Higher spin gravitational couplings: Ghosts in the Yang-Mills detour complex

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Gravitational interactions of higher spin fields are generically plagued by inconsistencies. There exists however, a simple framework that couples higher spins to a broad class of gravitational backgrounds (including Ricci flat and Einstein) consistently at the classical level. The model is the simplest example of a Yang-Mills detour complex and has broad mathematical applications, especially to conformal geometry. Even the simplest version of the theory, which couples gravitons, vectors and scalar fields in a flat background is rather rich, providing an explicit setting for detailed analysis of ghost excitations. Its asymptotic scattering states consist of a physical massless graviton, scalar, and massive vector along with a degenerate pair of zero norm photon excitations. Coherent states of the unstable sector do have positive norms, but their evolution is no longer unitary and amplitudes grow with time. The class of models proposed is extremely general and of considerable interest for ghost condensation and invariant theory.

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I. INTRODUCTION

Massless, massive, and partially massless free higher spin fields propagate consistently in maximally symmetric backgrounds (i.e., Minkowski, de Sitter and anti-de Sitter spaces) [1–3]. Allowing generic curved backgrounds introduces various inconsistencies. First, introducing general curvatures $R^{\#}_{\mu\nu} = [D_\mu, D_\nu]$ can destroy the gauge invariances or constraints which ensured the correct physical degree of freedom count in maximally symmetric backgrounds [4,5]. Second, even in benign backgrounds ensuring correct degrees of freedom, signals may propagate at superluminal speeds [6,7]. In this article we display a simple mechanism for maintaining the gauge invariances of higher spins in a broad class of gravitational backgrounds (we require only a harmonic curvature condition) and perform a detailed analysis of its simplest, spin 2, flat space incarnation.

The model is significant for various reasons. First is its simplicity, since our proposal centers on spin one fields coupled to a non-Abelian Yang-Mills background. The key point being that with appropriate choice of background Yang-Mills gauge group, this theory can actually describe higher spin excitations. Second, the model has deep mathematical ramifications, especially for conformal geometry and invariant theory [8]. Physically, there are two main directions that can be pursued. As we shall show in detail, the simplest formulation of the so-called "Yang-Mills detour complex" necessarily has ghost excitations. Therefore one can search for fundamental models by either studying subcomplexes (i.e., consistent truncations to physical degrees of freedom), or coupling infinite towers of physical fields transforming in unitary representations of the Yang-Mills gauge group. We plan to report on these searches in the future, but in this first article concentrate on the simplest version of the model which includes ghost excitations. Nonetheless, the model is sufficiently simple that we can characterize these in detail. Moreover, models of this type are interesting for studies of physical, yet unstable theories.

Much mathematical insight into the structure of manifolds has been gained by studying the equations of mathematical physics. Notable examples include the self-dual Yang-Mills equations and Donaldson's four manifold theory, and ensuing simplifications based on the monopole equations of its supersymmetrization [9]. In self-dual Yang-Mills theory an important *rôle* is played by a class of two operator complexes that are sometimes termed Yang-Mills complexes. In [8] it is observed that there is a closely related three operator complex for each full Yang-Mills connection. These are there termed Yang-Mills detour complexes since there are intimate links with conformal geometry and in dimension four the complexes fall into a class of complexes called conformal detour complexes [10]. The Yang-Mills detour complexes are related to an idea that has been extant in the Physics literature for some time. Namely, it is well known that massless vectors couple consistently to an on-shell Yang-Mills background if a nonminimal coupling is included [11]. Unwrapping this in mathematical terms yields a Yang-Mills detour complex. Here we study the Yang-Mills detour complex in one of its simplest possible settings and expose and explore its consistency at both the classical and quantum level. On a dimension 4 Lorentzian background we obtain a theory of higher spins by taking the Poincaré group as Yang-Mills gauge group and the vectors transforming in

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any finite dimensional representation. Moreover, the physical spectrum of the model equals the cohomology of the underlying complex.

The first objection, that this simple model mixes spacetime and internal symmetries, and so violates the Coleman-Mandula theorem [12], is actually evaded because we are only studying a theory of noninteracting free fields whereas the theorem pertains to triviality of an interacting S-matrix. The second complaint that finite dimensional representations of the Poincaré group, being noncompact, are nonunitary and therefore imply the likelihood of ghost states is, however, borne out.[1] (We note that an infinite dimensional unitary representation can yield an infinite tower of consistent higher spin interactions and comment further in the conclusions.) In the trivial field theory vacuum we indeed find a pair of degenerate, zero norm photons. Nonetheless, the model is of considerable interest because

(1) Ghost states can simply indicate instability of the trivial Lorentz invariant vacuum. The model is useful as both a laboratory to study these excitations plus there exists the possibility of finding a (possibly non-Lorentz invariant) stable vacuum (especially if interactions were included).
(2) The model can be used to study properties of the background manifold in which the higher spin fields propagate. Higher spin gauge invariances can provide new invariants of the background manifold [14]. Moreover, finding physical states amounts to computing the cohomology of the twisted Maxwell complex.
(3) Backgrounds other than the simplest Minkowski one may permit a physical scattering spectrum.

For the simplest nontrivial spin 2 example in a four dimensional Minkowski background we find the following spectrum[2]:

Spin	Mass	Norm
2	0	+ve
1	$\sqrt{2}m$	+ve
1	0	0
1	0	0
0	0	+ve

The Lorentz invariant Lagrangian for these excitations depends on (i) a 2-index symmetric tensor, (ii) a 2-form, and (iii) a vector field. However a detailed Hamiltonian helicity analysis is required to determine the graviton, massive vector, two photon, and massless scalar spectrum quoted above. Interestingly, the photon states correspond to generalized eigenvector solutions of the wave equations of motion. Physically this amounts to resonant states with amplitudes growing linearly in time. Moreover, in the unstable photon subspace of the Hilbert space, only zero norm states diagonalize the Hamiltonian. Coherent states of these photon excitations have norms which grow with time, in violation of unitarity, and signify the instability of the model.

This article is arranged as follows. In Sec. II we explain how to formulate higher spins as a complex and present the twisted Maxwell complex. This section presents a broad class of fairly simple physical models in a very general mathematical language which immediately implies that our model applies to arbitrarily high spins. In Sec. III we specialize the underlying vector matter fields to the fundamental representation of the Poincaré Yang-Mills gauge group and revert to a Minkowski background. At this point begins our study of the simplest, spin 2 version of the model. Its Hamiltonian analysis is given in Sec. IV while Sec. V concentrates on the dangerous (ghostlike) helicity one excitations. The quantization of the model is given in Sec. VI. In Sec. VII we compute coherent states and their evolution. Our conclusions and further speculations are given in Sec. VIII.

II. YANG-MILLS DETOUR COMPLEX

Non-Abelian Yang-Mills theory can be constructed iteratively by coupling vectors to vectors [11]. This implies that multiplets of Abelian vector fields V can be consistently (i.e., at the level of gauge invariance) coupled to background non-Abelian vector fields A so long as the field A is on shell. This information is summarized by the action (valid in any spacetime dimension and signature)

$$S = \frac{1}{2}\int_M V_\mu^T (g^{\mu\nu} D^\rho D_\rho - D^\nu D^\mu + F^{\mu\nu}) V_\nu, \qquad (1)$$

with gauge invariance $V_\mu \to V_\mu + D_\mu \alpha$ valid whenever $D^\mu F_{\mu\nu} = 0$ (suppressing indices corresponding to a representation R of the background Yang-Mills group carried by the dynamical Abelian vectors V). Here D_μ is the background Yang-Mills covariant derivative and $F_{\mu\nu}$ its curvature.

It is worthwhile reformulating this simple piece of physics in mathematical terms to exhibit its generality and to connect to that body of work (readers only interested in lower spin examples, for which current physical techniques rather than sophisticated mathematical machinery suffice, can safely skip the following two paragraphs): An obvious, yet powerful, observation is that in any dimension we can

[1]The failure of Yang-Mills theory to be unitary with the noncompact gauge group $Sl(2, \mathbb{C})$ has been studied in [13], which reinforces our finding.

[2]For flat backgrounds, the mass parameter m is freely tunable (save to vanishing values) and we will mostly set it to unity as it can be readily reinstated by a dimensional analysis. In general spaces it depends on the gravitational coupling. A parameter space study as in [7] is then required.

view a classically consistent higher spin gauge theory as a complex[3]

$$0 \rightarrow \left\{ \begin{array}{c} \text{Gauge} \\ \text{Parameters} \end{array} \right\} \xrightarrow{\mathcal{D}} \{\text{Fields}\} \xrightarrow{\mathcal{G}} \left\{ \begin{array}{c} \text{Field} \\ \text{Equations} \end{array} \right\} \xrightarrow{{}^*\mathcal{D}} \left\{ \begin{array}{c} \text{Bianchi} \\ \text{Identities} \end{array} \right\} \rightarrow 0. \quad (2)$$

The first cohomology of this complex equals the physical spectrum, a computation we will carry out later using physical methods. Here where we write "Field Equations" is really of course the vector bundle where these equations take values and a similar comment applies to the "Bianchi Identities" which give the integrability condition for the field equations. The simplest example is the Maxwell (detour) complex where the space of fields are 1-forms $V \in \Gamma(\Lambda^1 M)$, and $\mathcal{D} = d$ the Poincaré differential, its dual is ${}^*\mathcal{D} = \star d \star \equiv \delta$ ($\star$ denotes the Hodge dual) and Maxwell's equations are simply

$$\mathcal{G} V \equiv \delta d V = 0. \quad (3)$$

In this case the statement that (2) is a complex so that $\mathcal{G}\mathcal{D} = 0 = {}^*\mathcal{D}\mathcal{G}$, amounts to the gauge invariance $V \rightarrow V + d\alpha$ and the Bianchi identity $\delta \mathcal{G} V = 0$.

The Maxwell complex can be twisted by coupling to a vector bundle connection over the manifold M. In general then (2) fails to be a complex reflecting the usual problem of adding curvature to a flat theory. However, if the connection satisfies the Yang-Mills equations then remarkably it turns out that we still obtain a complex called a Yang-Mills detour complex [8]. Let us review, in our current notation and on a spacetime background, this simple construction. In this setting, the space of fields are one-forms taking values in a representation R of the Yang-Mills gauge group G. We work locally, so for the purposes of the calculations the manifold may be taken to be $\mathbb{R}^4$ and the bundle carrying the representation may be taken trivial (as a vector bundle). Let

$$D = d + A, \quad (4)$$

be the Yang-Mills connection (so the Yang-Mills potential A is a $\mathfrak{g}$-valued 1-form). Then we set

$$\mathcal{D} = D, \qquad {}^*\mathcal{D} = \star D \star, \qquad \mathcal{G} = \star D \star D - \star({}^*F), \quad (5)$$

where $F = D^2$ is the Yang-Mills curvature. Now we find that (2) is a complex so long as the Yang-Mills connection obeys the Yang-Mills equations

$$[D, {}^\star F] = 0. \quad (6)$$

This information is equivalently summarized by the action (1). The key point is the generality of this mechanism. The model (1) is a consistent one for a compact gauge group G and unitary representation R yet the sequence (2) is a complex for *any* gauge group and representation.

Our proposal is simply to relax compactness of G and set it to the spacetime Poincaré symmetry group. We begin our study with finite dimensional (and hence nonunitary) representations R. The ghost difficulties that the model faces are all hidden in the superscript "T" on the field V_μ in (1), indicating an inner product on vectors in the representation space R.

Nonetheless, the proposal is rather fruitful since taking the gauge group G to be the Poincaré one amounts to coupling the model to gravity. This idea is well known both in mathematics and physics (called the Cartan connection or Palatini formalism, respectively). Let us concentrate on four dimensions and adopt the 5×5 matrix representation of the Poincaré Lie algebra so that the background Yang-Mills potential reads

$$A = \begin{pmatrix} \omega^m{}_n & e^m \\ 0 & 0 \end{pmatrix}, \quad (7)$$

where indices $m, n, ..$ take values 0, 1, 2, 3 and are raised and lowered with the flat Minkowski metric $\eta_{mn} = \mathrm{diag}(-1, 1, 1, 1)_{mn}$. Here, we view e as the vierbein for the underlying spacetime and ω as the spin connection. The Yang-Mills curvature F then becomes

$$F = \begin{pmatrix} R^m{}_n & T^m \\ 0 & 0 \end{pmatrix}. \quad (8)$$

where $R = d\omega + \omega \wedge \omega$ is the Riemann curvature and $T = de + \omega \wedge e$ is the torsion of the connection. We may work either with torsion-free $T = 0$ spacetimes or include it according to the physics being probed. In the absence of torsion, the spin connection can be solved for as a function of the vierbein and the Yang-Mills equations become the equations of harmonic curvature

$$D^\mu R_{\mu\nu\rho\sigma} = 2D_{[\rho} R_{\sigma]\nu} = 0. \quad (9)$$

This requirement is weaker than Einstein's equations. Obvious solutions are Ricci flat, Einstein and self-dual backgrounds so the model clearly has a wide physical applicability.

Finally, now that the model couples to gravitational backgrounds, we obtain higher spin fields by taking the vector field V to be a tensor representation of the Poincaré group. These can be decomposed in terms of tensor representations of the Lorentz subgroup, so generically we find theories of higher spin fields $(f_\mu{}^{m_1 \ldots m_s}, v_\mu{}^{m_1 \ldots m_{s-1}}, \ldots, v_\mu)$.

[3]For the impatient physicist annoyed by this digression, note that the key point is that the image of each differential (the operators above the arrows) is contained in the kernel of its successor. This language is a powerful way to encode and study the information of gauge invariance most usually summarized by an action principle in Physics.

Even though our model can describe arbitrarily high spins and a broad class of backgrounds, in order to make a detailed and explicit opening analysis, we retreat to a flat and low spin $s \leq 2$ example. However, the main features we find there will of course apply to all of the above models.

III. MINKOWSKI TWISTED MAXWELL COMPLEX

We make two simplifications. The background space is Minkowski $\mathbb{R}^{3,1}$ and the representation R is the fundamental of the Yang-Mills Poincaré gauge group $G = SO(3,1) \ltimes \mathbb{R}^4$. In this case the Yang-Mills curvature vanishes so there is no nonminimal coupling in the detour operator $\mathcal{G}$. So we simply have what is known as a twisted Maxwell complex. The fundamental representation acts naturally on a 5-vector of 1-forms,

$$V = \begin{pmatrix} f^\rho \\ v \end{pmatrix} = \begin{pmatrix} f_\mu{}^\rho \\ v_\mu \end{pmatrix} dx^\mu, \tag{10}$$

and we no longer distinguish between flat (Lie algebra) and curved (spacetime) indices using the latter in both cases. Moreover, for a flat background the Riemann curvature, torsion, and spin connection all vanish and the Yang-Mills potential is simply

$$A = \begin{pmatrix} 0 & m\delta_\mu{}^\rho \\ 0 & 0 \end{pmatrix} dx^\mu. \tag{11}$$

A simple computation yields the Lagrangian

$$L = -\tfrac{1}{4}(F_{\mu\nu})^2 - \tfrac{1}{4}(G_{\mu\nu}{}^\rho)^2, \tag{12}$$

where the "Maxwell" curvatures (not to be confused with their background Yang-Mills counterpart in the previous section) are

$$\begin{aligned} F_{\mu\nu} &\equiv \partial_\mu v_\nu - \partial_\nu v_\mu, \\ G_{\mu\nu}{}^\rho &\equiv \partial_\mu f_\nu{}^\rho - \partial_\nu f_\mu{}^\rho + m(\delta^\rho{}_\mu v_\nu - \delta^\rho{}_\nu v_\mu). \end{aligned} \tag{13}$$

The gauge invariance $V \to V + D\alpha$ becomes[4]

$$f_\mu{}^\rho \to f_\mu{}^\rho + \partial_\mu \alpha^\rho + m\delta^\rho_\mu \beta, \qquad v_\mu \to v_\mu + \partial_\mu \beta. \tag{14}$$

In 4 dimensions there are 20 fields $(f_\mu{}^\rho, v_\mu)$ and five gauge invariances with parameters (α^ρ, β) so the model certainly describes a total of $20 - 2 \times 5 = 10$ physical degrees of freedom. (This is obvious from the standpoint of five massless Yang-Mills vector matter fields.) However, the partition of these modes into the irreducible Poincaré representations of Wigner [15] is hardly clear from the Lagrangian (12). To emphasize this point we expand this equation out as

$$\begin{aligned} L = &-\tfrac{1}{2}(\partial_\mu f_\nu{}^\rho)^2 + \tfrac{1}{2}(\partial.f^\rho)^2 - \tfrac{1}{2}(\partial_\mu v_\nu)^2 + \tfrac{1}{2}(\partial.v)^2 \\ &- m v^\nu(\partial^\rho f_{\nu\rho} - \partial_\nu f^\rho{}_\rho) + (d-1)m^2 v.v. \end{aligned} \tag{15}$$

The top line is a sum of Maxwell actions but the second line includes cross terms and an apparent mass term (here we have given the general result valid in d-dimensions). We have included a mass parameter m by naïve dimensional analysis. It can clearly take any value we so choose and we will work in units $m = 1$ for the remainder of the Article. It is important to note that this is a freedom peculiar to flat space. Upon considering more general curved backgrounds, the parameter m must be tuned to the gravitational coupling.[5]

For spins greater or equal to three, counting the number of degrees of freedom is again straightforward. For example, a spin 3 theory can be obtained by taking the representation R to be symmetric trace-free tensor which yields 56 fields $V_\mu{}^{MN}$ and 14 gauge invariances for a total of 28 physical degrees of freedom. Again, the Lagrangian is most compactly expressed in terms of curvatures

$$L = -\tfrac{1}{4}(F_{\mu\nu}{}^\rho)^2 - \tfrac{1}{4}(G_{\mu\nu}{}^{\rho\sigma})^2 - \tfrac{1}{4}(G_{\mu\nu}{}^\rho{}_\rho)^2, \tag{16}$$

where

$$\begin{aligned} G_{\mu\nu}{}^{\rho\sigma} &\equiv \partial_\mu f_\nu{}^{\rho\sigma} - \partial_\nu f_\mu{}^{\rho\sigma} - 2m(\delta^{(\rho}{}_\mu v_\nu{}^{\sigma)} \\ &\quad - \delta^{(\rho}{}_\nu v_\mu{}^{\sigma)}), \\ F_{\mu\nu}{}^\rho &\equiv \partial_\mu v_\nu{}^\rho - \partial_\nu v_\mu{}^\rho - m(\delta^\rho{}_\mu v_\nu - \delta^\rho{}_\nu v_\mu), \end{aligned} \tag{17}$$

although this sheds little light on the spectrum. Its leading form is of Maxwell type $L = -\frac{1}{2}(\partial_\mu f_\nu{}^{(\rho\sigma)})^2 + \frac{1}{2} \times (\partial.f^{(\rho\sigma)})^2 + \cdots$, and the leading excitation is a massless spin three field.[6] To determine the distribution of the remaining 26 physical degrees of freedom among massive and massless spins 0, 1 and 2 is a complicated task. However, since this amounts to computing the cohomology of the detour complex, it is a well-defined mathematical problem, which explains the effort we made in Section II to connect our work to this body of mathematics [8].

[4]To avoid confusion, note that the parameter α^ρ of the spin 2 gauge invariance has nothing to do with Poincaré translations which act in exactly the usual manner. It is amusing to note, however, that the appearance of β in the $f_\mu{}^\rho$ transformation is through the translation generators of the Poincaré Yang-Mills gauge group. The beauty of the detour complex, is precisely its prediction of these terms required for the higher spin gauge invariance.

[5]This could be either a curse or blessing, see [7] for a detailed analysis of this issue.

[6]One might wonder how the usual double-trace free constraint for massless fields arises. This is simply a matter of splitting the fields of the theory appropriately into their irreducible pieces and then determining which pieces form an helicity multiplet. It is not necessary to impose it by hand. Of course one might try to express the action only in terms of doubly traceless fields, but this would require first determining the spectrum, because many of the lower spin fields are of Stückelberg type. I.e., their *rôle* is auxiliary and they can be algebraically gauged away.

IV. HAMILTONIAN HELICITY ANALYSIS

To determine the spectrum of the model we make a Hamiltonian analysis and helicity decomposition. In mathematical terms, we are computing the cohomology of the complex (2) in the slot labeled "fields." We treat the time coordinate on a separate footing and denote spatial indices by $i, j, k, \ldots = 1, 2, 3$. The following computation is completely standard (excellent references are [16]), but we sketch some details for completeness.

Firstly, introduce canonical momenta P^j and $\pi^j{}_\rho$ by

$$P^j = \frac{\partial L}{\partial \dot{v}_j} = F_0{}^j = \dot{v}^j - \partial^j v_0,$$
$$\pi^j{}_\rho = \frac{\partial L}{\partial \dot{f}_j{}^\rho} = G_0{}^j{}_\rho = \dot{f}^j{}_\rho - \partial^j f_{0\rho} - 2\delta_\rho{}^{[0} v^{j]}. \tag{18}$$

Noting that the first order Lagrangian obtained by Legendre transformation must take the form $L^{(1)} = P^j F_{0j} + \pi^j{}_\rho F_{0j}{}^\rho - \hat{H}$ we rapidly find (suppressing spatial integrations $\int d^3x$)

$$L^{(1)} = P^j \dot{v}_j + \pi^j{}_\rho \dot{f}_j{}^\rho - H,$$
$$H = \tfrac{1}{2}[(P_j)^2 + (\pi_j{}^\rho)^2] + \tfrac{1}{4}[(F_{ij})^2 + (G_{ij}{}^\rho)^2] - \pi^j{}_0 v_j + v_0[\pi^j{}_j - \partial_j P^j] - f_0{}^\rho \partial_j \pi^j{}_\rho. \tag{19}$$

Clearly, v_0 and $f_0{}^\rho$ are Lagrange multipliers imposing primary constraints

$$\pi^j{}_j - \partial_j P^j = 0, \qquad \partial_j \pi^j{}_\rho = 0. \tag{20}$$

We now proceed by making a helicity decomposition, solving the constraints and computing an action principle for physical degrees of freedom only. Our helicity decomposition for general 1- and 2-index tensors is

$$Y_i = Y_i^T + \partial_i Y^L,$$
$$X_{ij} = X_{ij}^{Tt} + 2\partial_{(i} X_{j)}^T + \frac{1}{2}\Big(\delta_{ij} - \frac{\partial_i \partial_j}{\Delta}\Big) X^S + \frac{\partial_i \partial_j}{\Delta} X^L + \epsilon_{ijk}\partial^k X^A + 2\partial_{[i} X_{j]}^{AT}. \tag{21}$$

(where, for example, transverse objects are divergence free, so $\partial^i Y_i^T = 0$). We also heavily employ their inner products under a $\int d^3x$ integration

$$Y'_i Y^i = Y'^T{}_i Y^{Ti} - Y'^L \Delta Y^L,$$
$$X'_{ij} X^{ij} = X'^{Tt}{}_{ij} X^{Ttij} - 2X'^T{}_j \Delta X^{Tj} + \tfrac{1}{2} X'^S X^S + X'^L X^L - 2X'^A \Delta X^A - 2X'^{AT}_j \Delta X^{ATj}. \tag{22}$$

Here the negative definite operator $\Delta = \partial_i \partial^i$ denotes the spatial Laplacian which we take invertible. A useful mnemonic is that the number of indices on fields now labels their helicity. Written out helicity by helicity the primary constraints (20) are solved via

Helicity	Constraints
± 1	$\pi_k^{AT} = -\pi_k^T$
0	$\pi^L = 0$ $\pi_0^L = 0$ $P^L = \frac{1}{\Delta}\pi^S$

(23)

There are, of course, no constraints on the leading helicity ± 2 sector whose action reads

$$L^{(1)}_{\pm 2} = \pi^{Ttij} \dot{f}^{Tt}_{ij} - [\tfrac{1}{2}(\pi^{Ttij})^2 + \tfrac{1}{2} f^{Tt}_{ij}(-\Delta) f^{Ttij}]. \tag{24}$$

This consistently describes a physical massless spin two graviton. The helicity zero sector is not much more difficult. Upon substituting the constraints, f_0^L decouples and making field redefinitions

$$q_0 = \sqrt{\frac{-2}{\Delta}}\pi^S, \qquad p_0 = \sqrt{\frac{-\Delta}{2}}\Big(v^L - \frac{1}{2} f^S\Big),$$
$$\pi = \sqrt{\frac{-\Delta}{2}}\pi^A, \qquad \varphi = \sqrt{\frac{-\Delta}{2}} f^A, \tag{25}$$

we find

$$L^{(1)}_0 = \pi\dot{\varphi} - \tfrac{1}{2}[\pi^2 + \varphi(-\Delta)\varphi] + p_0 \dot{q}_0 - \tfrac{1}{2}[p_0^2 + q_0(-\Delta + 2) q_0]. \tag{26}$$

This describes a pair of physically consistent scalar fields, one massless and one with mass $\sqrt{2}$. As we shall see in the following section, the latter forms the zero helicity component of a physical massive vector field.

V. HELICITY 1 HAMILTONIAN ANALYSIS

The helicity 1 sector is more subtle. Although classically consistent, the model displays negative norm states when expanded about the trivial Lorentz invariant field theoretic background. Firstly we perform the classical constraint analysis.

Imposing the helicity ± 1 constraint as in (23), we find that the combination $f_j^T + f_j^{AT}$ decouples and

$$L^{(1)}_{\pm 1} = \Pi^t \dot{\Phi} - H^{(1)}_{\pm 1},$$
$$H^{(1)}_{\pm 1} = \tfrac{1}{2}(\Pi^t \tilde{M} \Pi + \Pi^t \tilde{N} \Phi + \Phi \tilde{P} \Phi), \tag{27}$$

where we have made field redefinitions packaged as a vector of $SO(2, 1)$

$$\Phi_j^T = \begin{pmatrix} v_j^T \\ f_{0j}^T \\ \sqrt{-\Delta}(f_j^T - f_j^{AT}) \end{pmatrix}, \qquad \Pi_j^T = \begin{pmatrix} P_j^T \\ \pi_{0j}^T \\ 2\sqrt{-\Delta}\pi_j^T \end{pmatrix}, \tag{28}$$

and

$$\tilde{M} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \tilde{N} = \begin{pmatrix} 0 & 0 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \tilde{P} = \begin{pmatrix} -\Delta + 2 & 0 & -\sqrt{-\Delta} \\ 0 & \Delta & 0 \\ -\sqrt{-\Delta} & 0 & -\Delta \end{pmatrix}. \tag{29}$$

Throughout this and the following sections we suppress the helicity ± 1 labels "${}^{T}_{j}$". The dynamics are most easily analyzed via the second order form[7] of the action (27)

$$L^{(2)}_{\pm 1} = \tfrac{1}{2}\dot{\Phi}^t M \dot{\Phi} + \dot{\Phi}^t N \Phi + \tfrac{1}{2}\Phi P \Phi, \tag{30}$$

where now

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad N = \begin{pmatrix} 0 & -\frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad P = \begin{pmatrix} \Delta - 3 & 0 & \sqrt{-\Delta} \\ 0 & -\Delta & 0 \\ \sqrt{-\Delta} & 0 & \Delta \end{pmatrix}. \tag{31}$$

The equations of motion

$$- M\ddot{\Phi} - 2N\dot{\Phi} + P\Phi = 0, \tag{32}$$

are a second order matrix ODE. Working in the eigenspace $\Delta = -k^2$ and considering wave solutions $\Phi = \lambda e^{i\omega t}$, then (32) becomes

$$(M\omega^2 - 2iN\omega + P)\lambda = 0. \tag{33}$$

The determinant of this matrix must vanish which yields

$$(k^2 - \omega^2 + 2)(k^2 - \omega^2)^2 = 0. \tag{34}$$

The zeros are precisely the relativistic dispersion relations of a single mass $\sqrt{2}$ and two massless vector fields. Observe that this mass eigenvalue agrees with that found in the zero helicity sector so we obtain a pair of photons and a massive vector. This is the spectrum quoted in the Introduction, we now analyze its quantization and stability.

VI. QUANTIZATION AND STABILITY

To quantize the model we expand the on-shell fields on plane wave solutions

$$\Phi = \sum_{i=1}^{3} (f_i \alpha_i^\dagger + \bar{f}_i \alpha_i), \tag{35}$$

where

$$f_1 = \frac{5}{4}\begin{pmatrix} 0 \\ 1 \\ i \end{pmatrix} e^{ikt}, \qquad f_2 = f_1 + ik\begin{pmatrix} 1 \\ \frac{1}{2}t \\ \frac{i}{2}t \end{pmatrix} e^{ikt}, \tag{36}$$

are photon solutions and the massive vector solution is

$$f_3 = \begin{pmatrix} 1 \\ -\frac{1}{2}i\sqrt{k^2+2} \\ -\frac{1}{2}k \end{pmatrix} e^{i\sqrt{k^2+2}\,t}. \tag{37}$$

As we shall see, the massive vector subspace of the Hilbert space is perfectly physical while the photon subspace is pathological. Already we see that the solution f_2 has amplitude growing linearly in time. Mathematically this is a generalized eigenvector solution to our system of PDEs. Physically it can be interpreted in terms of a resonance between highly tuned wave solutions and indicates an instability. Similar behavior has already been observed in the ghost condensation mechanism of [17] employed to obtain infrared modifications of Einstein gravity.

We now promote the Fourier coefficients $(\alpha_i, \alpha_i^\dagger)$ to operators in a Fock space. Positivity of the classical energy and in turn stability can be studied through the energy eigenvalues of single particle states. We will also analyze unitarity of the model by computing norms of quantum states.

Imposing canonical equal time commutation relations of the fields and their momenta

$$[\Pi, \Phi^t] = -i\mathbf{1}, \tag{38}$$

fixes the commutation relations of the creation and annihilation operators to

$$\Omega \equiv [\alpha, \alpha^\dagger] = \begin{pmatrix} -\frac{2k}{25} & -\frac{1}{5k} & 0 \\ -\frac{1}{5k} & 0 & 0 \\ 0 & 0 & \frac{1}{2\sqrt{k^2+2}} \end{pmatrix}, \tag{39}$$

(the right-hand side of this equation is the Wronskian of the solutions above). As promised this is block diagonal and positive definite in the massive vector block. The zero on the diagonal already signals the presence of zero norm states in the photonic Fock space.

The Hamiltonian may be expressed also in terms of Fock operators as

$$H = \alpha^\dagger \mathcal{M} \alpha, \tag{40}$$

with matrix

$$\mathcal{M} = \begin{pmatrix} 0 & -5k^2 & 0 \\ -5k^2 & 2k^2(k^2-1) & 0 \\ 0 & 0 & 2(k^2+2) \end{pmatrix}. \tag{41}$$

Taking into account the normalization of the symplectic form Ω we see that massive vectors states have both positive norms and energies with single particle, relativistic dispersion relation

[7]An interesting rewriting of this action is in terms of an $SO(2, 1)$ covariant derivative $D = d + MN$, so that

$$S^{(2)}_{\pm 1} = \frac{1}{2}\frac{D\Phi^t}{dt} M \frac{D\Phi}{dt} + \frac{1}{2}\Phi^t(P + NMN)\Phi.$$

The second term does, however, break the $SO(2, 1)$ invariance.

$$E = \sqrt{k^2 + 2}. \tag{42}$$

The photonic Fock space is much more subtle. Interestingly enough the eigenvalues of the matrix $\mathcal{M}$ can become negative but are actually bounded below. However, consider a single particle state

$$|1\rangle = \alpha^\dagger \lambda |0\rangle, \tag{43}$$

where $|0\rangle$ is the Fock vacuum and λ is some constant, complex 3-vector of coefficients. Requiring $|1\rangle$ to be an energy eigenstate implies that

$$H|1\rangle = \alpha^\dagger \mathcal{M} \alpha \alpha^\dagger \lambda |0\rangle = \alpha^\dagger \mathcal{M} \Omega \lambda |0\rangle = E|1\rangle \tag{44}$$

and in turn the equality

$$\mathcal{M} \Omega \lambda = E\lambda. \tag{45}$$

I.e., we must diagonalize the effective Hamiltonian matrix $\mathcal{H} \equiv \mathcal{M}\Omega$ rather than simply $\mathcal{M}$. Explicitly

$$\mathcal{H} = \begin{pmatrix} k & 0 & 0 \\ \frac{2k}{5} & k & 0 \\ 0 & 0 & \sqrt{k^2+2} \end{pmatrix}. \tag{46}$$

Again we see that the massive vector decouples with dispersion relation (42). While the only photon single particle energy eigenstate is

$$|\gamma\rangle \equiv a_2^\dagger |0\rangle, \tag{47}$$

with energy $E = k$ which is the correct Lorentz invariant dispersion relation for massless excitations. The norm of this state $\langle\gamma|\gamma\rangle = 0$, vanishes however.

We can also consider a general photonic single particle state $(\nu a_1^\dagger + \mu a_2^\dagger)|0\rangle$. Then denoting $\rho = \nu/\mu$ we find that states with ρ inside the disc

$$\left| \rho + \frac{5}{2k^2} \right| < \frac{5}{2k^2}, \tag{48}$$

have positive norm, those on the boundary zero norm and those exterior to the disc negative norm (the state $a_1^\dagger|0\rangle$ with $\rho = \infty$ also has negative norm). The only single particle state diagonalizing the Hamiltonian is the zero norm state $|\gamma\rangle$ corresponding to $\rho = 0$.

Observe that positivity properties of norms are improved in the nonrelativistic limit $k \to 0$, for which any ρ in the upper half plane solves (48). Nonetheless even in this limit the nonunitarity difficulty persists. Another mechanism available to cure the instability is to truncate the model by restricting physical states further to the cohomology of an appropriate nilpotent operator. Explicitly, call the top 2×2 block of the effective Hamiltonian in (46) $\hat{\mathcal{H}}$. Then since any matrix obeys its own characteristic polynomial, the matrix $\mathcal{N} \equiv \hat{\mathcal{H}} - k$ is nilpotent

$$\mathcal{N}^2 = 0, \tag{49}$$

and commutes with $\hat{\mathcal{H}}$. The cohomology of $\mathcal{N}$ in the malevolent photonic single particle Fock space is trivial, which is promising. We have not computed its cohomology for multiparticle states, but instead remark that this mechanism is unlikely to respect Lorentz invariance.

The presence of zero and negative norm states signals the breakdown of unitary evolution, as evidenced by the non-Hermitean effective Hamiltonian matrix $\mathcal{H}$, commensurate with resonant classical single particle wavefunctions growing linearly in time. Whether this instability indicates the existence of other stable but possibly non-Lorentz invariant vacua, or is a runaway instability is an open problem deserving further study. It seems likely that the addition of interparticle interactions is necessary to support a stable vacuum.

VII. COHERENT STATE EVOLUTION.

Let us consider coherent states in the photonic Fock space[8] Denoting $\hat{\alpha} = (\alpha_1, \alpha_2)$ and similarly employing hats to denote the top 2×2 photonic block for matrices, coherent states diagonalizing the annihilation operators

$$\hat{\alpha}|z\rangle = z|z\rangle, \tag{50}$$

are simply

$$|z\rangle = \exp(\hat{\alpha}^\dagger \hat{\Omega}^{-1} z)|0\rangle. \tag{51}$$

Here z is a complex 2-vector and the coherent state associated with the photon single particle state $|\gamma\rangle$ corresponds to

$$z = z_\gamma = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Its time evolution, given by[9]

$$|z(t)\rangle = e^{iHt}|z\rangle, \tag{52}$$

is easily computed to be

$$z(t) = \begin{pmatrix} z_1 \\ z_2 + \frac{2ikt}{5} z_1 \end{pmatrix} e^{ikt}, \tag{53}$$

which is the classical solution found above. Therefore, as usual, coherent states are maximally classical. The inner product for these states is

$$\langle w|z\rangle = \exp(w^\dagger \hat{\Omega}^{-1} z). \tag{54}$$

Since $\hat{\Omega}$ is a real symmetric matrix, norms of photonic coherent states

[8]This analysis is similar in spirit to [18], where models with wrong sign potentials and squeezed states are analyzed.

[9]In quantum mechanics coherent states evolve classically up to a phase corresponding to the zero point energy. As evidenced by (40), we have made the usual field theoretic normal ordering renormalization so this factor is absent.

$$\langle z|z\rangle = \exp(z^\dagger \hat{\Omega}^{-1} z), \tag{55}$$

are always positive. However, they are not conserved in time since evolution is no longer unitary (observe that the effective Hamiltonian $\mathcal{H}$ in (46) is not Hermitean). Instead we find that norms for the time evolved states $|z(t)\rangle$ obey

$$\langle z(t)|z(t)\rangle = \exp\left(z^\dagger \begin{pmatrix} \frac{8t^2k^5}{25} & -\frac{k(25+4ik^3t)}{5} \\ -\frac{k(25-4ik^3t)}{5} & 2k^3 \end{pmatrix} z\right). \tag{56}$$

Observe that the photon coherent state $|z_\gamma\rangle$ has a time independent norm $|||z_\gamma\rangle||^2 = \exp(2k^3)$. In general, however, unitary evolution is violated. In particular the state with

$$z = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

corresponding to $\rho = \infty$ in the notation of the previous section, has norm behaving as $\exp(4t^2k^5/25)$ for large times. This indicates that coherent combinations of the negative norm single particle states dominate the large time behavior of the model and are primarily responsible for its instability.

VIII. CONCLUSIONS

The Yang-Mills detour complex, obtained from an on-shell Poincaré Yang-Mills twist of the Maxwell complex along with a nonminimal coupling, yields a novel mechanism for coupling higher spins to gravitational backgrounds. Even the simplest, flat, fundamental representation version of the model, analyzed in depth here, has a rich spectrum though photon states have non-positive norms.

There are many open questions and directions the model can taken in. First, vacua other than the usual Lorentz invariant background, where all fields vanish, might be stable. Second, the Yang-Mills gauge group G can be enlarged. Obvious generalizations are to situations with conformal symmetry or supersymmetry where $\mathfrak{g}$ can be the conformal or super Poincaré algebra [8].

In general, given a complex, it often is possible to search for projections to a smaller one where the projections and differentials commute. (I.e., one forms a commutative diagram.) Hence, one can search for a smaller complex in which the zero norm and negative norm states are excised [8].

Another extremely interesting direction is to study models with infinite towers of fields by taking Maxwell fields labeled by infinite dimensional yet unitary representations of the Yang-Mills algebra $\mathfrak{g}$. These present the possibility of a fundamental theory with quantum consistency in the Lorentz invariant vacuum. Moreover, one might even hope that genuine interparticle interactions (rather than just ones to the background) would be possible with an infinite tower of fields threading a loophole in the Coleman-Mandula theorem.

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