Supersymmetric Q-lumps in the Grassmannian nonlinear sigma models

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We construct the $\mathcal{N} = 2$ supersymmetric Grassmannian nonlinear sigma model for the massless case and extend it to a massive $\mathcal{N} = 2$ model by adding an appropriate superpotential. We then study their Bogomol'nyi-Prasad-Sommerfield (BPS) equations leading to supersymmetric Q-lumps carrying both topological and Noether charges. These solutions are shown to be always time dependent even sometimes involving multiple frequencies. Thus we illustrate explicitly that the time dependence is consistent with remaining supersymmetries of solitons.

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I. INTRODUCTION

Q-lumps [1] are topological soliton solutions which also carry a conserved Noether charge Q [2] in a class of nonlinear sigma models of massive Kähler or hyper-Kähler models [3]. Unlike the pure topological solitons which are unstable against the size perturbation [4], these configurations are prevented from collapsing through time-dependent internal rotations and the size is determined by the conserved Noether charge. It is essential to have a potential term of specific form which is just a mass term in the linearized theory and it is known that, for a Kähler sigma model, Q-lumps can exist only if the target manifold has a continuous symmetry with at least one fixed point [3].

For any given value of Q, this potential term enables the existence of the Bogomol'nyi-Prasad-Sommerfield (BPS) bound, which guarantees that the Q-lumps minimize their energy. It is known that the specific potential naturally arises from the supersymmetric generalization of the bosonic nonlinear sigma model [3].

Q-lumps involve a nonvanishing kinetic contribution due to their time dependence and such configurations attracted a great deal of recent interest in the study of BPS solutions with nontrivial kinetic terms in both field theory and string theory [5–7]. On the other hand, more *explicit* construction of the supersymmetric generalization of the original massive $O(3)$ [1] and the Kähler [3] nonlinear sigma model solitons in $2 + 1$ dimensions seems to be lacking. In view of the growing interest on the subject, it would be desirable to investigate the roles of the supersymmetries in detail.

In this paper, we study the supersymmetric Grassmannian nonlinear sigma model in $2 + 1$ dimension [8] and its Q-lump solutions. We first construct a $\mathcal{N} = 2$ supersymmetric massless model from $\mathcal{N} = 1$ superfield formalism in constrained variable approach by eliminating the auxiliary fields. We then extend it to the massive $\mathcal{N} = 2$ model by adding an appropriate superpotential term.[1] The corresponding sets of BPS equations can be studied either by the method of completing squares or by directly finding conditions for the remaining supersymmetries. The former leads to sets of conditions for the saturation of the BPS energy bound by charges while the latter leads to conditions for preserving a fraction of supersymmetries. We shall find that the sets of BPS equations from the former are more general than those from the latter unlike the cases of previously known examples. The resulting Q-lump solutions are always time dependent. Thus this illustrates the consistency of time dependence of solutions and remaining supersymmetries. We also discuss the supersymmetric multiply charged Q-lump solution that involves time dependence of many frequencies. The existence of such solutions is highly nontrivial since the sectors of different frequencies are interacting with each other.

The kinetic energy due to the time dependence cannot be relaxed at least classically due to the BPS bound and the conservation of the topological and electric charges. When there are enough number of remaining supersymmetries, the kinetic energy can be protected even from the quantum corrections.

In Sec. II, we will set up our notations and introduce the $\mathcal{N} = 2$ massless Grassmannian nonlinear sigma model. In Sec. III, we extend the massless model to the massive one with $\mathcal{N} = 2$ supersymmetries by adding an appropriate superpotential. We then study the BPS equations by completing squares. In Sec. IV, we obtain conditions for the remaining supersymmetries leading to $1/2$ BPS states and discuss solutions of BPS equations. In Sec. V, the multicharged Q-lump solutions involving many frequencies are constructed. The last section is devoted to concluding remarks.

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[1]There have been some constructions of $d = 2$ $\mathcal{N} = 2$ massive nonlinear sigma models [9]. See also Ref. [10] for the $d = 4$ $\mathcal{N} = 2$ massive sigma models.

II. SUPERSYMMETRIC GRASSMANNIAN MODEL IN THREE DIMENSIONS

In this section we introduce the $\mathcal{N} = 2$ supersymmetric Grassmannian nonlinear sigma model for the massless case. It is basically a nonlinear sigma model with a target space of the Grassmannian manifold but also possesses the $\mathcal{N} = 2$ extended supersymmetries. To make this note self-contained, let us begin by setting up some notations. The superspace $Z = (x, \theta)$ is given by a spacetime coordinate x and an anticommuting coordinate θ which is a two-component Majorana spinor θ [11].

In 2 + 1 dimensions, the Dirac algebra is given by three 2×2 matrices γ^μ with

$$\gamma^0 = \sigma_2, \qquad \gamma^1 = i\sigma_3, \qquad \gamma^2 = i\sigma_1, \tag{2.1}$$

where the σ's denote the Pauli matrices and the index μ runs over 0, 1, 2. They satisfy the Clifford algebra

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}, \tag{2.2}$$

where $\eta^{\mu\nu}$ is the Minkowski metric of signature $(1, -1, -1)$. We introduce the two-component Majorana spinor θ

$$\theta = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}, \qquad \bar\theta = \begin{bmatrix} \bar\theta^1 & \bar\theta^2 \end{bmatrix}, \tag{2.3}$$

satisfying the anticommuting relation, $\{\theta_\alpha, \theta_\beta\} = 0$, where the adjoint of a spinor is, as usual, $\bar\theta = \theta^\dagger \gamma^0$. For bosonic variables, we use the notation $\bar b = b^\dagger$.

The Grassmannian manifold, $Gr(N, M)$, is the homogeneous space defined by $U(N + M)/U(N) \times U(M)$. The $\mathcal{N} = 1$ scalar superfield Φ for the Grassmannian model can be written in component form as

$$\Phi(x, \theta) = \phi(x) + \bar\theta\psi(x) + \tfrac{1}{2}\bar\theta\theta F(x), \tag{2.4}$$

where every component is an $(N + M) \times M$ matrix valued field. The action for the supersymmetric Grassmannian σ model is given by supersymmetrizing the bosonic model [8]

$$S_0 = \int d^3x d^2\theta \tfrac{1}{2} \mathrm{tr}\{\overline{\nabla\Phi}\nabla\Phi + 2\Sigma(\bar\Phi\Phi - I_{(M\times M)})\}, \tag{2.5}$$

where $\nabla_\alpha \Phi = D_\alpha \Phi - i\Phi A_\alpha$ is the gauge covariant derivative with $D_\alpha = \frac{\partial}{\partial\bar\theta^\alpha} - i(\gamma^\mu\theta)_\alpha \partial_\mu$ being the supercovariant derivative, and A_α is a real $M \times M$ matrix spinor gauge superfield given in the Wess-Zumino gauge by

$$A_\alpha = i(\gamma^\mu\theta)_\alpha A_\mu + \tfrac{1}{2}\bar\theta\theta\omega_\alpha. \tag{2.6}$$

The superfield Σ denotes an $M \times M$ matrix valued Lagrange multiplier

$$\Sigma_{(M\times M)} = \sigma + \bar\theta\xi + \tfrac{1}{2}\bar\theta\theta\alpha, \tag{2.7}$$

and the corresponding supersymmetric constraint is $\bar\Phi\Phi = I_{M\times M}$, where $I_{M\times M}$ is the $M \times M$ identity matrix. In component forms, this becomes $\bar\phi\phi = 1$, $\bar\psi\phi = \bar\phi\psi = 0$, $\bar\phi F + \bar F\phi - \bar\psi\psi = 0$, which we assume to hold throughout this paper.

The above action (2.5) is invariant under global $SU(N + M)$ transformation $\Phi \to G\Phi$, $G \in SU(N + M)$ as well as the gauge transformation given with $U \in U(M)$ as

$$\Phi \to \phi\bar U, \qquad A_\alpha \to UA_\alpha\bar U + iU\partial_\alpha\bar U. \tag{2.8}$$

The $U(M)$ covariant derivative is defined as $D_\mu\phi = \partial_\mu\phi + i\phi A_\mu$ with the field strength given by $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]$.

In component form, the supersymmetric Grassmannian model yields

$$\begin{aligned} S_0 = \int d^3x\, \mathrm{tr}\Big\{&(D_\mu\bar\phi)D^\mu\phi + i\bar\psi\gamma^\mu D_\mu\psi + \bar F F \\ &+ \frac{i}{2}\bar\psi\phi\omega - \frac{i}{2}\bar\omega\,\bar\phi\,\psi + \sigma(\bar\phi F - \bar\psi\psi + \bar F\phi) \\ &- \bar\phi\,\bar\xi\,\psi - \bar\psi\xi\phi + \alpha(\bar\phi\phi - 1)\Big\}. \end{aligned} \tag{2.9}$$

By the construction, the system is invariant under the supersymmetry transformations

$$\delta\phi = \bar\epsilon_M\psi, \qquad \delta\bar\phi = \bar\psi\epsilon_M, \tag{2.10}$$

$$\begin{aligned} \delta\psi &= \epsilon_M F - i\gamma^\mu\epsilon_M(D_\mu\phi), \\ \delta\bar\psi &= \bar F\bar\epsilon_M + i(D_\mu\bar\phi)\bar\epsilon_M\gamma^\mu, \end{aligned} \tag{2.11}$$

$$\begin{aligned} \delta F &= -i\bar\epsilon_M\gamma^\mu D_\mu\psi + \frac{i}{2}\bar\epsilon_M\phi\omega, \\ \delta\bar F &= i(D_\mu\bar\psi)\gamma^\mu\epsilon_M - \frac{i}{2}\bar\omega\,\bar\phi\,\epsilon_M, \end{aligned} \tag{2.12}$$

$$\delta A_\mu = \frac{i}{4}(\bar\epsilon_M\gamma_\mu\omega - \bar\omega\gamma_\mu\epsilon_M), \tag{2.13}$$

$$\delta\omega = \tfrac{1}{2}[\gamma^\mu, \gamma^\nu]\epsilon_M F_{\mu\nu}, \qquad \delta\bar\omega = -\tfrac{1}{2}F_{\mu\nu}\bar\epsilon_M[\gamma^\mu, \gamma^\nu], \tag{2.14}$$

where ϵ_M denotes an anticommuting Majorana spinor.

One may eliminate the auxiliary fields using their equations of motion,

$$\begin{aligned} F = -\sigma\phi, \qquad \sigma &= -\tfrac{1}{2}\bar\psi\psi, \\ A_\mu &= i\bar\phi\partial_\mu\phi + \tfrac{1}{2}\bar\psi\gamma_\mu\psi. \end{aligned} \tag{2.15}$$

The resulting action takes the form

$$\begin{aligned} S_0 &= \int d^3x\, \mathrm{tr}\Big\{|D_\mu\phi|^2 + i\bar\psi\gamma^\mu D_\mu\psi + \frac{1}{4}(\bar\psi\psi)^2\Big\} \\ &= \int d^3x\, \mathrm{tr}\Big\{|\partial_\mu\phi|^2 + i\bar\psi\gamma^\mu\partial_\mu\psi \\ &\quad - \Big(i\bar\phi\partial_\mu\phi + \frac{1}{2}\bar\psi\gamma_\mu\psi\Big)^2 + \frac{1}{4}(\bar\psi\psi)^2\Big\}. \end{aligned} \tag{2.16}$$

Here comes the observation of Refs. [12,13], which deal with the case of $CP(n)$ only. Namely the system in (2.16) has in fact $\mathcal{N}=2$ extended supersymmetries. The transformation rules are almost the same as before but the Majorana spinor ϵ_M is now replaced by a complex Dirac spinor ϵ. Explicitly the $\mathcal{N}=2$ SUSY transformation rules are

$$\begin{aligned}&\delta\phi=\bar{\epsilon}\psi,\qquad \delta\bar{\phi}=\bar{\psi}\epsilon,\\&\delta\psi=\tfrac{1}{2}\epsilon\phi\bar{\psi}\psi-i\gamma^{\mu}\epsilon\partial_{\mu}\phi+\gamma^{\mu}\epsilon\phi(i\bar{\phi}\partial_{\mu}\phi+\tfrac{1}{2}\bar{\psi}\gamma^{\mu}\psi),\\&\delta\bar{\psi}=\tfrac{1}{2}\bar{\psi}\psi\bar{\phi}\,\bar{\epsilon}+i\partial_{\mu}\bar{\phi}\,\bar{\epsilon}\,\gamma^{\mu}+(i\bar{\phi}\partial_{\mu}\phi+\tfrac{1}{2}\bar{\psi}\gamma^{\mu}\psi)\bar{\phi}\,\bar{\epsilon}\,\gamma^{\mu}.\end{aligned}\tag{2.17}$$

Note that, in this description, the constraint, $\bar{\phi}\psi=\bar{\psi}\phi=0$, is only solved implicitly but one can check that this condition is invariant under the transformations in (2.17).

For the study of the BPS equations, it is convenient to introduce the complex coordinates,

$$z=\tfrac{1}{\sqrt{2}}(x_1+ix_2),\qquad \bar{z}=\tfrac{1}{\sqrt{2}}(x_1-ix_2),\tag{2.18}$$

and $\partial_{\pm}=\frac{1}{\sqrt{2}}(\partial_1\mp i\partial_2)$, and $A_{\pm}=\frac{1}{\sqrt{2}}(A_1\mp iA_2)$, where $(+,-)$ refer to the holomorphic/antiholomorphic component of vectors in two spatial dimensions.

In considering BPS configurations, we shall consider purely bosonic configuration only by setting all the fermionic part to zero. By completing squares, the Hamiltonian H of the bosonic part only can be rearranged by

$$\begin{aligned}H&=\int d^2x\,\mathrm{tr}(|D_0\phi|^2+|D_+\phi|^2+|D_-\phi|^2)\\&=\int d^2x\,\mathrm{tr}(|D_0\phi|^2+2|D_{\mp}\phi|^2)\mp 2\pi T\geq 2\pi|T|,\end{aligned}\tag{2.19}$$

where the topological vortex number T, which can be written as a boundary term, is defined by

$$\begin{aligned}T&\equiv\frac{i}{2\pi}\int d^2x\,\mathrm{tr}(\epsilon^{ij}(D_i\phi)^{\dagger}D_j\phi)\\&=\frac{i}{2\pi}\oint_{\infty}dx_i\,\mathrm{tr}(\phi^{\dagger}\partial_i\phi).\end{aligned}\tag{2.20}$$

Thus for a given sector of the vortex number, the Hamiltonian is bounded from below. The saturation of the bound occurs if the BPS/anti-BPS equations,

$$\partial_0\phi=0,\qquad D_{\mp}\phi=0,\tag{2.21}$$

are satisfied where we refer the upper/lower signature for the BPS/anti-BPS sector, respectively.

The solutions of these equations correspond to the well-known holomorphic/antiholomorphic vortices of the two-dimensional nonlinear sigma model [8], which are obviously time independent as required by the BPS equations. We shall not discuss their properties any further here.

III. MASSIVE $\mathcal{N}=2$ MODEL AND TIME-DEPENDENT BPS SOLITONS

In this section, we would like to introduce first the massive Grassmannian sigma model with $\mathcal{N}=2$ supersymmetry. One can make the sigma model massive by the introduction of an appropriate superpotential as we shall explain below. We then find the corresponding BPS equations by the methods of completing squares of Hamiltonian as in the previous section.

We begin by introducing a superpotential of the following form,

$$W(\Phi)=\tfrac{1}{2}\mathrm{tr}\,\omega\bar{\Phi}P\Phi,\tag{3.1}$$

where P is the $(N+M)\times(N+M)$ Hermitian projection matrix satisfying $P^2=P$ and ω is a real positive number. Performing the θ integration, the action in a component form reads

$$S_1=\int d^3x\,\omega\,\mathrm{tr}(\bar{\phi}PF+\bar{F}P\phi-\bar{\psi}P\psi).\tag{3.2}$$

Then let us consider the total action $S=S_0+S_1$ with S_0 in (2.9). It has manifest $\mathcal{N}=1$ supersymmetry by construction. Eliminating the auxiliary field by using the equations of motion,

$$\begin{aligned}F=\omega P\phi-\phi\sigma,\qquad &\sigma=\omega\bar{\phi}P\phi-\tfrac{1}{2}\bar{\psi}\psi,\\&A_{\mu}=i\bar{\phi}\partial_{\mu}\phi+\tfrac{1}{2}\bar{\psi}\gamma_{\mu}\psi,\end{aligned}\tag{3.3}$$

the action becomes

$$\begin{aligned}S&=\int d^3x\,\mathrm{tr}\Big\{|D_{\mu}\phi|^2+i\bar{\psi}\gamma^{\mu}D_{\mu}\psi+\Big(\omega\bar{\phi}P\phi-\frac{1}{2}\bar{\psi}\psi\Big)^2\\&\quad-\omega^2\bar{\phi}P\phi+\omega\bar{\psi}P\psi\Big\}\\&=\int d^3x\,\mathrm{tr}\Big\{|\partial_{\mu}\phi|^2+i\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi\\&\quad-\Big(i\bar{\phi}\partial_{\mu}\phi+\frac{1}{2}\bar{\psi}\gamma_{\mu}\psi\Big)^2+\Big(\omega\bar{\phi}P\phi-\frac{1}{2}\bar{\psi}\psi\Big)^2\\&\quad-\omega^2\bar{\phi}P\phi+\omega\bar{\psi}P\psi\Big\}.\end{aligned}\tag{3.4}$$

One may then show that the above action is invariant under the following $\mathcal{N}=2$ supersymmetry transformations

$$\begin{aligned}&\delta\phi=\bar{\epsilon}\psi,\qquad \delta\bar{\phi}=\bar{\psi}\epsilon,\\&\delta\psi=\epsilon(\tfrac{1}{2}\phi\bar{\psi}\psi+\omega(P\phi-\phi\bar{\phi}P\phi))-i\gamma^{\mu}\epsilon\partial_{\mu}\phi\\&\qquad+\gamma^{\mu}\epsilon\phi(i\bar{\phi}\partial_{\mu}\phi+\tfrac{1}{2}\bar{\psi}\gamma^{\mu}\psi),\end{aligned}\tag{3.5}$$

$$\begin{aligned}\delta\bar{\psi}&=(\tfrac{1}{2}\bar{\psi}\psi\bar{\phi}+\omega(\bar{\phi}P-\bar{\phi}P\phi\bar{\phi}))\bar{\epsilon}+i\partial_{\mu}\bar{\phi}\,\bar{\epsilon}\,\gamma^{\mu}\\&\quad+(i\bar{\phi}\partial_{\mu}\phi+\tfrac{1}{2}\bar{\psi}\gamma^{\mu}\psi)\bar{\phi}\,\bar{\epsilon}\,\gamma^{\mu}\end{aligned}\tag{3.6}$$

by a straightforward computation. In addition, one may check that the constraint, $\bar{\phi}\psi=\bar{\psi}\phi=0$, is also invariant under the $\mathcal{N} = 2$ transformations, which insures the consistency of our approach.

Note that the theory possesses also a global symmetry defined by

$$\delta\phi = iP\phi, \qquad \delta\bar{\phi} = -i\bar{\phi}P. \tag{3.7}$$

The expression for the corresponding conserved current density is given by

$$J^\mu = i\,\mathrm{tr}((D^\mu\phi)\bar{\phi}P - P\phi(D^\mu\bar{\phi})), \tag{3.8}$$

and below we shall make use of the resulting Noether charge

$$Q = -i\int d^2x\,\mathrm{tr}((D_0\phi)\bar{\phi}P - P\phi(D_0\bar{\phi})). \tag{3.9}$$

Note that the Hamiltonian, including the contribution from the superpotential, takes a form,

$$H = \int d^2x\,\mathrm{tr}\{|D_0\phi|^2 + |D_i\phi|^2 + \omega^2(\bar{\phi}P\phi - (\bar{\phi}P\phi)^2)\}, \tag{3.10}$$

where only bosonic parts are turned on.

Again by the methods of completing squares, one part of Hamiltonian, H_1, can be rearranged by

$$\begin{aligned} H_1 &= \int d^2x\,\mathrm{tr}\{|D_0\phi|^2 + \omega^2(\bar{\phi}P\phi - (\bar{\phi}P\phi)^2)\} \\ &= \int d^2x\,\mathrm{tr}((1-\phi\bar{\phi})|\partial_0\phi \mp i\omega P\phi|^2) \pm \omega Q \geq \pm\omega Q, \end{aligned} \tag{3.11}$$

and the remaining part by

$$H_2 = \int d^2x\,\mathrm{tr}|D_i\phi|^2 = 2\int d^2x|D_\pm\phi|^2 \pm 2\pi T \geq \pm 2\pi T. \tag{3.12}$$

The inequality, $H \geq \pm 2\pi T \pm \omega Q$, then holds for any independent combination of the first and the second signatures in front of T and Q. Thus we conclude that the Hamiltonian is bounded from below by

$$H \geq 2\pi|T| + \omega|Q|. \tag{3.13}$$

The saturation of the bound occurs if the following equations

$$\partial_0\phi = \pm i\omega P\phi, \qquad D_\pm\phi = 0, \tag{3.14}$$

for any combination of signatures. Namely there are four branches of BPS equations saturating the bounds. From this purely bosonic consideration, the four branches are equally well served as a set of BPS equations, whose solutions are bounded from below by their topological and electric charges.

For instance, with $\partial_0\phi \mp i\omega P\phi = 0$, the Noether charge satisfies

$$\pm Q = 2\omega\int d^2x\,\mathrm{tr}[(1-\phi\bar{\phi})P\phi(P\phi)^\dagger] \geq 0, \tag{3.15}$$

and, in the right side of the inequality of (3.11), the contribution of the charge Q is always positive definite upon the saturation of the bound. Likewise, one can also show that, if $D_\pm\phi = 0$, $\pm T \geq 0$, respectively and, thus, the right-hand side of (3.12) is always positive definite whenever one has the saturation of the inequality.

Among the four branches in (3.14), only two combinations of signatures, $(-,+)$ and $(+,-)$, will be shown to be consistent with the remaining supersymmetries in the next section. The remaining two describe nonsupersymmetric solitons in a strict sense of remaining supersymmetries.

IV. REMAINING SUPERSYMMETRIES AND EXAMPLES

In this section, we would like to show that the time-dependent Q-ball of the previous section preserves 1/2 of the supersymmetries for some particular branches. Since the fermionic part of solutions are assumed to vanish, the variation of the fermionic part has to be zero for any such solutions preserving some supersymmetries. Thus we would like to show that the variation of the fermionic component

$$\begin{aligned} \delta\psi = (&-i\gamma^0 D_0\phi + \omega(P\phi - \phi\bar{\phi}P\phi))\epsilon \\ &- i(\gamma^1 D_1\phi + \gamma_2 D_2\phi)\epsilon \end{aligned} \tag{4.1}$$

vanishes for some nontrivial constant ϵ. The remaining supersymmetries are parametrized by $\epsilon_\pm = p_\pm\epsilon$ where we introduce the projection operators by

$$p_\pm = \frac{1 \mp i\gamma^1\gamma^2}{2} \tag{4.2}$$

that satisfies $p_\pm^2 = p_\pm$. Then, for this remaining space, the fermionic variation becomes

$$\begin{aligned} \delta\psi = (&i(1-\phi\bar{\phi})(\pm\partial_0\phi + i\omega P\phi) \\ &- i\gamma^1(D_1\phi \mp iD_2\phi))\epsilon_\pm. \end{aligned} \tag{4.3}$$

Thus, for this expression to be zero for nontrivial ϵ_+/ϵ_-, one needs

$$\partial_0\phi = \mp i\omega P\phi, \qquad D_\pm\phi = 0, \tag{4.4}$$

respectively for the upper/lower combination of signs. These are precisely the two branches of $(+,-)$ and $(-,+)$ combinations in (3.14). Thus we verified that only the two branches are really consistent with the remaining supersymmetries.

The solutions involve the nonzero electric charges together with the magnetic vortex charge. Because of the BPS equation, the solution has to be time dependent. The time dependence can be solved rather trivially by

$$P\phi = e^{\mp i\omega t} P\phi_0(x_1, x_2), \qquad P_\perp \phi = P_\perp \phi_0(x_1, x_2), \tag{4.5}$$

where $P_\perp = 1 - P$ and ϕ_0 is time independent. Thus we have shown that the supersymmetry can be preserved even for the configurations with an explicit time dependence. The solitons here are the supersymmetric Q-balls, whose description involves an explicit time dependence inevitably.

Note that the BPS solutions have nonvanishing electric and magnetic fluxes. It is worthwhile to give an explicit example, and we concentrate on the simple case of $CP(n)$, which corresponds to the Grassmannian manifold of $M = 1$ and $N = n$. Then explicitly one has $\phi^T = (\phi_1, \phi_2, \cdots, \phi_n, \phi_{n+1})$ and chooses the projection matrix P by

$$P = \mathrm{diag}(1, 1, 1, \cdots, 1, 0). \tag{4.6}$$

Let us introduce the projective coordinate ξ via

$$\phi^T = \frac{1}{\sqrt{1 + |\xi|^2}} (\xi_1, \xi_2, \cdots, \xi_n, 1). \tag{4.7}$$

The first equation of (3.14) is solved by

$$\xi = \exp^{\pm i\omega t} \xi_0, \tag{4.8}$$

where ξ_0 is time independent. The second equation is solved then simply by demanding ξ_0 as an antiholomorphic/holomorphic function (i.e. $\xi_0(\bar{z})/\xi_0(z)$) for $+/-$ signatures, respectively. The corresponding topological charge T and the Noether charge Q are expressed by

$$T = \frac{i}{2\pi} \int \mathrm{d}^2x \frac{\partial_{x_1}\xi^\dagger \partial_{x_2}\xi - \partial_{x_2}\xi^\dagger \partial_{x_1}\xi}{(1 + \xi^\dagger \xi)^2}, \tag{4.9}$$

$$Q = i \int \mathrm{d}^2x \frac{\partial_t \xi^\dagger \xi - \xi^\dagger \partial_t \xi}{(1 + \xi^\dagger \xi)^2}. \tag{4.10}$$

We note that the above solution reproduces the previous Q-lump solution of Refs. [1,3].

V. TIME-DEPENDENT Q-BALLS WITH MULTIFREQUENCY

The potential in (3.1) can be extended further by considering

$$W(\Phi) = \tfrac{1}{2} \mathrm{tr} \bar{\Phi} M \Phi, \tag{5.1}$$

where $M = \omega_k P_k$ with $P_k P_k = P_k$ and $P_k P_l = P_k \delta_{kl}$. Then after eliminating the auxiliary fields, the action with $\mathcal{N} = 2$ supersymmetries becomes

$$\begin{aligned} S &= \int d^3x\, \mathrm{tr}\Big\{ |D_\mu \phi|^2 + i\bar{\psi}\gamma^\mu D_\mu \psi + \Big(\bar{\phi} M \phi - \frac{1}{2}\bar{\psi}\psi\Big)^2 \\ &\quad - \bar{\phi} M^2 \phi + \bar{\psi} M \psi \Big\} \\ &= \int d^3x\, \mathrm{tr}\Big\{ |\partial_\mu \phi|^2 + i\bar{\psi}\gamma^\mu \partial_\mu \psi \\ &\quad - \Big(i\bar{\phi}\partial_\mu \phi + \frac{1}{2}\bar{\psi}\gamma_\mu \psi\Big)^2 + \Big(\bar{\phi} M \phi - \frac{1}{2}\bar{\psi}\psi\Big)^2 \\ &\quad - \bar{\phi} M^2 \phi + \bar{\psi} M \psi \Big\}. \end{aligned} \tag{5.2}$$

Then the analysis of the previous section can be repeated with these multiple projection operators. Since the analysis is pretty much the same, we shall present only the result here. One may show that the Hamiltonian is bounded from below by

$$H \geq 2\pi |T| + \sum_k |\omega_k| |Q_k|, \tag{5.3}$$

where the charge Q_k is defined by

$$Q_k = -i \int d^2x\, \mathrm{tr}((D_0\phi)\bar{\phi} P_k - P_k \phi (D_0 \phi)^\dagger). \tag{5.4}$$

The saturation of the bound leads to the BPS equations,

$$\partial_0 \phi = i \sum_k \epsilon_k \omega_k P_k \phi, \; D_\pm \phi = 0, \tag{5.5}$$

where ϵ_k can be either $+1$ or -1 denoting independent signatures.

One may further verify that, among these, only two combinations,

$$\partial_0 \phi = \mp i \sum_k \omega_k P_k \phi, \qquad D_\pm \phi = 0, \tag{5.6}$$

are consistent with the remaining supersymmetries.

As before, the time dependence of the BPS solution can be solved generically; for each sector of P_k, one has

$$P_k \phi = e^{i\epsilon_k \omega_k t} P_k \phi_0(x_1, x_2), \tag{5.7}$$

and, for the remaining part,

$$P_\perp \phi = P_\perp \phi_0(x_1, x_2), \tag{5.8}$$

where $P_\perp$ denotes now $1 - \sum_k P_k$. This time dependence is highly nontrivial in the sense that the sectors of fields defined by the projection P_k interact with each other nontrivially.

VI. DISCUSSIONS

In this paper, we construct the $\mathcal{N} = 2$ supersymmetric Grassmannian nonlinear sigma model for the massless case first and extend it to the massive $\mathcal{N} = 2$ by adding the appropriate superpotential. This massive model allows Q-lump solutions that carry both the topological and the Noether charges. We study the corresponding BPS equations via two methods. One is using the method of completing squares in the Hamiltonian, by which one may find that the Hamiltonian is bounded from below by the charges. The other is directly finding required conditions for the solutions to preserve a fraction of supersymmetries using the supersymmetry variation of the fermionic part. These two methods have been giving equivalent sets of BPS equations, within the present authors' knowledge, for any supersymmetric theories. In the present case, however, we find at the end of Sec. III that the former leads to the more general sets of BPS equations than those from the latter. In a strict sense, the latter is the precise way of getting supersymmetric solitons, whose supersymmetric multiplet has to be short by definition.

These Q-lumps are always time dependent as dictated by the BPS equations. Thus it is clear that the explicit time dependence can be consistent with the remaining supersymmetries. Supertubes [7], for instance, also carry the kinetic components corresponding to the nonvanishing electric fields but the solutions are time independent unlike the Q-lump solution discussed here. We also discuss the supersymmetric Q-lump solution that involves the time dependence of even multiple frequencies. The existence of such solutions is highly nontrivial since the sectors of different frequencies are interacting with each other in a nontrivial manner.

One curious question is whether the system allows $1/4$ BPS equations or not where the remaining supersymmetry is just one Majorana component. In the case of $\mathcal{N} = 4$ super Yang-Mills theories, the BPS equations corresponding to one Majorana component, i.e. $1/16$ supersymmetry, have been constructed [14]. For some $\mathcal{N} = 2$ theories, $1/8$ BPS states are classified as well [6]. Finally, in this paper, we did not investigate the detailed properties of the supersymmetric Q lump solutions including their moduli dynamics, interactions, and so on. These require a further study.

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