PHYSICAL REVIEW D **75,** 024036 (2007)

Transgressing the horizons: Time operator in two-dimensional dilaton gravity

Gabor Kunstatter[1,*] and Jorma Louko[2,†]

[1]*Department of Physics and Winnipeg Institute of Theoretical Physics, University of Winnipeg, Winnipeg, Manitoba, Canada R3B 2E9*
[2]*School of Mathematical Sciences, University of Nottingham, Nottingham NG7 2RD, United Kingdom*
(Received 4 September 2006; published 26 January 2007)

We present a Dirac quantization of generic single-horizon black holes in two-dimensional dilaton gravity. The classical theory is first partially reduced by a spatial gauge choice under which the spatial surfaces extend from a black or white hole singularity to a spacelike infinity. The theory is then quantized in a metric representation, solving the quantum Hamiltonian constraint in terms of (generalized) eigenstates of the ADM mass operator and specifying the physical inner product by self-adjointness of a time operator that is affinely conjugate to the ADM mass. Regularity of the time operator across the horizon requires the operator to contain a quantum correction that distinguishes the future and past horizons and gives rise to a quantum correction in the hole's surface gravity. We expect a similar quantum correction to be present in systems whose dynamics admits black hole formation by gravitational collapse.

DOI: 10.1103/PhysRevD.75.024036 PACS numbers: 04.60.Ds, 04.20.Fy, 04.60.Kz, 04.70.Dy

I. INTRODUCTION

Pure Einstein gravity in two spacetime dimensions is trivial, in the sense that Einstein's vacuum field equations are satisfied by any metric. Dynamically interesting two-dimensional gravity theories can however be constructed by including suitable matter, and some such two-dimensional theories are equivalent to a reduction of higher-dimensional Einstein gravity to spherical symmetry [1]. Quantization of two-dimensional gravity theories thus presents an interesting problem, both as a dynamically simplified setting for developing techniques that might be generalizable to higher dimensions, as well as a quantization of the spherically symmetric degrees of freedom of higher-dimensional Einstein black holes. In particular, the macroscopic geometric quantities that are associated with quantum black holes in the semiclassical limit, such as the surface gravity of the horizon, are all present in the two-dimensional setting. The quantization may therefore be of interest from the semiclassical point of view even if the fundamental building blocks of higher-dimensional gravity turn out to be strings, spin networks or other pregeometric quantities [2].

In this paper we quantize a class of two-dimensional dilaton gravity theories specified by a dilaton potential, under mild assumptions that guarantee the classical solutions with positive ADM mass to be black holes with a single, nondegenerate Killing horizon and suitable asymptotics. This class of theories includes, in particular, symmetry-reduced Einstein gravity in four or more spacetime dimensions.

We first partially reduce the theory classically by a spatial gauge choice [3,4] that allows the spatial surfaces to extend from a singularity to an infinity, crossing exactly one branch of the horizon, and we choose boundary conditions that imply positivity of the classical ADM mass, specify whether the singularity and horizon are those of a black hole or a white hole, and prescribe the Killing time evolution rate of the asymptotic ends of the spatial surfaces. We then Dirac quantize this partially reduced theory in a metric representation. The quantum Hamiltonian constraint is solved in terms of eigenstates of the quantum ADM mass operator, and a class of momentum-type quantum observables is constructed from classical observables that are related to the time difference between the asymptotic ends of the spatial surfaces. Transforming to a representation that allows the ADM mass eigenstates to be treated as non-normalizable states, we finally specify the inner product by requiring that a particular momentum observable, affinely conjugate to the ADM mass operator, is self-adjoint. The resulting spectrum of the ADM mass operator is continuous and consists of the positive real line.

The novel features of our quantum theory reside in the momentum observables. The classical momentum observables are constructed to be regular across the horizon that the spatial surfaces cross. As a consequence, when evaluated across the other horizon, they pick up an imaginary contribution inversely proportional to the hole's surface gravity. The corresponding quantum momentum observables are similarly constructed to be regular across the horizon that the spatial surfaces cross. When evaluated across the other horizon, they also turn out to pick up an imaginary contribution, and this contribution differs from that of the corresponding classical observable by a factor that approaches unity for masses much larger than Planck mass but is significantly smaller than unity near Planck mass and vanishes below Planck mass. The singular contributions in the momentum observables thus provide a definition of the inverse surface gravity operator in the quantum theory, with significant quantum corrections at the Planck scale. The presence of such quantum corrections can be understood as a consequence of the fluctuations that our Dirac quantization of the Hamiltonian

*Electronic address: g.kunstatter@uwinnipeg.ca
†Electronic address: jorma.louko@nottingham.ac.uk

constraint allows around the classical Hamiltonian constraint surface.

While the dynamical content of the system is limited in that the classical theory has no local propagating degrees of freedom [1,5–12], we expect a number of the features of the quantum theory to be generalizable upon inclusion of matter that gives the system local dynamics [4,13–16]. In particular, we expect the definition of regular quantum observables across the horizon to survive. Also, as our foliation extends to the singularity of the eternal hole, it may be possible in the presence of matter to introduce boundary conditions that allow the study of singularity formation in the quantum theory [4].

The paper is organized as follows. The partially reduced classical theory is presented in Sec. II, and the theory is quantized in Sec. III. The inverse surface gravity operator is constructed in Sec. IV. Section V contains a summary and a discussion.

II. CLASSICAL THEORY

A. Action and solutions

We work with the action

$$S[g, \phi] = \frac{1}{2G} \int d^2x \sqrt{-g}\left(\phi R(g) + \frac{V(\phi)}{l^2}\right), \tag{2.1}$$

which is, up to conformal reparametrizations of the metric, the most general two-dimensional second order, diffeomorphism invariant action involving a metric $g_{\mu\nu}$ and a scalar ϕ [1,5,6]. l is a positive constant of dimension length and G is the two-dimensional Newton's constant. We do not need to fix the physical dimension of G, but since GS is dimensionless, the physical dimensions of G and Planck's constant $\hbar$, to be introduced in Sec. III, are related so that $\hbar G$ is dimensionless.

The action (2.1) can be obtained from a class of gravitational actions in $2 + n$ dimensions by reduction to the spherically symmetric ansatz

$$ds^2_{2+n} = \frac{ds^2}{j(\phi)} + r^2 d\Omega_n^2, \tag{2.2}$$

where $n \geq 2$, $d\Omega_n^2$ is the line element on unit S^n, ds^2 is the two-dimensional line element that appears in (2.1), $j(\phi)$ satisfies $dj/d\phi = V(\phi)$ and the area-radius r is related to ϕ by $\phi = (r/l)^n$ and $dj/d\phi = V(\phi)$. The $(2 + n)$-dimensional action depends on the choice of the potential V and equals Einstein's action in the special case $V = \phi^{-1/n}$ [1,5,6].

As one may expect from the special case of symmetry-reduced Einstein gravity, the action (2.1) obeys a Birkhoff theorem [7]. Assuming that $V(\phi)$ is nowhere vanishing, the theorem states that the vector

$$k^\mu = \frac{1}{\sqrt{-g}} \epsilon^{\mu\nu} \partial_\nu \phi \tag{2.3}$$

is nonvanishing and Killing on every classical solution. Using ϕ as one of the coordinates, the solution can then be written in the Schwarzschild-like form

$$ds^2 = -[j(\phi) - 2lGM]dt_s^2 + [j(\phi) - 2lGM]^{-1} l^2 d\phi^2, \tag{2.4}$$

where t_s is the Schwarzschild time coordinate, the Killing vector (2.3) equals ∂_{t_s} and the integration constant M is the ADM mass. Note that the combination lGM is dimensionless. From now on we assume $M > 0$.

We assume the potential $V(\phi)$ to be positive and its small ϕ behavior to be such that $j(\phi)$ may be defined as

$$j(\phi) := \int_0^\phi d\tilde{\phi} V(\tilde{\phi}), \tag{2.5}$$

with the consequence that $j(\phi) \to 0$ as $\phi \to 0$. These assumptions hold, in particular, for symmetry-reduced Einstein gravity. It follows that the $(2 + n)$-dimensional metric (2.2) is generically singular at $\phi = 0$, and the two-dimensional metric (2.4) is generically singular at $\phi = 0$ for a range of theories, including symmetry-reduced Einstein gravity. We therefore regard $\phi = 0$ as a singularity that is not part of the spacetime. At $\phi \to \infty$, we assume that $j(\phi)$ grows without bound but so slowly that $\int^\infty [j(\phi)]^{-1/2} d\phi$ is infinite. Again, this holds for symmetry-reduced Einstein gravity. It follows that the metric (2.4) has at $\phi \to \infty$ an infinity, whose causal properties in terms of the null and spacelike infinities depend on whether $\int^\infty [j(\phi)]^{-1} d\phi$ is finite or infinite. The global structure of the spacetime can be found by standard techniques [17]. There is precisely one Killing horizon, which is bifurcate and located at $j(\phi) = 2lGM$ [18]. The Killing vector ∂_{t_s} is timelike in the exterior regions, where $j(\phi) > 2lGM$, and spacelike in the black and white hole regions, where $0 < j(\phi) < 2lGM$. Figure 1 shows a conformal diagram of the case in which $\int^\infty [j(\phi)]^{-1} d\phi$ is infinite.

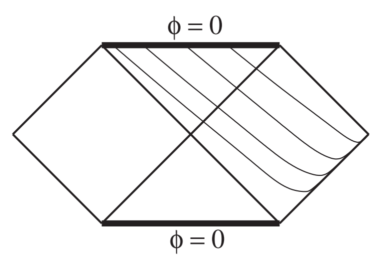

FIG. 1. Conformal diagram of the extended spacetime (2.4) with $M > 0$, in the case of infinite $\int^\infty [j(\phi)]^{-1} d\phi$ (which implies that the null infinities are distinct from the spacelike infinities). The thin lines show surfaces of constant Painlevé-Gullstrand T (2.7) with $\epsilon = 1$, assuming finite $\int_0 [j(\phi)]^{-1/2} d\phi$ (which determines the asymptotics near the singularity) and infinite $\int^\infty [j(\phi)]^{-3/2} d\phi$ (which determines the asymptotics near infinity). The diagram for $\epsilon = -1$ is obtained by up-down inversion.

We are interested in foliations that extend from $\phi = 0$ to an infinity at $\phi \to \infty$ and are regular across the horizon. A convenient example are the Painlevé-Gullstrand (PG) coordinates (T, Y) [19,20], related to the Schwarzschild coordinates (2.4) by

$$dY = \frac{l d\phi}{j(\phi)}, \tag{2.6a}$$

$$dT = dt_s + \epsilon\sqrt{\frac{2lGM}{j(\phi)}}\frac{l d\phi}{j(\phi) - 2lGM}, \tag{2.6b}$$

where $\epsilon = \pm 1$. The metric reads

$$ds^2 = j(\phi)\left[-dT^2 + \left(dY + \epsilon\sqrt{\frac{2lGM}{j(\phi)}}dT\right)^2\right] \tag{2.7}$$

and is clearly regular across the horizon. $\epsilon = 1$ (respectively $\epsilon = -1$) gives the ingoing (outgoing) PG metric, which covers the black (white) hole region and one exterior region. The asymptotic behavior of the constant T surfaces at $\phi \to 0$ and $\phi \to \infty$ depends on the asymptotic behavior of $j(\phi)$. Figure 1 shows a sketch of these surfaces in the case of finite $\int_0 [j(\phi)]^{-1/2} d\phi$ but infinite $\int^\infty [j(\phi)]^{-3/2} d\phi$, which occurs in symmetry-reduced Einstein gravity in four and five spacetime dimensions.

B. Hamiltonian analysis

For the Hamiltonian analysis, we parametrize the metric as

$$ds^2 = e^{2\rho}[-\sigma^2 dt^2 + (dx + N dt)^2], \tag{2.8}$$

where the rescaled lapse σ and rescaled shift N will play the role of Lagrange multipliers. From the action (2.1) we find that the momenta conjugate to ϕ and ρ are

$$\Pi_\phi = \frac{1}{G\sigma}(N\rho' + N' - \dot\rho), \tag{2.9a}$$

$$\Pi_\rho = \frac{1}{G\sigma}(N\phi' - \dot\phi), \tag{2.9b}$$

where dot denotes derivative with respect to t and prime denotes derivative with respect to x. The Hamiltonian action can be found by standard techniques [21,22] and reads

$$S = \int dt dx (\Pi_\rho \dot\rho + \Pi_\phi \dot\phi) - \int dt H, \tag{2.10}$$

where the total Hamiltonian is

$$H = \int dx (\sigma \mathcal{G} + N\mathcal{F}) + H_B, \tag{2.11}$$

the Hamiltonian constraint $\mathcal{G}$ and the momentum constraint $\mathcal{F}$ are given by

$$G\mathcal{G} := -G^2 \Pi_\rho \Pi_\phi + \phi'' - \phi'\rho' - \frac{1}{2l^2}e^{2\rho}V(\phi), \tag{2.12a}$$

$$\mathcal{F} := \rho'\Pi_\rho - \Pi_\rho' + \phi'\Pi_\phi, \tag{2.12b}$$

and H_B consists of boundary terms evaluated at the (asymptotic) upper and lower ends of the range of x.

The Hamiltonian equations of motion are the constraint equations $\mathcal{G} = 0 = \mathcal{F}$ enforced by the Lagrange multipliers, the momentum evolution equations

$$G\dot\Pi_\phi = -\sigma'' - (\sigma\rho')' + \frac{\sigma}{2l^2}e^{2\rho}\frac{dV}{d\phi} + (NG\Pi_\phi)', \tag{2.13a}$$

$$G\dot\Pi_\rho = (GN\Pi_\rho)' - (\sigma\phi')' + \sigma e^{2\rho}\frac{V(\phi)}{l^2}, \tag{2.13b}$$

and the relations (2.9). To obtain these equations of motion from the action (2.10), one needs to specify the boundary conditions and H_B so that the boundary terms in the variation of the action vanish. We shall address this issue within the partially reduced theory in Subsec. II D.

C. Spacetime reconstruction with the Painlevé-Gullstrand time

In this subsection we reconstruct from the canonical data $(\rho, \phi, \Pi_\rho, \Pi_\phi)$ the spacetime and the location of the spacelike surface on which the canonical data is defined. We follow closely Kuchař's analysis of spherically symmetric Einstein gravity in four dimensions [10], but we specify the location of the surface in terms of the PG time T (2.7), rather than in terms of the Schwarzschild time t_s (2.4). This will enable us to discuss the regularity of the horizon-crossing in the quantum theory in Sec. III.

To begin, we define the mass function $\mathcal{M}$ by

$$\mathcal{M} := \frac{1}{2lG}\{e^{-2\rho}[l^2G^2\Pi_\rho^2 - l^2(\phi')^2] + j(\phi)\}. \tag{2.14}$$

Differentiating with respect to x and using (2.12), we find

$$\mathcal{M}' = -le^{-2\rho}(\phi'\mathcal{G} + G\Pi_\rho\mathcal{F}). \tag{2.15}$$

When the constraints hold, $\mathcal{M}$ is therefore independent of x, and when all the equations of motion hold, $\mathcal{M}$ is also independent of t. Comparison with (2.4) or (2.7) shows that on a classical solution $\mathcal{M}$ is equal to the ADM mass M.

To find the location of the surface in the spacetime, we look for a coordinate transformation

$$T = T(x, t), \tag{2.16a}$$

$$\phi = \phi(x, t), \tag{2.16b}$$

that brings the metric (2.8) to the form

$$ds^2 = j(\phi)[-dT^2 + (dY + F dT)^2], \tag{2.17}$$

where

$$dY = \frac{l d\phi}{j(\phi)} \tag{2.18}$$

and F is initially unspecified. When the field equations hold, F will turn out to be related to the ADM mass as shown in (2.7).

Differentiating (2.16) yields

$$dT = \dot{T}dt + T'dx, \tag{2.19a}$$

$$d\phi = \dot{\phi}dt + \phi'dx. \tag{2.19b}$$

Substituting (2.19) in (2.17) and comparing with (2.8), we obtain

$$e^{2\rho} = j(\phi)[A^2 - (T')^2], \tag{2.20a}$$

$$e^{2\rho}(\sigma^2 - N^2) = j(\phi)(\dot{T}^2 - B^2), \tag{2.20b}$$

$$e^{2\rho}N = j(\phi)(AB - T'\dot{T}), \tag{2.20c}$$

where

$$A := \frac{l\phi'}{j} + FT', \tag{2.21a}$$

$$B := \frac{l\dot{\phi}}{j} + F\dot{T}. \tag{2.21b}$$

Solving (2.20) for N and σ, we find

$$N = \frac{AB - T'\dot{T}}{A^2 - (T')^2}, \tag{2.22a}$$

$$\sigma = \frac{A\dot{T} - BT'}{A^2 - (T')^2}. \tag{2.22b}$$

Note that the denominators in (2.22) are positive because of (2.20a). To arrive at (2.22b) from (2.20b), we have chosen the sign of the square root so that σ has the same sign as $\dot{T}$ when $T' = 0$. Assuming the metric to be invertible and both T and t to increase towards the future, it then follows by continuity that σ is everywhere positive.

So far no field equations have been used. To proceed, we substitute (2.22) in (2.9b). Writing ϕ' and $\dot{\phi}$ in terms of A and B from (2.21), we find that a cancellation occurs and allows the result to be written as

$$lG\Pi_\rho = j(\phi)(AF - T'). \tag{2.23}$$

Eliminating A from (2.21a) and (2.23) yields

$$T' = \frac{l(F\phi' - G\Pi_\rho)}{j(1 - F^2)}. \tag{2.24}$$

To find F, we substitute (2.24) in (2.20a) and (2.21a) and eliminate A. Using (2.14), we find

$$jF^2 = 2lG\mathcal{M}, \tag{2.25}$$

whose two solutions are

$$F = \pm\sqrt{\frac{2lG\mathcal{M}}{j}}. \tag{2.26}$$

Collecting, we finally obtain

$$T' = \frac{l}{j - 2lG\mathcal{M}}\left(-G\Pi_\rho \pm \sqrt{\frac{2lG\mathcal{M}}{j}}\phi'\right). \tag{2.27}$$

To summarize, Eqs. (2.14), (2.26), and (2.27) specify both the spacetime and the location of the surface in the spacetime. When the field equations hold, $\mathcal{M}$ (2.14) is the ADM mass, and comparison of (2.7) with (2.17) and (2.26) shows that for the upper (respectively lower) sign, T in (2.27) is the ingoing (outgoing) PG time. The embedding of the surface in the spacetime is determined by the canonical data by integrating (2.18) and (2.27), up to the isometries generated by the Killing vector $\partial/\partial T$. Note that the first term in (2.27) arises from the Schwarzschild time t_s (2.4) [6,10] and the second term arises from the transformation to the PG time. Note also from (2.14) that the zero in the denominator in (2.27) at the horizon is cancelled by a zero in the numerator to give a finite limit when the sign of Π_ρ is such that the surface crosses the horizon that the PG coordinates cover.

Although the spacetime interpretation of T' (2.27) relies on the field equations, Eq. (2.27) can be understood to define T' as a function on the phase space independently of the field equations [6,10]. We shall return to this in Subsec. II D after having performed a partial reduction and specified the boundary conditions.

D. Partial reduction

The Hamiltonian action (2.10) contains two constraints, the Hamiltonian constraint $\mathcal{G}$ and the spatial diffeomorphism constraint $\mathcal{F}$. We now eliminate $\mathcal{F}$ by a spatial gauge condition that fixes ϕ' to a given function of ϕ. For concreteness, we focus on the gauge [3]

$$l\phi' - j(\phi) = 0, \tag{2.28}$$

and postpone the discussion of other choices to Sec. V.

As the Poisson bracket of $\mathcal{F}$ and the left-hand side of (2.28) is nonzero, the gauge condition (2.28) is admissible [3]. Substituting (2.28) in the action (2.10), using (2.15) and introducing the rescaled lapse $\tilde{\sigma}$ by

$$\tilde{\sigma} := \frac{\sigma e^{2\rho}}{j}, \tag{2.29}$$

we obtain the action

$$S = \int dtdx(\Pi_\rho\dot{\rho} + \tilde{\sigma}\mathcal{M}') + S_B, \tag{2.30}$$

where S_B is a boundary action, to be specified shortly, and the mass function $\mathcal{M}$ is now given by

$$\mathcal{M} := \frac{1}{2lG}[e^{-2\rho}(l^2G^2\Pi_\rho^2 - j^2) + j]. \tag{2.31}$$

For notational convenience, we suppress the ϕ-dependence of j and continue to use for the mass function (2.31) the same symbol as in the unreduced theory.

The field equations read

$$\mathcal{M}' = 0 \tag{2.32}$$

and

$$\dot{\rho} = \tilde{\sigma}' e^{-2\rho} lG\Pi_\rho, \tag{2.33a}$$

$$lG\dot{\Pi}_\rho = \tilde{\sigma}' e^{-2\rho}(l^2G^2\Pi_\rho^2 - j^2). \tag{2.33b}$$

If desired, Π_ϕ and N can be recovered from the original equations of motion (2.9) and (2.13). In particular, preservation of the gauge condition (2.28) yields

$$N = \frac{\sigma lG\Pi_\rho}{j} = \tilde{\sigma} e^{-2\rho} lG\Pi_\rho. \tag{2.34}$$

We are now in a position to address the boundary conditions at $\phi \to \infty$ and $\phi \to 0$. We choose for concreteness a falloff that makes the foliation asymptotic to that of the PG coordinates (2.17) at each end and postpone the discussion of other choices to Sec. V. We also assume for concreteness the large ϕ behavior of $V(\phi)$ to be such that there exists a positive constant β for which the integral

$$I_\beta^+(\phi) := \int_\phi^\infty d\tilde{\phi}[j(\tilde{\phi})]^{-\beta-3/2} \tag{2.35}$$

is finite. This holds for any potential that satisfies $V(\phi) > C\phi^{\gamma-1}$ at large ϕ, where C and γ are positive constants, and holds therefore, in particular, for symmetry-reduced Einstein gravity. To control the surfaces at small ϕ, we choose a positive constant α for which the integral

$$I_\alpha^-(\phi) := \int_0^\phi d\tilde{\phi}[j(\tilde{\phi})]^{\alpha-1/2} \tag{2.36}$$

is finite. The finiteness of (2.5) shows that a choice with $\alpha \geq 1/2$ will work for all potentials.

Given the positive constants α and β, we impose at $\phi \to 0$ the falloff

$$\begin{aligned} e^{2\rho} &= j[1 + O(j^\alpha)], \\ lG\Pi_\rho &= \epsilon\sqrt{2lGM_0 j}[1 + O(j^\alpha)], \\ \tilde{\sigma} &= \sigma_0 + O(I_\alpha^-(\phi)), \end{aligned} \tag{2.37}$$

and at $\phi \to \infty$ the falloff

$$\begin{aligned} e^{2\rho} &= j[1 + O(j^{-\beta-1})], \\ lG\Pi_\rho &= \epsilon\sqrt{2lGM_\infty j}[1 + O(j^{-\beta})], \\ \tilde{\sigma} &= \sigma_\infty + O(I_\beta^+(\phi)), \end{aligned} \tag{2.38}$$

where ϵ equals either 1 or -1 and takes the same value in both (2.37) and (2.38). σ_0, σ_∞, M_0 and M_∞ are independent of x but may *a priori* depend on t. M_0 and M_∞ are assumed positive. The O-terms may depend on t, and we assume that they can be treated under algebraic manipulations and differentiation as series in powers of j. It can be verified that this falloff is consistent with the constraint (2.32) and preserved in time by the evolution Eqs. (2.33), where $\tilde{\sigma}$ remains freely specifiable apart from the falloff. Note that the O-terms in $\tilde{\sigma}$ generate time evolution that affects the O-terms in ρ and Π_ρ in precisely the order shown in (2.37) and (2.38). The evolution Eq. (2.33b) thus implies that M_0 and M_∞ are independent of t, the constraint (2.32) implies that M_0 and M_∞ are equal to each other, and it then follows from (2.31) that they are both equal to the ADM mass. The foliation is at $\phi \to 0$ and $\phi \to \infty$ asymptotic to the PG foliation (2.7), with the values of ϵ matching. σ_0 and σ_∞ remain freely specifiable functions of t, and they give the rate at which the asymptotic PG times evolve with respect to t. Finally, the action (2.30) and its variation under these conditions can be verified to be well-defined if we set

$$S_B = -\int dt(\sigma_\infty M_\infty - \sigma_0 M_0), \tag{2.39}$$

where σ_∞ and σ_0 are freely prescribable as functions of t but are considered fixed in the variation. Note that the total action can be written in the alternative form

$$S = \int dtdx(\Pi_\rho\dot{\rho} - \tilde{\sigma}'\mathcal{M}). \tag{2.40}$$

Consider now observables (or "perennials" [23]). The mass function $\mathcal{M}$ (2.31) has clearly a vanishing Poisson bracket with the single remaining constraint and is hence an observable. To find a second observable, we define

$$\Pi_\mathcal{M} := \frac{lG\Pi_\rho - \epsilon\sqrt{2lG\mathcal{M}j}}{j - 2lG\mathcal{M}}. \tag{2.41}$$

The right-hand side of (2.41) is not defined at the zeroes of the denominator, but if Π_ρ has the same sign as in the falloff region, it follows from (2.31) that $\Pi_\mathcal{M}$ can be written as

$$\Pi_\mathcal{M} = \frac{\epsilon(j - e^{2\rho})}{\sqrt{j^2 + e^{2\rho}(2lG\mathcal{M} - j)} + \sqrt{2lG\mathcal{M}j}}, \tag{2.42}$$

which is nonsingular at the zeroes of the denominator in (2.41). The phase space therefore contains a neighborhood of the classical solutions in which $\Pi_\mathcal{M}$ is well defined by (2.41), supplemented by (2.42) at the zeroes of the denominator. We restrict the attention to this neighborhood. As the notation suggests, $\Pi_\mathcal{M}$ is conjugate to $\mathcal{M}$,

$$\{\mathcal{M}(x), \Pi_\mathcal{M}(y)\} = \delta(x - y). \tag{2.43}$$

From (2.43) it follows that $\Pi_\mathcal{M}$ in its own right is not an observable. Consider, however the quantity

$$P := \int dx\Pi_\mathcal{M}(x), \tag{2.44}$$

where the convergence of the integral is guaranteed by the falloff (2.37) and (2.38). From (2.43) we find

$$\{\mathcal{M}(x), P\} = 1. \tag{2.45}$$

If $\lambda(x)$ is the infinitesimal parameter of a gauge transformation, vanishing at the upper and lower limits of x, the infinitesimal change in P under this transformation reads

$$\left\{P, \int dx \lambda'(x) \mathcal{M}(x)\right\} = -\int dx \lambda'(x) = 0. \quad (2.46)$$

Hence P is an observable.

When the equations of motion hold, Eqs. (2.27) and (2.28) show that $\Pi_{\mathcal{M}} = -T'$, where T is the PG time, ingoing for $\epsilon = 1$ and outgoing for $\epsilon = -1$. In terms of the spacetime geometry, P is therefore equal to the difference of the PG times at the left and right ends of the spatial surface. Note that this geometric interpretation is consistent with the equation of motion for P,

$$\dot{P} = \left\{P, \int dx \tilde{\sigma}'(x) \mathcal{M}(x)\right\} = \sigma_0 - \sigma_\infty. \quad (2.47)$$

The fully reduced theory can be obtained by taking the spatially constant value of $\mathcal{M}$ as a new phase space variable. Denoting this variable by M and proceeding as in [6,10], we find the fully reduced action

$$S_{\rm red} = \int dt [P\dot{M} - (\sigma_\infty - \sigma_0) M]. \quad (2.48)$$

P is therefore conjugate to the ADM mass in the fully reduced theory.

III. QUANTIZATION OF THE PARTIALLY REDUCED THEORY

In this section we quantize the partially reduced theory of Subsec. II D. Following Ashtekar's algebraic extension of Dirac quantization [24,25], we first find a vector space of solutions to the quantum constraint and then determine the physical inner product from the adjointness relations of a judiciously-chosen set of quantum observables.

A. Classical constraint

We begin with some observations about the classical constraint.

It is convenient to transform from the canonical pair (ρ, Π_ρ) to the pair (X, P_X), where $X = e^\rho$ and $P_X = e^{-\rho}\Pi_\rho$. The mass function (2.31) takes the form

$$\mathcal{M} = \frac{1}{2lG}\left(l^2 G^2 P_X^2 - \frac{j^2}{X^2} + j\right), \quad (3.1)$$

and the solutions to the classical constraint Eq. (2.32) can be written as

$$l^2 G^2 P_X^2 - \frac{j^2}{X^2} = 2lGM - j, \quad (3.2)$$

where the integration constant M is the value of $\mathcal{M}$, independent of the spatial coordinate x. The boundary conditions of Subsection II D imply that M is positive.

For each x, Eq. (3.2) can be understood as the classical energy conservation equation of a particle moving on the half line of positive X with the (true) Hamiltonian

$$H := l^2 G^2 P_X^2 - \frac{j^2}{X^2}, \quad (3.3)$$

which consists of a conventional quadratic kinetic term and the attractive potential well $-j/(X^2)$. The value of the energy is $2lGM - j$, which is positive (respectively negative) for those values of x that in the spacetime are inside (outside) the hole. We shall see that the oscillatory/exponential behavior of the solutions to the quantum constraint in Subsec. III B is in agreement with this classical picture.

We note in passing that the Poisson bracket algebra of H (3.3) and the functions

$$D := \frac{XP_X}{2}, \qquad K := \frac{X^2}{4l^2G^2}, \quad (3.4)$$

at fixed x is the $\mathfrak{o}(2, 1)$ algebra,

$$\{D, H\} = H; \qquad \{K, H\} = 2D; \qquad \{K, D\} = K. \quad (3.5)$$

In particular, the first of the brackets in (3.5) is equivalent to the observation that H is scale invariant: Under the scale transformation $(X, P_X) \rightarrow (\alpha X, P_X/\alpha)$, where α is a positive constant, H only changes by an overall multiplicative factor. In terms of the spacetime geometry, $K = e^{2\rho}$ is the conformal factor in the metric (2.8) and D can be related to the expansion of null geodesics [3]. The potential interest in this observation is that quantization of H, D and K forms the basis of conformal quantum mechanics [26], and it has been suggested that a near-horizon conformal symmetry could account for black hole microstates and black hole entropy [27,28]. There are however two obstacles to making progress from this observation in the present context. First, the classical $O(2, 1)$ symmetry generated by H, D and K cannot be promoted into a symmetry of conformal quantum mechanics—it develops an anomaly [29]. Second, as the classical system still has one constraint, the phase space functions H, D and K are not classical observables, and their quantization by the methods of conformal quantum mechanics would somehow need to accommodate a quantum version of the remaining constraint. We shall not pursue this line further here.

B. Quantum constraint

We quantize in a representation in which the quantum states are functionals of $X(x)$. The operator substitution in this representation at each x is

$$P_X \rightarrow -i\left(\frac{\hbar}{l}\right)\frac{\partial}{\partial X}, \quad (3.6)$$

where $\hbar$ is Planck's constant and the factor $1/l$ is required for dimensional consistency because of the functional dependence on x. Suppressing x, we promote the mass function (3.1) into the mass operator

$$\widehat{\mathcal{M}} := \frac{1}{2lG}\left(-\hbar^2 G^2 \frac{\partial^2}{\partial X^2} - \frac{j^2}{X^2} + j\right). \tag{3.7}$$

Note that the combination $\hbar G$ is dimensionless, as we observed in Subsection II A. Following Dirac's procedure [21,22], we then promote the classical constraint Eq. (2.32) into the quantum constraint equation

$$(\widehat{\mathcal{M}})'\Psi = 0. \tag{3.8}$$

We look for quantum states that are eigenstates of $\widehat{\mathcal{M}}$,

$$\widehat{\mathcal{M}}\Psi_M = M\Psi_M, \tag{3.9}$$

where the eigenvalue M is independent of x. As the classical boundary conditions of Subsection II D assume the ADM mass to be positive, we take $M > 0$. It is immediate from (3.8) that Ψ_M is annihilated by the quantum constraint. Using (3.7), Eq. (3.9) reads

$$\left(-\hbar^2 G^2 \frac{\partial^2}{\partial X^2} - \frac{j^2}{X^2}\right)\Psi_M = (2lGM - j)\Psi_M. \tag{3.10}$$

Note that (3.10) is the quantized version of (3.2). While (3.10) is still a functional differential equation in the variable $X(x)$, the absence of derivatives with respect to x implies that the different spatial points decouple, and we may separate the solution with the ansatz

$$\Psi_M(X(x)) = \prod_x \psi_M(X;x) := \exp\left\{\int \frac{dx}{l} \ln[\psi_M(X;x)]\right\}, \tag{3.11}$$

where the infinite product over x is defined via the integral expression. The factor $1/l$ in the integration measure is required for dimensional consistency. $\psi_M(X;x)$ then satisfies (3.10) as an ordinary differential equation at each x,

$$\left(-\hbar^2 G^2 \frac{\partial^2}{\partial X^2} - \frac{j^2}{X^2}\right)\psi_M(X;x) = (2lGM - j)\psi_M(X;x). \tag{3.12}$$

A solution to (3.12) for $2lGM - j \neq 0$ is

$$\psi_M^\nu(X;x) := \omega^{-\nu}\sqrt{X}J_\nu(\omega X), \tag{3.13}$$

where J_ν is the Bessel function of the first kind [30] and

$$\omega^2 = \frac{2lGM - j}{\hbar^2 G^2}, \tag{3.14}$$

$$\nu^2 = \frac{1}{4} - \frac{j^2}{\hbar^2 G^2}. \tag{3.15}$$

The branch point structure of J_ν implies that ψ_M^ν is independent of the sign taken in solving (3.14) for ω. For $2lGM - j = 0$, we take ψ_M^ν to be given by the $\omega \to 0$ limit of (3.13), $X^{\nu+1/2}/[2^\nu\Gamma(\nu+1)]$, which again is a solution to (3.12). ψ_M^ν is then regular as a function of x everywhere, including the zero of $2lGM - j$.

For $j \neq \hbar G/2$, the functions ψ_M^ν with the two values of ν (3.15) are linearly independent. The case $j = \hbar G/2$ is special since $\nu = 0$, and if a linearly independent second solution to (3.12) were desired, it could be given in terms of a Neumann function [30]. For our purposes, ψ_M^ν will suffice for all ν.

At $X \to \infty$, ψ_M^ν is oscillatory for $\omega^2 > 0$ and exponentially increasing for $\omega^2 < 0$. If (3.12) were interpreted as the time-independent Schrödinger equation for the quantization of the classical Hamiltonian (3.3) in the Hilbert space $L_2(\mathbb{R}_+, dX)$, the relevant solution for $\omega^2 < 0$ would therefore not be ψ_M^ν but instead the exponentially decreasing linear combination proportional to $\sqrt{X}K_\nu(\sqrt{-\omega^2}X)$, where K_ν is the modified Bessel function of the second kind [30]. The possible negative values of ω^2 would be discrete and determined by the self-adjointness boundary condition at $X \to 0$ (see Example 2.5.14 in [31]); in particular, for $\nu^2 < 0$ the spectrum of ω^2 would be unbounded from below with every choice of the boundary condition. The relevant solution for $\omega^2 > 0$ would similarly be determined by the self-adjointness boundary condition at $X \to 0$ and would coincide with ψ_M^ν only when $\nu^2 \geq 0$ and one of two special boundary conditions is chosen. In the present context, however, there is no reason to relate the solutions to $L_2(\mathbb{R}_+, dX)$, and we may continue to work with ψ_M^ν. A quantum regularity condition that will be imposed in Subsec. III C will in fact exclude linear combinations of ψ_M^ν with the two signs of ν.

C. Quantum observables

Recall that the classical observables $\mathcal{M}$ (2.31) and P (2.44) induce a global canonical chart on the fully reduced phase space. If f is a smooth function of a real variable, $f(\mathcal{M})$ and $f(\mathcal{M})P$ are thus classical observables, and the set of such observables is large enough to separate the fully reduced phase space. In this subsection we define corresponding quantum observables in the partially reduced quantum theory as linear operators on a vector space annihilated by the quantum constraint.

We begin with the "momentum" observables. As preparation, consider $\Pi_\mathcal{M}$ (2.41). In terms of the canonical pair (X, P_X), we have

$$\Pi_\mathcal{M} = \frac{lGXP_X - \epsilon\sqrt{2lG\mathcal{M}j}}{j - 2lG\mathcal{M}}. \tag{3.16}$$

We seek to define the corresponding operator $\widehat{\Pi_\mathcal{M}}$ on the mass eigenstates by

$$\widehat{\Pi_\mathcal{M}}\psi_M^\nu := \frac{-i\hbar G(X\partial_X + \eta) - \epsilon\sqrt{2lGMj}}{j - 2lGM}\psi_M^\nu, \tag{3.17}$$

where the factor ordering parameter η may depend on x but not on M. Since both $\Pi_\mathcal{M}$ and ψ_M^ν are regular as functions of x across $2lGM - j = 0$, we postulate also $\widehat{\Pi_\mathcal{M}}\psi_M^\nu$ to be regular as a function of x across $2lGM - j = 0$. Using identity 9.1.27 of [30] to write (3.17) as

$$\widehat{\Pi_{\mathcal{M}}}\psi_M^\nu = \frac{-i\hbar G(\nu + \frac{1}{2} + \eta) - \epsilon\sqrt{2lGMj}}{j - 2lGM}\psi_M^\nu - i\frac{X\psi_M^{\nu+1}}{\hbar G}, \tag{3.18}$$

where the last term is always regular across $2lGM - j = 0$, we see that this regularity condition implies

$$\eta = -\frac{1}{2} - \nu + i\frac{\epsilon j}{\hbar G}. \tag{3.19}$$

We further postulate that η remain bounded as $\hbar \to 0$, as expected of a factor ordering parameter. To achieve this, we choose the sign of ν for $j > \hbar G/2$ so that

$$\nu = i\epsilon\sqrt{\frac{j^2}{\hbar^2 G^2} - \frac{1}{4}}. \tag{3.20}$$

We leave the sign of ν for $j < \hbar G/2$ unspecified.

Given the classical observable $f(\mathcal{M})P$, we now define the corresponding operator $\widehat{fP}$ on the mass eigenstates by

$$\widehat{fP} := \int dx \widehat{\Pi_{\mathcal{M}}} \widehat{f(\mathcal{M})}. \tag{3.21}$$

A convenient phase choice for the mass eigenstates is

$$\Phi_M := \prod_x E_M\psi_M^\nu := \exp\left[\int \frac{dx}{l}\ln(E_M\psi_M^\nu)\right], \tag{3.22}$$

where

$$E_M := \exp\left\{-i\frac{\epsilon}{\hbar G}[\sqrt{2lGMj} - j\ln(\sqrt{j} + \sqrt{2lGM})]\right\}. \tag{3.23}$$

We then find

$$\begin{aligned}\widehat{fP}\Phi_M &= f(M)\int dx \widehat{\Pi_{\mathcal{M}}}\Phi_M \\ &= f(M)\Phi_M\left(\int dx\frac{\widehat{\Pi_{\mathcal{M}}}\Phi_M}{\Phi_M}\right) \\ &= f(M)\Phi_M\left[\int dx\frac{\widehat{\Pi_{\mathcal{M}}}(E_M\psi_M^\nu)}{E_M\psi_M^\nu}\right] \\ &= f(M)\Phi_M\left[\int dx\frac{i(\hbar/l)\partial_M(E_M\psi_M^\nu)}{E_M\psi_M^\nu}\right] \\ &= f(M)\Phi_M\left[i\hbar\partial_M\int\frac{dx}{l}\ln(E_M\psi_M^\nu)\right] \\ &= i\hbar f(M)\partial_M\Phi_M,\end{aligned} \tag{3.24}$$

where we have used the identity

$$\widehat{\Pi_{\mathcal{M}}}(E_M\psi_M^\nu) = i(\hbar/l)\partial_M(E_M\psi_M^\nu), \tag{3.25}$$

which follows by observing that $X\partial_X(\omega^{\nu+(1/2)}\psi_M^\nu) = \omega\partial_\omega(\omega^{\nu+(1/2)}\psi_M^\nu) = \frac{1}{2}M\partial_M(\omega^{\nu+(1/2)}\psi_M^\nu)$.

The "position" observables are straightforward: Given the classical observable $f(\mathcal{M})$, we define the corresponding quantum observable $\hat{f}$ on the mass eigenstates by

$$\hat{f}\Phi_M := f(M)\Phi_M. \tag{3.26}$$

To obtain an observable algebra that acts on a vector space, we extend formulas (3.24) and (3.26) to define the action of the momentum and position observables on more general functions of the variable $X(x)$ and the parameter M. Given this action, we then build the vector space $V := \mathcal{A}(\text{span}\{\Phi_M\})$, where $\mathcal{A}$ is the algebra generated by the momentum and position observables. V carries by construction a representation of $\mathcal{A}$, and viewing the derivative in (3.24) as the limit of a differential quotient provides by linearity a sense in which V is annihilated by the quantum constraint. One might thus attempt to define a quantum theory by introducing an inner product on V, or possibly on some subspace obtained by replacing span$\{\Phi_M\}$ by a suitable subspace and $\mathcal{A}$ by a suitable subalgebra. A quantum theory of this kind would be expected to contain mass eigenstates as normalizable states. While discrete black hole spectra have been encountered in a number of approaches (see [32–36] for a small selection and [35] for a more extensive bibliography), we shall modify the representation in a way that will lead to a continuous mass spectrum.

D. Physical Hilbert space

We look for a quantum theory in which the spectrum of $\widehat{\mathcal{M}}$ is continuous and consists of the positive half line. While the mass eigenstates Φ_M do then not exist as normalizable states, one expects there to exist a spectral decomposition in which any sufficiently well-behaved function $\alpha: \mathbb{R}_+ \to \mathbb{C}$ defines a normalizable state by the map

$$\alpha \mapsto \int_0^\infty \frac{dM}{M}\alpha(M)\Phi_M. \tag{3.27}$$

The factor $1/M$ in the integration measure is a convention that will simplify what follows. If formula (3.27) holds in a sense that allows integration by parts without boundary terms, the representation of $\mathcal{A}$ given by (3.24) and (3.26) then induces on the space of the sufficiently well-behaved functions the representation

$$(\hat{f}\alpha)(M) = f(M)\alpha(M), \tag{3.28a}$$

$$(\widehat{fP}\alpha)(M) = -i\hbar M\frac{d}{dM}\left[\frac{f(M)}{M}\alpha(M)\right]. \tag{3.28b}$$

To build a quantum theory with these properties, we adopt (3.28) as the *definition* of the $\mathcal{A}$-action on the space $C_0^\infty(\mathbb{R}_+)$ of smooth compactly-supported functions $\alpha: \mathbb{R}_+ \to \mathbb{C}$. This gives, in particular, the commutators

$$[\widehat{\mathcal{M}}, \hat{P}] = i\hbar, \tag{3.29a}$$

$$[\widehat{\mathcal{M}}, \widehat{\mathcal{M}P}] = i\hbar\widehat{\mathcal{M}}. \tag{3.29b}$$

We look on $C_0^\infty(\mathbb{R}_+)$ for an inner product $(\cdot, \cdot)$ of the form

$$(\alpha_2, \alpha_1) = \int_0^\infty dM \mu(M)\overline{\alpha_2(M)}\alpha_1(M), \tag{3.30}$$

where the overline denotes complex conjugation and the positive weight function μ is to be specified. For any real-valued function f, the corresponding operator $\hat{f}$ is then essentially self-adjoint. In particular, $\widehat{\mathcal{M}}$ is essentially self-adjoint and has spectrum $\mathbb{R}_+$. From this and the commutator (3.29a) it follows that $\hat{P}$ does not have self-adjoint extensions for any μ [37,38]. However, the affine commutation relation (3.29b) shows that $\widehat{\mathcal{M}P}$ can be made self-adjoint. Requiring $\widehat{\mathcal{M}P}$ to be symmetric, $(\alpha_2, \widehat{\mathcal{M}P}\alpha_1) = (\widehat{\mathcal{M}P}\alpha_2, \alpha_1)$, gives for μ a differential equation whose solution is $\mu(M) = c/M$, where the constant c can be set to 1 without loss of generality. Completion of $C_0^\infty(\mathbb{R}_+)$ in this inner product yields the Hilbert space $L_2(\mathbb{R}_+, dM/M)$, on which $\widehat{\mathcal{M}P}$ is essentially self-adjoint [31,37,38]. The mass eigenstates in the spectral decomposition (3.27) can be understood as non-normalizable states that satisfy

$$(\Phi_M, \Phi_{M'}) = M\delta(M, M'), \tag{3.31}$$

where δ is the Dirac delta-function.

The algebra $\mathcal{A}$ is by construction represented on the dense domain $C_0^\infty(\mathbb{R}_+) \subset L_2(\mathbb{R}_+, dM/M)$ and provides thus a large class of observables for the quantum theory.

IV. CROSSING THE QUANTUM HORIZON

The observables of the classical Hamiltonian theory contain information about the ADM mass of the spacetime and about the relative location of the asymptotic ends of the spatial surface, but no information about the spatial surface between its asymptotic ends. Similarly, operators in the quantum observable algebra $\mathcal{A}$ come with a geometric interpretation in terms of the ADM mass and the relative location of the asymptotic ends of the spatial surfaces, but not in terms of the local spacetime geometry. While this is to be expected, owing to the absence of local propagating degrees of freedom in the classical theory, we now show that the time-asymmetry built into the theory provides a way to introduce a quantum operator that is related to the surface gravity of the horizon.

Recall first that the spatial surfaces in the classical theory were chosen to extend from a singularity to an infinity, crossing the black hole horizon for $\epsilon = 1$ and the white hole horizon for $\epsilon = -1$. The classical momentum observables of the form $f(\mathcal{M})P$ depend explicitly on ϵ as seen from (2.41), and they have a geometric interpretation in terms of the ADM mass and the PG time difference between the left and right ends of the spatial surface.

Consider the classical theory with given ϵ, and denote P in this theory by P^ϵ to explicitly indicate its dependence on ϵ. Suppose that we attempt to introduce in this theory momentum observables of the form $fP^{-\epsilon}$. Proceeding for the moment formally, we obtain

$$f(\mathcal{M})P^{-\epsilon} = f(\mathcal{M})P^\epsilon + f(\mathcal{M}) \int dx \frac{2\epsilon\sqrt{2lG\mathcal{M}j}}{j - 2lG\mathcal{M}}. \tag{4.1}$$

The integral in (4.1) is clearly convergent at the lower end of x. The integral is convergent at $x \to \infty$ if $\int^\infty [j(\phi)]^{-3/2} d\phi$ is finite, which means geometrically that the surfaces of constant PG time asymptote to surfaces of constant Schwarzschild time. We assume this to be the case here and return to the question in Sec. V.

Let $\mathcal{M}$ take the spatially constant value M. The integral in (4.1) is then singular across $j = 2lGM$, the geometric reason being that the outgoing (respectively ingoing) PG time tends to ∞ $(-\infty)$ upon approaching the black (white) hole horizon from the exterior. However, the integral is well defined in the principal value sense [10], as well as in a contour integral sense [6] provided one specifies the half-plane in x to which the contour is deformed. If the contour circumvents the pole in the upper (lower) half of the complex x plane, and if f is real-valued, we thus obtain

$$\mathrm{Im}[fP^{-\epsilon}] = \mp \frac{\epsilon\pi f(M)}{\kappa(M)}, \tag{4.2}$$

where $\kappa(M)$ is the surface gravity of the horizon, given by $\kappa = (2l)^{-1}V(\phi_H)$, with ϕ_H denoting the value of ϕ at the horizon. In this sense, the inverse surface gravity of the horizon can be recovered from a controllable singularity in the observable $fP^{-\epsilon}$.

We note in passing that replacing P^ϵ in the fully reduced phase space action (2.48) by $P^{-\epsilon}$, defined by the contour integral, gives the action the imaginary contribution $\pm i\epsilon\pi \int \kappa^{-1}dM = \pm i\epsilon\pi \int d(\phi_H/G) = \pm i(\epsilon\hbar/2) \times \int dS_{\mathrm{BH}}$, where we have used the identity $dM = \kappa d(\phi_H/G)$ and the consequence that the Bekenstein-Hawking entropy is given by $S_{\mathrm{BH}} = 2\pi\phi_H/(\hbar G)$. This calculation has some similarity to the tunneling analyses that have led to the Bekenstein-Hawking entropy and to corrections thereof in the contexts of [6,39–45], including the numerical factor $\frac{1}{2}$, which leads to the expected exponential probability factor $\exp(\mp\epsilon \int dS_{\mathrm{BH}})$. For our system, this probability factor however equals unity when evaluated on a classical solution, since M and S_{BH} then do not evolve in time. We are therefore not aware of ways to develop this observation further in the present context.

Consider then the quantum theory of Sec. III with given ϵ. The operator counterpart of $fP^{-\epsilon}$ satisfies

$$\widehat{fP^{-\epsilon}}\Phi_M^\epsilon = \widehat{fP^\epsilon}\Phi_M^\epsilon + 2\epsilon f(M)\Phi_M^\epsilon \times \int dx \frac{\sqrt{2lGMj} + (\sqrt{j^2 - \hbar^2G^2/4} - j)}{j - 2lGM}, \tag{4.3}$$

where we have explicitly included the relevant superscripts $\pm\epsilon$ on the states and the operators. Compared with the classical relation (4.1), $\widehat{fP^{-\epsilon}}$ thus contains an additional term, which can be interpreted as a quantum correction. Taking the integral in (4.3) to be defined as a contour integral and assuming f to be real-valued, we find

$$[\mathrm{Im}(\widehat{fP^{-\epsilon}})]\Phi_M^\epsilon = \mp\epsilon\pi f(M)\widehat{\kappa^{-1}}\Phi_M^\epsilon, \tag{4.4}$$

where the operator $\widehat{\kappa^{-1}}$ is defined by

$$\widehat{\kappa^{-1}}\Phi_M^\epsilon := \frac{1}{\kappa(M)}\Theta\left(1 - \frac{\hbar^2}{16l^2M^2}\right)\sqrt{1 - \frac{\hbar^2}{16l^2M^2}}\Phi_M^\epsilon, \tag{4.5}$$

Θ being the Heaviside function. Comparison of (4.2) and (4.4) shows that we may regard $\widehat{\kappa^{-1}}$ as the inverse surface gravity operator in the quantum theory.

That $\widehat{\kappa^{-1}}$ differs from multiplication by the classical inverse surface gravity is a consequence of the fluctuations off the classical constraint surface that are present in our Dirac quantization of the Hamiltonian constraint. $\widehat{\kappa^{-1}}$ is close to the classical inverse surface gravity for $M \gg \hbar/(4l)$, but the difference becomes significant at the Planck scale, and $\widehat{\kappa^{-1}}$ vanishes on all states whose support is at $M \leq \hbar/(4l)$.

V. CONCLUSIONS

In this paper we have presented a Dirac quantization of generic single-horizon black holes in two-dimensional dilaton gravity, working under boundary conditions that allow the spatial surfaces to extend from a singularity to an infinity and eliminating the spatial reparametrization freedom by a spatial gauge choice at the classical level. The Hamiltonian constraint that remains was quantized in a metric representation. After finding a vector space of ADM mass eigenstate solutions to the quantum constraint, we transformed to a representation that allowed the mass spectrum to become continuous, and we chose the inner product by requiring self-adjointness of a time operator that is affinely conjugate to the ADM mass.

As the classical theory does not have local propagating degrees of freedom, one might not expect the quantum theory to have observables that correspond to localized geometric quantities in the spacetime. However, both the classical theory and the quantum theory were constructed under boundary conditions that distinguish future and past horizons, and we used this distinction to identify in the quantum theory an operator that corresponds to the inverse surface gravity of the horizon. The difference from the classical surface gravity is small for large ADM masses but becomes significant when the ADM mass approaches the Planck mass, and below (a numerical multiple of) the Planck mass the inverse surface gravity operator is identically vanishing.

For technical concreteness, we focused on boundary conditions under which the spatial surfaces asymptote to the PG foliation both at the singularity and at the infinity. While the technicalities of the spatial falloff depend on this choice, both the classical and the quantum analysis has a conceptually straightforward generalization to any asymptotics that retains the notion of freely specifiable asymptotic Killing time evolution. The only significant change in the classical observables is that $\Pi_{\mathcal{M}}$ (2.41) contains an additional term, which accounts for the transition from the PG time coordinate in (2.27) to the time coordinate that determines the new asymptotics. This term depends on j and any functions of ϕ that are introduced to specify the new foliation, but it depends on ρ and Π_ρ only through the combination $\mathcal{M}$. Assuming that we work with smooth foliations, the new term is also smooth. The operator $\widehat{\Pi_{\mathcal{M}}}$ (3.17) contains then the same additional term, but since this term is smooth, there is no change in the factor ordering parameter (3.19), and consequently there is no change in the singular part in (4.3). Hence the inverse surface gravity operator (4.5) is unchanged. Note that we can, in particular, choose the foliation near infinity to be asymptotic to the surfaces of constant Schwarzschild time, in which case the concerns of Sec. IV about the convergence of the integrals at $x \to \infty$ do not arise.

Similarly for technical concreteness, we focused on the spatial gauge choice (2.28) when eliminating the spatial reparametrization freedom in the classical theory. There is a straightforward generalization to gauge conditions of the form

$$l\phi' - g(\phi) = 0, \tag{5.1}$$

provided the positive gauge-fixing function g allows the spatial hypersurfaces to extend from a singularity to an infinity and suitable falloff conditions to be imposed. Assuming this is the case, the significant changes are that in the classical theory (2.41) is replaced by

$$\Pi_{\mathcal{M}} := \frac{lG\Pi_\rho - \epsilon g\sqrt{2lG\mathcal{M}/j}}{j - 2lG\mathcal{M}}, \tag{5.2}$$

and in the quantum theory (3.15) is replaced by

$$\nu^2 = \frac{1}{4} - \frac{g^2}{\hbar^2G^2}. \tag{5.3}$$

The inverse surface gravity operator then reads

$$\widehat{\kappa^{-1}} = \frac{1}{\kappa(M)} \Theta\left(1 - \frac{\hbar^2 G^2}{4[g(\phi_M)]^2}\right)\sqrt{1 - \frac{\hbar^2 G^2}{4[g(\phi_M)]^2}}, \quad (5.4)$$

where ϕ_M is the solution to

$$j(\phi_M) = 2lGM. \quad (5.5)$$

The inverse surface gravity operator therefore depends on the choice of g. To discuss this dependence further, one would need to develop a more quantitative control of the class of gs that are compatible with the boundary conditions of the classical theory.

Three points should be emphasized. First, the difference between the inverse surface gravity operator (5.4) and the classical inverse surface gravity $\kappa^{-1}(M)$ arises because the Hamiltonian constraint was *not* eliminated at the classical level but instead quantized in the Dirac sense as an operator. The regularity of the quantum observables across the future and past horizons was formulated in a way that hinges on the fluctuations off the classical constraint surface, and it was the distinction between regularity across the future horizon and past horizon that led to the identification of the inverse surface gravity operator.

Second, we chose to quantize the partially reduced theory in a "metric" representation. We introduced on the classical phase space a chart in which the variables are closely related to the local spacetime geometry, and the geometry of this chart then inspired the technical input in our quantization, leading, in particular, to the notion of regularity of the quantum observables across the Killing horizon. In comparison, it is possible to introduce in the (fully) unreduced classical theory a phase space chart that separates the constrained and unconstrained degrees of freedom: the unconstrained coordinates can be chosen as the ADM mass and the Killing time difference between the asymptotic ends of the spatial surfaces, whereas all the remaining information about the embedding of the spatial surfaces in the spacetime becomes encoded in the pure gauge degrees of freedom [8–11]. Quantum theories whose technical input is inspired by such a chart have been given [8–11], and these quantum theories can be specialized to boundary conditions that place one end of the spatial surfaces at a Killing horizon [46–53]. However, the geometry of such a phase space chart does not appear to suggest a horizon-crossing regularity condition in the quantum theory, and introducing an operator related to surface gravity would require other input. While it is well known that inequivalent quantum theories can arise from quantizations that draw their input from different phase space charts, the specific issue here may be related to the observation that geometrically nontransparent quantum variables can produce large quantum fluctuations in the spacetime geometry [54–56].

Third, the inverse surface gravity operator $\widehat{\kappa^{-1}}$ (5.4) depends on the partial gauge-fixing condition (5.1) in a way that has a geometric meaning. By (5.5), ϕ_M is the value of ϕ on the horizon of the classical solution with mass M. $\widehat{\kappa^{-1}}$ hence knows how the gauge choice makes the spatial surfaces cross the horizon but does not know what the surfaces do elsewhere. On the one hand, this is pleasing: the formalism relates the quantum-corrected surface gravity to the embedding of the spatial surfaces precisely where the surfaces cross the horizon. On the other hand, what is unsatisfactory is that the gauge choice was made already at the classical level. One would like first to quantize the theory in a gauge-invariant way, and if operators that pertain to specific foliations are desired, to introduce such operators only in the already-quantized theory. Unfortunately, our quantization technique relied in an essential way on the decoupling of the spatial points in the mass operator (3.7), and this decoupling only arose because the spatial diffeomorphism constraint was eliminated classically. If one attempted to treat also the spatial diffeomorphism constraint as a quantum constraint, one new issue would be how to preserve the constraint algebra in the quantum theory [23,57,58].

Given a function on the phase space of the fully reduced classical theory, one can explore the options of quantizing this function in our quantum theory via some interpretation of the rule "$P \mapsto -i\hbar\partial_M$ modulo factor ordering". However, there is no guarantee that a reasonable interpretation can be found for all functions of geometric interest. As an example, fix ϵ and consider the function

$$\lambda(M, P) := \epsilon e^{-\epsilon\kappa(M)P}. \quad (5.6)$$

By solving the geodesic equation on the horizon, it can be verified that $\lambda(M, P)$ is an affine parameter for the null geodesic that straddles the horizon, in a foliation that coincides with the PG coordinates except near the singularity and is at the singularity asymptotic to a single surface of constant PG time. Note that this means $\sigma_0 = 0$ and $\sigma_\infty = 1$ in (2.37) and (2.38). The affine parameter increases to the future and has been normalized so that it vanishes at the bifurcation point and equals ϵ on the surface $P = 0$. Now, if one had a self-adjoint operator version of $\kappa(M)P$, the operator exponential in (5.6) could be defined by spectral analysis. Suppose for concreteness that $\kappa(M)$ is proportional to $M^{1+2\gamma}$ with $\gamma \in \mathbb{R}$, which covers, in particular, symmetry-reduced gravity. The substitution $M^{1+2\gamma}P \mapsto -i\hbar M^{1+\gamma}\partial_M M^\gamma$ yields a symmetric operator, but analysis of the deficiency indices [31] shows that this operator has no self-adjoint extensions except when $\gamma = 0$, and $\gamma = 0$ is not consistent with the assumed asymptotic structure of the spacetime at $\phi \to \infty$. We have therefore not found a reasonable quantization of the affine parameter of the horizon in the present formalism.

As the classical system has no local propagating degrees of freedom, it seems unlikely that our inverse surface gravity operator could be used to make predictions in terms of Hawking radiation or black hole entropy. We expect however a number of the features of our quantum theory to be generalizable upon inclusion of matter with local dynamics, in particular, the way how regularity of quantum operators across the horizon is defined in the presence of quantum fluctuations off the classical constraint surface. Given a suitable adaptation of our boundary conditions to accommodate local dynamics [4], it may thus be possible to generalize our techniques to study both Hawking radiation and singularity formation in the quantum theory.

ACKNOWLEDGMENTS

We thank Ramin Daghigh, Jack Gegenberg, Viqar Husain and Oliver Winkler for useful discussions and an anonymous referee for helpful comments. G. K. thanks the University of Nottingham and J. L. thanks the Perimeter Institute for Theoretical Physics for hospitality. G. K. was supported in part by the Natural Sciences and Engineering Research Council of Canada. J. L. was supported in part by PPARC grant No. PP/D507358/1.

[1] D. Grumiller, W. Kummer, and D. V. Vassilevich, Phys. Rep. **369**, 327 (2002).
[2] C. Rovelli, In *Proceedings of the GR-15 Conference, Pune, India, 1997*, edited by N. Dadhich, J. Narlikar (IUCAA, Ganeshkhind, Pune, 1998).
[3] J. Gegenberg, G. Kunstatter, and R. D. Small, Class. Quant. Grav. **23**, 6087 (2006).
[4] R. G. Daghigh, J. Gegenberg, and G. Kunstatter, gr-qc/0607122.
[5] D. Louis-Martinez, J. Gegenberg, and G. Kunstatter, Phys. Lett. B **321**, 193 (1994).
[6] J. Gegenberg, G. Kunstatter, and D. Louis-Martinez, Phys. Rev. D **51**, 1781 (1995).
[7] D. Louis-Martinez and G. Kunstatter, Phys. Rev. D **49**, 5227 (1994).
[8] T. Thiemann and H. A. Kastrup, Nucl. Phys. **B399**, 211 (1993).
[9] H. A. Kastrup and T. Thiemann, Nucl. Phys. **B425**, 665 (1994).
[10] K. V. Kuchař, Phys. Rev. D **50**, 3961 (1994).
[11] M. Varadarajan, Phys. Rev. D **52**, 7080 (1995).
[12] M. Varadarajan, Phys. Rev. D **63**, 084007 (2001).
[13] K. V. Kuchař, J. D. Romano, and M. Varadarajan, Phys. Rev. D **55**, 795 (1997).
[14] V. Husain and O. Winkler, Phys. Rev. D **69**, 084016 (2004); Class. Quant. Grav. **22**, L127 (2005).
[15] V. Husain and O. Winkler, Phys. Rev. D **71**, 104001 (2005); **73**, 124007 (2006).
[16] V. Husain and O. Winkler, Class. Quant. Grav. **22**, L135 (2005).
[17] S. W. Hawking and G. F. R. Ellis, *The Large Scale Structure of Space-Time* (Cambridge University Press, Cambridge, England, 1973).
[18] R. M. Wald, *Quantum Field Theory in Curved Spacetime and Black Hole Thermodynamics* (University of Chicago Press, Chicago, 1994).
[19] P. Painlevé, C. R. Acad. Sci. (Paris) **173**, 677 (1921).
[20] A. Gullstrand, Ark. Mat. Astron. Fys. **16**, 1 (1922).
[21] P. A. M. Dirac, *Lectures on Quantum Mechanics* (Belfer Graduate School of Science, New York, 1964).
[22] M. Henneaux and C. Teitelboim, *Quantization of Gauge Systems* (Princeton University Press, Princeton, 1992).
[23] K. V. Kuchař, in *Proceedings of the 4th Canadian Conference on General Relativity and Relativistic Astrophysics*, edited by G. Kunstatter, D. E. Vincent, and J. G. Williams (World Scientific, Singapore, 1992).
[24] A. Ashtekar, *Lectures on Non-Perturbative Canonical Gravity* (World Scientific, Singapore, 1991).
[25] A. Ashtekar and R. S. Tate, J. Math. Phys. (N.Y.) **35**, 6434 (1994).
[26] V. de Alfaro, S. Fubini, and G. Furlan, Nuovo Cimento Soc. Ital. Fis. A **34**, 569 (1976).
[27] S. Carlip, Phys. Rev. Lett. **82**, 2828 (1999); **88**, 241301 (2002).
[28] S. N. Solodukhin, Phys. Lett. B **454**, 213 (1999).
[29] R. M. Cavalcanti and C. A. A. de Carvalho, J. Phys. A **31**, 2391 (1998); **32**, 6119 (1999); H. E. Camblong and C. R. Ordonez, Phys. Rev. D **68**, 125013 (2003); H. E. Camblong, L. N. Epele, H. Fanchiotti, and C. A. Garcia Canal, Ann. Phys. (N.Y.) **287**, 14 (2001); Phys. Rev. Lett. **85**, 1590 (2000); S. Weinberg, Physica A (Amsterdam) **96**, 327 (1979); Nucl. Phys. **B363**, 3 (1991); C. Ordonez, L. Ray, and U. van Kolck, Phys. Rev. C **53**, 2086 (1996).
[30] *Handbook of Mathematical Functions*, edited by M. Abramowitz and I. A. Stegun (Dover, New York, 1965).
[31] W. Thirring, *A Course in Mathematical Physics, Vol. 3: Quantum Mechanics of Atoms and Molecules* (Springer, New York, 1981).
[32] J. D. Bekenstein, Lett. Nuovo Cimento **11**, 467 (1974).
[33] A. Strominger and C. Vafa, Phys. Lett. B **379**, 99 (1996).
[34] A. Ashtekar, J. Baez, A. Corichi, and K. Krasnov, Phys. Rev. Lett. **80**, 904 (1998).
[35] J. Louko and J. Mäkelä, Phys. Rev. D **54**, 4982 (1996).
[36] A. Barvinsky and G. Kunstatter, Phys. Lett. B **389**, 231 (1996).
[37] J. R. Klauder, in: *Relativity*, edited by M. Carmeli, S. I. Fickler, and L. W. Witten (Plenum, New York, 1970).
[38] J. R. Klauder and E. W. Aslaksen, Phys. Rev. D **2**, 272 (1970).
[39] J. Gegenberg and G. Kunstatter, Phys. Rev. D **47**, R4192 (1993).
[40] M. K. Parikh and F. Wilczek, Phys. Rev. Lett. **85**, 5042 (2000).
[41] K. Srinivasan and T. Padmanabhan, Phys. Rev. D **60**, 024007 (1999); S. Shankaranarayanan, T. Padmanabhan, and K. Srinivasan, Class. Quant. Grav. **19**, 2671 (2002).
[42] E. C. Vagenas, Nuovo Cimento Soc. Ital. Fis. B **117**, 899 (2002); A. J. M. Medved and E. C. Vagenas, Mod. Phys. Lett. A **20**, 2449 (2005); **20**, 1723 (2005); M. Arzano, A. J. M. Medved, and E. C. Vagenas, J. High Energy Phys.

09 (**2005**) 037.
[43] M. Angheben, M. Nadalini, L. Vanzo, and S. Zerbini, J. High Energy Phys. 05 (2005) 014; M. Nadalini, L. Vanzo, and S. Zerbini, J. Phys. A **39**, 6601 (2006).
[44] R. Kerner and R. B. Mann, Phys. Rev. D **73**, 104010 (2006).
[45] E. T. Akhmedov, V. Akhmedova, and D. Singleton, Phys. Lett. B **642**, 124 (2006).
[46] J. Louko and B. F. Whiting, Phys. Rev. D **51**, 5583 (1995).
[47] S. Bose, J. Louko, L. Parker, and Y. Peleg, Phys. Rev. D **53**, 5708 (1996).
[48] J. Louko and S. N. Winters-Hilt, Phys. Rev. D **54**, 2647 (1996).
[49] J. Louko, J. Z. Simon, and S. N. Winters-Hilt, Phys. Rev. D **55**, 3525 (1997).
[50] G. Kunstatter, R. Petryk, and S. Shelemy, Phys. Rev. D **57**, 3537 (1998).
[51] C. Kiefer and J. Louko, Ann. Phys. (Leipzig) **8**, 67 (1999).
[52] A. J. M. Medved and G. Kunstatter, Phys. Rev. D **59**, 104005 (1999).
[53] A. J. M. Medved, Class. Quant. Grav. **20**, 2147 (2003).
[54] A. Ashtekar, Phys. Rev. Lett. **77**, 4864 (1996).
[55] R. Gambini and J. Pullin, Mod. Phys. Lett. A **12**, 2407 (1997).
[56] M. Varadarajan, Phys. Rev. D **57**, 3463 (1998).
[57] J. Lewandowski and D. Marolf, Int. J. Mod. Phys. D **7**, 299 (1998).
[58] R. Gambini, J. Lewandowski, D. Marolf, and J. Pullin, Int. J. Mod. Phys. D **7**, 97 (1998).