PHYSICAL REVIEW D **75,** 025010 (2007)

Description of 1/4 BPS configurations in minimal type IIB supergravity

Aristomenis Donos
Physics Department, Brown University Providence, Rhode Island 02912, USA
(Received 22 October 2006; published 12 January 2007)

In this paper we present an effort to extend the LLM construction of 1/2 BPS states in minimal IIB supergravity to configurations that preserve 1/4 of the total number of supersymmetries. Following the same techniques we reduce the problem to that of a single scalar which satisfies a nonlinear equation. In particular, the scalar is identified to be the Kahler potential with which a four dimensional base space is equipped.

DOI: 10.1103/PhysRevD.75.025010 PACS numbers: 04.65.+e, 11.30.Pb

I. INTRODUCTION

In supergravity, classical solutions that break some degree of supersymmetry have been studied extensively [1–5]. A particularly elegant study was given for type IIB 1/2 BPS in [5], where the phase space of nonrelativistic fermions was shown to emerge. Not only the conserved charges, angular momentum and five-form flux were shown to agree with energy and particle number but also the equivalence of the full dynamics was demonstrated in [6,7]. The fermionic description of 1/2 BPS states in $\mathcal{N} = 4$ SYM was obtained from matrix model reduction in [8,9]. The reduced matrix model and free fermion picture gave a successful description of giant graviton configurations [8,10–17].

The next step in constructing the full map between IIB string theory with $\mathrm{AdS}_5 \times S^5$ asymptotics and $\mathcal{N} = 4$ SYM theory is the consideration of the map between less supersymmetric states on both sides. On the field theory 1/4 BPS states have been constructed in [18–20], while even less supersymmetric brane configurations were considered in [21] and more recently in [22,23]. By reducing the number of preserved supersymmetries by a factor of 2 one is able to double the dimensionality of the phase space in which the states under consideration live.

Guided by the analysis of [5], where a two dimensional phase space was explicitly shown, we present an effort to enlarge the number of degrees of freedom by reducing the number of preserved supersymmetries to 8. Unfortunately the equation that we are called to solve in the end of our analysis is highly nontrivial and this is where we are forced to stop.

In the beginning of Sec. II we present our ansatz for the metric and the self-dual five-form which has an $SO(4) \times SO(2)$ symmetry. We then proceed to the dimensional reduction of the Killing spinor equation which leads to a differential equation and two projection equations for the Killing spinor. In the following subsection we study the constraints that the background geometry has to obey in order for the Killing spinor to exist. In the last section we argue that Einstein's equations are guaranteed to be satisfied, after having checked the Bianchi identities for the five-form, as a consequence of the integrability of the Killing spinor equation. We have also included two appendices that the reader might find helpful where, among other things, we list the Fierz identities that we used in the main text.

II. GEOMETRIES PRESERVING 1/4 OF THE SUPERSYMMETRIES

The method that we will follow for studying the constraints imposed on the geometry by supersymmetry was originally developed in [2–4,24,25]. Demanding the existence of a Killing spinor and constructing bilinear tensors from that spinor one can find first order equations that relate the fluxes to the metric and also impose constraints on the metric. Studying then the integrability conditions for the Killing spinor equation one can derive a set of field equations that the fluxes and the background geometry obey [26].

A. Reduction of the Killing spinor equation

Having in mind the extension of the LLM geometries [5] to include one more angular quantum number we make the following $SO(4) \times SO(2)$ symmetric ansatz for the metric and self-dual five-form field of minimal IIB SUGRA in ten dimensions

$$\begin{aligned} ds^2 &= g_{\mu\nu}dx^\mu dx^\nu + e^{H+G}d\hat{\Omega}_3^2 + e^{H-G}d\psi^2 \\ F &= \hat{F}_{\mu_1\mu_2}dx^{\mu_1} \wedge dx^{\mu_2} \wedge d\hat{\Omega}_3 + \tilde{F}_{\mu_1\cdots\mu_4}dx^{\mu_1} \wedge \cdots \\ &\quad \wedge dx^{\mu_4} \wedge d\psi. \end{aligned} \tag{1}$$

From the self-duality of the five-form $F = \star_{10} F$ and the Bianchi identity we obtain the relations

$$\hat{F} = \frac{4!}{2!} e^{2G+H} \star_6 \tilde{F} \qquad d\tilde{F} = 0 \qquad d\hat{F} = 0 \tag{2}$$

where the Hodge duality in (2) is meant to be taken with respect to the six dimensional metric $g_{\mu\nu}$ that appears in (1).

Following [5], the first step in the procedure is to dimensionally reduce the ten dimensional Killing spinor equation

$$\mathcal{D}_{\mathcal{M}}\eta = \nabla_M \eta + \frac{\imath}{480}\Gamma^{M_1\ldots M_5}F_{M_1\ldots M_5}\Gamma_M\eta = 0. \tag{3}$$

The product of this procedure will be a differential equation for the Killing spinor in six dimensions and two algebraic equations that come from the reductions on S^3 and S^1 respectively. We decompose the ten dimensional Dirac matrices with Lorentz indices as

$$\Gamma_\mu = \gamma_\mu \otimes \hat{\sigma}_1 \otimes \mathbb{I}_2, \qquad \mu = 1\ldots 6$$
$$\Gamma_{\hat{\mu}} = \mathbb{I}_8 \otimes \hat{\sigma}_2 \otimes \sigma_{\hat{\mu}}, \qquad \hat{\mu} = 7, 8, 9$$
$$\Gamma_{10} = \gamma_7 \otimes \hat{\sigma}_1 \otimes \mathbb{I}_2, \qquad \gamma_7 = \gamma_1\ldots\gamma_6$$

and the Weyl Killing spinor as

$$\eta = \epsilon \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \chi_\alpha = \varepsilon \otimes \chi_\alpha$$

where ϵ is an 8 component spinor on which the six dimensional gamma matrices γ_μ act and χ_α is a Killing spinor on a sphere of unit radius and it satisfies

$$\hat{\nabla}_{\hat{a}}\chi_\alpha = \frac{i\alpha}{2}\sigma_{\hat{a}}\chi_\alpha, \qquad a = \pm 1.$$

Under the above decomposition the ten dimensional Weyl condition reads

$$\Gamma_{11}\eta = \Gamma_1\ldots\Gamma_{10}\eta = \mathbb{I}_8 \otimes \hat{\sigma}_3 \otimes \mathbb{I}_2\eta = \eta \Rightarrow \hat{\sigma}_3\eta = \eta.$$

For the dependence on the coordinates we have

$$\varepsilon(x^\mu, \hat{\Omega}_3, \psi) = e^{(\imath/2)n\psi}\varepsilon(x^\mu, \hat{\Omega}_3)$$

which gives

$$\imath\partial_\psi\varepsilon = -\frac{n}{2}\varepsilon.$$

It is useful to rewrite the second term in (3) as

$$\begin{aligned} M &= \frac{\imath}{480}\Gamma^{M_1\ldots M_5}F_{M_1\ldots M_5} \\ &= \frac{\imath}{480}[10\Gamma^{M_1M_2}\hat{F}_{M_1M_2}e^{-(3/2)(G+H)}\Gamma^{\hat{a}\hat{b}\hat{c}}\varepsilon_{\hat{a}\hat{b}\hat{c}} \\ &\quad - 5\Gamma^{M_1M_2M_3M_4}\Gamma^\psi e^{-(1/2)(H-G)}\tilde{F}_{M_1M_2M_3M_4}] \\ &= -\frac{1}{8}\hat{\not{F}}\hat{\sigma}_2(1-\hat{\sigma}_3) \end{aligned}$$

where we used the duality (2) and the duality (A1) for the six dimensional gamma matrices. We note that the spin connection components that will be used in the reduction are given by

$$\omega_{\hat{\mu}m\hat{\nu}} = -\tfrac{1}{2}e^{(1/2)(H+G)}\hat{e}_{\hat{\mu}\hat{n}}e^\gamma_m\partial_\gamma(H+G)$$
$$\omega_{\psi m\psi} = -\tfrac{1}{2}e^{(1/2)(H-G)}e^\gamma_m\partial_\gamma(H-G).$$

The covariant derivatives on S^3 and S^1 are then given by the expressions

$$\nabla_{\hat{\mu}} = \hat{\nabla}_{\hat{\mu}} + \tfrac{1}{4}\Gamma_{\hat{\mu}}\Gamma^\lambda\partial_\lambda(G+H)$$
$$\nabla_\psi = \partial_\psi + \tfrac{1}{4}\Gamma_\psi\Gamma^\lambda\partial_\lambda(H-G).$$

where $\hat{\nabla}$ denotes the covariant derivative on the three sphere of unitary radius. Having collected all the necessary ingredients for the reduction we obtain a differential equation and two projection conditions for the six dimensional spinor ε

$$\mathcal{D}_\mu\epsilon = \nabla_\mu\varepsilon - \imath N\gamma_\mu\varepsilon = 0 \tag{4}$$

$$\mathcal{D}_{S^3}\epsilon = \left[\frac{\imath\alpha}{2}e^{-(1/2)(H+G)} - \frac{i}{4}\gamma^\lambda\partial_\lambda(H+G) + N\right]\varepsilon = 0 \tag{5}$$

$$\mathcal{D}_\psi\epsilon = \left[\frac{\imath n}{2}e^{-(1/2)(H-G)} + \frac{1}{4}\gamma_7\gamma^\lambda\partial_\lambda(H-G) - \imath\gamma_7 N\right]\varepsilon = 0 \tag{6}$$

where

$$N = -\tfrac{1}{4}\hat{\not{F}}e^{-(3/2)(G+H)}.$$

We observe that the form of our equations looks very similar to the ones that appear in [27] in the same context of reduction. It is convenient to linearly combine the two projectors (5) and (6) to obtain the equivalent ones

$$\begin{aligned} \mathcal{D}_H\varepsilon &= (\mathcal{D}_{S^3} - \imath\gamma_7\mathcal{D}_\psi)\varepsilon \\ &= \left[\frac{\imath\alpha}{2}e^{-(1/2)(H+G)} + \frac{n}{2}e^{-(1/2)(H-G)}\gamma_7 - \frac{i}{2}\gamma^\lambda\partial_\lambda H\right]\varepsilon \end{aligned} \tag{7}$$

$$\begin{aligned} \mathcal{D}_G\varepsilon &= (\mathcal{D}_{S^3} + \imath\gamma_7\mathcal{D}_\psi)\varepsilon \\ &= \left[\frac{\imath\alpha}{2}e^{-(1/2)(H+G)} - \frac{n}{2}e^{-(1/2)(H-G)}\gamma_7 - \frac{i}{2}\gamma^\lambda\partial_\lambda G + 2N\right]\varepsilon. \end{aligned} \tag{8}$$

As one can observe from (4), when computing the covariant derivative of any bilinear constructed from ε the result will contain the flux. We have, in many cases, found it useful to use (8) in order to eliminate the appearance of the flux from the computation of exterior derivatives of the bilinears. Equation (7) gives us linear relations between bilinears of different ranks.

B. Geometry constraints implied by supersymmetry

As pointed out earlier the study of Eqs. (4)–(6) is made simpler and more transparent with the introduction of bilinear forms which we list below

$$f_1 = \bar{\varepsilon}\gamma_7\varepsilon \tag{9}$$

$$f_2 = \imath\bar{\varepsilon}\varepsilon \tag{10}$$

$$K_\mu = \bar{\varepsilon}\gamma_\mu\varepsilon \tag{11}$$

$$L_\mu = \bar{\varepsilon}\gamma_\mu\gamma_7\varepsilon \tag{12}$$

$$Y_{\mu\lambda} = \imath\bar{\varepsilon}\gamma_{\mu\nu}\gamma_7\varepsilon \tag{13}$$

$$V_{\mu\nu} = \bar{\varepsilon}\gamma_{\mu\nu}\varepsilon \tag{14}$$

$$\Omega_{\mu\nu\lambda} = \imath\bar{\varepsilon}\gamma_{\mu\nu\lambda}\varepsilon. \tag{15}$$

and as we expect, since we are dealing with a spinor in six dimensions we have the appearance of two 2-forms and a 3-form. An equivalent set of (complex) bilinears, and more convenient when considering Fierz identities, is given by

$$Z^+ = \tfrac{1}{2}(-f_1 - \imath f_2) = \bar{\varepsilon_+}\varepsilon_- \tag{16}$$

$$Z^- = \tfrac{1}{2}(f_1 - \imath f_2) = \bar{\varepsilon_-}\varepsilon_+ \tag{17}$$

$$l^+_\mu = \tfrac{1}{2}(L_\mu + K_\mu) = \bar{\varepsilon_+}\gamma_\mu\varepsilon_+ \tag{18}$$

$$l^-_\mu = \tfrac{1}{2}(-L_\mu + K_\mu) = \bar{\varepsilon_-}\gamma_\mu\varepsilon_- \tag{19}$$

$$U^+_{\mu\nu} = \tfrac{1}{2}(V_{\mu\nu} - \imath Y_{\mu\nu}) = \bar{\varepsilon_+}\gamma_{\mu\nu}\varepsilon_- \tag{20}$$

$$U^-_{\mu\nu} = \tfrac{1}{2}(V_{\mu\nu} + \imath Y_{\mu\nu}) = \bar{\varepsilon_-}\gamma_{\mu\nu}\varepsilon_+ \tag{21}$$

$$q^\pm_{\mu\nu\lambda} = \imath\bar{\varepsilon_\pm}\gamma_{\mu\nu\lambda}\varepsilon_\pm \tag{22}$$

where as usually

$$\varepsilon_\pm = \tfrac{1}{2}(\mathbb{I}_8 \pm \gamma_7)\varepsilon \qquad \gamma_7\varepsilon_\pm = \pm\varepsilon_\pm.$$

Because of the duality relation (A1) that the Dirac matrices satisfy, the three-form q^+ is self-dual while the three-form q^- is antiself-dual.

Using the differential equation (4) one can obtain the differential identities that govern the forms listed in (9)–(15)

$$\begin{aligned}\nabla_\mu f_1 &= \frac{\imath}{4}\bar{\varepsilon}(\gamma_\mu\gamma_{\kappa\lambda}\gamma_7 + \gamma_{\kappa\lambda}\gamma_\mu\gamma_7)\varepsilon F^{\kappa\lambda}e^{-(3/2)(G+H)}\\ &= \frac{\imath}{2\cdot 3!}\varepsilon_{\mu\kappa\lambda\rho\sigma\tau}\bar{\varepsilon}(\gamma^{\rho\sigma\tau})\varepsilon F^{\kappa\lambda}e^{-(3/2)(G+H)}\\ &= \frac{1}{2\cdot 3!}\varepsilon_{\mu\kappa\lambda\rho\sigma\tau}\Omega^{\rho\sigma\tau}F^{\kappa\lambda}e^{-(3/2)(G+H)}\\ &= \frac{1}{2}\star(F\wedge\Omega)_\mu e^{-(3/2)(G+H)}. \end{aligned}\tag{23}$$

$$= -\frac{1}{3!}F_{\mu\rho\sigma\tau}\Omega^{\rho\sigma\tau}e^{(1/2)(G-H)} \tag{24}$$

$$\begin{aligned}\nabla_\mu f_2 &= \tfrac{1}{2}\bar{\varepsilon}(\gamma_\lambda g_{\mu\kappa} - g_{\mu\lambda}\gamma_\kappa)\varepsilon F^{\kappa\lambda}e^{-(3/2)(G+H)}\\ &= e^{-(3/2)(G+H)}F_{\mu\lambda}K^\lambda.\end{aligned}\tag{25}$$

$$\begin{aligned}\nabla_\mu K_\rho &= -\frac{\imath}{4}F^{\kappa\lambda}e^{-(3/2)(G+H)}\bar{\varepsilon}(\gamma_\rho\gamma_{\kappa\lambda}\gamma_\mu - \gamma_\mu\gamma_{\kappa\lambda}\gamma_\rho)\varepsilon\\ &= e^{-(3/2)(G+H)}f_2F_{\mu\rho}\\ &\quad - e^{-(3/2)(G+H)}\frac{1}{4}F^{\kappa\lambda}\varepsilon_{\mu\rho\pi\tau\kappa\lambda}Y^{\pi\tau}\end{aligned}\tag{26}$$

$$= e^{-(3/2)(G+H)}f_2F_{\mu\rho} + \tfrac{1}{2}e^{(1/2)(G-H)}F_{\mu\rho\pi\tau}Y^{\pi\tau} \tag{27}$$

$$\begin{aligned}\nabla_\mu L_\rho &= \frac{\imath}{4}F^{\kappa\lambda}e^{-(3/2)(G+H)}\bar{\varepsilon}(\gamma_\rho\gamma_{\kappa\lambda}\gamma_\mu + \gamma_\mu\gamma_{\kappa\lambda}\gamma_\rho)\gamma_7\varepsilon\\ &= e^{-(3/2)(G+H)}\left[F^\lambda_\mu Y_{\lambda\rho} + F^\lambda_\rho Y_{\lambda\mu} + \frac{1}{2}g_{\mu\rho}F^{\kappa\lambda}Y_{\kappa\lambda}\right]\end{aligned}\tag{28}$$

$$\begin{aligned}\nabla_\gamma V_{\delta\epsilon} &= -e^{-(3/2)(G+H)}[g_{\gamma[\epsilon}\Omega_{\delta]\alpha\beta}F^{\alpha\beta} - F^\beta_\gamma\Omega_{\beta\delta\epsilon}\\ &\quad + 2\Omega_{\alpha\gamma[\delta}F^\alpha_{\epsilon]}].\end{aligned}\tag{29}$$

As one expects we have the appearance of higher rank tensors on the right hand sides of the differential equations for the bilinears when comparing to the analysis of [5]. We now take a derivative of (15) giving us

$$\begin{aligned}\nabla_\kappa\Omega_{\mu\nu\lambda} &= -\bar{\varepsilon}(\gamma_{\mu\nu\lambda}N\gamma_\kappa - \gamma_\kappa N\gamma_{\mu\nu\lambda})\varepsilon\\ &= \tfrac{1}{4}e^{-(3/2)(G+H)}F^{\pi\rho}\bar{\varepsilon}(\gamma_{\mu\nu\lambda}\gamma_{\pi\rho}\gamma_\kappa\\ &\quad - \gamma_\kappa\gamma_{\pi\rho}\gamma_{\mu\nu\lambda})\varepsilon.\end{aligned}$$

After antisymmetrization we have that

$$d\Omega_{\kappa\lambda\mu\nu} = 4f_1e^{-(1/2)(H-G)}\tilde{F}_{\kappa\lambda\mu\nu}. \tag{30}$$

The above equation, as we will see later after fixing the form of f_1, fixes the 4-form and it gives us the Bianchi equation for it. One can then use the duality (2) to determine the 2-form $F_{\mu\nu}$, which at this point is not obvious why it will satisfy the Bianchi identity. For the dual 3-form we have the equation

$$\begin{aligned}\nabla_\kappa(\star\Omega)_{\mu\nu\lambda} &= -\bar{\varepsilon}(\gamma_{\mu\nu\lambda}\gamma_7N\gamma_\kappa - \gamma_\kappa N\gamma_{\mu\nu\lambda}\gamma_7)\varepsilon\\ &= -\tfrac{1}{4}e^{-(3/2)(G+H)}F^{\pi\rho}\bar{\varepsilon}(\gamma_{\mu\nu\lambda}\gamma_{\pi\rho}\gamma_\kappa\\ &\quad + \gamma_\kappa\gamma_{\pi\rho}\gamma_{\mu\nu\lambda})\gamma_7\varepsilon.\end{aligned}$$

Antisymmetrizing the last equation in κ, μ, ν, λ we obtain

$$(d\star\Omega)_{\kappa\mu\nu\lambda} = 2\bar{\varepsilon}(\gamma_{\mu\nu\lambda\kappa}N + N\gamma_{\kappa\mu\nu\lambda})\gamma_7\varepsilon \tag{31}$$

which as we see later constrains a 4-dimensional submanifold to a Kahler manifold.

The immediate consequences of the vector bilinears that we formed is the proof of the existence of a Killing vector for the six dimensional metric in (1) and a closed form. The above can be seen by considering the symmetric part of (26) which gives

$$\nabla_{(\mu}K_{\nu)} = 0$$

and the antisymmetric part of (28) which leads to

$$\nabla_{[\mu}L_{\nu]} = 0 \Rightarrow dL = 0.$$

As we show in the appendix the vectors $l^{\pm}$ are null and as a consequence we have that

$$L^2 = -K^2 = 2l^+ \cdot l^- \tag{32}$$

$$K \cdot L = 0. \tag{33}$$

We now follow an argument presented in [3] applied to our chiral spinors. As we prove in the appendix using Fierz identities we have the following relations

$$i_{l^+}q^+ = 0 \qquad i_{l^-}q^- = 0$$

and the dualities of $q^{\pm}$ imply that

$$l^{\pm} \wedge q^{\pm} = 0.$$

After the above observations we conclude that in a coordinate system where the metric takes the form

$$ds^2 = e^+e^- + \delta_{ab}e^a e^b, \qquad a, b = 1 \dots 4 \tag{34}$$

where

$$l^+ = e^+ \qquad l^- = e^-$$

the two three-forms can be written as

$$q^+ = l^+ \wedge I \tag{35}$$

$$q^- = l^- \wedge J \tag{36}$$

where the two-forms

$$I = \tfrac{1}{2}I_{ab}e^a \wedge e^b \tag{37}$$

$$J = \tfrac{1}{2}J_{ab}e^a \wedge e^b. \tag{38}$$

are anti-self-dual with respect to the metric

$$ds^2 = \delta_{ab}e^a e^b, \qquad a, b = 1 \dots 4 \tag{39}$$

with orientation defined by $\varepsilon^{+-abcd} = \varepsilon^{abcd}$. From Eqs. (A8)–(A11) one can also prove that

$$I^a_b I^b_c = -\delta^a_c \qquad J^a_b J^b_c = -\delta^a_c. \tag{40}$$

These two equation imply that the 2-forms I and J consist complex structures for the metric (39) which as we will later prove are in fact equal rendering the four dimensional manifold with metric $\delta_{ab}e^a e^b$ pre-Kahler. From the above considerations we see that

$$\begin{aligned} q^{+\pi\rho\sigma}q^-_{\pi\rho\sigma} &= 3[(l^+ \cdot l^-)(I^{ab}J_{ab}) + 2l^{+m}l^{-n}I_{nk}J^k_m] \\ &= 3(l^+ \cdot l^-)(I^{ab}J_{ab}) \end{aligned} \tag{41}$$

which in combination with (A5) can give us the product $l^+ \cdot l^-$ in terms of the functions Z^+ and Z^-, we will come back to this later.

We can combine Eq. (23) with (25) and (B7)with (B8) to relate f_1 and f_2 to G and H as

$$\partial_\mu f_1 = \tfrac{1}{2}f_1\partial_\mu(H - G) \Rightarrow f_1 = \lambda e^{(1/2)(H-G)} \tag{42}$$

and

$$\partial_\mu f_2 = \tfrac{1}{2}f_2\partial_\mu(H + G) \Rightarrow f_2 = \kappa e^{(1/2)(H+G)}. \tag{43}$$

Combining (23) with (25) and (B5)with (B6) we obtain the relation

$$\partial_\mu e^H = \frac{n}{\kappa}L_\mu \tag{44}$$

and the constrain

$$\frac{\alpha}{\lambda} = -\frac{n}{\kappa}. \tag{45}$$

Adding Eqs. (B12) and (B16) we obtain

$$\Omega_{\mu\nu\lambda}\partial^\lambda H = 0$$

which in combination with (44) yields

$$i_L\Omega = 0 \Rightarrow \tag{46}$$

$$i_{l^+}q^- = i_{l^-}q^+ \tag{47}$$

where we used (A6) and (A7). Equation (47) and the equation

$$i_{l^{\pm}}I = i_{l^{\pm}}J = 0$$

helps us relate the two 2-forms as

$$I = J. \tag{48}$$

which is one of the supersymmetry requirements following from the Killing spinor equation. After this observation we may write

$$\Omega = K \wedge I \tag{49}$$

$$\star\Omega = L \wedge I \tag{50}$$

and from Eqs. (40) and (41) we have that

$$q^{+\pi\rho\sigma}q^-_{\pi\rho\sigma} = 12l^+ \cdot l^-.$$

We now see from the Fierz identity (A5) that we did a little more work to recover the familiar result from [5]

$$L^2 = -K^2 = f_1^2 + f_2^2.$$

We now turn our attention to the algebraic relation (B17) which we will use to express the complex structure I in terms of other bilinears. For this reason we contract Eq. (50) with the vector L^μ to obtain

$$I = \frac{1}{f_1^2 + f_2^2}i_L\star\Omega.$$

We may now use Eqs. (A14)–(A17) to express

$$i_L V = f_1 K.$$

Contracting Eq. (B17) with L^μ yields a relation between the complex structure and the 2-form constructed from bilinears

$$I = -\frac{1}{k}e^{-(1/2)(H+G)}V + \frac{f_1 e^{-(1/2)(G+H)}}{f_1^2 + f_2^2} L \wedge K.$$

Having in mind the closure of the complex structure we are tempted to consider the external derivative of the above equation which in turn gives

$$dI = \frac{1}{2k}e^{-(1/2)(G+H)}d(G+H)\wedge V - \frac{1}{k}e^{-(1/2)(G+H)}dV - \frac{f_1 e^{-(1/2)(G+H)}}{f_1^2+f_2^2} L \wedge dK + L \wedge K \wedge d\left[\frac{f_1 e^{-(1/2)(G+H)}}{f_1^2+f_2^2}\right]. \tag{51}$$

In order to evaluate dV we go back to (29) and we observe that

$$\begin{aligned} dV_{\kappa\mu\nu} &= -\frac{3\imath}{4}e^{-(3/2)(G+H)}F^{\pi\rho}\bar{\varepsilon}(\gamma_{[\mu\nu}\gamma_{\pi\rho}\gamma_{\kappa]} - \gamma_{[\kappa}\gamma_{\pi\rho}\gamma_{\mu\nu]})\varepsilon \\ &= -\frac{3\imath}{12}e^{-(3/2)(G+H)}F^{\pi\rho}\bar{\varepsilon}(\gamma_{\kappa\mu\nu}\gamma_{\pi\rho} - \gamma_{\pi\rho}\gamma_{\kappa\mu\nu})\varepsilon \\ &= \imath\bar{\varepsilon}(\gamma_{\kappa\mu\nu}N - N\gamma_{\kappa\mu\nu})\varepsilon. \end{aligned}$$

As we described before we will exploit that the right hand of the previous equation take and we will use (8) to relate the derivative of V to fields that do not include the flux

$$dV = -\frac{1}{2}V \wedge dG + \frac{n}{2}e^{-(1/2)(H-G)}L \wedge I.$$

At this point we pause our analysis of (51) in order to fix our gauge. As in [5] we use the closed form L to identify a coordinate which we also call y with the analogous geometrical meaning of the product of the radii of S^3 and S^1 that appear in our ansatz (1). We make the gauge choice

$$\begin{aligned} e^+ &= X(dt + A_m dx^m) + Bdy \\ e^- &= -X(dt + A_m dx^m) + Bdy \\ K &= -Xdt - XA \qquad L = \gamma dy \end{aligned}$$

and from Eqs. (32) and (33) we draw the conclusion that

$$X = B^{-1} = h^{-2} = f_1^2 + f_2^2.$$

At this point the ten dimensional metric has the form

$$ds^2 = -\frac{1}{h^2}(dt+A)^2 + h^2 dy^2 + e^{-G-H}h_{mn}dx^m dx^n + \frac{\gamma n}{k} y e^G d\hat{\Omega}_3^2 + \frac{\gamma n}{k} y e^{-G} d\psi^2, \quad m = 1, \ldots, 4 \tag{52}$$

where we used (44) to fix

$$e^H = \frac{\gamma n}{k} y.$$

For later convenience we have rescaled the vielbein (34) as

$$e^a_\mu = e^{-(1/2)G-(1/2)H}\tilde{e}^a_\mu$$

and with this choice we have defined

$$h_{\mu\nu} = \delta_{ab}\tilde{e}^a_\mu \tilde{e}^b_\nu. \tag{53}$$

We also rescale the complex structure accordingly and define

$$\mathcal{J} = e^{G+H} I \tag{54}$$

which now satisfies

$$\mathcal{J}^m{}_p \mathcal{J}^p{}_n = -\delta^m_n$$

while raising of indices in the above formula is done using (53).

We now resume the analysis of (51) and express the derivative of K as

$$dK = \frac{f_2^2 - f_1^2}{f_2^2 + f_1^2} dG \wedge K + dH \wedge K - (f_2^2 + f_1^2)dA.$$

After a little algebra Eq. (51) takes the form

$$dI = -(dH + dG)\wedge I + f_1 e^{-(1/2)(G+H)} L \wedge dA. \tag{55}$$

In terms of $\mathcal{J}$ Eq. (55) reads

$$d\mathcal{J} = f_1 e^{(1/2)(G+H)} L \wedge dA. \tag{56}$$

It is instructive to split the operation of external differentiation as

$$d = \tilde{d} + d_y + d_t$$

defined as

$$\begin{aligned} \tilde{d}f &= \partial_{x^i} f \wedge dx^i, \qquad i = 1\ldots 4 \\ d_y f &= \partial_y f \wedge dy \qquad d_t f = \partial_t f \wedge dt. \end{aligned}$$

After the above splitting we observe from (56) that

$$\tilde{d}\mathcal{J} = 0 \qquad \partial_y \mathcal{J} = \frac{\gamma\lambda n}{k} y \tilde{d}A. \tag{57}$$

From the first equation we see that the four dimensional metric equipped with the metric (53) is Kahler for each y. We now consider (30) and using

$$\Omega = K \wedge I$$

we have that the four-form is given by

$$\begin{aligned} F &= \frac{e^{(1/2)(H-G)}}{4f_1} d\Omega = \frac{e^{(1/2)(H-G)}}{4f_1}[dK \wedge I - K \wedge dI] \\ &= \frac{e^{(1/2)(H-G)}}{4f_1}\left[\frac{f_2^2 - f_1^2}{f_2^2 + f_1^2} dG \wedge K \wedge I + dH \wedge K \wedge I - (f_2^2 + f_1^2) dA \wedge I\right] - \frac{e^{(1/2)(H-G)}}{4f_1} K \wedge [-(dG + dH) \wedge I \\ &\quad + f_1 e^{-(1/2)(G+H)} K \wedge L \wedge dA] \\ &= \frac{e^{(1/2)(H-G)}}{4f_1}\left[-\frac{2f_1^2}{f_1^2 + f_2^2} dG \wedge K \wedge I + f_1 e^{-(1/2)(G+H)} L \wedge K \wedge dA - (f_2^2 + f_1^2) dA \wedge I\right]. \end{aligned} \tag{58}$$

We now consider (23) and contract it with L^μ. This gives us

$$\begin{aligned} L^\mu \partial_\mu f_1^2 &= -\tfrac{1}{4} d\Omega_{\mu\rho\sigma\tau} L^\mu K^\rho I^{\sigma\tau} \\ \gamma f_1^2 \partial_y (G + H) &= \tfrac{1}{4} f_1 e^{-(1/2)(G+H)} (f_1^2 + f_2^2) I^{ij} \tilde{d} A_{ij}. \end{aligned}$$

The last term in the second equation can be found using (57) and noting that

$$\mathcal{J} \wedge \mathcal{J} = -2\sqrt{h} dz \wedge d\bar{z} \wedge dw \wedge d\bar{w}$$

where

$$\sqrt{h} = \left| \begin{matrix} \partial_z \partial_{\bar{z}} K & \partial_w \partial_{\bar{z}} K \\ \partial_z \partial_{\bar{w}} K & \partial_w \partial_{\bar{w}} K \end{matrix} \right|.$$

We also need to remember that the complex structure $\mathcal{J}$ is antiself-dual which leads us to the conclusion

$$\begin{aligned} -4\partial_y \sqrt{h} &= -2\frac{\gamma\lambda n}{k} y \sqrt{h} e^{-G-H} I^{ij} \tilde{d} A_{ij} \\ &= -8\sqrt{h} \frac{\lambda e^{-G} f_1 \partial_y (G+H)}{e^{-\frac{1}{2}(G+H)} (f_1^2 + f_2^2)} \end{aligned}$$

which in the end gives us the equation that related the volume of the four dimensional base space to the scalar G

$$\begin{aligned} \partial_y \ln\sqrt{h} &= \frac{2\lambda^2 e^{-2G} \partial_y G}{k^2 (1 + \frac{\lambda^2}{k^2} e^{-2G})} + \frac{2\lambda^2 e^{-2G} \partial_y H}{k^2 (1 + \frac{\lambda^2}{k^2} e^{-2G})} \\ \partial_y \ln\sqrt{h} &= -\partial_y \ln\left(1 + \frac{\lambda^2}{k^2} e^{-2G}\right) + (1 - z) \partial_y H \end{aligned} \tag{59}$$

where we have set

$$z = \frac{1 - \frac{\lambda^2}{k^2} e^{-2G}}{1 + \frac{\lambda^2}{k^2} e^{-2G}}.$$

At this point we would like to see what the duality (2) has to give. We evaluate the component

$$\begin{aligned} F_{tx} &= \frac{e^{2G+H}}{4!} \epsilon_{tx}{}^{\pi\rho\sigma\tau} F_{\pi\rho\sigma\tau} = \frac{e^{(3/2)(H+G)}}{4! \cdot 4f_1} \epsilon_{tx}{}^{\pi\rho\sigma\tau} d\Omega_{\pi\rho\sigma\tau} \\ &= \frac{1}{8} \frac{e^{(3/2)(G+H)}}{f_1} (f_1^2 + f_2^2)^2 \partial_y A_{x^1} I_{x^2 x^3} \epsilon_x{}^{x^1 x^2 x^3} \\ &= -\frac{1}{4} \frac{e^{(3/2)(G+H)}}{f_1} (f_1^2 + f_2^2)^2 \partial_y A_{x^1} I_x{}^{x^1}. \end{aligned}$$

Where we have used the antiself-duality of I. From (10) we have that

$$F_{tx} = -\tfrac{1}{2} e^{(3/2)(G+H)} f_2 \partial_x G.$$

Equating the right-hand sides of the above two equations we have

$$\begin{aligned} I_x{}^{x^1} \partial_y A_{x^1} &= \frac{2f_1 f_2}{(f_1^2 + f_2^2)^2} \partial_x G \\ \partial_y A_{x^1} &= -\frac{2f_1 f_2}{(f_1^2 + f_2^2)^2} I_x{}^{x^1} \partial_{x^1} G \end{aligned}$$

where we used (40). We can rewrite the above expression in the form

$$\begin{aligned} \partial_y A_{x^1} &= -\frac{2f_1 f_2}{(f_1^2 + f_2^2)^2} I_x{}^{x^1} \partial_{x^1} G \\ &= -2k\lambda e^{-H} \frac{1}{(\lambda^2 e^{-G} + k^2 e^G)^2} I_x{}^{x^1} \partial_{x^1} G \\ &= -\frac{e^{-H}}{2k\lambda^3} I_x{}^{x^1} \partial_{x^1} z. \end{aligned}$$

The last equation together with (57) gives us all the non-trivial components of dA. Imposing the consistency condition $d^2 A = 0$, the $x^1 x^2 x^3$ component of the equation gives us back the closure, while the y, x^1, x^2 component gives us another relation between the Kahler potential and the scalar G

$$2k\lambda^2 e^H \partial_y (e^{-H} \partial_y \mathcal{J}) + d(\mathcal{J} \cdot \partial z) = 0. \tag{60}$$

The last equation may be viewed as the analog of the Laplace equation that appears in the LLM construction of 1/2 BPS states. At this point we have reached two Eqs. (59) and (60) for two scalars, the Kahler potential and G that is used to parametrize the radii of S^3 and S^1 in our ansatz (1). From the definition of the complex structure we come to the conclusion that up to a harmonic function

$$z = -2y\partial_y \left(\frac{1}{y} \partial_y K\right). \tag{61}$$

Using the above equation we can now integrate Eq. (59) and give an equation that governs the Kahler potential[1]

[1]A version of this equation was obtained by O. Lunin [28]

$$\begin{vmatrix} \partial_z\partial_{\bar z}K & \partial_w\partial_{\bar z}K \\ \partial_z\partial_{\bar w}K & \partial_w\partial_{\bar w}K \end{vmatrix} = y\frac{e^{(2/y)\partial_y K}}{2}\left(-2y\partial_y\left(\frac{1}{y}\partial_y K\right)+1\right) \quad (62)$$

where we caution the reader that we have chosen to have a trivial "initial condition" for the differential Eq. (59).

It is important to check that the 2-form that one finds through (2) is closed. Here we list the components of the two-form that we find by dualizing the 4-form

$$F_{tx} = -\tfrac{1}{2}e^{(3/2)(G+H)}f_2\partial_x G$$
$$F_{ty} = -\tfrac{1}{4}\partial_y e^{2(G+H)}$$
$$F_{yx} = \tfrac{1}{4}A_x\partial_y e^{2(G+H)} - \tfrac{1}{8}e^{G+H}(e^G + e^{-G})\mathcal{J}_x^{x^1}\partial_{x^1}z$$
$$F_{x_1x_2} = \tfrac{1}{2}\mathcal{J}_{x_1x_2} - \tfrac{1}{4}y\partial_y\mathcal{J}_{x_1x_2} - \tfrac{1}{4}ye^{2G}\partial_y\mathcal{J}_{x_1x_2} - \tfrac{1}{4}y^2\partial_{[x_1}e^{2G}A_{x_2]}.$$

Checking then whether $dF = 0$ brings us in front of equations which we have already seen in previous considerations. During this check we do not need to use the specific forms (61) or (62) but only the differential equations (59) and (60).

At this point we would like to make some general comments about our solution which is summarized by the metric (52) and the four-form (58). One expects that for every supersymmetric solution that fits in our ansatz (1) we should be able to pick the correct Kahler potential that reproduces it. A particular interesting class of solutions is the half BPS states which were analyzed in [5]. In this case the Kahler potential can be written in terms of the scalar function $z(x_1, x_2, y)$ which was identified in [5]. The identification of coordinates is such that one can identify $z_1 = x_1 + \imath x_2$, $z_2 = r(x_1, x_2, y_{\mathrm{LLM}}, \theta)e^{\imath(\psi_2+t)}$ and $y = y_{\mathrm{LLM}}\sin\theta$ where x_1 and x_2 are the LLM two dimensional coordinates and θ with ψ_2 and the ψ that appears in our ansatz (1) will form the S^3 that appears in the LLM ansatz [5]. The details of this reduction will appear in a paper which is currently under preparation [29].

III. EINSTEIN'S FIELD EQUATIONS

In order to check Einstein's equation one considers the integrability conditions of the Killing spinor equation. In order to do this we could follow the analysis of [27] or [30] and do everything after the dimensional reduction. However, we will follow the discussion of [26] directly for the ten dimensional theory. We now present the argument here for completeness. The solutions that we have tried to describe up to now come from the requirement of existence of a commuting Weyl spinor for which the supercovariant derivative vanishes

$$\mathcal{D}_M\eta = \nabla_M\eta + \frac{\imath}{480}\Gamma^{M_1\ldots M_5}F_{M_1\ldots M_5}\Gamma_M\eta = 0.$$

Since the above relation is true we conclude that the commutator of two supercovariant derivatives should also be zero when applied on the same Killing spinor

$$\mathcal{R}_{MN}\eta = [\mathcal{D}_M, \mathcal{D}_N]\eta = 0.$$

The commutator of the supercovariant derivatives can be found in [31]. Contracting the commutator with the appropriate gamma matrices gives us the relation

$$\frac{1}{2}\Gamma_A^{MN}[\mathcal{D}_M, \mathcal{D}_N] = \frac{1}{2}E_{AM}\Gamma^M + \frac{\imath}{3!}\Gamma^{M_1M_2M_3}\nabla^B F_{AM_1M_2M_3B} \quad (63)$$

where

$$E_{MN} = R_{MN} - \tfrac{1}{2}g_{MN}R - \tfrac{1}{6}F_{MA_1A_2A_3A_4}F_N^{A_1A_2A_3A_4}.$$

One can relate the Bianchi identity of the five-form to its equation of motion through duality. Having this in mind and the fact that the geometries that we described demand a closed five-form we can conclude from (B12) that they satisfy Einstein's field equations.

IV. CONCLUSIONS AND SUMMARY

In this paper we have derived the equation that governs 1/4 BPS states in minimal IIB supergravity. After making an $SO(4) \times SO(2)$ symmetric ansatz for our fields we used the powerful techniques that were developed in [2–4,24,25] to find the constraints imposed on the background geometry by the existence of a Killing spinor. During this process we showed that every field can be solved for in terms of a single scalar function which appears in our equations as the Kahler potential of a four dimensional base space which is Kahler.

A very relevant future topic which needs to be understood is the physical moduli space which parametrizes regular 1/4 BPS solutions. This involves study of regularity conditions which will lead to further reduction of the space. An equally important problem is the recovery of the same moduli space from SYM theory. It is clear that it is related to the dynamics of more than one complex matrices.

ACKNOWLEDGMENTS

We would like to thank O. Lunin for the useful private discussions and his comments on this work. We would also like to thank A. Jevicki, R. McNees and R. de Mello Koch for many enlightening and useful discussions.

Note added:—After the appearance of this paper we became aware of [32] where the author, among other properties, has shown an interesting six dimensional Kahler structure for the case of BPS states containing a factor of AdS_3 (or S^3) in type IIB supergravity.

APPENDIX A: FIERZ IDENTITIES

In this appendix we list various identities for the six dimensional Clifford algebra that we used in the main text of the paper. We start by giving the duality relation for the gamma matrices which we heavily use

$$\gamma^{a_1\ldots a_n}=\frac{(-1)^{[n/2]+1}}{(6-n)!}\varepsilon^{a_1\ldots a_n b_{b+1}\ldots b_{6-n}}\gamma_{b_1\ldots b_{6-n}}\gamma_7. \qquad (A1)$$

The Fierz identities that we will use among the bilinear forms can be derived by the basic formula

$$\begin{aligned}\bar\psi_1\psi_2\bar\psi_3\psi_4 = \tfrac{1}{8}[&\bar\psi_1\psi_4\bar\psi_3\psi_2+\bar\psi_1\gamma_7\psi_4\bar\psi_3\gamma_7\psi_2\\ &-\tfrac{1}{2}\bar\psi_1\gamma_{\mu\nu}\psi_4\bar\psi_3\gamma^{\mu\nu}\psi_2\\ &-\tfrac{1}{2}\bar\psi_1\gamma_\mu\gamma_7\psi_4\bar\psi_3\gamma^{\mu\nu}\gamma_7\psi_2]\\ &+\tfrac{1}{8}[\bar\psi_1\gamma_\mu\psi_4\bar\psi_3\gamma^\mu\psi_2\\ &-\bar\psi_1\gamma_\mu\gamma_7\psi_4\bar\psi_3\gamma^\mu\gamma_7\psi_2]\\ &-\tfrac{1}{96}[\bar\psi_1\gamma_{\mu\nu\lambda}\psi_4\bar\psi_3\gamma^{\mu\nu\lambda}\psi_2\\ &-\bar\psi_1\gamma_{\mu\nu\lambda}\gamma_7\psi_4\bar\psi_3\gamma^{\mu\nu\lambda}\gamma_7\psi_2]. \qquad (A2)\end{aligned}$$

where we consider commuting spinors.

Using $\bar\psi_1=\bar\varepsilon_+\gamma_\mu$, $\psi_2=\varepsilon_+$, $\bar\psi_3=\bar\varepsilon_+$ and $\psi_4=\gamma_\mu\varepsilon_+$ in (A2) we obtain

$$l^+_\mu l^{+\mu}=0 \qquad (A3)$$

where we used the equations

$$\gamma_\mu\gamma_\nu\gamma^\mu=-4\gamma_\nu \qquad \gamma_\mu\gamma_{\mu\nu\lambda}\gamma^\mu=0.$$

In a similar manner one can obtain the relation

$$l^-_\mu l^{-\mu}=0. \qquad (A4)$$

Using $\bar\psi_1=\bar\varepsilon_+$, $\psi_2=\varepsilon_-$, $\bar\psi_3=\bar\varepsilon_-$ and $\psi_4=\varepsilon_+$ we obtain

$$Z^+Z^-=\tfrac{1}{4}l^+_\mu l^{-\mu}+\tfrac{1}{48}q^{+\pi\rho\sigma}q^-_{\pi\rho\sigma}. \qquad (A5)$$

Choosing $\bar\psi_1=\bar\varepsilon_+\gamma^\mu$, $\psi_2=\varepsilon_+$, $\bar\psi_3=\imath\bar\varepsilon_+$ and $\psi_4=\gamma_{\mu\nu\lambda}\varepsilon_+$ we have

$$l^{+\mu}q^+_{\mu\nu\lambda}=-l^{+\mu}q^+_{\mu\nu\lambda}\Rightarrow i_{l^+}q^+=0. \qquad (A6)$$

In a similar way one obtains

$$l^{-\mu}q^-_{\mu\nu\lambda}=-l^{-\mu}q^-_{\mu\nu\lambda}\Rightarrow i_{l^-}q^-=0. \qquad (A7)$$

Where we used the relations

$$\gamma^\mu\gamma_\rho\gamma_{\mu\nu\lambda}=4g_{\lambda\rho}\gamma_\nu-4g_{\nu\rho}\gamma_\lambda-2\gamma_{\nu\lambda\rho}$$

$$\gamma^\mu\gamma_{\rho\sigma\tau}\gamma_{\mu\nu\lambda}=-12\delta^{\alpha\beta}_{[\sigma\tau}\gamma_{\rho]}g_{\alpha\lambda}g_{\beta\nu}-2\gamma_{\lambda\nu\rho\sigma\tau}.$$

We consider (A2) with $\bar\psi_1=\bar\varepsilon_\pm\gamma^{\mu\nu}\gamma^\alpha$, $\psi_2=\varepsilon_\pm$, $\bar\psi_3=\bar\varepsilon_\pm$ and $\psi_4=\gamma_\alpha\gamma^{\gamma\delta}\varepsilon_\pm$ which gives

$$\begin{aligned}4\bar\varepsilon_\pm\gamma^{\mu\nu}\gamma^\alpha\varepsilon_\pm\bar\varepsilon_\pm\gamma_\alpha\gamma^{\gamma\delta}\varepsilon_\pm &= \bar\varepsilon_\pm\gamma^{\mu\nu}\gamma^\alpha\gamma_\rho\gamma_\alpha\gamma^{\gamma\delta}\varepsilon_\pm\bar\varepsilon_\pm\gamma^\rho\varepsilon_\pm\\ &\quad-\tfrac{1}{12}\bar\varepsilon_\pm\gamma^{\mu\nu}\gamma^\alpha\gamma_{\rho\sigma\tau}\gamma_\alpha\\ &\quad\times\gamma^{\gamma\delta}\varepsilon_\pm\bar\varepsilon_\pm\gamma^{\rho\sigma\tau}\varepsilon_\pm\\ &=-4\bar\varepsilon_\pm\gamma^{\mu\nu}\gamma_\rho\gamma^{\gamma\delta}\varepsilon_\pm\bar\varepsilon_\pm\gamma^\rho\varepsilon_\pm. \qquad (A8)\end{aligned}$$

We also list the identities

$$\gamma_{\mu\nu}\gamma_\alpha=g_{\alpha\nu}\gamma_\mu-g_{\alpha\mu}\gamma_\nu+\gamma_{\alpha\mu\nu} \qquad (A9)$$

$$\gamma_\alpha\gamma_{\gamma\delta}=g_{\alpha\gamma}\gamma_\delta-g_{\alpha\delta}\gamma_\gamma+\gamma_{\alpha\gamma\delta} \qquad (A10)$$

$$\begin{aligned}\gamma_{\mu\nu}\gamma_\alpha\gamma_{\gamma\delta}&=\gamma_{\alpha\gamma\delta\mu\nu}-6g_{\alpha\pi}g_{\gamma\rho}g_{\delta\sigma}\delta^{[\pi\rho}_{\mu\nu}\gamma^{\sigma]}\\ &\quad+4g_{\alpha\pi}g_{\mu\rho}g_{\nu\sigma}\delta^{\pi[\rho}_{\gamma\delta}\gamma^{\sigma]}\\ &\quad+6g_{\alpha\pi}g_{\gamma\rho}g_{\delta\sigma}\delta^{[\pi}_{[\nu}\gamma^{\rho\sigma]}{}_{\mu]}+2g_{\alpha[\gamma}\gamma_{\delta]\mu\nu} \qquad (A11)\end{aligned}$$

We now look at the Fierz identity involving $\bar\psi_1=\bar\varepsilon_+\gamma^{\gamma\delta}\gamma_\mu$, $\psi_2=\varepsilon_+$, $\bar\psi_3=\bar\varepsilon_-$ and $\psi_4=\gamma_\nu\gamma_{\gamma\delta}\varepsilon_-$ which gives after antisymmetrization in μ and ν

$$\begin{aligned}4\bar\varepsilon_+\gamma^{\gamma\delta}\gamma_{[\mu}\varepsilon_+\bar\varepsilon_-\gamma_{\nu]}\gamma_{\gamma\delta}\varepsilon_- &= -12\bar\varepsilon_+\gamma_{\mu\nu}\varepsilon_-\bar\varepsilon_-\varepsilon_+\\ &\quad+12\bar\varepsilon_+\varepsilon_-\bar\varepsilon_-\gamma_{\mu\nu}\varepsilon_+\\ &\quad-2\bar\varepsilon_+\gamma_{\mu\nu\rho\sigma}\varepsilon_-\bar\varepsilon_-\gamma^{\rho\sigma}\varepsilon_+, \qquad (A12)\end{aligned}$$

considering now the Fierz identity for $\bar\psi_1=\bar\varepsilon_+\gamma_\nu$, $\psi_2=\varepsilon_+$, $\bar\psi_3=\bar\varepsilon_-$ and $\psi_4=\gamma_\mu\varepsilon_-$ we have

$$\begin{aligned}\bar\varepsilon_+\gamma_{\mu\nu\rho\sigma}\varepsilon_-\bar\varepsilon_-\gamma^{\rho\sigma}\varepsilon_+ &= 8\bar\varepsilon_+\gamma_{[\nu}\varepsilon_+\bar\varepsilon_-\gamma_{\mu]}\varepsilon_-\\ &\quad+2\bar\varepsilon_+\gamma_{\mu\nu}\varepsilon_-\bar\varepsilon_-\varepsilon_+\\ &\quad-2\bar\varepsilon_+\varepsilon_-\bar\varepsilon_-\gamma_{\mu\nu}\varepsilon_+.\end{aligned}$$

Finally (A12) takes the form

$$\begin{aligned}\bar\varepsilon_+\gamma^{\gamma\delta}\gamma_{[\mu}\varepsilon_+\bar\varepsilon_-\gamma_{\nu]}\gamma_{\gamma\delta}\varepsilon_- &= -4\bar\varepsilon_+\gamma_{\mu\nu}\varepsilon_-\bar\varepsilon_-\varepsilon_+\\ &\quad+4\bar\varepsilon_+\varepsilon_-\bar\varepsilon_-\gamma_{\mu\nu}\varepsilon_+\\ &\quad-4\bar\varepsilon_+\gamma_{[\nu}\varepsilon_+\bar\varepsilon_-\gamma_{\mu]}\varepsilon_-. \qquad (A13)\end{aligned}$$

Another useful identity is generated by using the choice $\bar\psi_1=\bar\varepsilon_+\gamma_\mu$, $\psi_2=\varepsilon_+$, $\bar\psi_3=\bar\varepsilon_+$ and $\psi_4=\gamma^\mu\gamma_\nu\varepsilon_-$ which leads to

$$\begin{aligned}\bar\varepsilon_+\gamma_\mu\varepsilon_+\bar\varepsilon_+\gamma^\mu\gamma_\nu\varepsilon_-=0\Rightarrow\bar\varepsilon_+\gamma^\mu\varepsilon_+\bar\varepsilon_+\gamma_{\mu\nu}\varepsilon_-\\ =-\bar\varepsilon_+\varepsilon_-\bar\varepsilon_+\gamma_\mu\varepsilon_+l^{+\mu}U^+_{\mu\nu}\\ =-Z^+l^+_\nu. \qquad (A14)\end{aligned}$$

In a similar way one may also prove that

$$l^{-\mu}U^-_{\mu\nu}=-Z^-l^-_\nu. \qquad (A15)$$

Using (35), (36), and (48) in (A13) and after contracting with $\bar{\varepsilon}_+\gamma_\mu\varepsilon_+$ we can prove with the help of (A14) that

$$l^{+\mu}U^-_{\mu\nu} = Z^- l^+_\nu \tag{A16}$$

and in a similar way

$$l^{-\mu}U^+_{\mu\nu} = Z^+ l^-_\nu. \tag{A17}$$

APPENDIX B: ALGEBRAIC EQUATIONS FOR THE BILINEARS

1. Scalar identities

Multiplying (5) by $\imath\bar{\varepsilon}\gamma_7$ and (6) by $\bar{\varepsilon}$ we obtain the relations

$$-\frac{\alpha}{2}e^{-(1/2)(H+G)}f_1 - \frac{1}{4}L^\mu\partial_\mu(H+G) - \frac{1}{4}e^{-(3/2)(G+H)}Y^{\mu\nu}F_{\mu\nu} = 0$$

$$\frac{n}{2}e^{-(1/2)(H-G)}f_2 - \frac{1}{4}L^\mu\partial_\mu(H-G) + \frac{1}{4}e^{-(3/2)(G+H)}Y^{\mu\nu}F_{\mu\nu} = 0.$$

If we now multiply (5) by $\bar{\varepsilon}$ and (6) by $\imath\bar{\varepsilon}\gamma_7$ and equate the real and imaginary parts of the corresponding equations to zero we the relations

$$K^\mu\partial_\mu(H+G) = 0 \tag{B1}$$

$$2\alpha f_2 = e^{-(G+H)}V^{\mu\nu}F_{\mu\nu} \tag{B2}$$

$$K^\mu\partial_\mu(H-G) = 0 \tag{B3}$$

$$-2f_1 n = e^{-(H+2G)}V^{\mu\nu}F_{\mu\nu}. \tag{B4}$$

2. Vector identities

We now consider (5) and its conjugate again but this time we multiply by $-\imath\bar{\varepsilon}\gamma_\mu\gamma_7$ and $-i\gamma_\mu\gamma_7\varepsilon$ respectively and add them. The result of the operation reads

$$\alpha L_\mu e^{-(1/2)(G+H)} + \tfrac{1}{2}f_1\partial_\mu(G+H) + \tfrac{1}{2}\star(\Omega\wedge F)_\mu e^{-(3/2)(G+H)} = 0. \tag{B5}$$

Multiplying (6) and its conjugate by $\imath\bar{\varepsilon}\gamma_\mu\gamma_7$ respectively $\imath\gamma_\mu\gamma_7\varepsilon$ and adding the resulting equations we obtain

$$-ne^{-(1/2)(H-G)}L_\mu + \tfrac{1}{2}f_2\partial_\mu(H-G) - \tfrac{1}{2}e^{-(3/2)(G+H)}F^\lambda_\mu K_\lambda = 0. \tag{B6}$$

We now turn to (6) and we multiply it by $\bar{\varepsilon}\gamma_\mu$ giving back

$$-f_1\partial_\mu(H-G) + \star(F\wedge\Omega)_\mu e^{-(3/2)(G+H)} = 0. \tag{B7}$$

Multiplying (5) by $\bar{\varepsilon}\gamma_\mu$ we obtain

$$f_2\partial_\mu(H+G) - e^{-(3/2)(G+H)}F^\lambda_\mu K_\lambda = 0 \tag{B8}$$

3. Rank two identities

Multiplying (5) and (6) by $\bar{\varepsilon}\gamma_{\mu\nu}$ and $\bar{\varepsilon}\gamma_{\mu\nu}\gamma_7$, considering the identities

$$\gamma_{\mu\nu}\gamma_\lambda = 2g_{\lambda[\nu}\gamma_{\mu]} + \gamma_{\mu\nu\lambda}$$

$$\gamma_\lambda\gamma_{\mu\nu} = -2g_{\lambda[\nu}\gamma_{\mu]} + \gamma_{\mu\nu\lambda}$$

$$\gamma_{\mu\nu}\gamma_{\kappa\lambda} = -2g_{\mu[\kappa}g_{\lambda]\nu} + \gamma_{\mu\nu\kappa\lambda} + 2\gamma_{\kappa[\mu}g_{\nu]\lambda} - 2\gamma_{\lambda[\mu}g_{\nu]\kappa}$$

and taking separately the real and imaginary parts to zero we have

$$\frac{\alpha}{2}e^{-(1/2)(H+G)}Y_{\mu\nu} - \frac{1}{24}\epsilon_{\mu\nu\pi\rho\sigma\lambda}\Omega^{\pi\rho\sigma}\partial^\lambda(H+G) + \frac{1}{2}f_1 e^{-(3/2)(G+H)}F_{\mu\nu} + \frac{1}{8}\epsilon_{\mu\nu\kappa\lambda\pi\rho}F^{\kappa\lambda}V^{\pi\rho} = 0 \tag{B9}$$

$$L_{[\mu}\partial_{\nu]}(H+G) + e^{-(3/2)(G+H)}Y_{\kappa[\mu}F_{\nu]}{}^\kappa = 0 \tag{B10}$$

$$\frac{\alpha}{2}V_{\mu\nu}e^{-(1/2)(H+G)} - \frac{1}{4}K_{[\mu}\partial_{\nu]}(G+H) - \frac{1}{2}e^{-(3/2)(G+H)} \times F_{\mu\nu}f_2 - \frac{1}{8}\epsilon_{\kappa\lambda\mu\nu\pi\rho}Y^{\pi\rho}F^{\kappa\lambda}e^{-(3/2)(G+H)} = 0 \tag{B11}$$

$$\tfrac{1}{4}\Omega_{\mu\nu}{}^\lambda\partial_\lambda(G+H) - V_{\kappa[\mu}F_{\nu]}{}^\kappa e^{-(3/2)(G+H)} = 0 \tag{B12}$$

$$\frac{n}{2}e^{-(1/2)(H-G)}V_{\mu\nu} + \frac{1}{24}\epsilon_{\mu\nu\lambda\pi\rho\sigma}\partial^\lambda(H-G)\Omega^{\pi\rho\sigma} - \frac{1}{2}F_{\mu\nu}e^{-(3/2)(G+H)}f_1 - \frac{1}{8}\epsilon_{\mu\nu\kappa\lambda\pi\rho}F^{\kappa\lambda}V^{\pi\rho}e^{-(3/2)(G+H)} = 0 \tag{B13}$$

$$L_{[\nu}\partial_{\mu]}(H-G) + e^{-(3/2)(G+H)}Y_{\kappa[\mu}F_{\nu]}{}^\kappa = 0 \tag{B14}$$

$$\frac{n}{2}e^{-(1/2)(H-G)}Y_{\mu\nu} + \frac{1}{4}K_{[\nu}\partial_{\mu]}(H-G) - \frac{1}{2}F_{\mu\nu}f_2 e^{-(3/2)(G+H)} - \frac{1}{8}\epsilon_{\mu\nu\kappa\lambda\pi\rho}Y^{\pi\rho}F^{\kappa\lambda}e^{-(3/2)(G+H)} = 0 \tag{B15}$$

$$\tfrac{1}{4}\Omega_{\mu\nu\lambda}\partial^\lambda(H-G) + V_{\kappa[\mu}F_{\nu]}{}^\kappa e^{-(3/2)(G+H)} = 0. \tag{B16}$$

4. Rank three identities

We now multiply (7) by $\bar{\varepsilon}\gamma_{\mu\nu\kappa}$ and take the real and imaginary part separately

$$\begin{aligned} &\alpha e^{-(1/2)(G+H)}\Omega_{\mu\nu\kappa} + \tfrac{1}{2}\epsilon_{\mu\nu\kappa\rho\alpha\beta}Y^{\alpha\beta}\partial^\rho H = 0 \\ &ne^{-(1/2)(H-G)}\star\Omega_{\mu\nu\kappa} + 3\partial_{[\kappa}HV_{\mu\nu]} = 0. \end{aligned} \tag{B17}$$

Doing the same with (8) we obtain the equations

$$\alpha e^{-(1/2)(G+H)}\Omega_{\mu\nu\kappa} + \tfrac{1}{2}\epsilon_{\mu\nu\kappa\rho\alpha\beta}Y^{\alpha\beta}\partial^{\rho}G + 6e^{-(3/2)(G+H)}K_{[\kappa}F_{\mu\nu]} + e^{-(3/2)(G+H)}\epsilon_{\kappa\mu\nu\pi\rho\sigma}F^{\pi\rho}L^{\sigma} = 0$$
$$ne^{-(1/2)(H-G)} \star \Omega_{\mu\nu\kappa} - 3\partial_{[\kappa}GV_{\mu\nu]} - 2e^{-(3/2)(G+H)}F^{\rho}{}_{[\nu}\Omega_{\kappa\mu]\rho} = 0. \tag{B18}$$

[1] O. Lunin, J. M. Maldacena, and L. Maoz, hep-th/0212210.
[2] J. P. Gauntlett, J. B. Gutowski, C. M. Hull, S. Pakis, and H. S. Reall, Classical Quantum Gravity **20**, 4587 (2003).
[3] J. B. Gutowski, D. Martelli, and H. S. Reall, Classical Quantum Gravity **20**, 5049 (2003).
[4] J. P. Gauntlett, D. Martelli, J. Sparks, and D. Waldram, Classical Quantum Gravity **21**, 4335 (2004).
[5] H. Lin, O. Lunin, and J. M. Maldacena, J. High Energy Phys. 10 (2004) 025.
[6] L. Grant, L. Maoz, J. Marsano, K. Papadodimas, and V. S. Rychkov, J. High Energy Phys. 08 (2005) 025.
[7] L. Maoz and V. S. Rychkov, J. High Energy Phys. 08 (2005) 096.
[8] S. Corley, A. Jevicki, and S. Ramgoolam, Adv. Theor. Math. Phys. **5**, 809 (2002).
[9] D. Berenstein, J. High Energy Phys. 07 (2004) 018.
[10] J. McGreevy, L. Susskind, and N. Toumbas, J. High Energy Phys. 06 (2000) 008.
[11] M. T. Grisaru, R. C. Myers, and O. Tafjord, J. High Energy Phys. 08 (2000) 040.
[12] A. Hashimoto, S. Hirano, and N. Itzhaki, J. High Energy Phys. 08 (2000) 051.
[13] I. Bena and D. J. Smith, Phys. Rev. D **71**, 025005 (2005).
[14] V. Balasubramanian, M. Berkooz, A. Naqvi, and M. J. Strassler, J. High Energy Phys. 04 (2002) 034.
[15] S. R. Das, A. Jevicki, and S. D. Mathur, Phys. Rev. D **63**, 024013 (2000).
[16] S. R. Das, A. Jevicki, and S. D. Mathur, Phys. Rev. D **63**, 044001 (2001).
[17] R. de Mello Koch and R. Gwyn, J. High Energy Phys. 11 (2004) 081.
[18] A. V. Ryzhov, J. High Energy Phys. 11 (2001) 046.
[19] E. D'Hoker and A. V. Ryzhov, J. High Energy Phys. 02 (2002) 047.
[20] E. D'Hoker, P. Heslop, P. Howe, and A. V. Ryzhov, J. High Energy Phys. 04 (2003) 038.
[21] A. Mikhailov, J. High Energy Phys. 11 (2000) 027.
[22] I. Biswas, D. Gaiotto, S. Lahiri, and S. Minwalla, hep-th/0606087.
[23] G. Mandal and N. V. Suryanarayana, hep-th/0606088.
[24] J. P. Gauntlett, D. Martelli, S. Pakis, and D. Waldram, Commun. Math. Phys. **247**, 421 (2004).
[25] J. P. Gauntlett, D. Martelli, J. Sparks, and D. Waldram, Adv. Theor. Math. Phys. **8**, 711 (2004).
[26] U. Gran, J. Gutowski, G. Papadopoulos, and D. Roest, Classical Quantum Gravity **23**, 1617 (2006).
[27] J. T. Liu, D. Vaman, and W. Y. Wen, Nucl. Phys. **B739**, 285 (2006).
[28] O. Lunin, Brown University (unpublished).
[29] B. Chen, S. Cremonini, A. Donos, F.-L. Lin, H. Lin, J. T. Liu, D. Vaman, and W. Y. Wen (unpublished).
[30] J. T. Liu and D. Vaman, Phys. Lett. B **642**, 411 (2006).
[31] G. Papadopoulos and D. Tsimpis, Classical Quantum Gravity **20**, L253 (2003).
[32] N. Kim, J. High Energy Phys. 01 (2006) 094.