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# Covariant Hamiltonian dynamics

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We discuss the covariant formulation of the dynamics of particles with Abelian and non-Abelian gauge charges in external fields. Using this formulation we develop an algorithm for the construction of constants of motion, which makes use of a generalization of the concept of Killing vectors and tensors in differential geometry. We apply the formalism to the motion of classical charges in Abelian and non-Abelian monopole fields.

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## I. INTRODUCTION

A standard problem in the solution of classical and quantum mechanical systems is to identify the constants of motion associated with the system. There are many ways to obtain constants of motion, but two methods are fairly common in different branches of physics. In the canonical phase-space formulation of classical conservative systems we have to identify all dynamical quantities $Q$ which commute with the Hamiltonian in the sense of Poisson brackets[1]:

$$\{Q, H\} = 0. \tag{1}$$

The drawback of this prescription is, that for systems with gauge interactions this formulation is usually not manifestly gauge covariant.

On the other hand for systems with a nonflat configuration space, such as particles moving on a curved manifold (or space-time, in general relativity) the appropriate algorithm is to search for Killing vectors and their higher-rank generalizations. In Riemannian geometry these are covariant objects, but the procedure is only applicable for geodesic motion in the absence of nongeometrical external fields of force.

As observed in [1], for constants of motion to exist in the case of nongeodesic motion, e.g., for particles in external fields, the symmetries of the metric and those of the external fields have to match. In fact Killing vectors appear explicitly in the expressions for constants of motion linear in the momentum. In [2,3] a complete set of consistency conditions for the existence of constants of motion were derived for particles in arbitrary background geometries, using a covariant Hamiltonian phase-space approach including the contributions of spin. This procedure also applied to constants of motion which are higher-order polynomials in the momentum, as well as constants of motion which are Grassmann-odd expressions in the spin degrees of freedom, which generate standard or nonstandard supersymmetries on the worldline of the system.

In this paper we show how to extend this covariant phase-space approach to include the presence of external gauge fields. As in [1] our non-Abelian dynamics is based on the equations of motion postulated by Wong [4]. These equations were studied in a geometric setting, using the method of coadjoint orbits, for a similar purpose in [5], while a Lagrangian realization in terms of Grassmann variables was constructed in [6]. Having a completely covariant phase-space formulation we derive a set of generalized Killing equations, the solution of which produces all constants of motion in a manifestly covariant way. To avoid unnecessary complications, we formulate all our dynamical models in Euclidean or Riemannian space, but the generalization to Minkowskian or Lorentzian manifolds is straightforward.

## II. DYNAMICS OF POINT CHARGES

The classical dynamics of a point charge in a magnetic field is described by the Lorentz force law

$$m\dot{\mathbf{v}} = q\mathbf{v} \times \mathbf{B}. \tag{2}$$

In the standard canonical formulation this equation is derived from a Hamiltonian

$$H = \frac{1}{2m}(\mathbf{p} - q\mathbf{A})^2, \tag{3}$$

via Hamilton's equations:

$$\dot{\mathbf{r}} = \frac{\partial H}{\partial \mathbf{p}} = \frac{1}{m}(\mathbf{p} - q\mathbf{A}), \qquad \dot{\mathbf{p}} = -\frac{\partial H}{\partial \mathbf{r}} = \frac{q}{m}(\boldsymbol{\nabla}\mathbf{A}) \cdot (\mathbf{p} - q\mathbf{A}). \tag{4}$$

Therefore

$$\mathbf{p} = m\dot{\mathbf{r}} + q\mathbf{A} = m\mathbf{v} + q\mathbf{A}, \qquad \dot{\mathbf{p}} = q\boldsymbol{\nabla}\mathbf{A} \cdot \mathbf{v}, \tag{5}$$

and after substitution and the definition $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$ Eq. (2) follows. In terms of the field-strength tensor $\mathbf{F}$

$$F_{ij} = \varepsilon_{ijk} B_k = \nabla_i A_j - \nabla_j A_i, \tag{6}$$

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[1] In quantum mechanics, the Poisson bracket is replaced by the commutator; for brevity and to avoid irrelevant operator ordering complications, we stay with classical mechanics in this paper.

 

the equation for the Lorentz force takes the form

$$m\dot{v}_i = qF_{ij}v_j. \tag{7}$$

In this form the equation can be extended easily to relativistic particles in Minkowski space.

The above construction uses Cartesian coordinates $r_i$ and their canonical momenta $p_i$, such that the equations of motion an be written in terms of Poisson brackets

$$\{f, g\} = \sum_i \frac{\partial f}{\partial r_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial r_i}, \tag{8}$$

for phase-space functions $f(r_i, p_i, t)$ and $g(r_i, p_i, t)$. In term of these brackets

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \{f, H\}. \tag{9}$$

The phase-space coordinates $(r_i, p_i)$ are canonical as the only nontrivial fundamental bracket is

$$\{r_i, p_j\} = \delta_{ij}, \tag{10}$$

all others vanishing:

$$\{r_i, r_j\} = \{p_i, p_j\} = 0. \tag{11}$$

A disadvantage of this formulation is, that the canonical momenta are gauge dependent:

$$\mathbf{A}' = \mathbf{A} + \nabla\Lambda \Rightarrow \mathbf{p}' = \mathbf{p} + q\nabla\Lambda, \tag{12}$$

although this does not affect the fundamental brackets (10). As a result, the Hamiltonian equations of motion are not manifestly gauge covariant.

However, an alternative exists in which the dynamical variables of the particle are all gauge invariant, and which has the added advantage that the Hamiltonian takes a very simple form. Introduce the gauge-invariant momenta

$$\mathbf{\Pi} = \mathbf{p} - q\mathbf{A} = m\mathbf{v}. \tag{13}$$

Then the Hamiltonian takes the simple quadratic form

$$H = \frac{1}{2m}\mathbf{\Pi}^2. \tag{14}$$

This has the form of a free-particle Hamiltonian, but the dynamics is now manifest in the modified brackets:

$$\{f, g\} = \frac{\partial f}{\partial r_i}\frac{\partial g}{\partial \Pi_i} - \frac{\partial f}{\partial \Pi_i}\frac{\partial g}{\partial r_i} + qF_{ij}\frac{\partial f}{\partial \Pi_i}\frac{\partial g}{\partial \Pi_j}. \tag{15}$$

In particular the fundamental brackets are

$$\{r_i, \Pi_j\} = \delta_{ij}, \qquad \{r_i, r_j\} = 0, \qquad \{\Pi_i, \Pi_j\} = -qF_{ij}. \tag{16}$$

This shows, that the momenta $\mathbf{\Pi}$ are not canonical, but act like covariant derivatives, rather than ordinary partial derivatives; indeed, the last bracket is the Poisson-bracket version of the Ricci identity. As a result, we can derive the homogeneous Maxwell equations (the Bianchi identities) from the Jacobi identity:

$$\{\Pi_i, \{\Pi_j, \Pi_k\}\} + \{\Pi_j, \{\Pi_k, \Pi_i\}\} + \{\Pi_k, \{\Pi_i, \Pi_j\}\} = 0 \tag{17}$$

which implies

$$\nabla_i F_{jk} + \nabla_j F_{ki} + \nabla_k F_{ij} = 0 \Leftrightarrow \nabla \cdot \mathbf{B} = 0. \tag{18}$$

It remains to establish that the brackets and the Hamiltonian reproduce the correct equations of motion; this follows by direct computation:

$$\dot{r}_i = \{r_i, H\} = \frac{\Pi_i}{m}, \qquad \dot{\Pi}_i = \{\Pi_i, H\} = \frac{q}{m}F_{ij}\Pi_j. \tag{19}$$

The covariant phase-space formulation has been used by various authors in different contexts, see e.g. [1,3,5].

## III. SYMMETRIES AND CONSTANTS OF MOTION

The gauge-covariant formulation of Hamiltonian mechanics of charged particles is mathematically elegant; it is also most suited to study the existence of symmetries and constants of motion. Indeed, in the Hamiltonian framework a constant of motion $Q(\mathbf{r}, \mathbf{\Pi})$ is identified by the property that its bracket with the Hamiltonian vanishes:

$$\begin{aligned}\{Q, H\} = 0 \Rightarrow \Pi_i\left(\nabla_i Q - qF_{ij}\frac{\partial Q}{\partial \Pi_j}\right) \\ = \mathbf{\Pi} \cdot \left(\nabla Q + q\mathbf{B} \times \frac{\partial Q}{\partial}\right) = 0.\end{aligned} \tag{20}$$

A systematic procedure is to expand any constant of motion as a power series in $\mathbf{\Pi}$:

$$Q(\mathbf{r}, \mathbf{\Pi}) = C(\mathbf{r}) + C_i(\mathbf{r})\Pi_i + \tfrac{1}{2}C_{ij}(\mathbf{r})\Pi_i\Pi_j + \ldots. \tag{21}$$

Substitution gives a series of constraints

$$\begin{aligned}\nabla_i C &= qF_{ij}C_j, \\ \nabla_i C_j + \nabla_j C_i &= qF_{ik}C_{kj} + qF_{jk}C_{ki}, \\ \nabla_i C_{jk} + \nabla_j C_{ki} + \nabla_k C_{ij} &= qF_{il}C_{ljk} + qF_{jl}C_{lki} \\ &\quad + qF_{kl}C_{lij}, \ldots\end{aligned} \tag{22}$$

This series can be truncated whenever there is a Killing vector or tensor of flat space:

$$\nabla_{(i_1}C_{i_2 \ldots i_n)} = 0. \tag{23}$$

Then we can take $C_{i_1 \ldots i_p} = 0$ for all $p \geq n$, and the constant of motion takes the polynomial form

$$Q(\mathbf{r}, \mathbf{\Pi}) = \sum_{k=0}^{p-1} \frac{1}{k!} C_{i_1 \ldots i_k}(\mathbf{r})\Pi_{i_1} \ldots \Pi_{i_k}. \tag{24}$$

Note that it is always possible to add an arbitrary constant to the zeroth order coefficient $C(\mathbf{r})$. Therefore it is obvious that for particles in an electromagnetic background there are no nontrivial constants of motion corresponding to only a $C(\mathbf{r})$ with $C_i(\mathbf{r})$, $C_{ij}(\mathbf{r})$ and all higher coefficients vanishing. The first nontrivial case is therefore the set truncated at $p = 2$:

$$Q(\mathbf{r}, \mathbf{\Pi}) = C(\mathbf{r}) + \mathbf{C}(\mathbf{r}) \cdot \mathbf{\Pi}, \tag{25}$$

with $\mathbf{C}$ a Killing vector of flat space:

$$\nabla_i C_j + \nabla_j C_i = 0. \tag{26}$$

Such Killing vectors generate translations and rotations, and take the form

$$\mathbf{C} = \mathbf{m} + \mathbf{n} \times \mathbf{r}, \tag{27}$$

where $\mathbf{m}$ and $\mathbf{n}$ are arbitrary constant vectors. The lowest-order constraint equation now becomes

$$\nabla_i C = qF_{ij}C_j = q\nabla_i(A_jC_j) - qA_j\nabla_iC_j - qC_j\nabla_jA_i \tag{28}$$

Now use the Killing equation for $C_i$ to rewrite this equation as

$$\nabla_i(C - q\mathbf{C} \cdot \mathbf{A}) = q\mathbf{A} \cdot \mathbf{\nabla} C_i - q\mathbf{C} \cdot \mathbf{\nabla} A_i. \tag{29}$$

A very simple example a constant magnetic field:

$$\mathbf{A} = \frac{\beta}{2}\mathbf{C} \Rightarrow \mathbf{B} = \beta\mathbf{n}. \tag{30}$$

For such a field

$$C = q\mathbf{C} \cdot \mathbf{A} = \frac{q\beta}{2}\mathbf{C}^2 \qquad \text{(mod constant)}. \tag{31}$$

As a result the full constant of motion takes the form

$$Q = \mathbf{C} \cdot (q\mathbf{A} + \mathbf{\Pi}) = \mathbf{C} \cdot \mathbf{p} = \mathbf{m} \cdot \mathbf{p} + \frac{1}{\beta}\mathbf{B} \cdot (\mathbf{r} \times \mathbf{p}). \tag{32}$$

As $\mathbf{m}$ is arbitrary, all components of the momentum and the component of the angular momentum in the direction of $\mathbf{B}$ are conserved. This reflects the invariance under translations and transverse rotations in a constant magnetic field.

Starting from the general Killing vector (27) the equation for the Killing scalar becomes

$$\mathbf{\nabla} C = q\mathbf{m} \times \mathbf{B} + q(\mathbf{r}\mathbf{n} \cdot \mathbf{B} - \mathbf{n}\mathbf{r} \cdot \mathbf{B}). \tag{33}$$

As a nontrivial example, consider axially symmetric fields $\mathbf{B}$ with the axis defined by a unit vector $\mathbf{n}$:

$$\mathbf{B} = \frac{B(\rho)}{\rho}\mathbf{r} \times \mathbf{n} = \frac{B(\rho)}{\rho}\boldsymbol{\rho} \times \mathbf{n}, \qquad \boldsymbol{\rho} = \mathbf{r} - (\mathbf{r} \cdot \mathbf{n})\mathbf{n}, \tag{34}$$

where $\rho = |\boldsymbol{\rho}|$. Then Eq. (33) becomes

$$\mathbf{\nabla} C = q\mathbf{m} \times \mathbf{B} = \frac{qB}{\rho}(\boldsymbol{\rho}\mathbf{n} \cdot \mathbf{m} - \mathbf{n}\boldsymbol{\rho} \cdot \mathbf{m}). \tag{35}$$

The corresponding vector potential can be taken as

$$\mathbf{A} = g(\rho)\mathbf{n}, \tag{36}$$

provided we identify $B(\rho) = g'(\rho)$. Then Eq. (35) becomes

$$\begin{aligned}\mathbf{\nabla} C &= \frac{qg'(\rho)}{\rho}(\boldsymbol{\rho}\mathbf{n} \cdot \mathbf{m} - \mathbf{n}\boldsymbol{\rho} \cdot \mathbf{m}) \\ &= q(\mathbf{m} \cdot \mathbf{n})\mathbf{\nabla} g - q(\mathbf{m} \cdot \mathbf{\nabla} g)\mathbf{n}.\end{aligned} \tag{37}$$

This equation allows solutions for $\mathbf{m} = \lambda\mathbf{n}$, with $\lambda$ an arbitrary scalar parameter. As $\mathbf{n} \cdot \boldsymbol{\rho} = 0$ and $\mathbf{n}^2 = 1$, one finds

$$C = \lambda q g(\rho). \tag{38}$$

The full constant of motion then reads

$$\begin{aligned}Q &= \lambda q g(\rho) + (\lambda\mathbf{n} + \mathbf{n} \times \mathbf{r}) \cdot \mathbf{\Pi} \\ &= \lambda\mathbf{n} \cdot \mathbf{p} + \mathbf{n} \cdot (\mathbf{r} \times \mathbf{p}).\end{aligned} \tag{39}$$

As $\lambda$ is arbitrary, it follows that the components of the canonical momentum and the angular momentum in the direction $\mathbf{n}$ are independently conserved[2]

One can also search for constants of motion which are higher-order polynomials in the momentum, associated with flat-space Killing tensors. The simplest one is

$$C_{ij} = \delta_{ij}, \tag{40}$$

which has the special property that

$$F_{ij}C_{jk} + F_{kj}C_{ji} = 0. \tag{41}$$

Therefore the associated Killing vector and scalar can be taken to vanish, and the corresponding constant of motion is the Hamiltonian:

$$C = \tfrac{1}{2}\delta_{ij}\Pi_i\Pi_j = mH, \tag{42}$$

More complicated Killing tensors are of the form

$$C_{ij} = 2\delta_{ij}\mathbf{n} \cdot \mathbf{r} - (n_ir_j + n_jr_i), \tag{43}$$

for an arbitrary fixed unit vector $\mathbf{n}$, and

$$C_{ij} = \delta_{ij}\mathbf{r}^2 - r_ir_j, \tag{44}$$

which is the radial counterpart of (43). Any constants of motion associated with these Killing tensors are extensions of the $\mathbf{n}$-component of the Runge-Lenz vector

$$\mathbf{n} \cdot \mathbf{K} = \mathbf{n} \cdot (\mathbf{\Pi} \times \mathbf{L}) = \mathbf{n} \cdot \mathbf{r}^2 - \mathbf{n} \cdot \mathbf{\Pi}\mathbf{r} \cdot \mathbf{\Pi}, \tag{45}$$

or the total angular momentum

$$\mathbf{L}^2 = (\mathbf{r} \times \mathbf{\Pi})^2 = \mathbf{r}^2\mathbf{\Pi}^2 - (\mathbf{r} \cdot \mathbf{\Pi})^2 = \mathbf{r} \cdot \mathbf{K}. \tag{46}$$

Such constants of motion are associated with special field configurations, in particular, spherically symmetric ones. We discuss such fields in the next section.

[2]Actually, the components of the canonical and covariant angular momentum in the direction $\mathbf{n}$ are the same.

## IV. MAGNETIC MONOPOLES

A spherically symmetric magnetic solution of the Maxwell equations is the Dirac monopole:

$$\mathbf{B} = \frac{g\mathbf{r}}{r^3}. \tag{47}$$

For such a field equation (33) takes the form

$$\nabla C = \frac{qg}{r^3}(\mathbf{m} \times \mathbf{r} + (\mathbf{r} \cdot \mathbf{n})\mathbf{r} - r^2\mathbf{n}). \tag{48}$$

Now the first term is a curl, not a gradient; as a result we have to take $\mathbf{m} = 0$, and

$$C = -qg\frac{\mathbf{n} \cdot \mathbf{r}}{r}. \tag{49}$$

The result for the constants of motion based on the Killing vector (27) then is

$$Q = \mathbf{n} \cdot \left(-gq\frac{\mathbf{r}}{r} + \mathbf{r} \times \mathbf{\Pi}\right), \tag{50}$$

where $\mathbf{n}$ is an arbitrary vector; therefore all components of the gauge-covariant improved angular momentum

$$\mathbf{J} = \mathbf{L} - q\mathbf{r} \times \mathbf{A} - gq\frac{\mathbf{r}}{r}, \qquad \mathbf{L} = \mathbf{r} \times \mathbf{p}, \tag{51}$$

are conserved. We observe, that these quantities generate rotations and satisfy the standard $so(3)$ Lie algebra

$$\{J_i, J_j\} = \varepsilon_{ijk}J_k. \tag{52}$$

The Casimir invariant of $so(3)$ is the total angular momentum squared:

$$\mathbf{J}^2 = \mathbf{L}^2 + g^2q^2, \tag{53}$$

which is a constant of motion with

$$C_{ij} = \delta_{ij}\mathbf{r}^2 - r_ir_j, \qquad C_i = 0, \qquad C = g^2q^2. \tag{54}$$

It follows from (53), that the values of the total angular momentum in the classical theory satisfy the bound $\mathbf{J}^2 \geq C = g^2q^2$. A recent analysis relating many different formulations of this dynamical system is found in [7].

Remarkably, the Runge-Lenz vector can be extended to a constant of motion in another type of central magnetic field:

$$\mathbf{B} = \frac{g\mathbf{r}}{r^{5/2}}. \tag{55}$$

Indeed, in such a field there is a constant of motion

$$Q = \mathbf{n} \cdot \left(\mathbf{K} + \frac{2gq}{\sqrt{r}}\mathbf{L} - 2g^2q^2\frac{\mathbf{r}}{r}\right), \tag{56}$$

for any unit vector $\mathbf{n}$, with

$$\begin{aligned} C_{ij} &= 2\delta_{ij}\mathbf{n} \cdot \mathbf{r} - (n_ir_j + n_jr_i), \\ C_i &= \frac{2gq}{\sqrt{r}}\mathbf{n} \times \mathbf{r}, \\ C &= -2g^2q^2\frac{\mathbf{n} \cdot \mathbf{r}}{r}. \end{aligned} \tag{57}$$

In contrast, for such a central field there is no conserved angular momentum vector $\mathbf{J}$, although the total angular momentum $\mathbf{J}^2$ is conserved. Of course, the total energy of the magnetic field (55) diverges both at $r = 0$ and at $r \to \infty$, and the field does not satisfy the free Maxwell equations. Therefore it requires nontrivial magnetic sources and boundary conditions. However, even if the field would exist only in a restricted region of space, the constant of motion exists provided the orbit of the point charge is also restricted to this region.

## V. NON-ABELIAN POINT CHARGES

The gauge-covariant dynamics of point charges can be extended to non-Abelian point charges. The starting point is defined by the Wong equations [4], which can be written in the form

$$\mathbf{\Pi} = m\mathbf{v}, \qquad \dot{\mathbf{\Pi}} = gt_a\mathbf{v} \times \mathbf{B}_a, \qquad \dot{t}_a = gf_{abc}\mathbf{v} \cdot \mathbf{A}_bt_c, \tag{58}$$

where the $t_a$ are the non-Abelian gauge variables, and where the non-Abelian field strength is defined as

$$F_{ija} = \varepsilon_{ijk}B_{ka} = \nabla_iA_{ja} - \nabla_jA_{ia} - gf_{abc}A_{ib}A_{jc}. \tag{59}$$

A Lagrangian representation of the gauge variables in terms of Grassmann-odd degrees of freedom was shown to exist in Ref. [6]. The Hamiltonian formulation uses canonical coordinates and momenta, with a Hamiltonian

$$H = \frac{1}{2m}(\mathbf{p} - g\mathbf{A}_at_a)^2, \tag{60}$$

supplemented by the fundamental brackets

$$\{r_i, p_j\} = \delta_{ij}, \qquad \{t_a, t_b\} = -f_{abc}t_c, \tag{61}$$

with all other brackets vanishing. It is easily verified, that these brackets reproduce the equations of motion (58). A more transparent and gauge-covariant formulation is obtained by introduction of the covariant momentum

$$\mathbf{\Pi} = m\mathbf{v} = \mathbf{p} - g\mathbf{A}_at_a. \tag{62}$$

with the quasifree Hamiltonian

$$H = \frac{1}{2m}\mathbf{\Pi}^2. \tag{63}$$

The equations of motion (58) are now reobtained from the covariant brackets

$$\dot{f} = \{f, H\}, \tag{64}$$

where the brackets are defined explicitly by

$$\{f, h\} = \mathcal{D}_i f \frac{\partial h}{\partial \Pi_i} - \frac{\partial f}{\partial \Pi_i} \mathcal{D}_i h + g F_{ija} t_a \frac{\partial f}{\partial \Pi_i} \frac{\partial h}{\partial \Pi_j} - f_{abc} t_c \frac{\partial f}{\partial t_a} \frac{\partial h}{\partial t_b}. \quad (65)$$

Here the covariant phase-space derivative of a function $f(\mathbf{r}, \mathbf{\Pi}, t_a)$ appearing on the right-hand side is defined as

$$\mathcal{D}_i f = \nabla_i f - g f_{abc} t_a A_{ib} \frac{\partial f}{\partial t_c}. \quad (66)$$

As in the Abelian case we can now look for constants of motion by applying the Hamiltonian formalism:

$$\{Q, H\} = 0 \Rightarrow \Pi_i \left( \mathcal{D}_i Q - g F_{ija} t_a \frac{\partial Q}{\partial \Pi_j} \right) = 0, \quad (67)$$

or in vector notation

$$\mathbf{\Pi} \cdot \left( \mathbf{\nabla} Q - g t_a f_{abc} \mathbf{A}_b \frac{\partial Q}{\partial t_c} + g t_a \mathbf{B}_a \times \frac{\partial Q}{\partial \mathbf{\Pi}} \right) = 0. \quad (68)$$

Using a covariant momentum expansion

$$Q = C + C_i \Pi_i + \frac{1}{2} C_{ij} \Pi_i \Pi_j + \ldots, \quad (69)$$

we obtain a set of constraints to be satisfied

$$\begin{aligned} \mathcal{D}_i C &= g t_a F_{ija} C_j, \\ \mathcal{D}_i C_j + \mathcal{D}_j C_i &= g t_a (F_{ika} C_{kj} + g F_{jka} C_{ki}), \\ \mathcal{D}_i C_{jk} + \mathcal{D}_j C_{ki} + \mathcal{D}_k C_{ij} &= g t_a (F_{ila} C_{ljk} + F_{jla} C_{lki} \\ &\quad + F_{kla} C_{lij}), \ldots \end{aligned} \quad (70)$$

Clearly the condition for the series (69) to stop after a finite number of terms is the existence of a tensor satisfying the condition

$$\mathcal{D}_{(i_1} C_{i_2 \ldots i_n)} = 0, \quad (71)$$

a gauge-covariant generalization of the Killing equation. All of this is a direct non-Abelian generalization of the case of Maxwell-Lorentz theory.

## VI. 2-*D* YANG-MILLS THEORY

A simple example to illustrate the procedure of constructing constants of motion from generalized Killing-vectors and tensors is offered by 2-*D* $SU(2)$ Yang-Mills theory. In this theory the magnetic field strength is represented by a triplet of scalar fields $B^a$:

$$F_{ij}^a = B^a \varepsilon_{ij}, \quad (72)$$

which satisfies the euclidean Yang-Mills equations

$$D_i B^a = 0. \quad (73)$$

Thus the magnetic field is covariantly constant, and its modulus $B^2 = B^a B^a$ is a gauge-invariant real constant:

$$\nabla_i B^2 = 0. \quad (74)$$

As the effect of a local gauge transformation is a local rotation of $B^a$ in the internal space, it is possible to gauge transform the magnetic field locally into a constant, e.g.

$$B^a = (0, 0, B). \quad (75)$$

Such a constant magnetic field can be constructed from the linear gauge potential

$$A_i^a = -\tfrac{1}{2} B^a \varepsilon_{ij} r_j. \quad (76)$$

In a constant magnetic field there is translation and rotation invariance, hence we can look for Killing vectors of the type (27):

$$C_i = m_i - \lambda \varepsilon_{ij} r_j, \quad (77)$$

with $m_i$ and $\lambda$ arbitrary constants. It then only remains to solve for the generalize Killing scalar $C = C^a t_a$:

$$\begin{aligned} (\mathcal{D}_i C)^a &= \nabla_i C^a - g \epsilon^{abc} A_i^b C^c = g F_{ij}^a C_j \\ &= g B^a (\varepsilon_{ij} m_j + \lambda r_i). \end{aligned} \quad (78)$$

The straightforward solution to this equation with $A_i^a$ given by (76) is

$$C^a = g B^a \left( \varepsilon_{ij} r_i m_j + \frac{\lambda}{2} \mathbf{r}^2 \right). \quad (79)$$

With $m_i$ and $\lambda$ we thus associate two constants of motion; a gauge-improved momentum:

$$Q_i = \Pi_i - g B^a t_a \varepsilon_{ij} r_j = p_i + g A_i^a t_a, \quad (80)$$

and the canonical angular momentum

$$J = \varepsilon_{ij} r_i \Pi_j + \frac{g}{2} B^a t_a \mathbf{r}^2 = \varepsilon_{ij} r_i p_j. \quad (81)$$

As a next step, one can look for symmetric Killing tensors $C_{ij}$; however, the two candidates

$$C_{ij} = \delta_{ij}, \qquad C_{ij} = \delta_{ij} \mathbf{r}^2 - r_i r_j, \quad (82)$$

lead us back to the Hamiltonian (63) and the square of the angular momentum $J^2$, respectively. A nontrivial constant of motion of this type is a Runge-Lenz-like vector

$$\begin{aligned} K_i &= r_i^2 \mathbf{\Pi}^2 - \Pi_i \mathbf{\Pi} \cdot \mathbf{r} + g B^a t_a (\tfrac{1}{2} \varepsilon_{ij} \Pi_j \mathbf{r}^2 + r_i \varepsilon_{jk} r_k \Pi_k) \\ &\quad + \tfrac{1}{2} (g B^a t_a)^2 r_i \mathbf{r}^2 \\ &= r_i \mathbf{p}^2 - p_i \mathbf{p} \cdot \mathbf{r} + \tfrac{1}{4} g B^a t_a J r_i - \tfrac{1}{8} (g B^a t_a)^2 r_i \mathbf{r}^2. \end{aligned} \quad (83)$$

These constants of motion, described simultaneously by the arbitrary linear combination $\mathbf{n} \cdot \mathbf{K}$, are constructed in terms of the generalized Killing tensors

$$\begin{aligned} C_{ij} &= 2 \delta_{ij} \mathbf{r} \cdot \mathbf{n} - r_i n_j - r_j n_i, \\ C_i &= -g B^a t_a \varepsilon_{ij} (r_j \mathbf{r} \cdot \mathbf{n} + \tfrac{1}{2} n_j \mathbf{r}^2), \\ C &= \tfrac{1}{2} (g B^a t_a)^2 \mathbf{r}^2 \mathbf{r} \cdot \mathbf{n}. \end{aligned} \quad (84)$$

This solution of the 2-$D$ Yang-Mills equations can be embedded straightforwardly in 3-$D$ Yang-Mills theory by taking connections

$$A_x^a = -\tfrac{1}{2}B^a y, \qquad A_y^a = \tfrac{1}{2}B^a x, \tag{85}$$

while all other components of $A_\mu^a$ vanish. Actually, this amounts simply to embedding an Abelian solution of the type (30) into a non-Abelian model. In essence, the results of the Abelian case are reproduced.

## VII. THE NON-ABELIAN WU-YANG MONOPOLE

A genuinely non-Abelian static solution of the pure $SU(2)$ Yang-Mills equations in 3-$D$ space is the non-Abelian Wu-Yang monopole [8]; the monopole field is given by

$$A_i^a = \frac{1}{g}\frac{\varepsilon_{iak}r_k}{r^2}, \tag{86}$$

with the corresponding magnetic field strength

$$F_{ij}^a = \frac{1}{g}\frac{\varepsilon_{ijk}r_k r_a}{r^4}, \qquad D_j F_{ij}^a = 0. \tag{87}$$

In such a field a particle has a conserved charge invariant under combined spatial and isospin rotations

$$\mathcal{Q} = \frac{r^a t_a}{r} \Rightarrow \mathcal{D}_i \mathcal{Q} = 0. \tag{88}$$

In addition, there is a conserved angular momentum

$$\mathbf{n}\cdot\mathbf{J} = \mathbf{n}\cdot\left(\mathbf{r}\times\mathbf{\Pi} - \mathcal{Q}\frac{\mathbf{r}}{r}\right) \Leftrightarrow \mathbf{J} = \mathbf{r}\times\mathbf{p} - \mathbf{t} = \mathbf{L} - \mathbf{t}. \tag{89}$$

which is constructed from the Killing vector and associated scalar

$$C_i = (\mathbf{n}\times\mathbf{r})_i, \qquad C = -\mathcal{Q}\frac{\mathbf{n}\cdot\mathbf{r}}{r} = -\mathbf{n}\cdot\mathbf{r}\frac{r^a t_a}{r^2}. \tag{90}$$

It is easily established that the components of the angular momentum $\mathbf{J}$ generate the $so(3)$ Lie algebra (52). The contribution of isospin to the orbital angular momentum mixes gauge and spin degrees of freedom, a result well-known in the literature [9,10]. For point-particles carrying even-dimensional isospin representations (e.g., isodoublets) this turns the bound states into fermions [11].

Also in this case there exist constants of motion quadratic in the momenta: the Hamiltonian $H$ and the square of the total angular momentum:

$$\mathbf{J}^2 = \mathbf{r}^2\mathbf{\Pi}^2 - (\mathbf{r}\cdot\mathbf{\Pi})^2 + \left(\frac{r^a t_a}{r}\right)^2. \tag{91}$$

This constant of motion is constructed from the Killing tensor (44):

$$C_{ij} = \delta_{ij}\mathbf{r}^2 - r_i r_j,$$

with $C_i = 0$ and $C = \mathcal{Q}^2$. In contrast, a constant of motion polynomial in the momenta which generalizes the Runge-Lenz vector does not exist for a point-particle in the Wu-Yang monopole background.

As is well-known, there also exist Abelian monopole solutions in spontaneously broken non-Abelian gauge theories [12,13]. The motion of non-Abelian point particles in such a background has been studied in [14,15].

## VIII. CHARGED PARTICLES IN CURVED SPACE

The concept of Killing vector has its origin in differential geometry, where it arises as generator of an isometry. In the previous sections we have applied the concept in flat space, the isometries of which are translations and rotations. We have shown how the concept can be generalized in the presence of background gauge fields, Abelian as well as non-Abelian. Symmetries and constants of motion arise, in particular, when the isometries are matched by symmetries of the background fields. We have also discussed Killing tensors of higher rank, associated with constants of motion depending on higher powers of the momenta.

The generalizations can easily be extended to nonflat spaces. The Hamiltonian of a charged particle moving in a space with metric $g_{ij}(x)$ is

$$H = \frac{1}{2m}g^{ij}(x)\Pi_i\Pi_j. \tag{92}$$

For a particle without spin the covariant brackets are the same as in flat space:

$$\{x^i, \Pi_j\} = \delta_j^i, \qquad \{\Pi_i, \Pi_j\} = qF_{ij}. \tag{93}$$

In particular, the equations of motion become

$$\begin{aligned} \dot{x}^i &= \{x^i, H\} = \frac{1}{m}g^{ij}\Pi_j, \Leftrightarrow \Pi_i = mg_{ij}\dot{x}^j, \\ \dot{\Pi} &= \{\Pi_i, H\} = \frac{1}{m}g^{kl}(\Gamma_{ik}{}^j\Pi_l\Pi_j + qF_{ik}\Pi_l) \\ &\Leftrightarrow \frac{D\Pi_i}{Dt} = \dot{\Pi}_i - \dot{x}^k\Gamma_{ki}{}^j\Pi_j = qF_{ij}\dot{x}^j. \end{aligned} \tag{94}$$

By the first of these equations, the second one reduces to the Lorentz-Wong equations in curved space. As before, constants of motion are obtained by solving the equation

$$\{Q, H\} = 0, \tag{95}$$

with $Q$ a polynomial in the momenta

$$Q(x, \Pi) = C(x) + C^i(x)\Pi_i + \tfrac{1}{2}C^{ij}(x)\Pi_i\Pi_j + \ldots \tag{96}$$

Then the coefficients are solutions of the hierarchy of differential equations

$$\begin{aligned} D_i C &= \nabla_i C = qF_{ij}C^j, \\ D_i C_j + D_j C_i &= qF_{ik}C^k{}_j + qF_{jk}C^k{}_i, \\ D_i C_{jk} + D_j C_{ki} + D_k C_{ij} &= qF_{il}C^l{}_{jk} + qF_{jl}C^l{}_{ki} \\ &\quad + qF_{kl}C^l{}_{ij}, \ldots \end{aligned} \tag{97}$$

As usual, indices are raised and lowered with the metric, and the covariant derivative $D_i$ is constructed with the Levi-Civita connection

$$D_i C_j = \nabla_i C_j - \Gamma_{ij}{}^k C_k, \tag{98}$$

in the case of Abelian background gauge fields. In the case of non-Abelian background gauge fields we have to make the replacements

$$D_i \to \mathcal{D}_i = D_i - gf_{abc}t_a A_{ib}\frac{\partial}{\partial t_c}, \qquad qF_{ij} \to gt_a F_{ija}. \tag{99}$$

As an example we consider the motion of a charged particle on the unit sphere $S^2$ supplied with a constant magnetic field. The metric on the sphere is defined by the line element

$$ds^2 = g_{ij}dx^i dx^j = d\theta^2 + \sin^2\theta d\varphi^2. \tag{100}$$

The sphere $S^2$ admits a triplet of Killing vectors $C^i = (C^\theta, C^\varphi)$ satisfying the homogeneous form of the second Killing equation (97), with $C_{ij}$ and all higher Killing tensors vanishing:

$$\begin{aligned} C^i_{(1)} &= (-\sin\varphi, -\cot\theta\cos\varphi), \\ C^i_{(2)} &= (\cos\varphi, -\cot\theta\sin\varphi), \\ C^i_{(3)} &= (0, 1). \end{aligned} \tag{101}$$

These Killing vectors generate three independent rotations on $S^2$. The magnetic field with constant flux $B$ is described by the field strength

$$F_{ij}dx^i \wedge dx^j = B\sin\theta d\theta \wedge d\varphi. \tag{102}$$

Applying the vectors (101) in the right-hand side of the equation for the Killing scalars, we find that they take the form

$$\begin{aligned} C_{(1)} &= -qB\sin\theta\cos\varphi, \\ C_{(2)} &= -qB\sin\theta\sin\varphi, \\ C_{(3)} &= -qB\cos\theta. \end{aligned} \tag{103}$$

Therefore we find as constants of motion the components of the gauge-improved angular momentum $J_{(a)}$:

$$\begin{aligned} J_{(1)} &= -\sin\varphi\Pi_\theta - \cot\theta\cos\varphi\Pi_\varphi - qB\sin\theta\cos\varphi, \\ J_{(2)} &= \cos\varphi\Pi_\theta - \cot\theta\sin\varphi\Pi_\varphi - qB\sin\theta\sin\varphi, \\ J_{(3)} &= \Pi_\varphi - qB\cos\theta. \end{aligned} \tag{104}$$

As in Eq. (52), these constants of motion satisfy the $so(3)$ Poisson bracket algebra

$$\{J_{(a)}, J_{(b)}\} = \varepsilon_{abc}J_{(c)}. \tag{105}$$

Indeed, the present model is equivalent to a dimensional reduction of the monopole field from 3-$D$ flat space to the 2-$D$ unit sphere [7]. As a result, we also expect the existence of Killing tensors. First observe, that

$$\sum_a J^2_{(a)} = 2mH + q^2B^2, \tag{106}$$

which is a quadratic expression in the momenta with $C_{ij} = g_{ij}$. In addition, there two other independent symmetric Killing tensors:

$$\begin{aligned} C^{ij}_{(1)} &= \begin{pmatrix} 0 & \cos\varphi \\ \cos\varphi & -2\sin\varphi\cot\theta \end{pmatrix}, \\ C^{ij}_{(2)} &= \begin{pmatrix} 0 & \sin\varphi \\ \sin\varphi & 2\cos\varphi\cot\theta \end{pmatrix}. \end{aligned} \tag{107}$$

Inserting these expressions on the right-hand side of the second equation (97), and solving this equation, we find the associated generalized Killing vectors

$$\begin{aligned} C^i_{(1)} &= -\frac{qB}{\sin\theta}(\cos\theta\sin\theta\cos\varphi, -(\cos^2\theta - \sin^2\theta)\sin\varphi) \\ C^i_{(2)} &= -\frac{qB}{\sin\theta}(\cos\theta\sin\theta\sin\varphi, (\cos^2\theta - \sin^2\theta)\cos\varphi). \end{aligned} \tag{108}$$

The associated Killing scalars, the solution of the first equation (97), read

$$\begin{aligned} C_{(1)} &= q^2B^2\sin\theta\cos\theta\sin\varphi, \\ C_{(2)} &= -q^2B^2\sin\theta\cos\theta\cos\varphi. \end{aligned} \tag{109}$$

Combining these results we find the constants of motion

$$\begin{aligned} K_{(1)} &= \cos\varphi\Pi_\theta\Pi_\varphi - \cot\theta\sin\varphi\Pi^2_\varphi - qB\cos\theta\cos\varphi\Pi_\theta \\ &\quad + qB\frac{\cos2\theta\sin\varphi}{\sin\theta}\Pi_\varphi + q^2B^2\sin\theta\cos\theta\sin\varphi, \\ K_{(2)} &= \sin\varphi\Pi_\theta\Pi_\varphi + \cot\theta\cos\varphi\Pi^2_\varphi - qB\cos\theta\sin\varphi\Pi_\theta \\ &\quad - qB\frac{\cos2\theta\cos\varphi}{\sin\theta}\Pi_\varphi - q^2B^2\sin\theta\cos\theta\cos\varphi. \end{aligned} \tag{110}$$

Observe, that

$$K_{(1)} = \partial_\varphi K_{(2)}, \qquad K_{(2)} = -\partial_\varphi K_{(1)}. \tag{111}$$

These relations follow, because $\partial_\varphi D_i = D_i\partial_\varphi$.

## IX. SUPERSYMMETRY

Spinning particles whose internal angular momentum is described by Grassmann coordinates $\psi^i$ can have Grassmann-odd constants of motion, generating transformations in anticommuting coordinates. If their bracket closes on the Hamiltonian, they generate standard supersymmetries. In the case of charged particles in an external gauge field, the standard supercharge takes the form

$$\Omega = \Pi_i \psi^i, \tag{112}$$

while the nonzero covariant brackets are

$$\{x^i, \Pi_j\} = \delta^i_j, \qquad \{\Pi_i, \Pi_j\} = qF_{ij}, \qquad \{\psi^i, \psi^j\} = -i\delta^{ij}. \tag{113}$$

in the Abelian case, with appropriate modifications in the non-Abelian generalization. It follows, that the internal spin rotations are generated by the bilinears

$$s_i = -\frac{i}{2}\varepsilon_{ijk}\psi^j\psi^k, \qquad \{s_i, s_j\} = \varepsilon_{ijk}s_k. \tag{114}$$

The Hamiltonian in flat space reads

$$H = \frac{1}{2m}\mathbf{\Pi}^2 - \frac{q}{m}\mathbf{B}\cdot\mathbf{s}, \tag{115}$$

and satisfies the supersymmetric bracket relation

$$\{\Omega, \Omega\} = -2miH. \tag{116}$$

In such a theory any dynamical quantity of which the bracket with the supercharge vanishes, is automatically a constant of motion:

$$\{Q, \Omega\} = 0 \Rightarrow \{Q, H\} = 0, \tag{117}$$

owing to the Jacobi identity for the brackets. The reverse does not hold in general. Hence the class of superinvariants is a subclass of the constants of motion. For these superinvariants one can derive another more restrictive hierarchy of conditions which are sufficient, though in general not necessary, to obtain solutions of Eqs. (22) or their appropriate generalizations (70) or (97). These equations were derived in [3], hence it is not necessary to elaborate on them in detail. The generating equation is

$$-i\psi^i\left(\nabla_i Q - qF_{ij}\frac{\partial Q}{\partial \Pi_j}\right) + \Pi_i\frac{\partial Q}{\partial \psi_i} = 0, \tag{118}$$

obtained by writing out the bracket $\{\Omega, Q\}$. The hierarchy of square roots of the extended Killing equations is obtained by expanding $Q$ in a series in the momenta $\Pi_i$, and each coefficient $C_{i_1\ldots i_n}(x, \psi)$ in a (finite) polynomial in the Grassmann variables $\psi^i$. In some cases of physical interest the superinvariants do not only include known constants of motion, such as the angular momentum in the case of a monopole field, but also new Grassmann-odd invariants. The coefficients in the expressions for such conserved odd charges are generalizations of the so-called Killing-Yano tensors, rather than the Killing tensors proper. In the case of the magnetic monopole such an anticommuting constant of motion is the nonstandard supercharge [16]

$$\tilde{\Omega} = \varepsilon_{ijk}\left(x^i\Pi^j\psi^k - \frac{i}{3}\psi^i\psi^j\psi^k\right), \tag{119}$$

the bracket of which with itself closes on the square of the angular momentum, rather than on the Hamiltonian:

$$\{\tilde{\Omega}, \tilde{\Omega}\} = -i\left(\mathbf{J}^2 - 2gq\frac{\mathbf{r}\cdot\mathbf{s}}{r}\right), \tag{120}$$

with $\mathbf{J} = \mathbf{r}\times\mathbf{\Pi} + \mathbf{s}$. A generalization for the non-Abelian monopole was constructed in [17].

## X. SUMMARY AND DISCUSSION

In this paper we have developed an algorithm to construct constants of motion for conservative dynamical systems. The method, based on extensions of the Killing equations in differential geometry, works in the presence of gauge interactions and in nonflat geometries as well. It brings out, in particular, the importance of tuning the symmetries of the external fields with those of the geometry of the configuration space. The method has been illustrated with several examples, in particular, monopole-type solutions in Abelian and non-Abelian gauge theories. We have restricted ourselves to classical dynamical systems, but the use of a bracket formulation on phase space allows easy translation—modulo operator ordering—to the case of quantum systems. Extending the particle models with Grassmann variables to describe fermions opens the possibility to include supersymmetries in this framework.

We have discussed, in particular, constants of motion that are polynomial in the momenta. Not all constants of motion are necessarily of this form. In particular, any function $f(Q_\alpha)$ of constants of motion $Q_\alpha$ is itself a constant of motion. Such functions $f(Q_\alpha)$ can be algebraic and contain poles and other singularities. An example is the normalized Runge-Lenz vector

$$\mathbf{M} = \frac{\mathbf{K}}{|2H|}, \tag{121}$$

which is useful in the algebraic solution of the Kepler problem and the isotropic oscillator [18,19], and in the case of the magnetic monopole as well [15]. More generally, for constants of motion of the form

$$Q = \frac{U(\mathbf{r}, \mathbf{\Pi})}{V(\mathbf{r}, \mathbf{\Pi})}, \tag{122}$$

with $V \neq 0$, the condition $\{Q, H\} = 0$ leads to a generalization of Eq. (20) of the form

$$\Pi_i(V\nabla_i U - U\nabla_i V) = qF_{ij}\Pi_i\left(V\frac{\partial U}{\partial \Pi_j} - U\frac{\partial V}{\partial \Pi_j}\right), \tag{123}$$

for Abelian gauge theories, with a suitable generalization for non-Abelian theories.

Apart from the generic importance of constructing constants of motion, the classification of all dynamical variables commuting with the Hamiltonian is a starting point for the procedure of Hamiltonian reduction. This procedure provides an elegant way of constructing nontrivial integrable models; for a recent review see [20]. This technique was applied to derive $N = 4$ supersymmetric mechanics in a monopole background in [21]. The connection with Killing vectors and tensors and their generalizations discussed here provides a geometrical configuration-space description of this reduction process.

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