Teleparallel model for the neutrino

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(Received 4 April 2006; revised manuscript received 10 December 2006; published 10 January 2007)

The main result of the paper is a new representation for the Weyl Lagrangian (massless Dirac Lagrangian). As the dynamical variable we use the coframe, i.e. an orthonormal tetrad of covector fields. We write down a simple Lagrangian—wedge product of axial torsion with a lightlike element of the coframe—and show that variation of the resulting action with respect to the coframe produces the Weyl equation. The advantage of our approach is that it does not require the use of spinors, Pauli matrices, or covariant differentiation. The only geometric concepts we use are those of a metric, differential form, wedge product, and exterior derivative. Our result assigns a variational meaning to the tetrad representation of the Weyl equation suggested by Griffiths and Newing.

DOI: 10.1103/PhysRevD.75.025006 PACS numbers: 04.50.+h, 11.15.−q, 14.60.Lm

I. MAIN RESULT

Throughout this paper we work on a 4-manifold M equipped with prescribed Lorentzian metric g. The main construction presented in this paper is local so we do not make *a priori* assumptions on the geometric structure of spacetime.

In the following two subsections we describe two different models for the neutrino.

A. Traditional model

The accepted mathematical model for a neutrino field is the following linear partial differential equation on M known as the *Weyl equation*:

$$i\sigma^{\alpha}{}_{a\dot{b}}\{\nabla\}_{\alpha}\xi^{a} = 0. \tag{1}$$

The corresponding Lagrangian is

$$L_{\mathrm{Weyl}}(\xi) := \frac{i}{2}(\bar{\xi}^{\dot{b}}\sigma^{\alpha}{}_{a\dot{b}}\{\nabla\}_{\alpha}\xi^{a} - \xi^{a}\sigma^{\alpha}{}_{a\dot{b}}\{\nabla\}_{\alpha}\bar{\xi}^{\dot{b}}) * 1. \tag{2}$$

Here σ^{α}, $\alpha = 0, 1, 2, 3$, are Pauli matrices, ξ is the unknown spinor field, and $\{\nabla\}$ is the covariant derivative with respect to the Levi-Civita connection: $\{\nabla\}_{\alpha}\xi^{a} := \partial_{\alpha}\xi^{a} + \frac{1}{4}\sigma_{\beta}{}^{a\dot{c}}(\partial_{\alpha}\sigma^{\beta}{}_{b\dot{c}} + \{\Gamma\}^{\beta}{}_{\alpha\gamma}\sigma^{\gamma}{}_{b\dot{c}})\xi^{b}$, where $\{\Gamma\}^{\beta}{}_{\alpha\gamma}$ are Christoffel symbols uniquely determined by the metric.

B. Teleparallel model

The purpose of our paper is to give an alternative representation for the Weyl equation (1) and the Weyl Lagrangian (2). To this end, we follow [1] in introducing instead of the spinor field a different unknown—the so-called *coframe*. A coframe is a quartet of real covector fields ϑ^{j}, $j = 0, 1, 2, 3$, satisfying the constraint

$$g = o_{jk}\vartheta^{j} \otimes \vartheta^{k}, \tag{3}$$

where $o_{jk} = o^{jk} := \mathrm{diag}(1, -1, -1, -1)$. In other words, the coframe is a field of orthonormal bases with orthonormality understood in the Lorentzian sense. Of course, at every point of the manifold M the choice of coframe is not unique: there are 6 real degrees of freedom in choosing the coframe and any pair of coframes is related by a Lorentz transformation.

Let us stress that throughout the paper, and, in particular, in formula (3), the metric is assumed to be given (fixed). It is not necessarily the Minkowski metric.

We define an affine connection and corresponding covariant derivative $|\nabla|$ from the conditions

$$|\nabla|\vartheta^{j} = 0. \tag{4}$$

Let us emphasize that we follow [2–5] in employing holonomic coordinates, so in explicit form conditions (4) read $\partial_{\alpha}\vartheta^{j}_{\beta} - |\Gamma|^{\gamma}{}_{\alpha\beta}\vartheta^{j}_{\gamma} = 0$ giving a system of linear algebraic equations for the unknown connection coefficients $|\Gamma|^{\lambda}{}_{\mu\nu}$. The connection defined by the system of equations (4) is called the *teleparallel* or *Weitzenböck* connection.

Let l be a nonvanishing real lightlike teleparallel covector field ($l \cdot l = 0$, $|\nabla|l = 0$). Such a covector field can be written down explicitly as $l = l_{j}\vartheta^{j}$, where l_{j} are real constants (components of the covector l in the basis ϑ^{j}), not all zero, satisfying

$$o^{jk}l_{j}l_{k} = 0. \tag{5}$$

We define our Lagrangian as

$$L(\vartheta^{j}, l_{j}) = l_{i}o_{jk}\vartheta^{i} \wedge \vartheta^{j} \wedge d\vartheta^{k}, \tag{6}$$

where d stands for the exterior derivative. Note that $\frac{1}{3}o_{jk}\vartheta^{j} \wedge d\vartheta^{k}$ is the axial (totally antisymmetric) piece of torsion of the teleparallel connection. (The irreducible decomposition of torsion is described in Appendix B.2 of [6].) Let us emphasize that formula (6) does not explicitly involve connections or covariant derivatives.

The Lagrangian (6) is a rank 4 covariant antisymmetric tensor so it can be viewed as a 4-form and integrated over

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the manifold M to give an invariantly defined action $S(\vartheta^j, l_j) := \int L(\vartheta^j, l_j)$. Independent variation with respect to the coframe ϑ^j and parameters l_j subject to the constraints (3) and (5) produces a pair of Euler-Lagrange equations which we write symbolically as

$$\partial S(\vartheta^j, l_j)/\partial \vartheta^j = 0, \tag{7}$$

$$\partial S(\vartheta^j, l_j)/\partial l_j = 0. \tag{8}$$

Observe now that the Lagrangian (6) and constraints (3) and (5) are invariant under rigid (i.e. with constant coefficients) Lorentz transformations

$$(\vartheta^j, l_j) \mapsto (\Lambda^j{}_k \vartheta^k, (\Lambda^{-1})^k{}_j l_k), \tag{9}$$

where $o_{jk}\Lambda^j{}_p\Lambda^k{}_q = o_{pq}$ and $(\Lambda^{-1})^i{}_j\Lambda^j{}_k = \delta^i{}_k$. This means that any variation of the parameters l_j can be compensated by a rigid variation of the coframe ϑ^j. Hence, the field equation (8) is a consequence of the field equation (7). So further on we assume the parameters l_j to be fixed and study the field equation (7) only.

Remark 1.—The field equations (7) and (8) are written down explicitly in Appendix A. This explicit form is not used in the main text of the paper.

Remark 2.—Variation with respect to the parameters l_j is justified only when the integrals $\int \vartheta^i \wedge \vartheta^j \wedge d\vartheta^k$ are well defined as "proper" global integrals. This imposes severe restrictions on the geometry of spacetime and on the choice of coframes. For example, traveling-wave–type coframes in Minkowski space do not satisfy this condition as in this case the integrals $\int \vartheta^i \wedge \vartheta^j \wedge d\vartheta^k$ diverge. This, however, does not affect the main result of the paper, namely, the fact that Eqs. (1) and (7) are equivalent up to a change of variable (see Theorem 1 below), because here the result is purely local.

C. Equivalence of the two models

Let us define the spinor field ξ as the solution of the system of equations

$$|\nabla|\xi = 0, \tag{10}$$

$$\sigma_{\alpha a\dot{b}}\xi^a\bar{\xi}^{\dot{b}} = \pm l_\alpha = \pm l_j \vartheta^j_\alpha, \tag{11}$$

where $|\nabla|_\alpha \xi^a := \partial_\alpha \xi^a + \frac{1}{4}\sigma_\beta{}^{a\dot{c}}(\partial_\alpha \sigma^\beta{}_{b\dot{c}} + |\Gamma|^\beta{}_{\alpha\gamma}\sigma^\gamma{}_{b\dot{c}})\xi^b$ and the sign is chosen so that the right-hand side lies on the forward light cone. The system (10) and (11) determines the spinor field ξ uniquely up to a complex constant factor of modulus 1. This nonuniqueness is acceptable because we will be substituting ξ into the Weyl equation (1) and Weyl Lagrangian (2) which are both U(1)-invariant. We will call ξ the spinor field *associated* with the coframe ϑ^j.

The main result of our paper is the following.

Theorem 1.—For any coframe ϑ^j we have

$$L(\vartheta^j, l_j) = \pm 4 L_{\mathrm{Weyl}}(\xi), \tag{12}$$

where ξ is the associated spinor field. The coframe satisfies the field equation (7) if and only if the associated spinor field satisfies the Weyl equation (1).

Let us emphasize that all our constructions are local so we do not have to make assumptions on whether our spacetime is orientable, whether it admits a spin bundle, whether this bundle is trivial, etc. This means, of course, that Eq. (12) should be understood in the local sense.

The sign in Eq. (12) depends on the sign of the parameter l_0, on whether the covector $l = l_j\vartheta^j$ lies on the forward or backward light cone, and on the orientation of the coframe (eight different combinations).

The proof of Theorem 1 is given below. The crucial point is explained in Sec. IV, whereas technicalities are handled in a separate section. In the final section we discuss Theorem 1 within the context of known results from the theory of teleparallelism.

II. NOTATION

Our notation follows [2–5]. In particular, in line with the traditions of particle physics, we use Greek letters to denote tensor (holonomic) indices. Details of our spinor notation are given in Appendix A of [5].

We restrict changes of local coordinates on M to those preserving the locally defined orientation. This allows us to define the Hodge star $*$ in the usual way.

We define the forward light cone as the span of $\sigma_{\alpha a\dot{b}}\xi^a\bar{\xi}^{\dot{b}}$, $\xi \neq 0$. We also define

$$\sigma_{\alpha\beta ac} := (1/2)(\sigma_{\alpha a\dot{b}}\epsilon^{\dot{b}\dot{d}}\sigma_{\beta c\dot{d}} - \sigma_{\beta a\dot{b}}\epsilon^{\dot{b}\dot{d}}\sigma_{\alpha c\dot{d}}).$$

These "second order" Pauli matrices are polarized, i.e. $*\sigma = \pm i\sigma$ depending on the choice of "basic" Pauli matrices $\sigma_{\alpha a\dot{b}}$. We assume that $*\sigma = +i\sigma$.

III. EXCLUDING PARAMETER DEPENDENCE

We can always perform a restricted rigid Lorentz transformation (9) which turns an arbitrary set of parameters l_j into $l_j = (\pm 1, 0, 0, \pm 1)$. Our model is invariant under such transformations so it is sufficient to prove Theorem 1 for this particular choice of parameters. Moreover, by changing, if necessary, the sign of $L(\vartheta^j, l_j)$ we can always achieve

$$l_j = (1, 0, 0, 1). \tag{13}$$

Further on we assume the special choice of parameters (13) in which case our Lagrangian (6) takes the form

$$L(\vartheta^j, l_j) = (\vartheta^0 + \vartheta^3) \wedge (\vartheta^0 \wedge d\vartheta^0 - \vartheta^1 \wedge d\vartheta^1 - \vartheta^2 \wedge d\vartheta^2 - \vartheta^3 \wedge d\vartheta^3). \tag{14}$$

IV. B^2-INVARIANCE

The crucial step in the proof of Theorem 1 is the observation that our model is invariant under a certain class of local (i.e. with variable coefficients) Lorentz transformations of the coframe. In order to describe these transformations it is convenient to switch from the real coframe $(\vartheta^0, \vartheta^1, \vartheta^2, \vartheta^3)$ to the complex coframe $(l, m, \bar{m}, n)$, where

$$l := \vartheta^0 + \vartheta^3, \qquad m := \vartheta^1 + i\vartheta^2, \qquad n := \vartheta^0 - \vartheta^3 \tag{15}$$

[here the definition of l is in agreement with Eq. (13)]. In this new notation the Lagrangian (14) and constraint (3) take the form

$$L(\vartheta^j, l_j) = (1/2) l \wedge (n \wedge dl - \bar{m} \wedge dm - m \wedge d\bar{m}), \tag{16}$$

$$g = (1/2)(l \otimes n + n \otimes l - m \otimes \bar{m} - \bar{m} \otimes m). \tag{17}$$

Let us perform the linear transformation of the coframe

$$\begin{pmatrix} l \\ m \\ \bar{m} \\ n \end{pmatrix} \mapsto \begin{pmatrix} l \\ m + fl \\ \bar{m} + \bar{f} l \\ n + f\bar{m} + \bar{f} m + |f|^2 l \end{pmatrix}, \tag{18}$$

where $f: M \to \mathbb{C}$ is an arbitrary scalar function. It is easy to see that both the Lagrangian (16) and the constraint (17) are invariant under the transformation (18), hence the field equation (7) is also invariant.

Invariance of the field equation (7) means that solutions come in equivalence classes: two coframes are said to be equivalent if there exists a scalar function $f: M \to \mathbb{C}$ such that the transformation (18) maps one coframe into the other. In order to understand the group-theoretic nature of these equivalence classes, we note that at every point of the manifold M transformations (18) form a subgroup of the restricted Lorentz group. Moreover, this is a very special subgroup: it is the unique nontrivial Abelian subgroup of the restricted Lorentz group, see Appendix B. It is known that this subgroup, denoted B^2, is the subgroup preserving a given nonzero spinor. Our equivalence classes of coframes can be identified with cosets of B^2, hence they are equivalent to spinors.

Remark 3.—The rigorous statement is that a coset of the subgroup B^2 is equivalent to a spinor up to choice of sign, i.e. spinors ζ and $-\zeta$ correspond to the same coset. This nonuniqueness is acceptable because it is known (see, for example, Sec. 19 in [7] or Sec. 3.5 in [8]) that the sign of a spinor does not have a physical meaning.

Remark 4.—Our construction does not allow us to deal with the zero spinor.

V. TECHNICALITIES

Arguments presented in the previous section show that even though our field equation (7) has no spinors appearing in it explicitly, it is, in fact, a first order differential equation for an unknown spinor field. From this point it is practically inevitable that Eq. (7) is, up to a change of variable, Weyl's equation (1).

The actual proof of Theorem 1 is carried out by means of a straightforward (but lengthy) calculation. The calculation goes as follows.

The set of coframes has four connected components corresponding to two different orientations, $*(l \wedge m) = \pm i(l \wedge m)$, and to l lying on the forward or backward light cone. We assume for definiteness that we are working with coframes satisfying $*(l \wedge m) = +i(l \wedge m)$ and with l lying on the forward light cone.

It is easy to see that our transformation (18) preserves the tensor $l \wedge m$. Moreover, each equivalence class of coframes is completely determined by this tensor. Therefore, it is convenient to identify each equivalence class with a spinor field ζ in accordance with the formula

$$(l \wedge m)_{\alpha\beta} = \sigma_{\alpha\beta ab} \zeta^a \zeta^b. \tag{19}$$

The fact that a decomposable polarized antisymmetric tensor is equivalent to the square of a spinor is a standard one and was extensively used in [2–5].

Resolving Eq. (19) for the coframe $\{l, m, \bar{m}, n\}$, we get the following formulas: l is given by

$$l_\alpha = \sigma_{\alpha a\dot{b}} \zeta^a \bar{\zeta}^{\dot{b}}, \tag{20}$$

n is an arbitrary real (co)vector field satisfying

$$n \cdot n = 0, \qquad l \cdot n = 2, \tag{21}$$

and m is given by

$$m_\beta = (1/2) \sigma_{\alpha\beta ab} n^\alpha \zeta^a \zeta^b. \tag{22}$$

Formula (16) implies

$$* L(\vartheta^j, l_j) = (1/2)\sqrt{|\det g|}\, \varepsilon_{\alpha\beta\gamma\delta} l^\alpha (n^\beta \{\nabla\}^\gamma l^\delta - \bar{m}^\beta \{\nabla\}^\gamma m^\delta - m^\beta \{\nabla\}^\gamma \bar{m}^\delta), \tag{23}$$

where ε is the Levi-Civita symbol, $\varepsilon_{0123} := +1$. Here $\{\nabla\}$ stands for the Levi-Civita covariant derivative which should not be confused with the teleparallel covariant derivative $|\nabla|$. Substituting formulas (20) and (22) into formula (23) and using algebraic properties of Pauli matrices as well as conditions (21), we arrive at

$$* L(\vartheta^j, l_j) = -2i(\bar{\zeta}^{\dot{b}} \sigma^\alpha{}_{a\dot{b}} \{\nabla\}_\alpha \zeta^a - \zeta^a \sigma^\alpha{}_{a\dot{b}} \{\nabla\}_\alpha \bar{\zeta}^{\dot{b}}). \tag{24}$$

Formulas (19) and (24) show that our Lagrangian (16) is a function of $l \wedge m$ rather than of l and m separately. This is, of course, a consequence of the B^2-invariance described in the previous section.

Applying the Hodge star to Eq. (24) and comparing with Eq. (2), we get

$$L(\vartheta^j, l_j) = 4L_{\mathrm{Weyl}}(\zeta). \tag{25}$$

We have $|\nabla|(l \wedge m) = 0$, so formula (19) implies

$$|\nabla|\zeta = 0. \tag{26}$$

Comparing Eqs. (10) and (11) with Eqs. (20) and (26) we conclude that the spinor fields ξ and ζ coincide up to a complex constant factor of modulus 1. The Weyl Lagrangian is U(1)-invariant, so in Eq. (25) we can replace ζ by ξ, arriving at Eq. (12).

As we have established the identity (25) and as each equivalence class of coframes is equivalent to a spinor field ζ, our field equation (7) is equivalent to

$$i\sigma^\alpha{}_{a\dot{b}}\{\nabla\}_\alpha \zeta^a = 0. \tag{27}$$

The Weyl equation is U(1)-invariant, so in Eq. (27) we can replace ζ by ξ, arriving at Eq. (1). This completes the proof of Theorem 1.

The detailed calculation leading to Eq. (24) will be presented in a separate paper.

VI. DISCUSSION

The subject of teleparallelism has a long history dating back to the 1920s. Its origins lie in the pioneering works of Cartan, Einstein, and Weitzenböck. Modern reviews of the physics of teleparallelism are given in [9–15]. Note that Einstein's original papers on the subject are now available in English translation [16].

However, the construction presented in our paper differs from the traditional one. The crucial difference is our choice of Lagrangian (6) which is parameter-dependent and linear in torsion. The vast majority of publications on the subject deal with parameter-independent Lagrangians quadratic in torsion. One particular parameter-independent quadratic Lagrangian has received special attention as it leads to a teleparallel theory of gravity equivalent (in terms of the resulting metric) to general relativity; the explicit formula for this Lagrangian can be found, for example, in [6,11–15,17,18].

Another difference is that in teleparallelism it is traditional to vary the coframe without any constraints. This is because teleparallelism is usually viewed as a framework for alternative theories of gravity and in this setting the metric (3) has to be treated as an unknown. We, on the other hand, vary the coframe subject to the metric constraint (3). This is because we view teleparallelism as a framework for the reinterpretation of quantum electrodynamics and in this setting the metric plays the role of a given background.

It is interesting that our model exhibits similarities with Caroll-Field-Jackiw electrodynamics [19,20]. Both involve a covariantly constant covector field: in our model it is the lightlike covector field l which is covariantly constant with respect to the teleparallel connection, whereas in Caroll-Field-Jackiw electrodynamics it is a timelike covector field which is covariantly constant with respect to the Levi-Civita connection.

Our model also exhibits strong similarities with the "bumblebee model" discussed by Kostelecký [21]: our teleparallel lightlike covector field l plays a role similar to that of the "bumblebee field." Of course, in our case this covector field has a simple physical interpretation: according to Eq. (11) it is the neutrino current.

An interesting approach was previously suggested by Reifler who rewrote [22] the Weyl equation in terms of an isotropic 3-dimensional Euclidean vector field.

Finally, let us note that the fact that the Weyl *equation* can be rewritten in tetrad form is not in itself new as this was done by Griffiths and Newing [1]; see also Appendix A below. Our new result is the tetrad representation (6) for the Weyl *Lagrangian*.

ACKNOWLEDGMENTS

The author is grateful to F. W. Hehl, Y. Itin, J. B. Griffiths, E. W. Mielke, R. D. Morris, and F. J. Reifler for stimulating discussions.

APPENDIX A: EXPLICIT FORM OF THE FIELD EQUATIONS (7) AND (8)

The difficulty with writing down the field equations (7) and (8) explicitly is that one has to vary the coframe ϑ^j and parameters l_j subject to the constraints (3) and (5). The most straightforward way of implementing these constraints is by means of Lagrange multipliers. Such an approach is, however, impractical as it leads to the introduction of extra unknowns. Below we present a compact explicit form of the field equations (7) and (8) which does not use Lagrange multipliers.

Throughout this Appendix we assume the special choice of parameters (13). The general case reduces to (13) by means of a rigid Lorentz transformation (9). As in Sec. V, we assume for definiteness that $*(l \wedge m) = +i(l \wedge m)$ and that l lies on the forward light cone.

The explicit form of the field equation (7) is

$$v = 0, \tag{A1}$$

where

$$v_\alpha := \{\nabla\}^\beta (l \wedge m)_{\alpha\beta} - m^\beta \{\nabla\}_\alpha l_\beta$$

and l and m are elements of the complex coframe (15). Equation (A1) is taken from Griffiths' and Newing's paper [1]: see formula (2.2) in the latter. Of course, Eq. (A1) can also be derived by varying our action $S(\vartheta^j, l_j)$ with respect to the coframe ϑ^j; this calculation is lengthy and will be presented in a separate paper.

Let us examine Eq. (A1) so as to establish the actual number of independent "scalar" equations contained in it and the actual number of independent scalar unknowns. It would seem that (A1) is a system of 4 complex scalar equations (4 being the number of components of the covector v) for 6 real scalar unknowns (6 being the dimension of the Lorentz group). However, it is easy to see that we *a priori* have $l^\alpha v_\alpha = m^\alpha v_\alpha = 0$ so Eq. (A1) is equivalent to the pair of scalar complex equations $\bar{m}^\alpha v_\alpha = n^\alpha v_\alpha = 0$. It is also easy to see that v is invariant under the action of the transformation (18), hence the set of solutions to Eq. (A1) is invariant under this transformation, which means that we are dealing with a pair of complex scalar unknowns (see argument in Sec. IV). Thus, Eq. (A1) is a system of 2 complex scalar equations for 2 complex scalar unknowns, as expected of the Weyl equation.

Note that the scalar $\bar{m}^\alpha v_\alpha$ is also invariant under the action of the transformation (18) and has the following geometric meaning: $\mathrm{Im}(\bar{m}^\alpha v_\alpha) = *L$ where L is the Lagrangian (14), whereas $\mathrm{Re}(\bar{m}^\alpha v_\alpha) = 2\{\nabla\}_\alpha l^\alpha$.

Let us now derive the explicit form of Eq. (8). In doing this one should be careful in setting the values of l_j to (13) only *after* the variations of l_j have been carried out.

Derivation of the explicit form of Eq. (8) means finding critical points of the linear function $S(\vartheta^j, l_j)$ of four real variables l_j subject to the quadratic constraint (5). Elementary arguments show that (13) is a critical point if and only if

$$o_{jk}\int(\vartheta^0 + \vartheta^3)\wedge\vartheta^j\wedge d\vartheta^k = 0,$$

$$o_{jk}\int(\vartheta^1 + i\vartheta^2)\wedge\vartheta^j\wedge d\vartheta^k = 0.$$

Switching to the complex coframe (15) and replacing 4-forms by equivalent scalars we get

$$\mathrm{Im}\int \bar{m}^\alpha v_\alpha\sqrt{|\det g|}dx^0dx^1dx^2dx^3 = 0, \tag{A2}$$

$$\int n^\alpha v_\alpha\sqrt{|\det g|}dx^0dx^1dx^2dx^3 = 0, \tag{A3}$$

where x^α are local coordinates on the manifold M; note that in deriving (A3) we had to integrate by parts in order to get rid of the term with dn. The system (A2) and (A3) is the explicit form of the field equation (8).

In the end of Sec. I B we outlined a general argument explaining why Eq. (8) is a consequence of Eq. (7). Examination of the explicit form of these equations confirms this general observation: Eqs. (A2) and (A3) are indeed a consequence of Eq. (A1).

APPENDIX B: SUBGROUPS OF THE LORENTZ GROUP

A subgroup of the restricted (proper orthochronous) Lorentz group is said to be *weakly irreducible* if the only nondegenerate (with respect to the metric) invariant subspaces of the tangent space are $\{0\}$ and the tangent space itself. The complete list of weakly irreducible subgroups of the restricted Lorentz group is given, with a number of misprints, in Sec. 10.122 of [23].

Let us now look for Abelian subgroups of the restricted Lorentz group. Of course, any 1-dimensional subgroup is Abelian, so further on we will only be interested in Abelian subgroups of dimension greater than 1. One can easily construct a 2-dimensional Abelian subgroup as follows: decompose the tangent space into an orthogonal sum of two nondegenerate 2-dimensional subspaces (one Lorentzian the other Euclidean), consider 1-dimensional subgroups acting on the subspaces (Lorentzian boosts and Euclidean rotations), and then take the product of these two 1-dimensional subgroups. However, such a 2-dimensional Abelian subgroup is not particularly interesting as it is not weakly irreducible.

We call an Abelian subgroup of the restricted Lorentz group *nontrivial* if it has dimension greater than 1 and is weakly irreducible. Examination of the list from Sec. 10.122 of [23] shows that the restricted Lorentz group does indeed have a nontrivial Abelian subgroup and that this nontrivial Abelian subgroup is unique up to conjugation. In $\mathrm{SL}(2,\mathbb{C})$ notation the subgroup in question is written as

$$B^2 := \left\{\begin{pmatrix}1 & f\\ 0 & 1\end{pmatrix}\middle| f\in\mathbb{C}\right\}.$$

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