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Black hole solutions of dimensionally reduced Einstein-Gauss-Bonnet gravity with a cosmological constant

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We study the phase space of the spherically symmetric solutions of the system obtained from the dimensional reduction of the six-dimensional Einstein-Gauss-Bonnet action with a cosmological constant. We show that all the physical solutions have anti-de Sitter asymptotic behavior.

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I. INTRODUCTION

In a recent paper [1], one of us has investigated the black hole solutions of a dimensionally reduced gravity model with Gauss-Bonnet (GB) corrections to the Einstein-Hilbert (EH) lagrangian of general relativity. Using global methods it was possible to classify the solutions in terms of their asymptotic behavior, in analogy with what done in [2] in the pure Einstein-Hilbert case and in [3] in the presence of a cosmological constant (CC). The results of the work of [1] showed a remarkable similarity between the EH-GB and the EH-CC system. It is therefore interesting to extend that investigation to the case where both the GB term and a cosmological constant are added to the EH lagrangian.

As it is well known, GB terms arise as a natural extension to higher dimensions of the EH Lagrangian, sharing most properties of general relativity [4]. They have therefore found many applications in Kaluza-Klein theories [5]. Of course, the properties of black hole solutions in dimensionally reduced models are of great interest. In the case of pure Einstein gravity they were studied in [2], where it was shown that the only solution of physical interest is the four-dimensional Schwarzschild metric with flat internal space. When a cosmological constant is added, physically relevant solutions have anti-de Sitter asymptotics and negative-curvature internal space [3]. The case of a six-dimensional EH-GB model without cosmological constant was studied in [1]. It displayed some similarities with the case of EH-CC, in particular, the existence of asymptotically anti-de Sitter black holes.[1]

In this paper, our aim is to extend these investigations to the case of the EH-CC-GB system, classifying all the solutions having the form of a direct product of a four-dimensional spherically symmetric black hole with a maximally symmetric internal space. Since a general discussion would be too involved, we shall limit ourselves to the case of six dimensions, where the only relevant GB correction is quadratic in the curvature and has the form $\mathcal{S} = \mathcal{R}^{\mu\nu\rho\sigma}\mathcal{R}_{\mu\nu\rho\sigma} - 4\mathcal{R}^{\mu\nu}\mathcal{R}_{\mu\nu} + \mathcal{R}^2$.

In Refs. [1–3], this topic was studied by considering the phase space of the solutions of the field equations. In fact, much information on the physics can be obtained from this investigation. For example, the classification of the critical points of the dynamical systems at infinity permits to know the asymptotic behavior of all possible solutions of the field equations, while the behavior of the solutions near the critical points at finite distance determines if the corresponding metrics admit a horizon. However, some problem arise when a GB term is present in the action. It is well known that in this case the field equations remain second order and linear in the second derivatives, but they are no longer quadratic in the first derivatives. By consequence, the potential of the dynamical system is no longer polynomial, but presents poles for some values of the variables.

The result of our investigation is that all the relevant solutions, i.e. all spherically symmetric solutions with the radius of the internal space going to constant at spatial infinity, have anti-de Sitter asymptotic behavior, as in the EH-CC case. However, in our case the possibility emerges of a flat or positive-curvature internal space, which is not allowed in the EH-CC case.

Let us consider the $(n + 4)$-dimensional action

$$I = \int \sqrt{-g} d^{(n+4)}x (2\beta + \mathcal{R}^{(n+4)} + \alpha \mathcal{S}^{(n+4)}), \qquad (1.1)$$

where $\mathcal{R}^{(n+4)}$ is the curvature scalar, $\mathcal{S}^{(n+4)}$ the quadratic GB term of the manifold, 2β the cosmological constant, and α a coupling parameter of dimension $[L]^2$.

We perform a dimensional reduction which casts the metric in the form of a direct product of a four-dimensional manifold with an n-dimensional space of constant curvature, whose size is parametrized by a scalar field ϕ. As discussed in [1], it is not possible to find an ansatz for the metric of the EH-GB system that completely disentangles the scalar field ϕ from the curvature in the dimensionally reduced action, except when the internal space is flat. Therefore we maintain the usual ansatz

$$ds^2_{(n+4)} = e^{-n\phi} ds^2_{(4)} + e^{2\phi} g^{(n)}_{ab} dx^a dx^b, \qquad (1.2)$$

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[1]Black holes in higher-dimensional EH-GB theories admitting higher-dimensional spherical symmetry have been extensively studied in the literature [6].

where $ds^2_{(4)}$ is the line element of the four-dimensional spacetime and $g^{(n)}_{ab}$ is the metric of the n-dimensional maximally symmetric internal space, with $\mathcal{R}^{(n)}_{ab} = \lambda_i g^{(n)}_{ab}$. The action is dimensionally reduced to

$$I = \int \sqrt{-g} d^4x \bigg[(1 + 2\alpha\lambda_i e^{-2\phi})\mathcal{R}^{(4)} + \alpha e^{n\phi}\mathcal{S}^{(4)} + 4n\alpha e^{n\phi}\mathcal{G}^{(4)}_{\mu\nu}\nabla^\mu\phi\nabla^\nu\phi + \left(\frac{n(n+2)}{2} - (n^2 - 2n - 12)\alpha\lambda_i e^{-2\phi}\right)(\nabla\phi)^2 - \frac{n(n+2)(n^2+n-3)}{3}\alpha e^{n\phi}(\nabla\phi)^4 + 2\beta e^{-n\phi} + \lambda_i e^{-(n+2)\phi} + (n-2)(n-3)\alpha\lambda_i^2 e^{-(n+4)\phi} \bigg] \tag{1.3}$$

The ground state of the theory is assumed to have the form of a direct product of a four dimensional and an n-dimensional maximally symmetric space, i.e. $\mathcal{R}^{(4)}_{\mu\nu\rho\sigma} = \Lambda_e(g^{(4)}_{\mu\rho}g^{(4)}_{\nu\sigma} - g^{(4)}_{\mu\sigma}g^{(4)}_{\nu\rho})$, $\mathcal{R}^{(n)}_{\mu\nu\rho\sigma} = \Lambda_i(g^{(n)}_{\mu\rho}g^{(n)}_{\nu\sigma} - g^{(n)}_{\mu\sigma}g^{(n)}_{\nu\rho})$. Substituting this ansatz into the field equations derived from (1.1), one obtains

$$\begin{aligned} &\alpha[(n-1)(n-2)(n-3)(n-4)\Lambda_i^2 + 24\Lambda_e^2 + 24(n-1)(n-2)\Lambda_e\Lambda_i] + (n-1)(n-2)\Lambda_i + 12\Lambda_e + 2\beta = 0, \\ &\alpha[n(n-1)(n-2)(n-3)\Lambda_i^2 + 12n(n-1)\Lambda_i\Lambda_e] + n(n-1)\Lambda_i + 6\Lambda_e + 2\beta = 0. \end{aligned} \tag{1.4}$$

In the case of interest, $n = 2$, if $\alpha\beta \leq 3/4$ and $\alpha\beta \neq 5/12$, the system admits two solutions

$$\Lambda_e = \frac{1}{4\alpha}\left(-1 \pm \sqrt{1 - \frac{4\alpha\beta}{3}}\right), \qquad \Lambda_i = \frac{3(-1 \pm \sqrt{1 - \frac{4\alpha\beta}{3}}) - 4\alpha\beta(-1 \pm 3\sqrt{1 - \frac{4\alpha\beta}{3}})}{4\alpha(5 - 12\alpha\beta)}. \tag{1.5}$$

If $\alpha > 0$ and $\beta > 0$, the values of Λ_e are both negative, corresponding to anti-de Sitter spacetime, if $\alpha < 0$ and $\beta < 0$, both positive (de Sitter spacetime); finally, if α and β have opposite sign one solution is positive and one negative. The values of Λ_i are both negative, corresponding to internal space H^2, if $\alpha > 0$, and $0 < \alpha\beta < 5/12$, both positive (internal space S^2) if $\alpha < 0$ and $0 < \alpha\beta < 5/12$, one positive and one negative otherwise. An interesting limit case arises when $\alpha\beta = 3/4$. In this limit the internal space is flat.

Consequently, for a range of values of α and β black hole solutions of (1.1) may have anti-de Sitter behavior at spatial infinity (we are not interested in de Sitter spacetimes since they do not have an asymptotic region). In the limit $\beta \to 0$ one recovers the solutions with vanishing cosmological constant of Ref. [1]. The limit $\alpha \to 0$ is instead singular: in absence of GB corrections the unique solution of (1.4) is given by $\Lambda_e = -\beta/6$, $\Lambda_i = -\beta/2$ and is therefore AdS $\times H^2$ for $\beta > 0$, or $dS \times S^2$ for $\beta < 0$.

II. THE DYNAMICAL SYSTEM

In [1] the dynamical system associated to the spherically symmetric solutions of the model was derived when $\beta = 0$. We now review that derivation when a cosmological constant is added to the Lagrangian. For the four-dimensional metric we adopt the ansatz [1–3]

$$ds^2_{(4)} = -e^{2\nu}dt^2 + \sigma^{-2}e^{4\zeta-2\nu}d\xi^2 + e^{2\zeta-2\nu}g_{ij}dx^i dx^j, \tag{2.1}$$

where ν, ζ and σ as well as ϕ are functions of ξ and g_{ab} is the metric of a two-dimensional maximally symmetric space, with $\mathcal{R}_{ij} = \lambda_e g_{ij}$. This ansatz enforces radial symmetry and is convenient for the following discussion, but of course the case of physical interest is that of spherical symmetry, $\lambda_e > 0$.

Defining the new variables [1]

$$\chi = 2\zeta - \nu - \phi, \qquad \eta = 2\zeta - \nu - 2\phi, \tag{2.2}$$

and substituting the ansatz (1.2) and (2.1) into the action, after factoring out the internal space the action can be cast in the form

$$\begin{aligned} I = -8\pi \int d^4x \bigg\{ &\sigma[6\chi'^2 + 3\zeta'^2 + 3\eta'^2 - 8\chi'\zeta' - 8\chi'\eta' + 4\zeta'\eta'] - \frac{1}{\sigma}(\lambda_e e^{2\zeta} + \lambda_i e^{2\eta} + \beta e^{2\chi}) \\ &+ 4\alpha e^{-2\chi}\bigg[\sigma(\eta' - \chi')(4\zeta' + 3\eta' - 5\chi')\lambda_e e^{2\zeta} + \sigma(\zeta' - \chi')(3\zeta' + 4\eta' - 5\chi')\lambda_i e^{2\eta} \\ &- \sigma^3(\zeta' - \chi')(\eta' - \chi')(11\chi'^2 + 4\zeta'^2 + 4\eta'^2 + 7\zeta'\eta' - 13\chi'\zeta' - 13\chi'\eta') - \lambda_e\lambda_i\frac{e^{2(\zeta+\eta)}}{\sigma}\bigg]\bigg\}. \end{aligned} \tag{2.3}$$

As usual, the action (2.3) does not contain derivatives of σ, which acts therefore as a Lagrangian multiplier enforcing the Hamiltonian constraint. Moreover, in spite of the presence of the higher-derivative GB term, it contains only first derivatives of the fields, although up to the fourth power, and therefore gives rise to second order field equations. Finally, the action is invariant under the interchange of ζ and η.

One can now vary (2.3) and then write the field equations in first order form in terms of the new variables,

$$W = \chi', \qquad X = \zeta', \qquad Y = \eta', \qquad U = e^{\chi}, \qquad Z = e^{\zeta}, \qquad V = e^{\eta}, \tag{2.4}$$

which satisfy

$$U' = WU, \qquad Z' = XZ, \qquad V' = YV. \tag{2.5}$$

Varying with respect to σ and then choosing the gauge $\sigma = 1$, one obtains the Hamiltonian constraint

$$\begin{aligned} E \equiv{}& P^2 + \lambda_e Z^2 + \lambda_i V^2 + \beta U^2 \\ &+ \frac{4\alpha}{U^2}[\lambda_e\lambda_i Z^2 V^2 + \lambda_e Z^2(Y - W)A \\ &+ \lambda_i V^2(X - W)B - 3(X - W)(Y - W)C^2] = 0, \end{aligned} \tag{2.6}$$

where

$$\begin{aligned} P^2 &= 6W^2 + 3X^2 + 3Y^2 - 8WX - 8WY + 4XY, \\ C^2 &= 11W^2 + 4X^2 + 4Y^2 + 7XY - 13WX - 13WY, \\ A &= 4X + 3Y - 5W, \\ B &= 3X + 4Y - 5W. \end{aligned}$$

Variation with respect to χ, ζ and η gives rise to the other field equations

$$\begin{aligned} &2X' + 2Y' - 3W' + \Big\{\frac{2\alpha}{U^2}[\lambda_e Z^2(2X + 4Y - 5W) + \lambda_i V^2(4X + 2Y - 5W) + 22W^3 - 2X^3 - 2Y^3 - 36W^2X - 36W^2Y \\ &\quad - 12X^2Y - 12Y^2X + 17WX^2 + 17WY^2 + 44XYW]\Big\}' \\ &= \frac{\beta}{2}U^2 + \frac{2\alpha}{U^2}[-\lambda_e\lambda_i Z^2V^2 + \lambda_e Z^2(Y - W)A + \lambda_i V^2(X - W)B - (X - W)(Y - W)C^2], \end{aligned} \tag{2.7}$$

$$\begin{aligned} &X' + 2Y' - 2W' + \Big\{\frac{4\alpha}{U^2}[\lambda_e Z^2(2X + 2Y - 3W) - (X - W)(10W^2 + 2X^2 + 5Y^2 + 6XY - 9WX - 14WY - \lambda_i V^2)]\Big\}' \\ &= \lambda_e Z^2 + \beta U^2 + \frac{4\alpha}{U^2}[\lambda_i V^2(X - W)B - (X - W)(Y - W)C^2], \end{aligned} \tag{2.8}$$

$$\begin{aligned} &2X' + Y' - 2W' + \Big\{\frac{4\alpha}{U^2}[\lambda_i V^2(2X + 2Y - 3W) - (Y - W)(10W^2 + 5X^2 + 2Y^2 + 6XY - 14WX - 9WY - \lambda_e Z^2)]\Big\}' \\ &= \lambda_i V^2 + \beta U^2 + \frac{4\alpha}{U^2}[\lambda_e Z^2(Y - W)A - (X - W)(Y - W)C^2], \end{aligned} \tag{2.9}$$

In the variables (2.4), the problem takes the form of a six-dimensional dynamical system, subject to a constraint. Notice that the function E defined in (2.6) is a constant of the motion of the system (2.5), (2.7), (2.8), and (2.9), whose value vanishes by virtue of the Hamiltonian constraint.

The Einstein limit

In the Einstein limit $\alpha = 0$ one recovers the results of [3]. We summarize them in terms of the variables introduced above: when $\alpha = 0$, the dynamical system reduces to Eqs. (2.5) and

$$\begin{aligned} 2X' + 2Y' - 3W' &= \frac{\beta}{2}U^2, \\ X' + 2Y' - 2W' &= \lambda_e Z^2 + \beta U^2, \\ 2X' + Y' - 2W' &= \lambda_i V^2 + \beta U^2, \end{aligned} \tag{2.10}$$

subject to the constraint

$$E = P^2 + Z^2 + V^2 + \beta U^2 = 0. \tag{2.11}$$

The physical trajectories lie on the four-dimensional hypersurface $E = 0$.

The critical points at finite distance correspond to the short radius limit of the solutions. They lie on the surface $U_0 = Z_0 = V_0 = P_0 = 0$, but only points with $X_0 = Y_0 = W_0$ correspond to regular horizons, while the others give rise to naked singularities. The eigenvalues of the linearized equations around the critical points are 0, with degeneracy 3, X_0, Y_0 and W_0.

The asymptotic properties of the solutions are related to the structure of the phase space at infinity. This can be investigated defining new variables

$$t = \frac{1}{W}, \qquad x = \frac{X}{W}, \qquad y = \frac{Y}{W}, \qquad u = \frac{U}{W}, \qquad z = \frac{Z}{W}, \qquad v = \frac{V}{W}. \tag{2.12}$$

In terms of these variables, the field equations at infinity are then obtained for $t \to 0$, and read

$$\begin{aligned}
\dot{t} &= -\left(2v^2 + 2z^2 + \frac{5}{2}\beta u^2\right)t,\\
\dot{u} &= \left(1 - 2v^2 - 2z^2 - \frac{5}{2}\beta u^2\right)u,\\
\dot{x} &= z^2 + 2v^2 + 2\beta u^2 - \left(2v^2 + 2z^2 + \frac{5}{2}\beta u^2\right)x\\
\dot{z} &= \left(x - 2v^2 - 2z^2 - \frac{5}{2}\beta u^2\right)z,\\
\dot{y} &= 2z^2 + v^2 + 2\beta u^2 - \left(2v^2 + 2z^2 + \frac{5}{2}\beta u^2\right)y,\\
\dot{v} &= \left(y - 2v^2 - 2z^2 - \frac{5}{2}\beta u^2\right)v,
\end{aligned} \tag{2.13}$$

where a dot denotes $td/d\xi$. The critical points at infinity are found at $t_0 = 0$ and

(a) $\beta u_0^2 = \lambda_i v_0^2 = \lambda_e z_0^2 = 0$, $x = x_0$, $y = y_0$, with $3x_0^2 + 3y_0^2 + 4x_0y_0 - 8x_0 - 8y_0 + 6 = 0$.
(b) $\beta u_0^2 = \lambda_i v_0^2 = 0$, $\lambda_e z_0^2 = 1/4$, $x_0 = 1/2$, $y_0 = 1$.
(c) $\beta u_0^2 = \lambda_e z_0^2 = 0$, $\lambda_i v_0^2 = 1/4$, $x_0 = 1$, $y_0 = 1/2$.
(d) $\beta u_0^2 = 0$, $\lambda_i v_0^2 = \lambda_e z_0^2 = 3/16$, $x_0 = y_0 = 3/4$.
(e) $\lambda_e z_0^2 = 0$, $\lambda_i v_0^2 = -1/3$, $\beta u_0^2 = 2/3$, $x_0 = 2/3$, $y_0 = 1$.
(f) $\lambda_i v_0^2 = 0$, $\lambda_e z_0^2 = -1/3$, $\beta u_0^2 = 2/3$, $x_0 = 1$, $y_0 = 2/3$.
(g) $\lambda_i v_0^2 = \lambda_e z_0^2 = -1$, $\beta u_0^2 = 2$, $x_0 = y_0 = 1$.
(h) $\lambda_i v_0^2 = \lambda_e z_0^2 = 0$, $\beta u_0^2 = 2/5$, $x_0 = y_0 = 4/5$

Points (a) are the endpoints of the hypersurface $U = V = Z = 0$, points (b) of the hypersurface $U = V = 0$, points (c) of the hypersurface $U = Z = 0$. Clearly, points (e)–(h) exist only for $\beta > 0$.

The eigenvalues of the linearized equations around the critical points and their degeneracies are:

(a) $0(3)$, 1, x_0, y_0.
(b, c) $-1/2(3)$, -1, $1/2(2)$.
(d) $-3/4(2)$, $-3/2$, $1/4$, $-\frac{1}{8}(3 \pm i\sqrt{15})$.
(e, f) $-1(2)$, $-1/3$, -2, $-\frac{1}{6}(3 \pm \sqrt{33})$.
(g) -1, $-2(3)$, $1(2)$.
(h) $-1(3)$, $-1/5(2)$, -2.

From the study of the eigenvalues and eigenvectors of the linearized equations one can deduce the structure of phase space at infinity. It results that points (a) attract only unphysical trajectories on the surface at infinity, while points (b)–(d) attract only trajectories with $\beta = 0$. The relevant critical points are therefore (e)–(h). Of these, (e) attracts trajectories with $\lambda_e \geq 0$, $\lambda_i < 0$, (f) attracts trajectories with $\lambda_e < 0$, $\lambda_i \geq 0$, (g) attracts trajectories with both $\lambda_e < 0$, $\lambda_i < 0$ and (h) trajectories with $\lambda_e \geq 0$, $\lambda_i \geq 0$.

The asymptotic behavior of the solutions can be deduced from the location of the critical points at infinity [2]. Excluding points (a) that do not correspond to physical trajectories, one has, in terms of a radial variable r:

(b) $ds^2 \sim -dt^2 + dr^2 + r^2 d\Omega_+^2$, $e^{2\phi} \sim$ const.
(c) $ds^2 \sim -r^2dt^2 + r^2dr^2 + r^2 d\Omega_0^2$, $e^{2\phi} \sim r^2$.
(d) $ds^2 \sim -rdt^2 + dr^2 + r^2 d\Omega_+^2$, $e^{2\phi} \sim r$.
(e) $ds^2 \sim -r^2dt^2 + r^{-2}dr^2 + r^2 d\Omega_+^2$, $e^{2\phi} \sim$ const.
(f) $ds^2 \sim -r^4dt^2 + dr^2 + r^2 d\Omega_-^2$, $e^{2\phi} \sim r^2$.
(g) $ds^2 \sim -r^2dt^2 + r^{-2}dr^2 + d\Omega_-^2$, $e^{2\phi} \sim$ const.
(h) $ds^2 \sim -r^2dt^2 + r^{-1}dr^2 + r^2 d\Omega_+^2$, $e^{2\phi} \sim r$.

We have denoted with $d\Omega_+^2$ the metric of a unitary 2-sphere, with $d\Omega_-^2$ that of a 2-dimensional space of constant negative curvature, and with $d\Omega_0^2$ that of a flat 2-plane. The solutions ending at points (b) are asymptotically flat, those ending at (e) asymptotically anti-de Sitter, while the others have more exotic behavior. Exact solutions displaying the asymptotic behaviors listed above are presented in Appendix A.

From a Kaluza-Klein point of view, the only solutions with physical relevance are those with $\lambda_e > 0$ and $e^{2\phi} \to$ const, namely, those ending at (e), which have a negative-curvature internal space. The solutions ending at (h) can also have $\lambda_e > 0$, but decompactify for $r \to \infty$. It follows that, as one could have guessed, the only significant solutions of this model are the asymptotically anti-de Sitter solutions (e), which asymptote to the exact ground state discussed at the end of Sec. I.

III. THE EH-CC-GB PHASE SPACE

As discussed in Sec. I, for a range of values of α and β Eqs. (1.4) admit the ground state (1.5), and therefore black holes with anti-de Sitter asymptotic behavior may be expected. The phase space of the system can be studied by the same methods that were used in the Einstein case. However, as usual in the presence of GB terms, some problems arise because of the poles in the field equations for $U = 0$ [1,7]. Special care must therefore be taken when approaching this limit.

Equations (2.7), (2.8), and (2.9) have to be solved for the variables X', Y' and W' in order to put the system in its canonical form. One can then find the critical points at finite distance by requiring the vanishing of the derivatives of the fields. As in the EH-CC case, they lie on the hypersurface $U_0 = Z_0 = V_0 = 0$. However, in the present case, the other variables must satisfy the constraint $W_0 = X_0 = Y_0$, or $X_0 = \frac{4\pm\sqrt{5}}{5}W_0$, $Y_0 = \frac{4\mp\sqrt{5}}{5}W_0$, in order to avoid singularities of the field equations. Only the first instance corresponds to regular horizons. In that case the eigenvalues of the linearized equations near the critical points are identical to those found in the Einstein limit.

TABLE I. Location of the critical points at infinity.

	x_0	y_0	βt_0^2	$\lambda_e z_0^2$	$\lambda_i v_0^2$
(a)	1	1	0	0	0
(b)	1/2	1	0	1/4	0
(c)	1	1/2	0	0	1/4
(e)	2/3	1	$\frac{3\pm\sqrt{9-12\gamma}}{9}$	0	$-\frac{9-12\gamma\pm2\sqrt{9-12\gamma}}{9(5-12\gamma)}$
(f)	1	2/3	$\frac{3\pm\sqrt{9-12\gamma}}{9}$	$-\frac{9-12\gamma\pm2\sqrt{9-12\gamma}}{9(5-12\gamma)}$	0
(g)	1	1	$1\pm\sqrt{1-4\gamma}$	-1	-1
(h)	4/5	4/5	$\frac{5\pm\sqrt{25-60\gamma}}{25}$	0	0
(i)	2/3	1	0	1/3	0
(l)	1	2/3	0	0	1/3
($m_\pm$)	$\frac{4\pm\sqrt{5}}{5}$	$\frac{4\mp\sqrt{5}}{5}$	0	0	0

The critical points at infinity are obtained by introducing the variables (2.12) in the dynamical system and requiring the vanishing of their derivatives as $t \to 0$. They are listed in Table I, where we have set $\gamma = \alpha\beta$.

Of course, the critical points (e), (f) can only exist if $\gamma \leq 3/4$, the points (g) if $\gamma \leq 1/4$ and the points (h) if $\gamma \leq 5/12$. For $\gamma < 3/4$, the location of the critical points is rather similar to that obtained in the $\alpha = 0$ limit, except for the point d), that has disappeared and the new points i)–$m_\pm$), that are typical of the GB theory [1]. In the limit $\gamma \to 0$ one recovers the critical points of the EH-CC model.

A stronger similarity is however present with the phase space of the EH-GB limit $\beta = 0$. In fact, one has exactly the same critical points as for nonvanishing β, with only the values of u_0, z_0 and v_0 shifted. As we shall see, however, the nature of the critical points may be different in the two cases. Moreover, the limit $\beta \to 0$ is not trivial.

In Table II are listed the eigenvalues of the linearized equations near the critical points with their degeneracy. In the last eigenvalues at the critical points (g), Σ is a cumbersome function of γ. It turns out that the real part of both these eigenvalues is negative for $\gamma < -3/4$, while for $-3/4 < \gamma < 1/4$, one of the eigenvalues has positive real part.

TABLE II. The eigenvalues of the linearized equations near the critical points at infinity and their eigenvalues.

	Eigenvalues (*with degeneracy*)
(a)	$0(2),\ 1(3),\ 4\gamma$
(b), (c)	$-\frac{1}{2},\ -\frac{1}{2}+4\gamma,\ \frac{1}{2}(2),\ -\frac{3-8\gamma\pm\sqrt{1+16\gamma-32\gamma^2}}{4}$
(e), (f)	$-1(2),\ -2,\ -\frac{1}{3},\ -\frac{3\pm\sqrt{33-32\gamma}}{6}$
(g)	$-1,\ -2(2),\ 1,\ -\frac{1}{2}(1\pm\sqrt{\Sigma})$
(h)	$-1(3),\ -2,\ -\frac{1}{5}(2)$
(i), (l)	$-\frac{2}{3},\ -\frac{1}{3},\ -1,\ \frac{1}{3}(3)$
($m_\pm$)	$-\frac{2}{3}(2),\ 0,\ \frac{1}{3},\ \frac{2\pm3\sqrt{5}}{15}$

From the study of the linearized equations, one can deduce that point a) attracts only trajectories lying on the surface at infinity, while the points (i), (l), $m_\pm$) attract only trajectories on the hypersurface $U = 0$, which corresponds to the limit $\alpha \to \infty$ of pure GB gravity [1]. Therefore, these points are not of interest for our problem. Moreover, points (b), (c) are endpoints only of trajectories with $\beta = 0$. The other points can attract trajectories with nonvanishing β. In particular, (e) attracts trajectories with $\lambda_i = 0$, (f) trajectories with $\lambda_e = 0$, (g) trajectories with $\lambda_e < 0$, $\lambda_i < 0$, and (h) trajectories with any values of λ_e, λ_i.

The critical points (a)–(h) generalize those found in the EH-CC case, and have the same asymptotic behavior as the corresponding ones. We do not discuss the asymptotic behavior of the new points (i), (l), ($m_\pm$), since they correspond to the limit $\alpha \to \infty$. Special exact solutions of the EH-CC-GB system with asymptotic behavior (e)–(h) are listed in Appendix B.

Of particular interest are the solutions that end at the critical point (e). These asymptote to the exact ground state solution discussed in Sec. I, namely $\mathrm{AdS}^4 \times S^2$, if $\alpha > 0$ and $5/12 < \gamma < 3/4$ or $\alpha < 0$ and $\beta > 0$, or to $\mathrm{AdS}^4 \times H^2$, if $\alpha > 0$ and $\gamma < 5/12$. Contrary to the EH-CC case, solutions with anti-de Sitter asymptotics can therefore exist also if $\beta < 0$. In the present coordinates they take the form

$$ds^2 = -(|\Lambda_e|r^2 + 1)dt^2 + (|\Lambda_e|r^2 + 1)^{-1}dr^2 + r^2 d\Omega_+^2,$$
$$e^{2\phi} = |\Lambda_i|^{-1}, \qquad (3.1)$$

where Λ_e, Λ_i are given by (1.5).

Also interesting is the solution (g), that asymptotes to the exact solution $\mathrm{AdS}^2 \times H^2 \times H^2$. Its four-dimensional section is analogous to a Bertotti-Robinson metric. The other solutions have less common asymptotic behavior.

The structure of the solutions of the EH-GB-CC model is more complicated than that of pure EH-CC, although less critical points are available for $\beta \neq 0$. The trajectories start at the points $U = V = Z = 0$, $W = X = Y$ and can terminate at one of the points (e)–(h), depending on the value of λ_e, λ_i and on the value of γ. From the Kaluza-

Klein point of view, the relevant solutions are those with $\lambda_e > 0$ and $e^{2\phi} \longrightarrow$ const at infinity. As in the $\beta = 0$ case, these are solutions ending at (e), but if $\beta \neq 0$, the internal space can be flat, if $\beta = \frac{3}{4\alpha}$, or compact, if $\frac{5}{12\alpha} < \beta < \frac{3}{4\alpha}$.

IV. CONCLUSIONS

We have shown that regular black holes in EH-CC-GB theory have the same asymptotic behavior as the maximally symmetric ground states. In contradistinction to the EH-CC case, the internal space can have positive, negative or vanishing curvature.

Although the calculation are easier in our six-dimensional model, we believe that the situation is essentially unchanged in higher dimensions, except that in case of three or more internal dimensions ground states with flat spacetime can exist for some values of the parameters.

APPENDIX A: EXACT SOLUTIONS IN THE EINSTEIN LIMIT

In the Einstein limit, $\alpha = 0$, the field equations reduce to

$$\begin{aligned} \chi'' &= 2\lambda_e e^{2\zeta} + 2\lambda_i e^{2\eta} + \frac{5\beta}{2} e^{2\chi}, \\ \zeta'' &= \lambda_e e^{2\zeta} + 2\lambda_i e^{2\eta} + 2\beta e^{2\chi}, \\ \eta'' &= 2\lambda_e e^{2\zeta} + \lambda_i e^{2\eta} + 2\beta e^{2\chi}, \end{aligned} \tag{A1}$$

subject to the constraint

$$6\chi'^2 + 3\zeta'^2 + 3\eta'^2 - 8\chi'\zeta' - 8\chi'\eta' + 4\zeta'\eta' + \lambda_e e^{2\zeta} + \lambda_i e^{2\eta} + \beta e^{2\chi} = 0. \tag{A2}$$

Some exact solutions of the system (A1) and (A2) can be found in special cases. The limit $\beta = 0$ has been discussed in [1,2]: when $\lambda_i = 0$ one obtains the Schwarzschild metric with constant scalar field,

$$ds^2 = -\left(1 - \frac{2m}{r}\right)dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 d\Omega_+^2, \qquad e^{2\phi} = \text{const},$$

with m a free parameter. This is a special case of solutions (b) of Sec. II.

For $\lambda_e = 0$, one has instead a solution of the form

$$ds^2 = -r^2\left(1 - \frac{2m}{r}\right)dt^2 + r^2\left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 d\Omega_0^2, \qquad e^{2\phi} = r^2,$$

which corresponds to the asymptotic behavior (c).

Finally, if $\eta' = \zeta'$, one has

$$ds^2 = -r\left(\frac{4}{27} - \frac{2m}{r^{3/2}}\right)dt^2 + \left(\frac{4}{27} - \frac{2m}{r^{3/2}}\right)^{-1} dr^2 + r^2 d\Omega^2, \qquad e^{2\phi} = r,$$

which corresponds to (d).

In the following, it will be useful to write the solutions in their six-dimensional Schwarzschild-like form

$$ds^2 = -e^{2\lambda} dT^2 + e^{-2\lambda} dR^2 + e^{2\rho} d\Omega_e^2 + e^{2\sigma} d\Omega_i^2, \tag{A3}$$

where

$$e^{2\lambda} = e^{4\zeta + 4\eta - 6\chi}, \qquad \epsilon^{2\rho} = e^{2\chi - 2\zeta}, \qquad \epsilon^{2\sigma} = e^{2\chi - 2\eta}. \tag{A4}$$

In these coordinates the previous solutions read respectively

$$ds^2 = -\left(1 - \frac{2M}{R}\right)dT^2 + \left(1 - \frac{2M}{R}\right)^{-1} dR^2 + R^2 d\Omega_+^2 + c^2 d\Omega_0^2,$$

$$ds^2 = -\left(1 - \frac{2M}{R}\right)dT^2 + \left(1 - \frac{2M}{R}\right)^{-1} dR^2 + c^2 d\Omega_0^2 + R^2 d\Omega^2,$$

$$ds^2 = -\left(1 - \frac{2M}{R^3}\right)dT^2 + \left(1 - \frac{2M}{R^3}\right)^{-1} dR^2 + \frac{R^2}{3}(d\Omega_i^2 + d\Omega_e^2),$$

with c constant.

We pass now to consider the case when $\beta \neq 0$.

1. $\lambda_e = 0$, $\lambda_i < 0$, $\eta' = \chi'$.

The field equations (A1) reduce to

$$e^{2\eta} = \frac{\beta}{2} e^{2\chi}, \qquad \chi'' = \frac{3}{2}\beta e^{2\chi}, \qquad \zeta'' = \frac{2}{3}\chi''.$$

Integrating,

$$e^{2\chi} = \frac{2}{3\beta}\frac{a^2}{\sinh^2 a\xi}, \qquad e^{2\zeta} = A e^{2(2\chi + b\xi)/3},$$

for constant a and b. Substituting in the constraint (A2) and requiring the presence of a regular horizon, one obtains the condition $b = a$. Then, defining $R = [c/(1 - e^{2a\xi})]^{1/3}$, with c a positive constant, one gets

$$e^{2\chi} = AR^3(R^3 - c), \qquad e^{2\zeta} = BR(R^3 - c),$$

where $A = 8a^2/3\beta c^2$ and B is an integration constant. Finally, choosing $B = A$, rescaling the time coordinate, and defining $M = \beta c/12$, one obtains

$$e^{2\lambda} = \frac{\beta}{6}R^2 - \frac{2M}{R}, \qquad e^{2\rho} = R^2, \qquad e^{2\sigma} = \frac{2}{\beta}. \tag{A5}$$

In four-dimensional coordinates the solution reads

$$ds^2 = -\left(\frac{\beta^2}{12}r^2 - \frac{2m}{r}\right)dt^2 + \left(\frac{\beta^2}{12}r^2 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 d\Omega_0^2, \qquad e^{2\phi} = \frac{2}{\beta},$$

and has therefore the asymptotic behavior (e).

The solution (A5) can also be generalized to the case of positive λ_e, although the solution is not trivial in the coordinates ζ, η, χ. It reads

$$-\left(\frac{\beta}{6}R^2+1-\frac{2M}{R}\right)dT^2+\left(\frac{\beta}{6}R^2+1-\frac{2M}{R}\right)^{-1}dR^2+R^2d\Omega_+^2+\frac{2}{\beta}d\Omega_-^2,$$

and is the direct product of AdS^4 with a two-dimensional space of constant negative curvature H^2.

2. $\lambda_i = 0$, $\lambda_e < 0$, $\zeta' = \chi'$

This system is identical to that of the previous case, after interchanging ζ and η (and hence ρ and σ). Proceeding as before, one obtains

$$e^{2\lambda}=\frac{\beta}{6}R^2-\frac{2M}{R},\qquad e^{2\rho}=\frac{2}{\beta},\qquad e^{2\sigma}=R^2. \tag{A6}$$

In four-dimensional coordinates (2.1), the solution reads

$$ds^2=-r^4\left(\frac{1}{3}-\frac{2m}{r^3}\right)dt^2+\left(\frac{1}{3}-\frac{2m}{r^3}\right)^{-1}dr^2+r^2d\Omega_-^2,$$
$$e^{2\phi}=r^2,$$

and has therefore the asymptotic behavior (f).

In analogy with the previous case, the solution (A6) can be generalized to positive λ_i,

$$-\left(\frac{\beta}{6}R^2+1-\frac{2M}{R}\right)dT^2+\left(\frac{\beta}{6}R^2+1-\frac{2M}{R}\right)^{-1}dR^2+\frac{2}{\beta}d\Omega_-^2+R^2d\Omega_+^2.$$

3. $\lambda_e = \lambda_i = 0$, $\eta' = \zeta'$

In this case, the field equations reduce to

$$\chi''=\frac{5}{2}\beta e^{2\chi},\qquad \zeta''=\frac{4}{5}\chi''.$$

Integrating,

$$e^{2\chi}=\frac{2}{5\beta}\frac{a^2}{\sinh^2 a\xi},\qquad e^{2\zeta}=e^{2\eta}=Ae^{2(4\chi+b\xi)/5},$$

for constant a and b. Substituting in the constraint, and requiring the presence of a regular horizon, one obtains the condition $b=a$. Defining $R=[c/(1-e^{2a\xi})]^{1/5}$, for constant c, one gets

$$e^{2\chi}=AR^5(R^5-c),\qquad e^{2\zeta}=BR^3(R^5-c),$$

where $A=8a^2/5\beta c^2$ and B is an integration constant. Finally, choosing $B=A$, rescaling T, and defining $M=5\beta c/4$, one obtains

$$e^{2\lambda}=\frac{5\beta}{2}R^2-\frac{2M}{R^3},\qquad e^{2\rho}=e^{2\sigma}=R^2. \tag{A7}$$

In four-dimensional coordinates the solution can be written

$$-r^2\left(10\beta-\frac{2m}{r^{5/2}}\right)dt^2+\frac{1}{r}\left(10\beta-\frac{2m}{r^{5/2}}\right)^{-1}dr^2+r^2d\Omega_0^2,$$
$$e^{2\phi}=r,$$

and has therefore the asymptotic behavior (h). In this form, the solution was obtained in [2].

The solution (A7) can also be generalized to the case of positive λ_e and λ_i,

$$-\left[\frac{5\beta}{2}R^2+1-\frac{2M}{R^3}\right]dT^2+\left[\frac{5\beta}{2}R^2+1-\frac{2M}{R^3}\right]^{-1}dR^2+\frac{R^2}{3}(d\Omega_+^2+d\Omega_+^2).$$

In this form it generalizes the six-dimensional Tangherlini–anti-de Sitter metric [8], with S^4 replaced by $S^2\times S^2$.

4. $\lambda_e < 0$, $\lambda_i < 0$, $\eta' = \zeta' = \chi'$

The field equations yield

$$\chi''=\frac{\beta}{2}e^{2\chi},\qquad e^{2\zeta}=e^{2\eta}=\frac{\beta}{2}e^{2\chi},$$

and hence

$$e^{2\chi}=\frac{2}{\beta}\frac{a^2}{\sinh^2 a\xi}.$$

Defining a variable $R=2a/(1-e^{2a\xi})$, one gets

$$e^{2\lambda}=\frac{\beta}{2}R(R-2M),\qquad e^{2\rho}=e^{2\sigma}=\frac{2}{\beta}, \tag{A8}$$

where $M=a$. The metric is clearly the direct product AdS$^2\times H^2\times H^2$. After dimensional reduction,

$$ds^2=-r(r-2m)dt^2+\frac{dr^2}{r(r-2m)}+\frac{4}{\beta^2}d\Omega_-^2,$$
$$e^{2\phi}=\frac{2}{\beta},$$

which corresponds to the asymptotic behavior (g).

APPENDIX B: EXACT SOLUTIONS OF THE EH-CC-GB SYSTEM

Some of the exact solutions of the previous appendix can be extended to the GB case. We write them in six-dimensional form, since the duality is more apparent.

(e) $ds^2=-(|\Lambda_e|r^2+1)dt^2+(|\Lambda_e|r^2+1)^{-1}dr^2+r^2d\Omega_+^2+\frac{1}{|\Lambda_i|}d\Omega_-^2$, or $ds^2=-|\Lambda_e|r^2dt^2+(|\Lambda_e|r^2)^{-1}dr^2+r^2d\Omega_0^2+\frac{1}{|\Lambda_i|}d\Omega_-^2$

(f) $ds^2=-(|\Lambda_e|r^2+1)dt^2+(|\Lambda_e|r^2+1)^{-1}dr^2+\frac{1}{|\Lambda_i|}d\Omega_-^2+r^2d\Omega_+^2$, or $ds^2=-|\Lambda_e|r^2dt^2+(|\Lambda_e|r^2)^{-1}dr^2+\frac{1}{|\Lambda_i|}d\Omega_-^2+r^2d\Omega_0^2$.

(g) $ds^2 = -(\frac{r^2}{\Delta} - m)dt^2 + (\frac{r^2}{\Delta} - m)^{-1}dr^2 + \Delta d\Omega_-^2 + \Delta d\Omega_-^2$.

(h) $ds^2 = -\frac{r^2}{\Gamma}dt^2 + \frac{\Gamma}{r^2}dr^2 + r^2 d\Omega_0^2 + r^2 d\Omega_0^2$. where Λ_e and Λ_i are given by (1.5) and

$$\Delta = \frac{1 \pm \sqrt{1 - 4\alpha\beta}}{2\beta}, \qquad \Gamma = \frac{5(1 \pm \sqrt{1 - 12\alpha\beta/5})}{2\beta}.$$

An interesting two-parameter exact solution for special values of α and β has recently been given in [9].

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