifies, whose coordinates we denote by $\vec{y}$ and we get the pp-wave metric

$$ds^2 = 2dudv + \cos^2 u d\phi^2 + \frac{1}{(G(u))^2}d\vec{x}^2 + \sin^2 u(dq^2 + d\vec{y}^2), \qquad (7.9)$$

where we have defined

$$G(u) = \frac{e^{-(1/2)F(-(1/\sqrt{2})\cot u)}}{\sin u}. \qquad (7.10)$$

This metric may be brought into Brinkmann form by the coordinate transformations

$$u = U,$$
$$v = V - \frac{1}{2}\xi^2 \tan U + \frac{1}{2}(p^2 + \vec{Z}^2)\cot U - \frac{1}{2}\frac{\partial_U G}{G}\vec{X}^2,$$
$$\phi = \frac{1}{\cos U}\xi, \qquad q = \frac{1}{\sin U}p,$$
$$\vec{y} = \frac{1}{\sin U}\vec{Z}, \qquad \vec{x} = \frac{e^{-F/2}}{\sin U}\vec{X}, \tag{7.11}$$

[here $G(U) \equiv G(u(U))$] and the pp-wave metric becomes (in Einstein frame)

$$ds^2 = 2dUdV - [H(U)\vec{X}^2 + \vec{Y}^2](dU)^2 + d\vec{X}^2 + d\vec{Y}^2, \tag{7.12}$$

where we have defined $\vec{Y} = (\vec{Z}, p, \xi)$ and the function $H(U)$ is defined by

$$H = \partial_U\left(\frac{\partial_U G}{G}\right) - \left(\frac{\partial_U G}{G}\right)^2. \tag{7.13}$$

After some algebra this may be written in terms of the original function $F(z^+)$ where $z^+ = -\frac{1}{\sqrt{2}}\cot U$

$$H(U) = 1 - \frac{[1 + 2(z^+)^2]^2}{4}\left[\frac{d^2F}{(dz^+)^2} + \frac{1}{2}\left(\frac{dF}{dz^+}\right)^2\right]$$
$$= 1 + \frac{[1 + 2(z^+)^2]^2}{8}\left(\frac{d\Phi}{dz^+}\right)^2, \tag{7.14}$$

where we have expressed this in terms of the dilaton $\Phi(U(z^+))$. In addition there is a fiveform field strength which becomes, in the Penrose limit

$$F_5 = dU \wedge d\xi \wedge dp \wedge dX^2 \wedge dX^3 + dU \wedge dZ^1 \wedge dZ^2 \wedge dZ^3 \wedge dZ^4. \tag{7.15}$$

It is interesting to examine the nature of the function $H(U)$ for some specific backgrounds considered in the previous sections. Consider, for example, the background given by (2.5). In this case

$$z^+ = X^+ - \tanh X^+. \tag{7.16}$$

To obtain the form of the function $H(U)$ we need to invert X^+ and express this as a function of z^+ and so obtain $F(z^+)$. Finally we have to substitute $z^+ = -\frac{1}{\sqrt{2}}\cot U$ and calculate various quantities. Let us examine the nature of this function near the singularity at $X^+ = 0$. This corresponds to $z^+ = 0$ and near the singularity $z^+ \sim \frac{(X^+)^3}{3}$. Now, the whole range of values of U cover the range of z^+ from $-\infty$ to $+\infty$ multiple number of times. Consider one such domain $0 < U < \pi$ which covers $-\infty < z^+ < \infty$. It is then straightforward to see that near the point $z^+ = 0$ which means $U \to \pi/2$ we have

$$H(U) \sim \frac{1}{(U - \frac{\pi}{2})^2}, \qquad e^{\Phi(U)} \sim (U - \frac{\pi}{2})^{\sqrt{8}/3}. \tag{7.17}$$

Thus the Penrose limit of our original space-time is singular as well.

In fact, it is easy to see by comparing Eq. (7.14) [or, rather, because of the change of coordinates, (2.4)] to Eq. (2.9) that the pp-wave is singular if and only if the pre-Penrose limit original space-time is singular. That the Penrose limit of a singular space-time is also singular was demonstrated in far more generality than this in Ref. [83]. Interestingly the $\frac{1}{U^2}$-type singular profile seen in Eq. (7.17) was shown in Ref. [83] to be quite typical of the Penrose limit of cosmological singularities, and is amenable to perturbative string theory analysis [84].

B. Matrix membrane action

In the Penrose limit there are spacelike isometries, which may be made manifest by choosing a different set of coordinates [85] in which the Einstein frame metric becomes

$$ds^2 = 2dUdV - 4Y^5dY^6dU - [H(U)\vec{X}^2 + (Y^1)^2 + \cdots + (Y^4)^2](dU)^2 + d\vec{X}^2 + d\vec{Y}^2. \tag{7.18}$$

Consider now the above background with both V and Y^6 compact with radii R_V and R_B respectively. The usual construction of the matrix theory dual of the sector of the theory with momentum $P_V = J/R_V$ along V involves

(1) A T duality along Y^6 to obtain a IIA theory;
(2) A lift to M theory by introducing a new direction Y^7;
(3) KK reduction of this M theory along V to yield another IIA theory;
(4) Performing two T-dualities along Y^6 and Y^7 on this IIA theory.

Then, following the usual DLCQ logic, the dual theory is a 2 + 1 dimensional $SU(J)$ Yang-Mills theory living on a torus. the action of this theory is obtained by following the same steps as in e.g. Refs. [37,85]. Here we quote the final form of the action

$$S = \int d\tau \int_0^{2\pi((l_B^2)/R)} d\sigma \int_0^{2\pi((g_B l_B^2)/R)} d\rho \mathcal{L}, \tag{7.19}$$

where l_B, g_B are the string length and string coupling of the original IIB theory. The Lagrangian density is

$$L = \mathrm{Tr}\frac{1}{2}\Big\{[(D_\tau\chi^\alpha)^2 - e^{\Phi(\tau)}(D_\sigma\chi^\alpha)^2 - e^{-\Phi(\tau)}(D_\rho\chi^\alpha)^2] + \frac{1}{G_{\mathrm{YM}}^2}[e^{\Phi(\tau)}F_{\sigma\tau}^2 + e^{-\Phi(\tau)}F_{\rho\tau}^2 - F_{\rho\sigma}^2] - H(\tau)[(\chi^1)^2 + (\chi^2)^2] - (\chi^3)^2 - \cdots - (\chi^6)^2 - 4(\chi^7)^2 + \frac{G_{\mathrm{YM}}^2}{2}[\chi^\alpha, \chi^\beta]^2 + 2iG_{\mathrm{YM}}\chi^7[\chi^5, \chi^6] + \frac{4}{G_{\mathrm{YM}}}\chi^7F_{\sigma\rho}\Big\}, \tag{7.20}$$

where we have defined a new field χ^α, $\alpha = 1\cdots7$ and

$\chi^i = X^i$, $i = 1, 2$ while $\chi^{i+2} = Y^i$, $i = 1 \cdots 5$. Along with the steps outlined above, this Lagrangian follows by taking $\tau = U$ gauge, and so $\Phi(\tau)$ here is $\Phi(\tau = U(u(z^+(X^+))))$ of Sec. VII A. The Yang-Mills coupling constant is determined in terms of the quantities of the IIB theory as with

$$G_{\rm YM} = \sqrt{\frac{RR_B^2}{g_B l_B^4}}. \tag{7.21}$$

Unlike the matrix membrane actions in Ref. [37], the Yang-Mills coupling defined by the cubic and quartic commutator terms in the action (7.20) is time independent. Naively it appears that so long as $g_B \ll 1$ the theory is strongly coupled *at all times* and the fields collapse to diagonal fields in a suitable gauge. Furthermore the radius of the ρ direction becomes small and the theory reduces to a $1 + 1$ dimensional theory. This becomes the light-cone worldsheet action after a dualization of the gauge field in terms of a new scalar χ^8. The matrix membrane Lagrangian density becomes in this limit

$$L = \tfrac{1}{2}\{[(\partial_\tau \chi^\alpha)^2 + (\partial_\tau \chi^8)^2 - e^{\Phi(\tau)}(\partial_\sigma \chi^\alpha)^2 - e^{\Phi(\tau)}(\partial_\sigma \chi^8)^2] - H(\tau)[(\chi^1)^2 + (\chi^2)^2] - (\chi^3)^2 - \cdots - (\chi^6)^2 + 4\chi^7 \partial_\tau \chi^8\}. \tag{7.22}$$

The details of the dualization are provided in Appendix C. Using the procedure of Sec. VI it is straightforward to see that this is the light-cone gauge worldsheet Lagrangian in the background (Einstein frame) metric (7.18), precisely as it should be.

The Lagrangian (7.20) was written in terms of a flat worldvolume metric. Let us instead introduce the worldvolume metric γ, whose line element is

$$ds^2 = -d\tau^2 + e^{-\Phi(\tau)} d\sigma^2 + e^{\Phi(\tau)} d\rho^2. \tag{7.23}$$

Then the bosonic part of the matrix membrane action can be written (almost) covariantly as

$$S = \int d\tau \int_0^{2\pi(\ell_B^2/R)} d\sigma \int_0^{2\pi g_B(\ell_B^2/R)} d\rho \sqrt{-\gamma}\Big\{-\frac{1}{2G_{\rm YM}^2}\gamma^{ac}\gamma^{bd}F_{ab}F_{cd} - \frac{1}{2}\gamma^{ab}D_a X^\alpha D_b X^\alpha - H(\tau)[(\chi^1)^2 + (\chi^2)^2] - (\chi^3)^2 - \ldots - (\chi^6)^2 - 4(\chi^7)^2 + \frac{G_{\rm YM}^2}{2}[\chi^\alpha, \chi^\beta]^2 + 2iG_{\rm YM}\chi^7[\chi^5, \chi^6] + \frac{4}{G_{\rm YM}}\chi^7 F_{\sigma\rho}\Big\}, \tag{7.24}$$

where $a, b, \ldots$ are worldvolume indices. Thus the matrix membrane theory may be considered to live on a curved space, albeit with time-dependent mass terms. The curved space, (7.23), is, however, typically singular at $\tau = 0$; for example, the Ricci scalar is

$$R = \tfrac{1}{2}\Phi'(\tau)^2, \tag{7.25}$$

which, via (7.14), is singular if the pp-wave is singular. Since $\sigma = \rho = \text{const}$ is a geodesic, the $\tau = 0$ singularity is at finite affine parameter.

This story is almost exactly the same as that in Ref. [37] in which the matrix membrane, for the type IIB maximally supersymmetric pp-wave deformed to a big bang-type singularity by a null dilaton, lived on a singular worldvolume. The difference between that work and this work is that there the metric was conformally flat, but singular; here the metric is singular but is not conformally flat. This is therefore a further extension of Refs. [22,28] which discussed matrix *strings* which ended up living on Milne space, and for which, therefore, the "big bang" singularity turned into an orbifold singularity.

Indeed, like Ref. [37] but unlike Refs. [22,28], we expect that the matrix membrane theory exhibits mode production. Note that this is *not* particle production in the target space theory, which we have shown to be absent. Mode production in the matrix membrane theory is an extension of a similar phenomenon of mode production on the light-cone worldsheet. Here we consider a *fixed* number of strings with a fixed nonzero value of k_-. Consider, for example, a single closed string so that the extent of the worldsheet σ direction is $0 \leq \sigma \leq 2\pi l_s^2 k_-$. The higher worldsheet modes are higher oscillator modes of this single string with the same values of k_i, k_-. Worldsheet Lagrangians which are explicitly time dependent would naturally evolve a state in the oscillator ground state to higher oscillator states with the same target space momenta. In our matrix membrane theory the oscillators of χ^i are labeled by the quantized momenta (m, n) in the (σ, ρ) directions, respectively, which correspond to oscillator states of single (p, q) strings. The presence of τ-dependent factors in front of the $D_\sigma \chi^i$ and $D_\rho \chi^i$ then imply, as in Ref. [37], that if we start with the oscillator vacuum in the past, the state near the singularity would be a squeezed state of higher oscillator modes of a (p, q) string. In Ref. [37] higher modes of a pure F string ($n = 0$ modes) were not produced. In the present case, excited states of a F string are produced as well. This is in accord with our analysis of the worldsheet string theory.

The question whether matrix membrane theory resolves the singularity of the pp-wave background is related to the question whether the worldsheet action in this background leads to a well-defined perturbative string theory. We have not yet been able to address this question directly. However, the connection of the pp-wave with a sector of a $3 + 1$ dimensional gauge theory suggests that there could be a nonsingular description.

Recall that the IIB pp-wave discussed above with compact V, Y^6 is the Penrose limit of $AdS_5 \times \frac{S^5}{Z_{M_1} \times Z_{M_2}}$ [51–53]. In this limit, $R, M_1, M_2 \rightarrow \infty$ and the finite radii R_V, R_B are given by

$$R_V = \frac{R^2}{4M_2}, \qquad R_B = \frac{R}{M_1}. \tag{7.26}$$

States in the pp-wave background with finite P_V and P_6 descend from states in the original background with large angular momenta along the S^5. The matrix membrane is supposed to provide a nonperturbative description of string theory in the original $AdS_5 \times \frac{S^5}{Z_{M_1} \times Z_{M_2}}$ in this large angular momentum sector, and the momentum modes along σ and ρ directions are the F-string and D-string oscillators.

On the other hand, the AdS/CFT correspondence then implies that there is a usual dual $3 + 1$ dimensional gauge theory description along the lines of Ref. [54], which turns out to be a large quiver with $M_1 M_2$ nodes, each having a $U(N)$ gauge theory [52,53]. The oscillators of the F-string now appear in this gauge theory as operators with many scalar insertions.

It is natural to expect that this chain of correspondences persist with our time-dependent deformation. Correlation functions of gauge invariant operators in terms of suitably redefined fields are therefore expected to be nonsingular. This might indicate that the theory could be nonsingular when a correct choice of dynamical variables is made. However we do not know at this moment how to make such a choice directly in the worldsheet or matrix membrane theory.

In fact, the present paper indicates that different holographic descriptions are useful to analyze what happens to string theory near such null singularities. For backgrounds of the type analyzed in Refs. [22–44], the matrix string theory or the matrix membrane theory provided a transparent explanation of how null singularities may be "resolved." For the kind of backgrounds we have focussed in this paper, the AdS/CFT type of correspondence is more suitable.

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APPENDIX A: THE CONFORMALLY COUPLED SCALAR

We consider the light-cone quantization of the conformally coupled scalar, whose Lagrangian is

$$S = -\int d^4x \sqrt{-g}\left[\frac{1}{2}(\partial\varphi)^2 + \frac{1}{6}R\varphi^2\right] \tag{A1}$$

in a background metric, Eq. (4.17).

Mode expanding the scalar we get

$$\varphi(x) = e^{-f/2}\int d^2k \int_0^\infty dk_- \frac{1}{\sqrt{(2\pi)^3 2k_-}} \times [a(k_i, k_-)e^{-i(k_i x^i + k_- X^- + (k_i^2/2k_-)X^+)} + a^\dagger(k_i, k_-)e^{i(k_i x^i + k_- X^- + (k_i^2/2k_-)X^+)}]. \tag{A2}$$

The momentum conjugate to φ is $2e^f \partial_{X^-}\varphi$. One can see that if the creation and annihilation operators satisfy the standard commutation relation

$$[a(k_i, k_-), a^\dagger(k_i', k_-')] = 2\pi^3 \delta^2(k_i - k_i')\delta(k_- - k_-'), \tag{A3}$$

then φ satisfies the standard comutation relation with its conjugate momentum.

The conformal vacuum is defined as the state which satisfies the condition [77]

$$a(k_i, k_-)|0\rangle = 0. \tag{A4}$$

APPENDIX B: CALCULATION OF THE BULK TWO-POINT FUNCTION

The equation of motion for a minimally coupled massive scalar in the metric (5.1) is given by

$$\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\varphi) - m^2\varphi = 0. \tag{B1}$$

Solutions can be found using the method of separation of variables. Substituting the ansatz,

$$\varphi(z, x^\mu) = e^{g(X^+)}e^{ik_- X^- + ik_i x^i}\zeta(z), \tag{B2}$$

we find that $g(X^+)$ and $\zeta(z)$ satisfy the equation

$$e^{-f}[-ik_-(2g' + f') - k_i^2] - \frac{m^2}{z^2} + \frac{\zeta''}{\zeta} - \frac{3\zeta'}{z\zeta} = 0. \tag{B3}$$

Setting the first (z-independent) expression equal to a constant $-\omega^2$, give

$$z^2\zeta'' - 3z\zeta' + (-\omega^2 z^2 - m^2)\zeta = 0, \quad -ik_-(2g' + f') - k_i^2 = -\omega^2 e^f. \tag{B4}$$

The first equation for the z-part is the same as in AdS_5 and its solution are Bessel functions, whose solutions are given by Eq. (5.4). The second equation in Eq. (B4) for g can be solved easily, yielding the final solution (5.3).

Before proceeding we note that the modes, Eq. (5.3), satisfy the completeness relation

$$\int d^4x e^{2f} \frac{u_{(k_i,k_-,\omega)}(z=\epsilon, x^\mu)}{\zeta_\omega(\epsilon)} \frac{u(k_i', k_-', \omega')(z=\epsilon, x^\mu)}{\zeta_\omega(\epsilon)} = (2\pi)^4 \delta(k_- + k_-')\delta^2(k_i + k_i')\delta(\omega^2 - \omega'^2)|2k_-|. \tag{B5}$$

1. Calculation from boundary action

Denote $\int d\vec{k} = \int_{-\infty}^{\infty} d^2k_i \int_{-\infty}^{\infty} dk_- \int_{-\infty}^{\infty} \frac{d\omega^2}{|2k_-|}$. Substituting Eq. (5.7) in the action, Eq. (5.2), and using the equation of motion Eq. (B1), the bulk action reduces to a boundary term

$$S = -\int d^4x \sqrt{-g} g^{zz} \varphi(\vec{x}, z)\partial_z \varphi(\vec{x}, z)|_{z=\epsilon} \tag{B6}$$

$$= \int d\vec{k} d\vec{k}' \varphi(k_i, k_-, \omega^2)\varphi(k_i', k_-', \omega'^2)\delta(k_- + k_-') \times \delta^{(2)}(k_i + k_i')2|k_-|\delta(\omega^2 - \omega'^2)\omega^{2\nu}C(\nu), \tag{B7}$$

where $C(\nu)$ is a coefficient which depends on ν, and we have dropped the terms that are singular in ϵ (c.f. Ref. [62]). This can be further simplified to yield Eq. (5.8).

We would like to convert Eq. (5.8) into a position space correlator. Using the completeness relation, Eq. (B5), we have

$$\varphi(k_i, k_-, \omega^2) = \frac{\epsilon^{-\Delta_-}}{(2\pi)^4} \int d^4x e^{2f} \varphi(x^\mu, \epsilon) \times \frac{u(-k_i, -k_-, \omega^2)(\epsilon, x^\mu)}{\zeta_\omega(\epsilon)}. \tag{B8}$$

In the discusion below, we denote

$$\varphi(x) \equiv \epsilon^{-\Delta_-} \varphi(x, \epsilon) \tag{B9}$$

from Eqs. (5.3) and (5.7), we see that $\varphi(x)$ is independent of ϵ.

Then, Eq. (5.8) can be written as

$$S = \int d^4x d^4x' e^{3f(X^+)/2} e^{3f(X'^+)/2} \varphi(\vec{x})\varphi(\vec{x}')G(\vec{x}, \vec{x}'), \tag{B10}$$

where

$$G(\vec{x}, \vec{x}') = \frac{1}{(2\pi)^8} \int d\vec{k} e^{-ik_-\Delta X^- - ik_i\Delta x^i} e^{-(ik_i^2/2k_-)\Delta X^+} \times e^{(i\omega^2/2k_-)\Delta\lambda} \omega^{2\nu}, \tag{B11}$$

with $\Delta x^\mu = x^\mu - x'^\mu$.

The momentum integrals in $G(\vec{x}, \vec{x}')$ can be carried out and gives the final action in position space, (5.9).

2. Calculation using light front operator quantization

The *normalized* solutions of the Klein-Gordon equation are

$$\varphi_{\alpha,\vec{k},k_-}(\vec{x}, x^\pm, z) = \frac{1}{\sqrt{2(2\pi)^3}} \sqrt{\frac{\alpha}{2k_-}} [z^2 J_\nu(\alpha z)] \times [e^{-f(x^+)/2} G_{\vec{k},k_-,\alpha}(x^+)] e^{i(\vec{k}\cdot\vec{x} + k_- x^-)}, \tag{B12}$$

where

$$G_{\vec{k},k_-,\alpha}(x^+) = \exp\left[-i\frac{\vec{k}^2 + \alpha^2}{2k_-}\lambda(x^+) + i\frac{\vec{k}^2}{2k_-}\kappa(x^+)\right], \tag{B13}$$

where

$$\lambda(x^+) = \int^{x^+} dy e^{f(y)}, \qquad \kappa(x^+) = \lambda(x^+) - x^+. \tag{B14}$$

These modes are normalized according to a Klein-Gordon norm on an $x^+ = \text{const}$ surface, which is defined as

$$(\varphi_1, \varphi_2) = -i \int dz d\vec{x} dx^- \sqrt{-g} g^{+-} (\varphi_1 \partial_- \varphi_2^\star - \varphi_2^\star \partial_- \varphi_1). \tag{B15}$$

In this norm we have

$$(\varphi_{\alpha,\vec{k},k_-}, \varphi_{\alpha',\vec{k}',k_-'}) = \delta^{(2)}(\vec{k} - \vec{k}')\delta(\alpha - \alpha')\delta(k_- - k_-'). \tag{B16}$$

We have the mode expansion

$$\varphi(x^\pm, z, \vec{x}) = \int_0^\infty d\alpha \int_0^\infty dk_- \int d^2k [\varphi_{\alpha,\vec{k},k_-}(x^\pm, z, \vec{x}) \times a(\vec{k}, k_-, \alpha) + \varphi^\star_{\alpha,\vec{k},k_-}(x^\pm, z, \vec{x}) a^\dagger(\vec{k}, k_-, \alpha)]. \tag{B17}$$

The commutation relations are

$$[a(\vec{k}, k_-, \alpha), a^\dagger(\vec{k}', k_-', \alpha')] = \delta(\alpha - \alpha')\delta^2(\vec{k} - \vec{k}') \times \delta(k_- - k_-'). \tag{B18}$$

The Feynman propagator is

$$G_F(x^\pm, z, \vec{x}; x^{\pm\prime}, z', \vec{x}') = \int_0^\infty d\alpha \int_0^\infty dk_- \int \frac{d^2k}{2(2\pi)^3}\left(\frac{\alpha}{k_-}\right)(zz')^2 J_\nu(\alpha z)J_\nu(\alpha z') \times [e^{i\vec{k}\cdot(\vec{x}-\vec{x}')+ik_-(x^- x^{-\prime})} e^{-f(x^+)/2} G_{\vec{k},k_-,\alpha}(x^+) e^{-f(x^{+\prime})/2} G^\star_{\vec{k},k_-,\alpha}(x^{+\prime})\theta(x^+ - x^{+\prime}) + e^{-i\vec{k}\cdot(\vec{x}-\vec{x}')-ik_-(x^- x^{-\prime})} e^{-f(x^+)/2} G^\star_{\vec{k},k_-,\alpha}(x^+) e^{-f(x^{+\prime})/2} G_{\vec{k},k_-,\alpha}(x^{+\prime})\theta(x^{+\prime} - x^+)]. \tag{B19}$$

This may be represented as

$$G_F(x^\pm, z, \vec{x}; x^{\pm\prime}, z', \vec{x}') = e^{-(1/2)[f(x^+)+f(x^{+\prime})]}(zz')^2 \int_0^\infty d\alpha \int_{-\infty}^\infty dk_- \int_{-\infty}^\infty dk + \int d^2k \mathcal{F}(\vec{k}, k_\pm, \alpha). \tag{B20}$$

where

$$\mathcal{F}(\vec{k}, k_\pm, \alpha) = e^{i\vec{k}\cdot(\vec{x}-\vec{x}')} e^{ik_-(x^- x^{-\prime})} e^{-ik_+[\lambda(x^+)-\lambda(x^{+\prime})]} e^{i(\vec{k}^2/2k_-)[\kappa(x^+)-\kappa(x^{+\prime})]} \frac{\alpha J_\nu(\alpha z)J_\nu(\alpha z')}{2k_+k_- - (\vec{k}^2 + \alpha^2 - i\epsilon)}. \tag{B21}$$

Since $\lambda(x^+)$ is a monotonic function of x^+, the integral over k_+ in Eq. (B21) will reproduce Eq. (B19).

The integral over α in Eq. (B20) may be carried out using the integrals

$$\int_0^\infty d\alpha \alpha \frac{J_\nu(\alpha z)J_\nu(\alpha z')}{2k_+k_- - (\vec{k}^2 + \alpha^2 - i\epsilon)} = I_\nu(pz)K_\nu(pz'), \qquad z < z', \\ = K_\nu(pz)I_\nu(pz'), \qquad z' < z, \tag{B22}$$

where

$$p = \sqrt{\vec{k}^2 - 2k_+k_-}. \tag{B23}$$

Thus, for $z > z'$ the momentum space propagator may be written as

$$G_{F,\vec{k},k_-} = e^{-(1/2)[f(x^+)+f(x^{+\prime})]}(zz')^2 \int_{-\infty}^\infty dk_+ e^{-ik_+[\lambda(x^+)-\lambda(x^{+\prime})]} e^{i(\vec{k}^2/2k_-)[\kappa(x^+)-\kappa(x^{+\prime})]} I_\nu(z'p)K_\nu(zp), \tag{B24}$$

In the limit $z' \to 0$ with fixed z the function $I_\nu(pz') \sim (pz')^\nu$. The bulk boundary propagator is then obtained by dividing by the leading z' dependence of the Feynman propagator. Finally the boundary correlator is obtained by taking the limit $z \to 0$. The result is easily seen to agree with Eq. (B11).

APPENDIX C: FROM MATRIX MEMBRANE ACTION TO WORLDSHEET ACTION

Let us start with the abelian version of the "matrix" membrane action (7.20). We wish to find the Green-Schwarz string action. To do this, we follow the usual procedure of first dualizing the gauge field and then employing dimensional reduction along ρ. For this, we really only care about the sub-Lagrangian,

$$\mathcal{L}_1 = \frac{1}{G_{\rm YM}^2}[e^{\Phi(\tau)}F_{\sigma\tau}^2 + e^{-\Phi(\tau)}F_{\rho\tau}^2 - F_{\rho\sigma}^2] + \frac{4}{G_{\rm YM}}\chi^7 F_{\sigma\rho} - 4(\chi^7)^2. \tag{C1}$$

$\mathcal{L}_1$ is that part of the Lagrangian which involves F, plus a term that will disappear when we complete a square.

To dualize the gauge field, we add an auxiliary field χ^8, and write

$$\mathcal{L}_1 = \frac{1}{G_{\rm YM}^2}[e^{\Phi(\tau)/2}F_{\sigma\tau} + G_{\rm YM}e^{-\Phi(\tau)/2}\partial_\rho\chi^8]^2 + \frac{1}{G_{\rm YM}^2}[e^{-\Phi(\tau)/2}F_{\rho\tau} - G_{\rm YM}e^{\Phi(\tau)/2}\partial_\sigma\chi^8]^2 - \frac{1}{G_{\rm YM}^2}[F_{\rho\sigma} - G_{\rm YM}\partial_\tau\chi^8]^2 + \frac{4}{G_{\rm YM}}\chi^7 F_{\sigma\rho} - 4(\chi^7)^2 + (\partial_\tau\chi^8)^2 - e^{\Phi(\tau)}(\partial_\sigma\chi^8)^2 - e^{-\Phi(\tau)}(\partial_\rho\chi^8)^2. \tag{C2}$$

Note that the last three terms ensure that the action is actually linear in χ^8; the equation of motion for χ^8, in fact, is just the Bianchi identity $dF = 0$. Therefore, we can treat F as an independent field—the χ^8 equation of motion is solved by taking $F = dA$—and instead of integrating out χ^8, we integrate out F. Since F appears quadratically, we can integrate out F by solving its equations of motion and plugging the solutions back into the Lagrangian. The equations of motion for F are

$$F_{\sigma\tau} = -G_{\rm YM} e^{-\Phi(\tau)} \partial_\rho \chi^8, \tag{C3a}$$

$$F_{\rho\tau} = G_{\rm YM} e^{\Phi(\tau)} \partial_\sigma \chi^8, \tag{C3b}$$

$$F_{\sigma\tau} = G_{\rm YM} \partial_\tau \chi^8 + 2G_{\rm YM} \chi^7. \tag{C3c}$$

Therefore, $\mathcal{L}_1$ is equivalent to the action

$$\begin{aligned} \mathcal{L}_1 &= 0 + 0 - 4(\chi^7)^2 + 8(\chi^7)^2 + 4\chi^7 \partial_\tau \chi^8 - 4(\chi^7)^2 + (\partial_\tau \chi^8)^2 - e^{\Phi(\tau)}(\partial_\sigma \chi^8)^2 - e^{-\Phi(\tau)}(\partial_\rho \chi^8)^2 \\ &= 4\chi^7 \partial_\tau \chi^8 + (\partial_\tau \chi^8)^2 - e^{\Phi(\tau)}(\partial_\sigma \chi^8)^2 - e^{-\Phi(\tau)}(\partial_\rho \chi^8)^2. \end{aligned} \tag{C4}$$

Thus, the matrix membrane action (7.20) reduces to the string action (7.22).

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