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Dispersion relations in noncommutative theories

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We present a detailed study of plane waves in noncommutative abelian gauge theories. The dispersion relation is deformed from its usual form whenever a constant background electromagnetic field is present and is similar to that of an anisotropic medium with no Faraday rotation nor birefringence. When the noncommutativity is induced by the Moyal product we find that for some values of the background magnetic field no plane waves are allowed when time is noncommutative. In the Seiberg-Witten context no restriction is found. We also derive the energy-momentum tensor in the Seiberg-Witten case. We show that the generalized Poynting vector obtained from the energy-momentum tensor, the group velocity and the wave vector all point in different directions. In the absence of a constant electromagnetic background we find that the superposition of plane waves is allowed in the Moyal case if the momenta are parallel or satisfy a sort of quantization condition. We also discuss the relation between the solutions found in the Seiberg-Witten and Moyal cases showing that they are not equivalent.

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I. INTRODUCTION

The fact that coordinates and momenta do not commute in quantum theory leads naturally to the proposal that coordinates should also be noncommuting. This would introduce a new scale in the theory which could be used to regulate the divergences in quantum field theory [1] but the success of the renormalization program lead to the dismissal of proposals like this. More recently, however, noncommuting coordinates were found in several settings involving string theory. In particular, there is a decoupling limit of open strings in the presence of D-branes where the effective gauge field theory is defined in a noncommutative space-time induced by the Moyal product [2]

$$A(x) \star B(x) = e^{(i/2)\theta^{\mu\nu}\partial_\mu^x\partial_\nu^y} A(x)B(y)|_{y\to x}, \tag{1.1}$$

where $\theta^{\mu\nu}$ is the noncommutativity parameter [3]. The effect of noncommutativity in quantum field theory is to add phase factors in the vertices which produce a mixture of infrared and ultraviolet divergences usually breaking down renormalizability [4]. The only theories which are known to be free of such a mixing are the supersymmetric ones [5].

In this context the action for an Abelian gauge field is

$$S = -\frac{1}{4}\int d^4x \hat{F}^{\mu\nu} \star \hat{F}_{\mu\nu}, \tag{1.2}$$

where $\hat{F}_{\mu\nu} = \partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu - i[\hat{A}_\mu, \hat{A}_\nu]$ and the brackets denote a Moyal commutator. This action is invariant under a non conventional gauge transformation

$$\delta\hat{A}_\mu = \partial_\mu \hat{\lambda} - i[\hat{A}_\mu, \hat{\lambda}]. \tag{1.3}$$

It is possible to use the Seiberg-Witten map [2]

$$\hat{A}_\mu = A_\mu - \frac{1}{2}\theta^{\alpha\beta}A_\alpha(\partial_\beta A_\mu + F_{\beta\mu}), \tag{1.4}$$

to get a gauge field A_μ with the conventional gauge transformation and an action written in terms of the conventional field strength. In this picture, the action is expressed as a power series in the noncommutativity parameter and, to first order in θ, it is given by

$$S = -\frac{1}{4}\int d^4x\Big[F^{\mu\nu}F_{\mu\nu} + 2\theta^{\mu\rho}F_\rho{}^\nu\Big(F_\mu{}^\sigma F_{\sigma\nu} + \frac{1}{4}\eta_{\mu\nu}F^{\alpha\beta}F_{\alpha\beta}\Big)\Big]. \tag{1.5}$$

In the same way that plane waves can be found in ordinary non-Abelian gauge theories [6] they can also be found in noncommutative theories [7]. A discussion of waves in more general noncommutative space-times can be found in [8,9]. Noncommutativity breaks Lorentz invariance spontaneously due to the existence of a constant matrix $\theta^{\mu\nu}$ and this means that light waves may no longer travel with the velocity of light. In the absence of a background electromagnetic field the usual dispersion relation is found, whether the noncommutativity is induced by the Moyal product or by the Seiberg-Witten map. If a constant electromagnetic background is present the dispersion relation is changed [10–13]. This clearly opens a new window to detect Lorentz violations effects due to noncommutativity. There are several proposals to find out Lorentz violation and use them as evidence for quantum gravity effects [14]. In particular, Lorentz violation due to noncommutativity and quantum gravity effects can be found in

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the standard model [15], high energy gamma ray bursts [16], Cerenkov [17] and synchrotron [18] radiation, among many other examples. It is relevant now to study systematically the modifications induced by noncommutativity in the dispersion relations and that is the aim of this paper.

In the next section we will study plane wave solutions and the corresponding dispersion relations in the Seiberg-Witten map context. We will obtain the complete dispersion relation when an electromagnetic background is present. We find that group and wave velocities have the same magnitude and that the group velocity is not in the direction of the wave vector. To find out the direction in which the energy is being propagated we compute the energy-momentum tensor in Sec. III. We choose the energy-momentum tensor which is conserved and gauge invariant but is neither symmetric nor traceless. We then show that the group velocity and the generalized Poynting vector obtained from the energy-momentum tensor are not in the same direction. All these effects are similar to those characteristic of an anisotropic medium. No analogue of Faraday rotation or birefringence is found since the polarizations travel with the same velocity.

In Sec. IV we look for plane wave solutions in the Moyal product context. We show that there are plane wave solutions to all orders in θ and derive the dispersion relation. The anisotropic effects also show up in this case. Now we find that the background and the noncommutativity are no longer arbitrary and that there are restrictions when the noncommutativity involves time. In the next section we discuss the equivalence of both pictures. We show that plane waves in one picture do not correspond to plane waves in the other one if the background is the same. We also show what is the Moyal picture solution corresponding to plane waves in the Seiberg-Witten context.

Next we show that two plane waves in the Moyal picture case can obey the superposition principle if their four-momenta satisfy $\theta^{\alpha\beta}p_\alpha k_\beta = 2n\pi$, with n an integer. They obey the usual dispersion relation. In particular, if the momenta are in the same direction they can form a wave packet. Finally, in the last section, we present some conclusions and further discussions.

II. SEIBERG-WITTEN MAP PICTURE

In this section we will study some exact solutions to the field equation coming from the action (1.5), that is,

$$\partial_\mu F^{\mu\nu} + \theta^{\alpha\beta}F_\alpha{}^\mu(\partial_\beta F_\mu{}^\nu + \partial_\mu F_\beta{}^\nu) = 0. \qquad (2.1)$$

Clearly, a constant background $F_{\mu\nu}(x) = B_{\mu\nu} =$ constant is a solution. For a plane wave we assume that $F_{\mu\nu}(x) = \tilde{F}_{\mu\nu}(kx)$. Then the Bianchi identity contracted with k^μ tell us that

$$k^2\tilde{F}'_{\mu\nu} + k^\lambda k_\nu \tilde{F}'_{\lambda\mu} - k^\lambda k_\mu \tilde{F}'_{\lambda\nu} = 0, \qquad (2.2)$$

where F' denotes differentiation with respect to kx. Now using the field Eq. (2.1) we get

$$k^2\tilde{F}'_{\mu\nu} + 2\theta^{\alpha\beta}\tilde{F}_\alpha{}^\rho k_\beta k_\rho \tilde{F}'_{\mu\nu} = 0. \qquad (2.3)$$

The field Eq. (2.1) implies that $k^\mu\tilde{F}'_{\mu\nu}$ is of order θ and since F and F' differ by a factor of i, the second term in (2.3) can be disregarded. Then $k^2\tilde{F}'_{\mu\nu} = 0$. We then conclude that for a plane wave the usual dispersion relation $k^2 = 0$ holds. After using the Bianchi identity back in the field equation we get $k^\mu\tilde{F}'_{\mu\nu} = 0$ showing that the plane wave is transversal like in the commutative case.

Let us now consider the case of a superposition of a constant background $B_{\mu\nu}$ and a plane wave $\tilde{F}_{\mu\nu}(kx)$. Then the field Eq. (2.1) becomes

$$k_\mu\tilde{F}'^{\mu\nu} + \theta^{\alpha\beta}(B_\alpha{}^\mu + \tilde{F}^\mu_\alpha)(k_\beta\tilde{F}'^\nu_\mu + k_\mu\tilde{F}'^\nu_\beta) = 0. \qquad (2.4)$$

The quadratic terms in $\tilde{F}$ can be disregarded once we use the Bianchi identities to turn them into the form $k_\mu\tilde{F}'^{\mu\nu}$ and then using the fact that it is of order θ. The equation of motion then reduces to

$$\tilde{k}_\mu\tilde{F}'^{\mu\nu} = 0, \qquad (2.5)$$

where

$$\tilde{k}_\mu = k_\mu + \theta^{\alpha\beta}B_{\alpha\mu}k_\beta - \theta_\mu{}^\alpha B_\alpha{}^\beta k_\beta. \qquad (2.6)$$

This is quite interesting since the Bianchi identity is written with respect to k_μ as

$$k_\mu\tilde{F}'_{\nu\rho} + k_\nu\tilde{F}'_{\rho\mu} + k_\rho\tilde{F}'_{\mu\nu} = 0, \qquad (2.7)$$

while the field equation is written with respect to the modified wave vector $\tilde{k}_\mu$. If we now contract the Bianchi identity with $\tilde{k}^\rho$ we get $k_\mu\tilde{k}^\mu = 0$ or

$$k^2 = -2\theta^{\alpha\beta}B_\alpha{}^\rho k_\beta k_\rho. \qquad (2.8)$$

Since $k_\mu\tilde{k}^\mu = 0$ we can use (2.6) to get $\tilde{k}^2 = -k^2$. Notice that (2.8) is the first sign that the plane wave velocity may not be equal to the velocity of light in the presence of a background.

To solve (2.8) we take $k^\mu = (\omega, \vec{k})$ so that $\vec{k}$ can be used to decompose all vectors in components parallel and perpendicular to it, $\vec{V} = V_L\hat{k} + \vec{V}_T$ with $\vec{k}\cdot\vec{V}_T = 0$, and $\hat{k} = \vec{k}/|\vec{k}|$. We also introduce the vectors $\vec{\theta}$ and $\vec{\tilde{\theta}}$ as $\theta^{ij} = \epsilon^{ijk}\theta^k$ and $\theta^{0i} = \tilde{\theta}^i$, respectively, and use the vectors $\vec{\mathcal{E}}$ and $\vec{\mathcal{B}}$ for the background $B^{0i} = \mathcal{E}^i$ and $B^{ij} = \epsilon^{ijk}\mathcal{B}^k$, respectively. An analogous decomposition is used for $\tilde{F}^{\mu\nu}$. With this notation (2.8) takes the form

$$\frac{\vec{k}^2}{\omega^2} = 1 - 2\Big[\vec{\mathcal{E}}_T\cdot\vec{\tilde{\theta}}_T + \frac{1}{\omega}\vec{k}\cdot(\vec{\mathcal{B}}_T\times\vec{\tilde{\theta}}_T)\Big] - 2\Big[\vec{\mathcal{B}}_T\cdot\vec{\theta}_T - \frac{1}{\omega}\vec{k}\cdot(\vec{\mathcal{E}}_T\times\vec{\theta}_T)\Big], \qquad (2.9)$$

which gives the dispersion relation

$$\omega = |\vec{k}|[1 + \vec{\mathcal{E}}_T \cdot \vec{\tilde{\theta}}_T + \vec{\mathcal{B}}_T \cdot \vec{\theta}_T + \hat{k} \cdot (\vec{\mathcal{B}}_T \times \vec{\tilde{\theta}}_T - \vec{\mathcal{E}}_T \times \vec{\theta}_T)]. \tag{2.10}$$

This reproduces the results found in [10–12,19] for several particular cases.

Notice that the frequency is now dependent on the direction of wave vector, a characteristic of anisotropic media. We can also compute the phase and group velocities for each mode. The phase velocity can be found to be

$$v_p = 1 + \vec{\mathcal{E}}_T \cdot \vec{\tilde{\theta}}_T + \vec{\mathcal{B}}_T \cdot \vec{\theta}_T + \hat{k} \cdot (\vec{\mathcal{B}}_T \times \vec{\tilde{\theta}}_T - \vec{\mathcal{E}}_T \times \vec{\theta}_T) = \frac{\omega}{|\vec{k}|}, \tag{2.11}$$

and also depends on the wave vector direction. The group velocity is given by

$$\vec{v}_g = (1 + \vec{\mathcal{E}}_T \cdot \vec{\tilde{\theta}}_T + \vec{\mathcal{B}}_T \cdot \vec{\theta}_T)\hat{k} - \tilde{\theta}_L \vec{\mathcal{E}}_T - \mathcal{E}_L \vec{\tilde{\theta}}_T - \theta_L \vec{\mathcal{B}}_T - \mathcal{B}_L \vec{\theta}_T + \vec{\mathcal{B}} \times \vec{\tilde{\theta}} - \vec{\mathcal{E}} \times \vec{\theta}, \tag{2.12}$$

and it is not in the same direction as the wave vector. It has a component in the direction of the wave vector which has the same magnitude of the phase velocity and a transversal component which is first order in θ. Then, both phase and group velocities have the same magnitude $v_g = v_p$ to order θ. Notice also that (2.6) defines a modified wave vector

$$\vec{\tilde{k}} = |\vec{k}|[(1 - 2\mathcal{E}_L \tilde{\theta}_L + 2\vec{\mathcal{B}}_T \cdot \vec{\theta}_T)\hat{k} - \tilde{\theta}_L \vec{\mathcal{E}}_T - \mathcal{E}_L \vec{\tilde{\theta}}_T - \theta_L \vec{\mathcal{B}}_T - \mathcal{B}_L \vec{\theta}_T - \vec{\mathcal{E}} \times \vec{\theta} + \vec{\mathcal{B}} \times \vec{\tilde{\theta}}]. \tag{2.13}$$

We can compute its vector product with the group velocity up to order θ^2. The result is nonvanishing meaning that the modified wave vector and the group velocity are not in the same direction. It remains to be seen whether the plane wave energy is transported along the direction of $\vec{v}_g$. This will be done in next section.

Since $k_\mu \tilde{k}^\mu = 0$ the field Eq. (2.5) reduces to $\tilde{k}_\mu \tilde{A}^\mu = 0$ so that the polarization is orthogonal to $\tilde{k}$. To have a better understanding of this point let us rewrite the Bianchi identity (2.7) in vectorial form as

$$\vec{k} \cdot \vec{\tilde{B}} = 0, \tag{2.14}$$

$$\vec{k} \times \vec{\tilde{E}} - \omega \vec{\tilde{B}} = 0, \tag{2.15}$$

and the field Eq. (2.5) as

$$\vec{\tilde{k}} \cdot \vec{\tilde{E}} = 0, \tag{2.16}$$

$$\vec{\tilde{k}} \times \vec{\tilde{B}} + \tilde{\omega} \vec{\tilde{E}} = 0. \tag{2.17}$$

From (2.14) we learn that the magnetic field is transversal to $\vec{k}$ and can be determined by (2.15) in terms of the transverse electric field $\vec{\tilde{E}}_T$. Then (2.16) tell us that the vector field is transverse to $\vec{\tilde{k}}$ so that its longitudinal component with respect to $\vec{k}$, $\tilde{E}_L$, can be found in terms of $\vec{\tilde{E}}_T$. Finally, (2.17) just reproduces the dispersion relation $\omega\tilde{\omega} - \vec{k} \cdot \vec{\tilde{k}} = 0$, so that $\vec{\tilde{E}}_T$ is not determined. We thus find that the plane wave is transversal and has 2 degrees of freedom and both polarizations travel with the same velocity.

To untangle the relative directions of the several vectors involved let us notice that $\vec{\tilde{E}}$ and $\vec{\tilde{B}}$ are orthogonal to each other and we can use their directions to define two orthogonal directions. The third orthogonal direction is then defined by $\vec{\tilde{E}} \times \vec{\tilde{B}}$. Then taking the scalar product of $\vec{\tilde{B}}$ with (2.15) we find that $\vec{k}$ has a component along $\vec{\tilde{E}} \times \vec{\tilde{B}}$. A similar conclusion holds for $\vec{\tilde{k}}$. From (2.14) we find that $\vec{k}$ can have a component along $\vec{\tilde{E}}$ and similarly from (2.16) $\vec{\tilde{k}}$ can have a component along $\vec{\tilde{B}}$. Notice that in the pure plane wave case, without any background, (2.14), (2.15), (2.16), and (2.17) reduce to the same relations found in the absence of noncommutativity. Then $\vec{\tilde{E}}$, $\vec{\tilde{B}}$ and $\vec{k} = \vec{\tilde{k}}$ are mutually orthogonal vectors.

The next task is to find out the direction in which energy is being transported. To do so we will compute the energy-momentum tensor.

III. THE ENERGY-MOMENTUM TENSOR

The usual properties of the energy-momentum tensor usually do not hold in noncommutative field theories due to the presence of $\theta^{\mu\nu}$. A theory which is invariant under rigid translations gives rise to a conserved energy-momentum tensor $T^{\mu\nu}$ which may not be symmetric. However, it can be symmetrized by the Belinfante procedure. Lorentz invariance, on the other hand, also gives rise to a conserved tensor, $M^{\mu\nu\rho}$, such that $\partial_\rho M^{\rho\mu\nu} = T^{\mu\nu} - T^{\nu\mu}$. In a Lorentz invariant theory $M^{\mu\nu\rho}$ is conserved and $T^{\mu\nu}$ is symmetric but in noncommutative theories we expected to find out an antisymmetric part for $T^{\mu\nu}$. Alternatively, we could enforce a symmetric $T^{\mu\nu}$ in noncommutative theories but then its conservation is compromised [20]. Also, the energy-momentum tensor obtained before and after the Seiberg-Witten map may not be the same [20–24]. Other properties are discussed in [25].

We are interested in finding the direction where the energy is flowing so we need a locally conserved energy-momentum tensor. After the Seiberg-Witten map, the canonical energy-momentum tensor is [22]

$$T^c_{\mu\nu} = 2\Pi_\mu{}^\alpha \partial_\nu A_\alpha - \eta_{\mu\nu}\mathcal{L}, \tag{3.1}$$

where $\Pi_{\mu\nu} = \frac{\delta S}{\delta F_{\mu\nu}}$, S is the action (1.5) and $\mathcal{L}$ its Lagrangian. Notice that $T^c_{\mu\nu}$ is neither symmetric nor traceless. It is conserved on-shell but it is not gauge invariant. We can apply a sort of Belinfante procedure [22] and add a total derivative to $T^c_{\mu\nu}$ in order to get

$$T_{\mu\nu} = 2\Pi_\mu{}^\alpha F_{\nu\alpha} - \eta_{\mu\nu}\mathcal{L}, \tag{3.2}$$

which is also neither symmetric nor traceless, but is gauge invariant. It is also conserved $\partial_\mu T^{\mu\nu} = 0$ if the equations of motion are used. Its explicit form is

$$\begin{aligned} T_{\mu\nu} = {}& \left(1 - \frac{1}{2}\theta^{\alpha\beta}F_{\alpha\beta}\right)F_\mu{}^\rho F_{\rho\nu} - F_\mu{}^\alpha \theta_\alpha{}^\beta F_\beta{}^\gamma F_{\gamma\nu} \\ & - F_\nu{}^\alpha \theta_\alpha{}^\beta F_\beta{}^\gamma F_{\gamma\mu} - \theta_\mu{}^\alpha F_\alpha{}^\beta F_\beta{}^\gamma F_{\gamma\nu} \\ & - \frac{1}{4}\theta_\mu{}^\alpha F_{\alpha\nu}F^2 - \eta_{\mu\nu}\mathcal{L}, \end{aligned} \tag{3.3}$$

and agrees with the results of [22,23]. After a lengthy calculation we can find its components

$$\begin{aligned} T^{00} = {}& \frac{1}{2}(1 + \vec\theta\cdot\vec B)(\vec E^2 + \vec B^2) - (\vec{\tilde\theta}\cdot\vec E)\vec E^2 \\ & - (\vec\theta\cdot\vec E)(\vec E\cdot\vec B), \end{aligned} \tag{3.4}$$

$$\begin{aligned} T^{0i} = {}& (1 - \vec{\tilde\theta}\cdot\vec E + \vec\theta\cdot\vec B)(\vec E\times\vec B)^i - \frac{1}{2}(\vec E^2 - \vec B^2)(\vec{\tilde\theta}\times\vec B)^i \\ & - (\vec E\cdot\vec B)(\vec\theta\times\vec B)^i, \end{aligned} \tag{3.5}$$

$$\begin{aligned} T^{i0} = {}& (1 - \vec{\tilde\theta}\cdot\vec E + \vec\theta\cdot\vec B)(\vec E\times\vec B)^i + \frac{1}{2}(\vec E^2 - \vec B^2)(\vec\theta\times\vec E)^i \\ & - (\vec E\cdot\vec B)(\vec{\tilde\theta}\times\vec E)^i, \end{aligned} \tag{3.6}$$

$$\begin{aligned} T^{ij} = {}& -(1 + \vec\theta\cdot\vec B - \vec{\tilde\theta}\cdot\vec E)(E^iE^j + B^iB^j) \\ & + \frac{1}{2}(\vec E^2 - \vec B^2)(\tilde\theta^iE^j + B^i\theta^j) - (\vec E\cdot\vec B)(B^i\tilde\theta^j - \theta^iE^j) \\ & + \delta^{ij}\left[\frac{1}{2}(1 - \vec{\tilde\theta}\cdot\vec E)(\vec E^2 + \vec B^2) + (\vec\theta\cdot\vec B)\vec B^2 \right. \\ & \left. + (\vec{\tilde\theta}\cdot\vec B)(\vec E\cdot\vec B)\right]. \end{aligned} \tag{3.7}$$

Some pieces were already known, in particular, cases. For instance, when $\vec{\tilde\theta} = 0$, (3.4) agrees with the result in [23].

The presence of an antisymmetric part in the energy-momentum tensor requires some care. We can still interpret T^{i0} as a sort of generalized Poynting vector and T^{00} as an energy density because $T^{\mu\nu}$ is locally conserved. Notice that T^{00} does not seem to be positive definite. The noncommutative contributions proportional to $\vec E^2$ and $\vec B^2$ are harmless because $\theta^{\mu\nu}F_{\mu\nu} \ll 1$ and $\frac{1}{2}(\vec E^2 + \vec B^2)$ is larger than them. The noncommutative term with $\vec E\cdot\vec B$ is also small than the commutative term $\frac{1}{2}(\vec E^2 + \vec B^2)$ because $\vec E\cdot\vec B \le \frac{1}{2}(\vec E^2 + \vec B^2)$. So, somewhat surprisingly, the energy density (3.4) is positive definite for small noncommutativity.

To find out the direction of the energy flux let us manipulate (2.14), (2.15), (2.16), and (2.17). We can use (2.15) to find that

$$\vec{\tilde B}^2 = \frac{\vec k^2}{\omega^2}\vec{\tilde E}^2 - \frac{1}{\omega^2}(\vec k\cdot\vec{\tilde E})^2. \tag{3.8}$$

By (2.9) we know that $\vec k^2/\omega^2 = 1 + \mathcal{O}(\theta)$. From (2.17) we find that $\vec k\cdot\vec{\tilde E} = -\frac{1}{\tilde\omega}\vec{\tilde B}\cdot(\vec k\times\vec{\tilde k})$, but from (2.13) we get that $\vec k\times\vec{\tilde k}$ is of order θ and so is $\vec k\cdot\vec{\tilde E}$. Then, from (3.8) we find that also $\vec{\tilde E}^2 - \vec{\tilde B}^2$ is of order θ.

Consider first the case of vanishing background. Since $\vec{\tilde E}$ and $\vec{\tilde B}$ are orthogonal to each other and $\vec{\tilde E}^2 - \vec{\tilde B}^2 = 0$ only the first term of T^{i0} contributes and the energy flux is in the direction of $\vec{\tilde E}\times\vec{\tilde B}$. We can now take a time average and only the quadratic terms will survive. This means that all noncommutative contributions vanish and we get the commutative Poynting vector as a result. We can also take the time average of T^{00}. All noncommutative contributions are cubic in the fields and vanish. We get the same energy density as in the commutative case. It is quite interesting that in the absence of a background the noncommutative plane wave behaves like in the commutative case.

Let us return to the case where the background is present. Now $\vec E^2 - \vec B^2$ is no longer of order θ but proportional to the background fields and the plane wave. Also, $\vec E\cdot\vec B$ no longer vanishes because of the background contribution. So T^{i0} will in general have all terms present. Even if we take a time average many terms will survive. This means that the direction of the energy flux will be the direction $\vec E\times\vec B$ plus small noncommutative corrections. Notice also that both $\vec E$ and $\vec B$ depend on the background so the direction of the energy flux will be background dependent.

We can now check whether the energy flux is in the direction of the group velocity. We can take the vector product of the time averaged T^{i0} with the group velocity to order θ^2 and verify that it does not vanish. Therefore, the direction of the Poynting vector and the group velocity do not coincide. Also, the vector product with either the wave vector $\vec k$ or the modified wave vector $\vec{\tilde k}$ does not vanish confirming the anisotropic properties produced by the background. Since the polarizations travel with the same velocity neither Faraday rotation nor birefringence is present.

IV. MOYAL PRODUCT PICTURE

In the Moyal product picture the field equation derived from (1.2) is

$$\hat D_\mu \hat F^{\mu\nu} = \partial_\mu \hat F^{\mu\nu} - i[\hat A_\mu, \hat F^{\mu\nu}] = 0. \tag{4.1}$$

The solution for a constant background is [7]

$$\hat A_\mu = -\frac{1}{2}B_{\mu\nu}x^\nu, \tag{4.2}$$

with $B_{\mu\nu}$ again constant. Notice that $B_{\mu\nu}$ does not need to be antisymmetric. The field strength is given by

$$\hat{F}_{\mu\nu} = B_{\mu\nu} + \frac{1}{4}\theta^{\alpha\beta}B_{\mu\alpha}B_{\nu\beta}, \tag{4.3}$$

and it satisfies the field Eq. (4.1) to all orders in θ. Notice also that the field strength can vanish by an appropriate choice of the background.

For a plane wave we choose [7]

$$\hat{A}_\mu(x) = \tilde{A}_\mu(kx), \tag{4.4}$$

and we find that $\hat{F}_{\mu\nu} = k_\mu \tilde{A}'_\nu - k_\nu \tilde{A}'_\mu$ to all orders in θ since the commutator term in (4.1) does not give any contribution. The field equation then reads

$$k^2\tilde{A}'^\mu - k^\mu k_\nu \tilde{A}'^\nu = 0, \tag{4.5}$$

and we find a solution if $k^2 = 0$ and $k^\mu \hat{A}_\mu = 0$. Then a transversal plane wave is also a solution to all orders in θ.

Remarkably, the superposition of a constant background (4.2) and a plane wave (4.4) also constitutes a solution to all orders. To show this we first notice that the field strength is given to all orders in θ by

$$\hat{F}_{\mu\nu} = B_{\mu\nu} + \frac{1}{4}\theta^{\alpha\beta}B_{\mu\alpha}B_{\nu\beta} + \bar{k}_\mu \tilde{A}'_\nu - \bar{k}_\mu \tilde{A}'_\mu, \tag{4.6}$$

where

$$\bar{k}_\mu = k_\mu - \frac{1}{2}\theta^{\alpha\beta}B_{\mu\alpha}k_\beta. \tag{4.7}$$

This means that the effect of the background on the wave vector is to replace it by $\bar{k}^\mu$. Now, by applying the covariant derivative $\hat{D}^\rho$ to the Bianchi identity

$$\hat{D}_\rho \hat{F}_{\mu\nu} + \hat{D}_\nu \hat{F}_{\rho\mu} + \hat{D}_\mu \hat{F}_{\nu\rho} = 0, \tag{4.8}$$

and using the equation of motion (4.1) we find

$$\hat{D}^2\hat{F}_{\mu\nu} - i[\hat{F}_\mu{}^\rho, \hat{F}_{\rho\nu}] + i[\hat{F}_\nu{}^\rho, \hat{F}_{\rho\mu}] = 0. \tag{4.9}$$

For our solution we find, using (4.6), that the commutator terms vanish so that $\hat{D}^2\hat{F}_{\mu\nu} = 0$.

On the other side, taking the covariant derivative of (4.6) we find to all orders in θ that

$$\hat{D}_\rho \hat{F}_{\mu\nu} = \bar{k}_\rho \tilde{F}_{\mu\nu}, \tag{4.10}$$

where $\tilde{F}_{\mu\nu} = \bar{k}_\mu \tilde{A}'_\nu - \bar{k}_\mu \tilde{A}'_\mu$, so that $\hat{D}^2\hat{F}_{\mu\nu} = \bar{k}^2\tilde{F}_{\mu\nu}$. Taking into account that $\hat{D}^2\hat{F}_{\mu\nu} = 0$ we find that $\bar{k}^2 = 0$ or

$$k^2 = 2k_\mu V^\mu - V_\mu V^\mu, \tag{4.11}$$

where

$$V_\mu = \frac{1}{2}\theta^{\alpha\beta}B_{\mu\alpha}k_\beta. \tag{4.12}$$

Going back to the field equation we find, using (4.10), that

$$\hat{D}_\mu \hat{F}^{\mu\nu} = -\bar{k}^\nu \bar{k}_\mu \tilde{A}'^\mu, \tag{4.13}$$

so that $\bar{k}_\mu \tilde{A}^\mu = 0$.

Let us now focus on the plane wave contribution. Its field strength is given by $\tilde{F}_{\mu\nu} = \bar{k}_\mu \tilde{A}'_\nu - \bar{k}_\nu \tilde{A}'_\mu$ and taking into account (4.10) it satisfies

$$\hat{D}_\rho \tilde{F}_{\mu\nu} = \bar{k}_\rho \tilde{F}_{\mu\nu}. \tag{4.14}$$

This means that the Bianchi identity (4.8) for $\tilde{F}$ now reduces to

$$\bar{k}_\mu \tilde{F}'_{\nu\rho} + \bar{k}_\nu \tilde{F}'_{\rho\mu} + \bar{k}_\rho \tilde{F}'_{\mu\nu} = 0, \tag{4.15}$$

while the equation of motion becomes $\bar{k}_\mu \tilde{F}^{\mu\nu} = 0$. This means that the electric and magnetic components are orthogonal to $\vec{\bar{k}}$ and not to $\vec{k}$.

To find the dispersion relation from (4.11) we must first notice that it gives a second degree equation for ω. This means that the solution will depend on the value of the discriminant

$$\Delta = (2\alpha\vec{\bar{k}} - V^0\vec{b})^2 - |\vec{\bar{k}} \times \vec{b}|^2, \tag{4.16}$$

where

$$\vec{b} = \vec{\tilde{\theta}} \times \vec{\mathcal{B}}, \qquad \alpha = 1 - \frac{1}{2}\vec{\mathcal{E}} \cdot \vec{\tilde{\theta}}. \tag{4.17}$$

We could not find a closed form for a generic value of Δ so we will analyze the possible solutions according to the noncommutativity which is present. Since the second degree equation has in general two solutions we choose the one which reproduces the usual dispersion relation in the commutative limit.

In the noncommutative magnetic case, that is when $\vec{\tilde{\theta}} = 0$, the discriminant (4.16) is always positive and it is easy to find the solution

$$\omega = |\vec{k}|\left(\left| \left(1 + \frac{1}{2}\vec{\mathcal{B}}.\vec{\theta}\right)\hat{k} - \frac{1}{2}(\hat{k}\cdot\vec{\mathcal{B}})\vec{\theta} \right| - \frac{1}{2}\hat{k}\cdot(\vec{\mathcal{E}} \times \vec{\theta})\right). \tag{4.18}$$

This result agrees with [26].

In the electric case, $\vec{\theta} = 0$, the discriminant is not positive definite so we have to consider the effect of the background. If the background is purely electric, $\vec{\mathcal{B}} = 0$, then the discriminant is always positive and we have

$$\omega = |\vec{k}| \frac{|\hat{k} - \frac{1}{2}(\vec{k}\cdot\vec{\tilde{\theta}})\vec{\mathcal{E}}|}{1 - \frac{1}{2}\vec{\tilde{\theta}}\cdot\vec{\mathcal{E}}}. \tag{4.19}$$

In the case where the background is purely magnetic, $\vec{\mathcal{E}} = 0$, we find

$$\omega = \frac{|\vec{k}|}{(1 - \frac{1}{4}(\tilde{\vec{\theta}} \times \vec{\mathcal{B}})^2)}\left[\sqrt{1 - \frac{1}{4}|\hat{k} \times (\tilde{\vec{\theta}} \times \vec{\mathcal{B}})|^2} - \hat{k} \cdot (\tilde{\vec{\theta}} \times \vec{\mathcal{B}})\right], \tag{4.20}$$

and the plane wave solution exists only when $1 - \frac{1}{4}|\hat{k} \times (\tilde{\vec{\theta}} \times \vec{\mathcal{B}})|^2 \geq 0$. Then, in the case of space-time noncommutativity we find that there is a restriction for the existence of plane waves in a purely magnetic background. It is curious that when time is noncommutative the quantum theory is problematic since unitarity is lost [27] while here we find a restriction already at the classical level. In the other cases there are no restriction for the existence of plane waves. Notice also that all dispersion relations depend on the wave vector direction so in all cases the presence of an electromagnetic background simulates an anisotropic medium.

V. EQUIVALENCE OF BOTH PICTURES

The Seiberg-Witten map (1.4) is a change of variables which preserves the gauge orbits. Since the physics can not depend on the choice of variables we expect that the results obtained in the two pictures should be equivalent. However, taking the noncommutative dispersion relation (4.11) to order θ, and assuming that the background is the same in both cases, we get $k^2 = 2k_\mu V^\mu$, and not $k^2 = 4k_\mu V^\mu$ as required by (2.8). We can also take the Moyal picture solution of a constant background and a plane wave (4.6) and apply the Seiberg-Witten map to it. The resulting field strength is not of the form $B_{\mu\nu} + \tilde{F}_{\mu\nu}(kx)$ but has extra pieces linear in x^μ so it does not correspond to a superposition of a background with a plane wave in the Seiberg-Witten picture. Hence, the field configurations are not equivalent. Since $F_{\mu\nu}$ is gauge invariant the terms linear in x^μ can not be removed by a gauge transformation. This shows that the solutions are inequivalent and it does not make any sense to try to compare the results in different pictures.

However we can find the solution in the Moyal picture which is equivalent to the superpositions of a background plus a plane wave in the Seiberg-Witten picture. It is given by

$$\hat{A}_\mu(x) = \tilde{A}_\mu(kx) - V_\nu x^\nu \tilde{A}_\mu(kx) - \frac{1}{2}\hat{B}_{\mu\nu}x^\nu, \tag{5.1}$$

with V_μ given by (4.12). Using the Seiberg-Witten map we get

$$A_\mu = \tilde{A}^{(c)}_\mu(kx) - \frac{1}{2}B_{\mu\nu}x^\nu, \tag{5.2}$$

where

$$\tilde{A}^{(c)}_\mu(kx) = [1 + \theta^{\alpha\beta}\tilde{A}_\alpha(kx)k_\beta]\tilde{A}_\mu(kx) + \theta^{\alpha\beta}\tilde{A}_\alpha(kx)\hat{B}_{\beta\mu},$$
$$B_{\mu\nu} = \hat{B}_{\mu\nu} - \frac{3}{4}\hat{B}_{\mu\alpha}\hat{B}_{\nu\beta}. \tag{5.3}$$

This means that $F_{\mu\nu}$ has the form $B_{\mu\nu} + \tilde{F}_{\mu\nu}(kx)$ and describes a plane wave. Notice that $\hat{A}_\mu$ and A_μ in (5.1) and (5.2), respectively, are related by the Seiberg-Witten map if the momentum is also transformed as $\hat{k}^\mu = k^\mu - V^\mu$. Now we get the correct dispersion relation and polarization condition for $\tilde{A}_\mu$ in the Seiberg-Witten picture.

VI. SUPERPOSITION OF PLANE WAVES

We now consider the superposition of two plane waves with different momenta

$$\hat{A}_\mu(x) = \hat{A}_{1\mu}(kx) + \hat{A}_{2\mu}(px), \qquad p^\mu \neq k^\mu. \tag{6.1}$$

The field strength is easily found to be

$$\hat{F}_{\mu\nu} = k_\mu\hat{A}_{1\nu} - k_\nu\hat{A}_{1\mu} + p_\mu\hat{A}_{2\nu} - p_\nu\hat{A}_{2\mu} + 2\sin\left(\frac{k\theta p}{2}\right)(\hat{A}_{1\mu}\hat{A}_{2\nu} - \hat{A}_{1\nu}\hat{A}_{2\mu}), \tag{6.2}$$

where $k\theta p = k_\alpha\theta^{\alpha\beta}p_\beta$. The equation of motion takes the form

$$k_\mu k^{[\mu}\hat{A}_1^{\nu]} + p_\mu p^{[\mu}\hat{A}_2^{\nu]} + 2\sin\left(\frac{k\theta p}{2}\right)(k_\mu + p_\mu)\hat{A}_1^{[\mu}\hat{A}_2^{\nu]} + 2\sin\left(\frac{k\theta p}{2}\right)\left(\hat{A}_{1\mu}p^{[\mu}\hat{A}_2^{\nu]} - \hat{A}_{2\mu}k^{[\mu}\hat{A}_1^{\nu]} + 2\sin\left(\frac{k\theta p}{2}\right)(\hat{A}_{1\mu} - \hat{A}_{2\mu})\hat{A}_1^{[\mu}\hat{A}_2^{\nu]}\right) = 0, \tag{6.3}$$

and it is easily seen that there is a nontrivial solution if

$$k\theta p = 2n\pi, \qquad k^2 = p^2 = 0, \qquad k_\mu\hat{A}_1^\mu = p_\mu\hat{A}_2^\mu = 0, \tag{6.4}$$

where n is an integer. This shows that it is possible to have a superposition of two transversal plane waves in a noncommutative theory if the wave vectors k^μ and p^μ are parallel or satisfy $k\theta p = 2n\pi$. The dispersion relation is the same as in the commutative case. In fact, this can be easily generalized to a finite number of plane waves.

In the Seiberg-Witten picture there is no solution corresponding to a superposition of plane waves. This is due to the nonlinear terms present in the field Eq. (2.1).

VII. CONCLUSIONS

We have shown the existence of plane wave solutions in noncommutative abelian gauge theories. In both pictures they present a deformed dispersion relation in the presence of a electromagnetic background. In the Seiberg-Witten picture the dispersion relation is given by (2.10) while in the Moyal picture it is given by (4.18), (4.19), and (4.20). The dispersion relation depends on the wave vector direction and presents similar properties to those found when we consider the propagation of light in an anisotropic medium. It is worth noticing that if $\tilde{\theta}$ is not vanishing, that is, when the noncommutativity involves time, there are restrictions on the background for the existence of plane wave solutions in the Moyal picture but not in the Seiberg-Witten one. Remarkably, the Moyal picture allows solutions involving a superposition of plane waves. In this case the momenta are either parallel or satisfy (6.4).

In the Seiberg-Witten picture we also discussed the energy-momentum tensor. It can be used to define a generalization of the Poynting vector and energy density to the noncommutative case. The Poynting vector, the group velocity, the wave vector and the modified wave vector all point in different directions. Even so, the generalized Poynting vector represents the transport of energy since it obeys a continuity equation. This means that the effect of the background electromagnetic field in the presence on noncommutativity can be interpreted as an anisotropic medium which presents neither Faraday rotation nor birefringence effects.

We also showed that plane waves in one picture does not correspond to plane waves in the other picture. This means that extreme care must be taken when comparing results in different pictures. Since there are many proposed tests for Lorentz violation in several settings it is very important to understand the noncommutative contribution to them. The results presented here are just the first steps in this direction.

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