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Initial data sets for the Schwarzschild spacetime

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A characterization of initial data sets for the Schwarzschild spacetime is provided. This characterization is obtained by performing a $3+1$ decomposition of a certain invariant characterization of the Schwarzschild spacetime given in terms of concomitants of the Weyl tensor. This procedure renders a set of necessary conditions—which can be written in terms of the electric and magnetic parts of the Weyl tensor and their concomitants—for an initial data set to be a Schwarzschild initial data set. Our approach also provides a formula for a static Killing initial data set candidate—a KID candidate. Sufficient conditions for an initial data set to be a Schwarzschild initial data set are obtained by supplementing the necessary conditions with the requirement that the initial data set possesses a stationary Killing initial data set of the form given by our KID candidate. Thus, we obtain an algorithmic procedure of checking whether a given initial data set is Schwarzschildean or not.

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I. INTRODUCTION

The Schwarzschild and Kerr spacetimes occupy a privileged role among the asymptotically flat solutions to the Einstein field equations. Although they are, respectively, static and stationary, they are models for dynamical black hole spacetimes. Moreover, they are believed to describe, in a certain sense, the asymptotic state of the dynamical black hole spacetimes.

It is generally agreed that in the next years, the main source of understanding of dynamical black hole spacetimes will come from numerical simulations of these solutions to the Einstein field equations. Normally, these numerical simulations make use of a $3+1$ splitting of the spacetime. Thus, it becomes of relevance acquiring an understanding of the $3+1$ features of the Schwarzschild and, eventually, the Kerr solutions. Some relevant research in this direction has been carried out in [1–9].

When performing a $3+1$ decomposition of the Schwarzschild spacetime—or essentially any other exact solution to the Einstein equations—the relative simplicity of the solution gets blurred by the choice of gauges which do not reflect the symmetries of the spacetime or its algebraic structure, as in the case of the Petrov type. In this context, invariant characterizations of the spacetime become very useful, in particular, those involving scalars.

Precisely in this set of mind, in Ref. [9], the following question was raised: when an initial data set for the Einstein vacuum field equations corresponds to a slice in the Schwarzschild spacetime? By *the Schwarzschild spacetime* it will be understood the Schwarzschild-Kruskal maximal extension, $(\mathcal{M}, g_{\mu\nu})$, of the Schwarzschild spacetime [10]. An initial data set for the Einstein vacuum field equations is a triplet $(\mathcal{S}, h_{ij}, K_{ij})$ consisting of a Riemannian manifold $(\mathcal{S}, h_{ij})$ and a symmetric tensor field K_{ij} in $\mathcal{S}$—the first and second fundamental form, or alternatively, the initial 3-metric and extrinsic curvature—satisfying the Einstein vacuum constraint equations. Under these circumstances the manifold $\mathcal{S}$ can be isometrically embedded in a four dimensional Lorentzian manifold $(\mathcal{M}, g_{\mu\nu})$, a spacetime, which is a solution of Einstein vacuum field equations. Such spacetime is called the *development* of the initial data set. It is well-known that a *maximal development* of an initial data set exists and is unique up to isometry [11]. In this paper we establish conditions under which the development of an initial data set is a subset of Schwarzschild spacetime. Data sets satisfying this property are called Schwarzschild initial data. We shall assume $\mathcal{S}$ to be either a Cauchy hypersurface or an hyperboloid—that is, a spacelike hypersurface which is asymptotically null. It is noted that in Ref. [12], a similar question, restricted to the case of spherically symmetric 3-geometries was raised.

A partial answer to the question of the characterization of Schwarzschild initial data sets was provided in [9]. It made use of the fact that for a vacuum spacetime, $(\mathcal{M}, g_{\mu\nu})$, the relation

$$\nabla^{\sigma}\nabla_{\sigma}C_{\mu\nu\lambda\rho} = fC_{\mu\nu\lambda\rho}, \tag{1}$$

which in the sequel we will call the *Zakharov property*, where $C_{\mu\nu\lambda\rho}$ denotes the Weyl tensor of $g_{\mu\nu}$ and $f \neq 0$ is a function, implies that the spacetime is of Petrov type D [13]. All the vacuum type D spacetimes are known, so a direct calculation reveals that the only Petrov type D spacetimes satisfying the Zakharov property are those which

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have an hypersurface orthogonal timelike Killing vector—thus, for example, the Kerr solution does not satisfy the Zakharov property, but the Schwarzschild spacetime does. The Zakharov property projects nicely under a 3 + 1 decomposition, making it amenable for a characterization of Schwarzschild initial data. The characterization given in [9] was completed by the addition of two extra ingredients: the asymptotic flatness—asymptotic Euclideanity—of the initial data and the nonvanishing of the ADM mass. An important point which was left open was the question of the propagation of the Zakharov condition: does the development of an initial data satisfying the Zakharov property satisfy the property at latter times?

In the present article we pursue an alternative characterization of Schwarzschild data. In particular, we want to avoid the use of the two global ingredients used in [9]: the asymptotic flatness and the nonvanishing of the ADM mass. Our starting point is a certain invariant characterization of the Schwarzschild spacetime in terms of concomitants of the Weyl tensor which was given in [14]. Necessary conditions for a pair (h_{ij}, K_{ij}) satisfying the vacuum constraint equations to be Schwarzschild data are obtained by making a 3 + 1 splitting of the characterization of Ref. [14]. Now, in order to answer the more subtle question whether the necessary conditions are also sufficient ones, one has to confront in one way or the other the issue of their *propagation*. In our context the propagation of the conditions resulting from the 3 + 1 splitting of the characterization of [14] would imply that the initial data under question are indeed Schwarzschildean. If the conditions do not propagate, then we may need to add extra conditions in order to construct Schwarzschild initial data. This issue is dealt with by observing that the invariant characterization of [14] also yields a formula for the static Killing vector of the Schwarzschild spacetime. This formula is again given in terms of concomitants of the Weyl tensor, and can be split in *lapse* and *shift* terms. These shift and lapse can be pull-backed to the manifold $\mathcal{S}$ and if they satisfy certain conditions which we shall call *the Killing initial data conditions*—the KID conditions—then the development of the initial data possesses a Killing vector. Now, if the restriction of the resulting Killing vector to the initial hypersurface $\mathcal{S}$ is timelike for an open subset $\mathcal{U} \subset \mathcal{S}$, then by continuity, there will be a timelike Killing vector in, at least, a small portion of the domain of dependence of $\mathcal{U}$, $\mathcal{D}(\mathcal{U})$.

We can take advantage of the existence of a static timelike Killing vector in $\mathcal{D}(\mathcal{U})$ to show that the conditions obtained from the 3 + 1 splitting of the invariant characterization mentioned above can be propagated, at least in the small portion of $\mathcal{D}(\mathcal{U})$, where the Killing vector is stationary, thereby obtaining a full set of conditions ensuring that this part of $\mathcal{D}(\mathcal{U})$ is a subset of Schwarzschild spacetime. This full set of conditions consists of the 3 + 1 splitting of the Schwarzschild invariant characterization and the KID conditions. A novelty in our approach lies in providing an explicit Ansatz—*a Killing candidate*—for the solution of these KID conditions, thus yielding an algorithmic procedure to verify the existence of a timelike vector in the development of the initial data.

The article is structured as follows: in Sec. II we detail the local invariant characterization of the Schwarzschild spacetime presented in [14]. This will be our starting point. Section III deals with essential concepts used in the formulation of the initial value problem in general relativity and the 3 + 1 splitting of geometric quantities. Using these tools we explain in Sec. IV how to decompose the characterization of Ferrando and Sáez which leads us to a set of necessary conditions for initial data to be a Schwarzschild initial data set—see theorem 2. A further analysis of these conditions is performed in Sec. VI where an algebraic classification of the electric and the magnetic parts of Weyl tensor is performed providing canonical forms for them in each type of Schwarzschild initial data set. In Sec. VII we investigate how to enlarge the set of necessary conditions in order to obtain a set of geometric conditions guaranteeing that the initial data are indeed Schwarzschild initial data. These extra conditions are given by the KID equations. The full set of conditions are put together in theorem 3 which is the most important result of this paper. A practical application of theorem 3 is presented in the appendix.

A great deal of the algebraic calculations of this paper were undertaken with the MATHEMATICA package *xTensor* [15]. This package is specially adapted to calculations involving abstract indexes and among its many features it includes algorithms to handle efficiently the 3 + 1 decomposition.

II. A LOCAL CHARACTERIZATION OF THE SCHWARZSCHILD SPACETIME

We begin by introducing some notation and by fixing conventions. As in the introduction, $(\mathcal{M}, g_{\mu\nu})$ denotes a 4-dimensional spacetime. We shall use the signature $(-+++)$. The Greek indices $\mu, \nu, \ldots$ are abstract spacetime indices while the Latin indices $i, j, \ldots$ are abstract spatial ones. Boldface Latin indices $\boldsymbol{a}, \boldsymbol{b}, \ldots$ will be reserved to denote components with respect to a frame and they will range $0, \ldots, 3$ in a spacetime frame and $1, 2, 3$ in a spatial frame.

Let $U_{\mu\nu\lambda\psi}$ and $V_{\mu\nu\lambda\psi}$ be two rank 4 tensors antisymmetric in the first and second pair of indices and symmetric under the interchange of these pair of indices. We define the $\star$-product of them via [16],

$$(U \star V)_{\mu\nu\lambda\psi} = \tfrac{1}{2} U_{\mu\nu}{}^{\kappa\pi} V_{\kappa\pi\lambda\psi}. \qquad (2)$$

Let $C_{\mu\nu\lambda\psi}$ denote the Weyl tensor associated to the metric $g_{\mu\nu}$. Then

$$(C \star C \star C)_{\mu\nu\lambda\psi} = \tfrac{1}{4} C_{\mu\nu}{}^{\sigma\tau} C_{\sigma\tau}{}^{\kappa\pi} C_{\kappa\pi\lambda\psi}, \tag{3a}$$

$$\mathrm{tr}(C \star C \star C) = \tfrac{1}{2} (C \star C \star C)^{\mu\nu}{}_{\mu\nu}. \tag{3b}$$

Crucial objects in the subsequent analysis are the following scalars:

$$\rho = \left(\tfrac{1}{12} \mathrm{tr}(C \star C \star C)\right)^{1/3}, \tag{4a}$$

$$\alpha = \frac{1}{9\rho^2} g^{\mu\nu} \nabla_\mu \rho \nabla_\nu \rho + 2\rho. \tag{4b}$$

Finally, we also write

$$(g \wedge g)_{\mu\nu\lambda\sigma} = 2(g_{\mu\lambda} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\lambda}), \tag{5}$$

and define the following tensors:

$$S_{\mu\nu\lambda\sigma} = \frac{1}{3\rho}\left(C_{\mu\nu\lambda\sigma} + \frac{1}{2}\rho (g \wedge g)_{\mu\nu\sigma\lambda}\right), \tag{6a}$$

$$P_{\mu\nu} = C^*_{\lambda\mu\sigma\nu} \nabla^\lambda \rho \nabla^\sigma \rho \tag{6b}$$

$$Q_{\mu\nu} = S_{\lambda\mu\sigma\nu} \nabla^\lambda \rho \nabla^\sigma \rho. \tag{6c}$$

Here, and in the sequel, $C^*_{\mu\nu\lambda\sigma}$ denotes the *(Hodge) dual* of the Weyl tensor,

$$C^*_{\mu\nu\lambda\sigma} = \tfrac{1}{2} C_{\mu\nu\kappa\pi} \epsilon^{\kappa\pi}{}_{\lambda\sigma}. \tag{7}$$

In terms of the above concomitants of the Weyl tensor, Ferrando and Sáez's characterization of the Schwarzschild spacetime states the following, see [14]. Note that there are slight differences between our conventions in the definition of the concomitants and those of Ferrando and Sáez.

Theorem 1 (Ferrando and Sáez, 1998). Necessary and sufficient conditions for a vacuum spacetime $(\mathcal{M}, g_{\mu\nu})$ to be locally isometric to the Schwarzschild spacetime are

$$\rho \neq 0, \tag{8a}$$

$$(S \star S)_{\mu\nu\sigma\lambda} - S_{\mu\nu\sigma\lambda} = 0, \tag{8b}$$

$$P_{\mu\nu} = 0, \tag{8c}$$

$$2Q_{\mu\nu} v^\mu v^\nu + Q^\mu{}_\mu \leq 0, \tag{8d}$$

$$\alpha > 0, \tag{8e}$$

where v^μ is an arbitrary unit timelike vector. Moreover, the Schwarzschild mass is given by

$$m = \frac{\rho}{\alpha^{3/2}}, \tag{9}$$

and a timelike Killing vector in one of the static regions by

$$\xi^\mu = \frac{1}{\rho^{4/3}\sqrt{|Q_{\lambda\nu} v^\lambda v^\nu|}} Q^\mu{}_\nu v^\nu. \tag{10}$$

Remarks.

(1) The conditions (8a) and (8b) are equivalent (locally) to having static, type D spacetimes—these were subdivided by Ehlers and Kundt [17] into the classes A, B and C; the class A containing the Schwarzschild metric. If in addition one has that $P_{\mu\nu} \neq 0$, then the spacetime corresponds to the C-metric, the only representative of the class C. On the other hand, if $P_{\mu\nu} = 0$ and $2Q_{\mu\nu} v^\mu v^\nu + Q^\mu{}_\mu > 0$ one has a type A metric. The condition $\alpha > 0$ is the one that selects the Schwarzschild metric among the 3 contained in the Ehlers-Kundt class A.

(2) Theorem 1 can be reformulated replacing condition (8d) by

$$Q_{\mu\nu} = -9\alpha\rho^2 \xi_\mu \xi_\nu. \tag{8d$'$}$$

where ξ_μ is a smooth vector field which is a Killing vector in those regions where it is timelike. To show this we first note that (8d′) entails (8d) and hence $\mathcal{M}$ is locally Schwarzschild. Reciprocally if $\mathcal{M}$ is a subset of Schwarzschild spacetime then, an explicit computation in a coordinate system covering $\mathcal{M}$ enables us to derive (8d′). In our calculation we chose the coordinate chart given in [18] which covers the full Schwarzschild-Kruskal analytical extension and hence it can be used to construct a coordinate patch covering any subset $\mathcal{M}$ of such extension. In this paper we shall use theorem 1 with condition (8d′) instead of condition (8d) because the former is better suited for our calculations.

III. STANDARD RESULTS FROM THE 3 + 1 DECOMPOSITION

As mentioned in the introduction, the first part of our analysis will be concentrated with obtaining a 3 + 1 splitting of the conditions given in theorem 1. Let $(\mathcal{S}, h_{ij})$ be a 3-dimensional connected Riemannian manifold. The map $\phi: \mathcal{S} \to \mathcal{M}$ is an isometric embedding if $\phi^* g_{\mu\nu} = h_{ij}$ where as usual ϕ^* denotes the pullback of tensor fields from $\mathcal{M}$ to $\mathcal{S}$. In the framework of the 3 + 1 decomposition it is more advantageous to work with *foliations*. This is a family of submanifolds $\{S_t\}_{t\in I}$ of $\mathcal{M}$ whose union is an open subset $\mathcal{N} \subset \mathcal{M}$ and no two submanifolds of the family have common points. Each submanifold of the foliation is called a leaf. When $\mathcal{N} = \mathcal{M}$ then $\{S_t\}_{t\in I}$ is called a foliation of $\mathcal{M}$. The foliations relevant for us are those formed by leaves which are 3-dimensional spacelike hypersurfaces of $\mathcal{M}$. In this case we can define a unit 1-form n_μ on $\mathcal{N}$ by the property that n_μ is normal to any of the leaves of the foliation. Since the leaves are spacelike we have $n^\mu n_\mu = -1$. The tensor fields $h_{\mu\nu}$ and $K_{\mu\nu}$ of $\mathcal{N}$ defined by the relations

$$h_{\mu\nu} \equiv g_{\mu\nu} + n_\mu n_\nu, \qquad K_{\mu\nu} \equiv -\tfrac{1}{2}\mathcal{L}_n h_{\mu\nu}, \tag{11}$$

play the role of the first and second fundamental form, respectively, for any of the leaves. Here $\mathcal{L}$ denotes the Lie derivative. From now on we will assume that $\phi(\mathcal{S})$ belongs to a given foliation. It is then clear that $\phi^* h_{\mu\nu} = h_{ij}$. Further, we may define the tensor field $K_{ij} \equiv \phi^* K_{\mu\nu}$.

The 3 + 1 splitting of a timelike vector field t^μ can be also easily achieved in terms of its *lapse*, N, and *shift*, N^μ:

$$t^\mu = Nn^\mu + N^\mu, \qquad N \equiv -t_\mu n^\mu, \qquad N^\mu \equiv h^\mu{}_\nu t^\nu. \tag{12}$$

Less trivially, we consider now the 3 + 1 splitting of the Weyl tensor—and its dual—in terms of its electric and magnetic parts with respect to n^μ:

$$C_{\mu\nu\lambda\sigma} = 2(l_{\mu[\lambda}E_{\sigma]\nu} - l_{\nu[\lambda}E_{\sigma]\mu} - n_{[\lambda}B_{\sigma]\tau}\epsilon^\tau{}_{\mu\nu} - n_{[\mu}B_{\nu]\tau}\epsilon^\tau{}_{\lambda\sigma}), \tag{13a}$$

$$C^*_{\mu\nu\lambda\sigma} = 2(l_{\mu[\lambda}B_{\sigma]\nu} - l_{\nu[\lambda}B_{\sigma]\mu} + n_{[\lambda}E_{\sigma]\tau}\epsilon^\tau{}_{\mu\nu} + n_{[\mu}E_{\nu]\tau}\epsilon^\tau{}_{\lambda\sigma}), \tag{13b}$$

where

$$E_{\tau\sigma} \equiv C_{\tau\nu\sigma\lambda}n^\nu n^\lambda, \qquad B_{\tau\sigma} \equiv C^*_{\tau\nu\sigma\lambda}n^\nu n^\lambda, \tag{14}$$

denote the *n-electric* and *n-magnetic* parts, respectively, and $l_{\mu\nu} \equiv h_{\mu\nu} + n_\mu n_\nu$, while $\epsilon_{\tau\lambda\sigma} \equiv \epsilon_{\nu\tau\lambda\sigma}n^\nu$. The tensors $E_{\mu\nu}$ and $B_{\mu\nu}$ are symmetric, traceless, and spatial: $n^\mu E_{\mu\nu} = n^\mu B_{\mu\nu} = 0$. The tensor $\epsilon_{\mu\nu\sigma}$ is fully antisymmetric and when a foliation is present it can be put in correspondence with the volume element of any of its leaves.

Let D_μ denote the operator obtained from the spacetime connection ∇_μ by:

$$D_\mu T^{\alpha_1\ldots\alpha_p}{}_{\beta_1\ldots\beta_q} \equiv h^{\alpha_1}{}_{\rho_1}\ldots h^{\alpha_p}{}_{\rho_p}h^{\sigma_1}{}_{\beta_1}\ldots h^{\sigma_q}{}_{\beta_q}h^\lambda{}_\mu\nabla_\lambda \times T^{\rho_1\ldots\rho_p}{}_{\sigma_1\ldots\sigma_q}, \tag{15}$$

$p, q \in \mathbb{N}$, where $T^{\alpha_1\ldots\alpha_p}{}_{\beta_1\ldots\beta_q}$ is any tensor field in $\mathcal{M}$. We have the important property:

$$\phi^*(D_\mu T^{\alpha_1\ldots\alpha_p}{}_{\beta_1\ldots\beta_q}) = D_i(\phi^* T^{\alpha_1\ldots\alpha_p}{}_{\beta_1\ldots\beta_q}), \tag{16}$$

the operator D_i being the Levi-Civita connection associated with the 3-metric h_{ij}:

$$D_j h_{ik} = 0. \tag{17}$$

For later use we need to find the 3 + 1 splitting of $\nabla_\mu E_{\mu\lambda}$ and $\nabla_\mu B_{\mu\lambda}$ when Einstein vacuum equations hold. This can be done by finding the 3 + 1 splitting of the contracted Bianchi identity $\nabla_\rho C^\rho{}_{\mu\nu\sigma} = 0$, valid when $R_{\mu\nu} = 0$ and use (13a) and (13b). As is well known (see e.g. [19]) the result is ($a^\mu \equiv n^\rho\nabla_\rho n^\mu$)

$$\mathcal{L}_n E_{\mu\nu} = 2a_\lambda B_{\sigma(\nu}\varepsilon_{\mu)}{}^{\sigma\lambda} - 3E_\nu{}^\sigma K_{\mu\sigma} - 2E_\mu{}^\sigma K_{\nu\sigma} + 2E_{\mu\nu}K^\sigma{}_\sigma + E^{\sigma\lambda}K_{\sigma\lambda}h_{\mu\nu} + \varepsilon_\mu{}^{\sigma\lambda}(D_\lambda B_{\nu\sigma}) \tag{18a}$$

$$\mathcal{L}_n B_{\mu\nu} = -2a_\lambda E_{\sigma(\nu}\varepsilon_{\mu)}{}^{\sigma\lambda} - 3B_\nu{}^\sigma K_{\mu\sigma} - 2B_\mu{}^\sigma K_{\nu\sigma} + 2B_{\mu\nu}K^\sigma{}_\sigma + B^{\sigma\lambda}K_{\sigma\lambda}h_{\mu\nu} - \varepsilon_\mu{}^{\sigma\lambda}(D_\lambda E_{\nu\sigma}), \tag{18b}$$

which are *evolution equations* and

$$D^j E_{ji} + B_{jk}K^j{}_l\epsilon^{kl}{}_i = 0, \tag{19a}$$

$$D^j B_{ji} - E_{jk}K^j{}_l\epsilon^{kl}{}_i = 0, \tag{19b}$$

which are constraint equations. Expanding the Lie derivative in terms of the Levi-Civita covariant derivative we find the desired 3 + 1 decomposition of $\nabla_\mu E_{\mu\lambda}$ and $\nabla_\mu B_{\mu\lambda}$.

Initial data sets for the vacuum equations

As explained in the introduction, $(\mathcal{S}, h_{ij}, K_{ij})$ is an initial data set for the vacuum Einstein field equations if h_{ij}, K_{ij} satisfy the constraint equations:

$$r + K^2 - K^{ij}K_{ij} = 0, \tag{20a}$$

$$D^j K_{ij} - D_i K = 0, \tag{20b}$$

on $\mathcal{S}$ where r is the Ricci scalar of h_{ij}, and we write $K = K^i{}_i$. If $(\mathcal{M}, g_{\mu\nu})$ is the initial data development and ϕ: $\mathcal{S} \to \mathcal{M}$ the isometric embedding of the initial data, then we have the property $K_{ij} = \phi^* K_{\mu\nu}$, where $K_{\mu\nu}$ is defined by (11) (of course a foliation of $\mathcal{M}$ containing $\phi(\mathcal{S})$ must be constructed first). From the initial data $(\mathcal{S}, h_{ij}, K_{ij})$ we may define the tensor fields E_{ij}, B_{ij} by the relations

$$E_{ij} \equiv r_{ij} + KK_{ij} - K_{ik}K^k{}_j, \tag{21a}$$

$$B_{ij} \equiv \epsilon^{kl}{}_{(i}D_{|k}K_{l|j)}. \tag{21b}$$

We note that the symmetrization in the last equation is not needed if the constraints (20a) and (20b) are satisfied, which we shall assume. Thus, if the vacuum constraints hold then E_{ij}, B_{ij} are both symmetric and traceless. Defining in the data development the electric and magnetic parts of the Weyl tensor as shown in (14) we have $E_{ij} = \phi^* E_{\mu\nu}$, and $B_{ij} = \phi^* B_{\mu\nu}$.

IV. 3 + 1 DECOMPOSITION OF THE CHARACTERIZATION BY FERRANDO AND SÁEZ

We proceed now to a 3 + 1 decomposition of the conditions appearing in the invariant characterization of theorem 1. This leads naturally to a first set of necessary conditions for a given initial data set $(\mathcal{S}, h_{ij}, K_{ij})$ to be a Schwarzschild initial data set. We study each condition of the theorem separately in the forthcoming subsections.

A. Decomposition of Eq. (8a)

First, we note that:

$$\mathrm{tr}\,(C \star C \star C) = 6B_\mu{}^\nu B^{\mu\sigma}E_{\nu\sigma} - 2E_{\mu\nu}E_\sigma{}^\nu E^{\mu\sigma}, \tag{22}$$

from where the scalar ρ, see Eq. (4a), can be calculated yielding

$$\rho = (\tfrac{1}{2}B_\mu{}^\nu B^{\mu\sigma}E_{\nu\sigma} - \tfrac{1}{6}E_{\mu\nu}E_\sigma{}^\nu E^{\mu\sigma})^{1/3}. \tag{23}$$

Using other conditions of theorem 1 we will be able to get a simpler expression for ρ, see (26) below.

B. Decomposition of Eq. (8b)

Using the formula for the Weyl tensor in terms of its electric and magnetic parts, Eq. (13a), it is not difficult to decompose (8b). This decomposition renders the condition (8b) of theorem 1 in the form

$$4E_{\mu[\delta}E_{\lambda]\nu} + 2h_{\nu[\lambda}(B_{\delta]\pi}B_{\mu}{}^{\pi} + E_{\delta]\pi}E_{\mu}{}^{\pi} - \rho E_{\delta]\mu}) + 2h_{\mu[\delta}(B_{\lambda]}{}^{\pi}B_{\nu\pi} + E^{\pi}_{\lambda]}E_{\nu\pi} - \rho E_{\lambda]\nu}) + 2h_{\mu[\lambda}h_{\delta]\nu}(B_{\pi\kappa}B^{\pi\kappa} + 2\rho^2) = 0, \tag{24a}$$

$$B_{\nu}{}^{\sigma}E_{\delta}{}^{\alpha}\epsilon_{\lambda\sigma\alpha} + B^{\sigma\alpha}E_{\nu\sigma}\epsilon_{\lambda\delta\alpha} - B_{\nu}{}^{\sigma}E_{\lambda}{}^{\alpha}\epsilon_{\delta\sigma\alpha} + \rho B_{\nu}{}^{\sigma}\epsilon_{\lambda\delta\sigma} = 0, \tag{24b}$$

$$E_{\lambda}{}^{\alpha}E_{\mu\alpha} - B_{\lambda}{}^{\alpha}B_{\mu\alpha} + \rho(E_{\mu\lambda} - 2\rho h_{\mu\lambda}) = 0. \tag{24c}$$

From here we obtain the important relations

$$B_{\mu\nu} = -\frac{1}{\rho}(B_{\nu}{}^{\lambda}E_{\mu\lambda} + B_{\mu}{}^{\lambda}E_{\nu\lambda}), \tag{25a}$$

$$E_{\mu\nu} = \frac{1}{\rho}(B_{\nu}{}^{\lambda}B_{\mu\lambda} - E_{\nu}{}^{\lambda}E_{\mu\lambda}) + 2\rho h_{\mu\nu}. \tag{25b}$$

The conditions (25a) and (25b) are essentially the same ones which were obtained in [9] from a 3 + 1 splitting of the Zakharov property (1).

Taking the trace of condition (25a) on finds that $E_{\alpha\beta}B^{\alpha\beta} = 0$, a well know property of the Schwarzschild spacetime. The condition (25b) enables us to obtain a simple expression for ρ:

$$\rho^2 = \frac{(E_{\lambda\alpha}E^{\lambda\alpha} - B_{\lambda\alpha}B^{\lambda\alpha})}{6}, \qquad E_{\lambda\alpha}E^{\lambda\alpha} \geq B_{\lambda\alpha}B^{\lambda\alpha}. \tag{26}$$

It should be mentioned that $E_{\lambda\alpha}E^{\lambda\alpha} - B_{\lambda\alpha}B^{\lambda\alpha}$ and $E_{\lambda\alpha}B^{\lambda\alpha}$ are, respectively, the real and imaginary parts of I, one of the invariants of the Weyl tensor, see e.g. [20,21]. From here, the 4-dimensional covariant derivative of ρ can be obtained which is

$$\nabla_{\mu}\rho = \frac{1}{6\rho}(E^{\lambda\alpha}\nabla_{\mu}E_{\lambda\alpha} - B^{\lambda\alpha}\nabla_{\mu}B_{\lambda\alpha}). \tag{27}$$

Replacing the derivatives of the electric and magnetic parts by their 3 + 1 splitting [Eqs. (18a) and (18b) must be used to find such splitting] we obtain the 3 + 1 splitting of $\nabla_{\mu}\rho$. We shall write it in shift and lapse parts as follows,

$$\nabla_{\mu}\rho = Pn_{\mu} + P_{\mu}, \tag{28}$$

where

$$P \equiv -\frac{1}{2}E^{\alpha\lambda}K_{\alpha\lambda} - \rho K^{\alpha}{}_{\alpha} - \frac{1}{6\rho}\epsilon^{\beta\sigma}{}_{\alpha}(E^{\alpha\lambda}D_{\sigma}B_{\lambda\beta} + B^{\alpha\lambda}D_{\sigma}E_{\lambda\beta}), \tag{29a}$$

$$P_{\mu} \equiv \frac{1}{6\rho}(-B^{\kappa\lambda}D_{\mu}B_{\kappa\lambda} + E^{\kappa\lambda}D_{\mu}E_{\kappa\lambda}) = D_{\mu}\rho. \tag{29b}$$

Equations (25a), are equivalent to (24b) and (24c). If $B_{\mu\nu} \neq 0$ —the case $B_{\mu\nu} = 0$ is studied separately in Sec. VI B—then (24a) can be transformed into

$$E_{\lambda[\nu}E_{\mu]\rho} + E_{\rho\sigma}E^{\sigma}_{[\mu}h_{\nu]\lambda} + E_{\lambda}{}^{\sigma}E_{\sigma[\nu}h_{\mu]\rho} + \tfrac{1}{2}E_{\sigma\alpha}E^{\sigma\alpha}h_{\lambda[\mu}h_{\nu]\rho} = 0. \tag{30}$$

Lemma 1. The equation (30) is an identity.

Proof : We start with the identity

$$C_{[\mu\nu}{}^{[\alpha\beta}\delta^{\sigma]}{}_{\pi]} = 0, \tag{31}$$

which holds only in dimension four, see e.g. [22]. Multiplying this equation by $C^{\mu\nu}{}_{\sigma\lambda}$ and expanding we get

$$2C^{\alpha\mu}{}_{\pi}{}^{\nu}C_{\lambda\mu}{}^{\beta}{}_{\nu} + C^{\alpha\beta\mu\nu}C_{\lambda\pi\mu\nu} - 2C^{\alpha\mu}{}_{\lambda}{}^{\nu}C^{\beta}{}_{\nu\pi\mu} + \delta_{\lambda}{}^{\beta}C^{\alpha\mu\nu\sigma}C_{\pi\mu\nu\sigma} - \delta^{\alpha}{}_{\lambda}C^{\beta\mu\nu\sigma}C_{\pi\mu\nu\sigma} = 0.$$

Replacing the Weyl tensor by (13a) we obtain an expression which can be written in the form

$$c^{\alpha\beta}{}_{\lambda\pi} + n^{[\alpha}d^{\beta]}{}_{\lambda\pi} + n_{[\lambda}t^{\alpha\beta}{}_{\pi]} = 0, \tag{32}$$

where $c^{\alpha\beta}{}_{\lambda\pi}$ is the tensor appearing on the right-hand side of (30) with two indexes raised and $d^{\beta}{}_{\lambda\pi}$, $t^{\alpha\beta}{}_{\pi}$ are spatial tensors. The previous equation implies that $c^{\alpha\beta}{}_{\lambda\pi} = 0$ thus proving the lemma. ■

C. Decomposition of Eq. (8c)

In order to alleviate the notation in our equations it is convenient to introduce the following definitions

$$\tilde{E}_\mu \equiv E_{\mu\nu}P^\nu, \qquad \tilde{B}_\mu \equiv B_{\mu\nu}P^\nu,$$
$$\gamma^2 \equiv P_\nu P^\nu, \qquad \Omega \equiv E_{\mu\nu}P^\mu P^\nu. \tag{33}$$

In terms of these quantities and 3 + 1 quantities, the tensor $P_{\mu\nu}$ can be decomposed as:

$$P_{\mu\nu} = (P^2 + \gamma^2)B_{\mu\nu} + 2Pn_{(\mu}\tilde{B}_{\nu)} + 2PE_{\tau(\mu}\epsilon^\tau{}_{\nu)\lambda}P^\lambda - 2P_{(\mu}\tilde{B}_{\nu)} + l_{\mu\nu}\tilde{B}_\sigma P^\sigma - 2n_{(\mu}\tilde{E}_{|\tau}\epsilon^\tau{}_{\sigma|\nu)}P^\sigma. \tag{34}$$

Consequently, the condition (8c) renders:

$$0 = P_{\mu\nu}n^\mu n^\nu = \tilde{B}_\nu P^\nu, \tag{35a}$$
$$0 = n^{\mu'}P_{\mu'\nu'}h^{\nu'}{}_\nu = -P\tilde{B}_\mu + \tilde{E}_\tau\epsilon^\tau{}_{\sigma\mu}P^\sigma, \tag{35b}$$
$$0 = P_{\mu'\nu'}h^{\mu'}{}_\mu h^{\nu'}{}_\nu = (P^2 + \gamma^2)B_{\mu\nu} + 2PE_{\tau(\mu}\epsilon^\tau{}_{\nu)\lambda}P^\lambda - 2P_{(\mu}\tilde{B}_{\nu)} + h_{\mu\nu}\tilde{B}_\lambda P^\lambda. \tag{35c}$$

D. Decomposition of Eq. (8d′)

To find the decomposition of (8d′) we need to decompose first $Q_{\mu\nu}$ and the vector field ξ_μ. The latter is trivially decomposed in the form

$$\xi_\mu = Yn_\mu + Y_\mu$$

where Y_μ is spatial and Y is a scalar. The decomposition of $Q_{\mu\nu}$ is far more involved and it is not shown. Inserting all the decompositions just mentioned in (8d′) we obtain the conditions

$$27\alpha\rho^3Y^2 = \gamma^2\rho - \Omega, \tag{36a}$$
$$27\alpha\rho^3YY_\nu = \epsilon_\nu{}^{\lambda\sigma}P_\lambda\tilde{B}_\sigma - P\tilde{E}_\nu + \rho PP_\nu, \tag{36b}$$
$$2B_{\sigma(\nu}\epsilon_{\mu)}{}^{\sigma\lambda}P_\lambda P - 2P_{(\nu}\tilde{E}_{\mu)} + E_{\mu\nu}(P^2 + \gamma^2) - \rho P_\mu P_\nu + h_{\mu\nu}((-P^2 + \gamma^2)\rho + \Omega) = -27\alpha\rho^3Y_\mu Y_\nu. \tag{36c}$$

E. Decomposition of Eq. (8e)

Finally, we note that the inequality (8e) can be written as

$$\alpha = \frac{1}{9\rho^2}(\gamma^2 - P^2) + 2\rho > 0. \tag{37}$$

V. SCHWARZSCHILD INITIAL DATA: NECESSARY CONDITIONS

Now, we proceed to pull back the conditions (25a), (25b), (35b), (35c), and (36c) and the inequality (37) to the manifold $\mathcal{S}$. In such a way one obtains a first set of *necessary* conditions for a initial data set to be a Schwarzschildean initial data set.

Theorem 2. Let (h_{ij}, K_{ij}) be an initial data set and define from it the following quantities

$$\rho = \left(\tfrac{1}{2}B_i{}^jB^{il}E_{jl} - \tfrac{1}{6}E_{ij}E_l{}^jE^{il}\right)^{1/3}, \tag{38a}$$
$$P = -\frac{1}{2}E^{ij}K_{ij} - \rho K - \frac{1}{6\rho}\epsilon^{jk}{}_i(E^{il}D_kB_{lj} + B^{il}D_kE_{lj}), \tag{38b}$$
$$P_i = D_i\rho, \qquad \tilde{E}_i = E_{ij}P^j, \qquad \tilde{B}_i = B_{ij}P^j, \tag{38c}$$
$$\gamma^2 = P_iP^i, \qquad \Omega = \tilde{E}_iP^i, \tag{38d}$$
$$\alpha = \frac{\gamma^2 - P^2}{9\rho^2} + 2\rho, \qquad Y^2 = \frac{\gamma^2\rho - \Omega}{27\alpha\rho^3}, \tag{38e}$$
$$27\alpha\rho^3YY_i = \epsilon_i{}^{jk}P_j\tilde{B}_k - P\tilde{E}_i + \rho PP_i, \tag{38f}$$

where E_{ij} and B_{ij} are to be calculated from the initial data h_{ij} *and* K_{ij} using the formulas (21a) and (21b). Necessary conditions for an initial data set (h_{ij}, K_{ij}), satisfying the Einstein vacuum constraints, Eqs. (20a) and (20b), on a manifold $\mathcal{S}$ to be a Schwarzschild initial data set are:

$$B_{ij} = -\frac{1}{\rho}(B_i{}^k E_{kj} + B_j{}^k E_{ki}), \tag{39a}$$

$$E_{ij} = \frac{1}{\rho}(B_i{}^k B_{kj} - E_i{}^k E_{kj}) + 2\rho h_{ij}, \tag{39b}$$

$$\tilde{B}_j P^j = 0, \tag{39c}$$

$$\tilde{E}_k \epsilon^k{}_{li} P^l - P\tilde{B}_i = 0, \tag{39d}$$

$$(P^2 + \gamma^2)B_{ij} + 2PE_{k(i}\epsilon^k{}_{j)l}P^l - 2P_{(i}\tilde{B}_{j)} = 0, \tag{39e}$$

$$2B_{l(i}\epsilon_{j)}{}^{lk}P_k P - 2P_{(i}\tilde{E}_{j)} + E_{ji}(P^2 + \gamma^2) - \rho P_j P_i + h_{ji}((-P^2 + \gamma^2)\rho + \Omega) = -27\alpha\rho^3 Y_j Y_i, \tag{39f}$$

$$(\tfrac{1}{2}B_i{}^j B^{il} E_{jl} - \tfrac{1}{6}E_{ij}E_l{}^j E^{il})^{1/3} \neq 0, \tag{39g}$$

$$\frac{1}{9\rho^2}(\gamma^2 - P^2) + 2\rho > 0. \tag{39h}$$

VI. ALGEBRAIC CLASSIFICATION OF SCHWARZSCHILD INITIAL DATA

Using the conditions found in theorem 2 we can determine the possible types of Schwarzschild initial data one can have, according to the possible admissible algebraic structure of the tensors E_{ij} and B_{ij}. In order to do this, we will make use of adapted bases in certain subspaces of the tangent space of a point $p \in \mathcal{M}$. This subspace plays the role of the tangent space of $\phi(\mathcal{S})$ at p. We will establish the possible *canonical forms* for the matrices of the components of the tensors E_{ij} and B_{ij} in these canonical bases. This is achieved by means of algebraic computations involving the conditions of theorem 2. Therefore the canonical forms obtained here are only valid at p and we should prove *a posteriori* that initial data with E_{ij} and B_{ij} adopting a given canonical form do in fact exist. This is a clear consequence of the fact that conditions supplied by theorem 2 are only necessary. More generally, one may construct initial data sets such that the *algebraic types* deduced here for E_{ij} and B_{ij} vary on $\mathcal{S}$. Indeed, some of the algebraic types calculated here are attached to points lying in a specific region of Schwarzschild-Kruskal spacetime and cannot be found outside these regions.

An important result which we must bear in mind when developing the classification is the fact that ρ and α never vanish in the Kruskal-Schwarzschild spacetime, as can be checked by explicit calculations in a coordinate system. Similarly one can show that the vector ξ^μ of (8d′) is spacelike only for points outside the static regions.

A. Initial data with $Y = 0$

The condition $Y = 0$ implies that the vector ξ^μ of (8d′) is spacelike so the canonical forms considered in this subsection only arise for submanifolds $\mathcal{S}$ whose points are in the exterior of the static regions. We consider two subcases defined by $P^i = 0$ and $P^i \neq 0$.

In the first case, the condition $P^i = 0$ only occurs when $\nabla_a \rho$ is causal, which, consistently with the above remarks, is only possible for points outside the static regions as can be explicitly checked by a calculation in a coordinate system. In this case the initial data hypersurface $\mathcal{S}$ belongs to the foliation defined by $\rho = \text{const}$. A not very long calculation using the conditions of theorem 2 leads us to

$$B_{ij} = 0, \qquad E_{ij} = \rho h_{ij} - \frac{27\alpha\rho^3 Y_i Y_j}{P^2}, \qquad P \neq 0. \tag{40}$$

If $P^i \neq 0$ then the canonical forms are somewhat more complex

$$E_{ij} = \frac{-3\rho}{P^2 - \gamma^2}P_i P_j + \frac{\rho(-2\gamma^4 + P^2(P^2 + \gamma^2))}{(P^2 - \gamma^2)^2}h_{ij} - \frac{27\alpha\rho^2(P^2 + \gamma^2)}{(P^2 - \gamma^2)^2}Y_i Y_j, \tag{41}$$

$$B_{ij} = \frac{27\alpha\rho^3 P_k P Y_l(\epsilon_i{}^{kl}Y_j + \epsilon_j{}^{kl}Y_i)}{(P^2 - \gamma^2)^2}, \tag{42}$$

where Y^j is orthogonal to P^j and

$$Y^j Y_j = \frac{P^2 - \gamma^2}{9\alpha\rho^2}. \tag{43}$$

Of course these canonical forms are only valid for points with $P^2 - \gamma^2 \neq 0$ (points not intersecting the event horizon).

B. Initial data with $Y \neq 0$

The condition $Y \neq 0$ implies that $P^i \neq 0$ and thus $\gamma \neq 0$. Also note that if $Y \neq 0$ then Y and Y^j are fixed by the initial data by means of Eqs. (62a) and (62b). We divide up these initial data in two cases: those with $\tilde{B}^i P = 0$ and those with $\tilde{B}^i P \neq 0$.

Case $P\tilde{B}^i = 0$

We find next the form which conditions of theorem 2 take when $P\tilde{B}^i = 0$. The conditions (39a)–(39e) in proposition 2 will be interpreted as defining a certain *canonical* *orthonormal basis* for our problem. In particular, if $P\tilde{B}^i = 0$, then condition (39d) reduces to

$$\tilde{E}_k \epsilon^k{}_{li} P^l = 0. \tag{44}$$

The latter condition implies that $\tilde{E}^k$ has to be proportional to P^k—that is, P^k is an eigenvector of $E_j{}^k$,

$$E_j{}^k P^j = \lambda P^k. \tag{45}$$

In view of the above, it is natural to consider an orthonormal basis $\{(e_a)^i\} = \{(e_1)^i, (e_2)^i, (e_3)^i\}$, such that

$$P^i = \gamma(e_1)^i, \qquad \tilde{E}^i = \Xi(e_1)^i, \qquad \Xi = \pm\sqrt{\tilde{E}_j\tilde{E}^j}. \tag{46}$$

The remaining frame vectors $(e_2)^i$ and $(e_3)^i$ are left undetermined. Now, the conditions (39d) and (39e) can be used to make further statements about the matrices whose entries are the components of the tensors E_{ab} and B_{ab} in the frame $\{(e_a)^i\}$. These matrices are defined by $E_{ab} = E_{ij}(e_a)^i(e_b)^j$, $B_{ab} = B_{ij}(e_a)^i(e_b)^j$. One finds that

$$E_{12} = E_{13} = 0, \qquad E_{11} = \frac{\Xi}{\gamma} \tag{47a}$$

$$B_{22} = \frac{2E_{23}P\gamma}{P^2 + \gamma^2} = -B_{33}, \qquad B_{23} = \frac{(E_{33} - E_{22})P\gamma}{P^2 + \gamma^2}, \tag{47b}$$

$$B_{11} = B_{12} = 0 \tag{47c}$$

From here we easily deduce $\tilde{B}^a = 0$. These expressions can be further restricted if we apply (39f)—previously we need to use (38f) to find Y and Y^j in our chosen basis, see (51) below. As a result we find that all the components of the magnetic part are zero, purely electric initial data sets, whereas the electric part takes the form

$$E_{23} = 0, \qquad E_{22} = E_{33} = -\frac{\rho(\gamma^2 - P^2) + \Omega}{P^2 + \gamma^2}, \tag{48a}$$

with

$$\Xi^2 = \frac{4\gamma^2(\rho(\gamma^2 - P^2) + \Omega)^2}{(P^2 + \gamma^2)^2}. \tag{48b}$$

Finally, contracting (39b) with $(e_a)^i(e_b)^j$ we get

$$E_{11} = -2\rho, \qquad E_{22} = E_{33} = \rho. \tag{49}$$

Therefore the electric part can be written in abstract index notation as

$$E_{ij} = \rho\left(h_{ij} - \frac{3}{\gamma^2}P_iP_j\right). \tag{50}$$

This last equation comprise the most general form for the electric part which fulfills conditions (39a)–(39e) whenever $\tilde{B}^a = 0$ (and as a consequence $B_{ij} = 0$ as proven before). It can be readily checked that condition (24a) is automatically satisfied by (50), and thus provides no further information.

The class of purely electric initial data sets includes a number of relevant examples. Time-symmetric initial data sets, i.e., those for which $K_{ij} = 0$, are trivially contained in this class. Another example of purely electric data sets is given by those having spherical symmetry, see example 1.

Using the above results we can also supply canonical expressions for Y and Y^j

$$Y = \frac{\gamma}{3\rho\sqrt{\alpha}}, \qquad Y^i = \frac{P}{3\rho\sqrt{\alpha}}(e_1)^i. \tag{51}$$

These expressions will be important later, for they give rise to *Killing Initial Data candidates* which, under certain conditions, enable us to show that initial data whose electric and magnetic part fulfill the conditions obtained in this subsection are actually Schwarzschildean initial data.

Example 1. Spherically symmetric data. In the so-called proper distance gauge spherically symmetric first and second fundamental forms look like

$$h_{ij}dx^idx^j = dl^2 + R^2(l)(d\theta^2 + \sin^2\theta d\varphi^2), \tag{52a}$$

$$K_{ij}dx^idx^j = K_L(l)dl^2 + K_R(l)(d\theta^2 + \sin^2\theta d\varphi^2) \tag{52b}$$

where u^i is the outward-pointing normal to the spheres with constant R, l is a coordinate measuring the proper distance between spheres of fixed radius and $K_L(l)$, $K_R(l)$ are scalar functions, see [12]. The Hamiltonian and momentum constraints for these data are given, respectively, by

$$K_R(K_R + 2K_L) - \frac{1}{R^2}(2RR'' + R'^2 - 1) = 0, \tag{53a}$$

$$K_R' + \frac{R'}{R}(K_R - K_L) = 0, \tag{53b}$$

where the prime denotes the derivative with respect to l. If these equations hold then it can be explicitly checked that $B_{ij} = 0$. As a result, conditions (39a)–(39e) are automatically satisfied. This can be regarded as a 3-dimensional manifestation of Birkhoff's theorem because the development of the data given by (52a) and (52b) is a vacuum spherically symmetric spacetime and thus, by Birkhoff's theorem, a patch of Schwarzschild spacetime. Therefore the data (52a) and (52b) are automatically Schwarzschild initial data if the vacuum constraints hold. It is noteworthy to point out that the approach in [12] is somehow different in as much that they do not assume *a priori* that the constraint equations are satisfied.

C. Initial data sets with $\tilde{B}^i \neq 0$

In this case, the most general one of our analysis, P^i ceases to be an eigenvalue of $E_i{}^j$, however, it will be still an element of our canonical basis. We write as before, $P^i = \gamma(e_1)^i$ so that again $B_{11} = (e_1)^i(e_1)^j B_{ij} = 0$. Now, the condition (39c) suggest to consider a second frame vector $(e_2)^i$ parallel to $\tilde{B}^i$. The latter choice fixes automatically $(e_3)^i$, which has to be parallel to $\epsilon^i{}_{jk}P^j\tilde{B}^k$. In what follows, we will write

$$\tilde{B}^i = \beta(e_2)^i, \qquad \beta = \pm\sqrt{\tilde{B}_k\tilde{B}^k}. \tag{54}$$

Using the frame $\{(e_{\boldsymbol{a}})^i\}$ thus constructed, one can use the conditions (39d) and (39e) to express the components $B_{\boldsymbol{ab}} = B_{ij}(e_{\boldsymbol{a}})^i(e_{\boldsymbol{b}})^j$ in terms of $E_{\boldsymbol{ab}} = E_{ij}(e_{\boldsymbol{a}})^i(e_{\boldsymbol{b}})^j$, and the scalars β, γ and P. More precisely, one has that

$$E_{12} = 0, \qquad E_{13} = \frac{P\beta}{\gamma^2} \tag{55}$$

and

$$B_{12} = \frac{\gamma}{P}E_{13} = \frac{\beta}{\gamma}, \tag{56a}$$

$$B_{13} = 0, \tag{56b}$$

$$B_{22} = \frac{2\gamma P E_{23}}{P^2 + \gamma^2}, \tag{56c}$$

$$B_{33} = -\frac{2\gamma P E_{23}}{P^2 + \gamma^2}, \tag{56d}$$

$$B_{23} = \frac{\gamma P(E_{33} - E_{22})}{P^2 + \gamma^2}. \tag{56e}$$

As in subsection VI B 1, we now resort to conditions (39f), (39a), and (39b) to obtain further restrictions on the components of E_{ij} and B_{ij}. We spare the reader from the intermediate long calculations and just provide the final result for the components of the electric and magnetic parts which are

$$(E_{\boldsymbol{ab}}) = \begin{pmatrix} -\frac{\Omega}{\gamma^2} & 0 & \frac{P}{\gamma^2}\sqrt{\frac{(\Omega-\gamma^2\rho)(2\gamma^2\rho+\Omega)}{\gamma^2-P^2}} \\ 0 & \frac{\rho(P^2+\gamma^2)+\Omega}{-\gamma^2+P^2} & 0 \\ \frac{P}{\gamma^2}\sqrt{\frac{(\Omega-\gamma^2\rho)(2\gamma^2\rho+\Omega)}{\gamma^2-P^2}} & 0 & -\frac{\gamma^4\rho+P^2(\gamma^2\rho+\Omega)}{\gamma^2(P^2-\gamma^2)} \end{pmatrix}, \tag{57a}$$

$$(B_{\boldsymbol{ab}}) = \begin{pmatrix} 0 & \frac{P}{\gamma^2}\sqrt{\frac{(\Omega-\gamma^2\rho)(2\gamma^2\rho+\Omega)}{\gamma^2-P^2}} & 0 \\ \frac{P}{\gamma^2}\sqrt{\frac{(\Omega-\gamma^2\rho)(2\gamma^2\rho+\Omega)}{\gamma^2-P^2}} & 0 & \frac{P(2\gamma^2\rho+\Omega)}{\gamma(P^2-\gamma^2)} \\ 0 & \frac{P(2\gamma^2\rho+\Omega)}{\gamma(P^2-\gamma^2)} & 0 \end{pmatrix}. \tag{57b}$$

These canonical forms assume the conditions $P^2 - \gamma^2 \neq 0$ so they only hold for points which are not in the event horizon. Also, as in previous case, canonical expressions for Y and Y^j can be obtained

$$Y = \frac{\sqrt{\gamma^2\rho - \Omega}}{3\rho^{3/2}\sqrt{3\alpha}}, \tag{58}$$

$$Y^i = (e_1)^i\frac{P\sqrt{\gamma^2\rho - \Omega}}{3\gamma\rho^{3/2}\sqrt{3\alpha}} - (e_3)^i\frac{\sqrt{(P^2 - \gamma^2)(2\gamma^2\rho + \Omega)}}{3\gamma\rho^{3/2}\sqrt{3\alpha}}. \tag{59}$$

VII. SUFFICIENT CONDITIONS FOR SCHWARZSCHILD INITIAL DATA

In order to discuss whether or not the conditions given in proposition 2 are also sufficient conditions for a pair (h_{ij}, K_{ij}) of symmetric tensors on a 3-dimensional manifold $\mathcal{S}$ which satisfy the Einstein constraint equations to be a Schwarzschild initial data set, one is confronted with the issue of the propagation of the conditions (39a)–(39h). One has to show that the development of initial data satisfying our conditions—or an extended set of them—is a subset of Schwarzschild spacetime. More precisely, assume that on a given 3-dimensional manifold $\mathcal{S}$, the initial data (h_{ij}, K_{ij}) satisfies the conditions presented in theorem 2, and set $S_0 \equiv \phi(\mathcal{S})$ with $\phi\colon \mathcal{S} \to \mathcal{M}$ an isometric embedding. Then if one is able to prove—under, perhaps, some extra conditions—that conditions (8a)–(8e) given in theorem 1 are also satisfied in an open subset $\mathcal{N}$ of the development which contains S_0, it follows that $\mathcal{N}$ would be a portion of the Schwarzschild spacetime. Note that due to the uniqueness of the maximal development of the initial data, see e.g. [11], this would prove that the maximal development is also a patch of Schwarzschild spacetime. A similar idea could be applied if one is considering a subset $\mathcal{U} \subset \mathcal{S}$.

To address the problem explained in previous paragraph we will require that the development of our initial data, or at least a subset $\mathcal{U} \subset \mathcal{S}$ thereof, possesses a timelike Killing vector ξ^μ. In that case, denoting by ϕ_t the local flow generated by ξ^μ then one may seek conditions under which (39a)–(39e) hold in the family of hypersurfaces $\phi_t(S_0)$, $t \in (-\epsilon, \epsilon)$ which would imply that conditions (8a)–(8e) of theorem 1 are also satisfied in an open subset $\mathcal{N}$ of the development.

In what follows, it is assumed that the initial data set (h_{ij}, K_{ij}) besides fulfilling the conditions (39a)–(39h), is also sufficiently regular and possesses the required asymptotic decay so that at least an open set, $\mathcal{N}$, of development exists.

A. Killing initial data sets

A Killing initial data set (KID) associated to the initial data $(\mathcal{S}, h_{ij}, K_{ij})$ is a pair (Y, Y^i) consisting of a scalar Y and a vector Y^i defined on $\mathcal{S}$ satisfying the following system of linear partial differential equations, *the KID equations*

$$D_{(i}Y_{j)} - YK_{ij} = 0, \tag{60a}$$

$$D_iD_jY - \mathcal{L}_{Y^l}K_{ij} = Y(r_{ij} + KK_{ij} - 2K_{il}K^l_j). \tag{60b}$$

For any KID there exists a Killing vector ξ^μ in the development of $(\mathcal{S}, h_{ij}, K_{ij})$ whose 3 + 1 decomposition when pull-backed to $\mathcal{S}$ is precisely the KID [23–25]. More precisely, if we consider any foliation of the development with $\phi(\mathcal{S})$ as one of its leaves then the 3 + 1 decomposition of ξ^μ reads

$$\xi^\mu = Yn^\mu + Y^\mu, \tag{61}$$

where n^μ is the normal vector to the leaves. In this case ϕ^*Y and $Y_i = \phi^*Y_\mu$ satisfy the Eqs. (60a) and (60b) on $\mathcal{S}$.

Determining whether an initial data set possesses a KID is in general a nontrivial endeavour which entails finding solutions of the system (60a) and (60b). Fortunately, theorem 1 and theorem 2 enables us to find an Ansatz to solve these equations:

$$Y = \sqrt{\frac{\gamma^2\rho - \Omega}{27\alpha\rho^3}}, \tag{62a}$$

$$Y^i = \frac{\rho PP^i - \tilde{E}^iP + \tilde{B}^kP_l\epsilon^{il}{}_k}{\sqrt{27\alpha\rho^3(\gamma^2\rho - \Omega)}}. \tag{62b}$$

The pair (Y, Y^i) defined by the above equations will be called a *KID candidate*. Clearly a KID candidate can be always constructed from any initial data set with $\alpha \neq 0$, $\rho \neq 0$ and $\gamma^2\rho - \Omega > 0$. It is plausible to conjecture that for an initial data set satisfying the conditions given in theorem 2, the KID candidate satisfies the KID equations (60a) and (60b). A possible approach to how this conjecture could be attacked is discussed in the appendix for the simpler case of time-symmetric data. In view of the computational difficulties in proving this assertion, we shall adopt here a pragmatic perspective and require that in addition to satisfying the conditions in theorem 2 the initial data set (h_{ij}, K_{ij}) is such that the KID candidate solves the KID equations, so that at least a portion of the resulting spacetime possesses a timelike Killing vector; namely, that portion in the domain of dependence of the subset $\mathcal{U} \subset \mathcal{S}$ where $Y^iY_i - Y^2 < 0$ (timelike KID).

A simpler KID candidate can be found from (62a) and (62b) if we eliminate α using (9). Since m, given by (9) has to be a constant for Schwarzschildean initial data sets, it can be also removed from the expression for the KID yielding

$$Y = \frac{\sqrt{\gamma^2\rho - \Omega}}{\rho^{11/6}}, \qquad Y^i = \frac{\rho PP^i - \tilde{E}^iP + \tilde{B}^kP_l\epsilon^{il}{}_k}{\rho^{11/6}\sqrt{\gamma^2\rho - \Omega}}. \tag{63}$$

B. Main result

The following theorem can be regarded as a converse of theorem (2).

Theorem 3. Let $(\mathcal{S}, h_{ij}, K_{ij})$ be an initial data set satisfying the vacuum Einstein constraint equations (20a) and (20b) and in addition the conditions (39a)–(39h) of theorem 2. Further, let the quantities

$$Y = \frac{\sqrt{\gamma^2\rho - \Omega}}{\rho^{11/6}}, \qquad \gamma^2\rho - \Omega > 0, \tag{64}$$

$$Y^i = \frac{\rho PP^i - \tilde{E}^iP + \tilde{B}^kP_l\epsilon^{il}{}_k}{\rho^{11/6}\sqrt{\gamma^2\rho - \Omega}}, \tag{65}$$

solve the KID equations (60a) and (60b) in an open subset $\mathcal{U} \subset \mathcal{S}$ with

$$Y^iY_i - Y^2 < 0, \tag{66}$$

then $(\mathcal{U}, h_{ij}, K_{ij})$ is isometric to a region of an initial data set $(\mathcal{S}', h'_{ij}, K'_{ij})$ for the Schwarzschild spacetime.

Proof : According to standard results, the development $\mathcal{D}(\phi(\mathcal{S}))$ is a vacuum solution of Einstein equations so we may define its Weyl tensor $C_{\mu\nu\lambda\sigma}$ in the standard fashion. Conditions (60a) and (60b) imply that a Killing vector ξ^μ exists in $\mathcal{N} \cap \mathcal{D}(\phi(\mathcal{U}))$ where $\mathcal{N}$ is an open set containing $\phi(\mathcal{U})$. Let n^μ be the normal vector of a foliation of $\mathcal{N}$ adapted to $\phi(\mathcal{U})$ and define the 3 + 1 splitting of ξ^μ as in (61). By continuity, $\mathcal{N}$ can be chosen in such a way that $Y^\mu Y_\mu - Y^2 < 0$ on $\mathcal{N} \cap \mathcal{D}(\phi(\mathcal{U}))$ if $Y^iY_i - Y^2 < 0$ on $\mathcal{U}$ from which we deduce that ξ^μ is timelike in $\mathcal{N} \cap \mathcal{D}(\phi(\mathcal{U}))$.

Now, $\mathcal{L}_\xi C_{\mu\nu\sigma\lambda} = 0$ and any concomitant constructed exclusively from the Weyl tensor $C_{\mu\nu\sigma\lambda}$ will satisfy a similar property. Therefore we deduce

$$\mathcal{L}_\xi((S \star S)_{\mu\nu\sigma\lambda} - S_{\mu\nu\sigma\lambda}) = 0, \qquad \mathcal{L}_\xi P_{\mu\nu} = 0, \\ \mathcal{L}_\xi(Q_{\mu\nu} + 9\alpha\rho^2\xi_\mu\xi_\nu) = 0. \tag{67}$$

Also the conditions (39a)–(39h) entail

$$((S \star S)_{\mu\nu\sigma\lambda} - S_{\mu\nu\sigma\lambda})|_{\phi(\mathcal{U})} = 0, \qquad P_{\mu\nu}|_{\phi(\mathcal{U})} = 0, \tag{68}$$

and (64) and (65) imply

$$(Q_{\mu\nu} + 9\alpha\rho^2\xi_\mu\xi_\nu)|_{\phi(\mathcal{U})} = 0. \tag{69}$$

Equations (67)–(69) can be regarded as a Cauchy initial value problem for a first order Linear system of PDE's if we take $(S \star S)_{\mu\nu\sigma\lambda} - S_{\mu\nu\sigma\lambda}$, $P_{\mu\nu}$ and $Q_{\mu\nu} + 9\alpha\rho^2\xi_\mu\xi_\nu$ as the unknowns. Therefore the uniqueness of its solutions implies

$$(S \star S)_{\mu\nu\sigma\lambda} - S_{\mu\nu\sigma\lambda} = 0, \qquad P_{\mu\nu} = 0, \qquad Q_{\mu\nu} + 9\alpha\rho^2\xi_\mu\xi_\nu = 0 \tag{70}$$

in $\mathcal{N}' \cap \mathcal{D}(\phi(\mathcal{U}))$, where $\mathcal{N}'$ is an open set containing $\phi(\mathcal{U})$. Similarly, we can construct the system

$$\mathcal{L}_\xi \rho = 0, \qquad \mathcal{L}_\xi \alpha = 0, \tag{71}$$

whose initial data, by (39g) and (39h), have the properties

$$\rho|_{\phi(\mathcal{U})} \neq 0, \qquad \alpha|_{\phi(\mathcal{U})} > 0 \tag{72}$$

The properties of the initial data clearly guarantee that the solutions of the system are such that $\rho \neq 0$ and $\alpha > 0$ in $\mathcal{N}'' \cap \mathcal{D}(\phi(\mathcal{U}))$ where $\mathcal{N}''$ is an open set with similar properties as $\mathcal{N}'$. Thus, we conclude that conditions of theorem 1 hold on $\mathcal{N} \cap \mathcal{N}' \cap \mathcal{N}'' \cap \mathcal{D}(\phi(\mathcal{U}))$ and accordingly this set is isometric to a portion of the Schwarzschild spacetime. ■

Remarks. If a vacuum initial data set $(\mathcal{S}, h_{ij}, K_{ij})$ satisfies just conditions (39a)–(39h) in some open region of $\mathcal{S}$, then it is possible in principle for the initial data set to be Schwarzschildean. One would be forced, for example, to show the existence of a timelike KID by other means. However, it is tempting to conjecture that if a vacuum initial data set satisfying (39a)–(39h) has a timelike KID, it has to be of the form given by (64) and (65).

VIII. CONCLUSIONS

In this work we have formulated necessary and sufficient conditions for initial data set to be a Schwarzschild initial data set. The set of necessary conditions and the set of sufficient conditions are not the same but they have common features. The difference between both sets of conditions is the presence of the KID equations in the set of sufficient conditions. Ideally, one would expect that the necessary conditions obtained out of the invariant characterization of [14] should imply that the KID candidate given by (62a) and (62b) actually solves KID equations. The proof of this conjecture in the general case remains unknown. Therefore we have adopted in this work the pragmatic and algorithmic perspective of requiring as part of our sufficiency conditions that the KID candidate is *actually* a timelike KID, at least for an open region $\mathcal{U}$ of the initial hypersurface. Because of the correspondence between KIDs and Killing vectors in the development, this last hypothesis implies the existence of a timelike Killing vector in the development of the region $\mathcal{U}$. This is a standard hypothesis in some characterizations of exact solutions, see e.g. [26–28].

The results presented in this work may be of use for numerical relativists working in the simulation of black hole spacetimes. In particular, one could wonder whether the formulas (62a) and (62b) which can be calculated at every time step of a numerical simulation can be used as some sort of *symmetry seeking gauge*, i.e. lapse and shift, in evolutions where the late stages are describable by the Schwarzschild solution, like in case of the head-on collision of black holes. Further, one could ask if out of the necessary conditions presented in this work it is possible to construct a measure telling how a given pair of intrinsic metric and extrinsic curvature, (h_{ij}, K_{ij}), differs from a Schwarzschild initial data. Such a construct should be of value to analyze the validity of perturbation schemes. These, and related issues will be discussed elsewhere.

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APPENDIX: PRACTICAL EXAMPLE: TIME-SYMMETRIC INITIAL DATA

In this appendix we explain how to use theorem 3 to the case of time-symmetric initial data sets. In such case one has $K_{ij} = 0$ which entails

$$E_{ij} = r_{ij}, \qquad B_{ij} = 0, \qquad P = 0, \tag{A1}$$

and the Hamiltonian constraint reduces to $r = 0$. Also we can take advantage of the results of subsection VI B 1 and specially of the expressions for the electric part written in (50). With these assumptions one can easily show that conditions (39a)–(39f) of theorem 2 hold. The only nontrivial conditions which must be checked are (39g) and (39h) and the fact that the quantities defined by (64) and (65) solve the KID conditions (60a) and (60b) and satisfy (66).

The conditions $P = 0$, $B_{ij} = 0$ imply that the shift Y^i of a KID candidate vanishes. Consequently the first KID equation (60a) is satisfied automatically while (60b) reduces to

$$D_iD_jY = YE_{ij}. \tag{A2}$$

For data with $B_{ij} = 0$ we have $\Omega = -2\gamma^2\rho$ and thus Eq. (64) becomes

$$Y = \frac{\gamma}{\rho^{4/3}}. \tag{A3}$$

Clearly condition (66) is fulfilled. Replacing (A3) in (A2) we obtain after some algebra

$$-28\gamma P_iP_j + 9E_{ij}\gamma\rho^2 + 12\rho(P_jD_i\gamma + P_iD_j\gamma) + 12\gamma\rho D_jP_i - 9\rho^2D_jD_i\gamma = 0. \tag{A4}$$

Now, if we are to prove that certain time-symmetric initial data are actually Schwarzschild initial data then we need to show that (A4) becomes an identity. To illustrate how this works in practise, we shall consider an initial data set such that the metric h_{ij} is given by

$$h_{ij}dx^idx^j = dx^2 + \frac{1}{F^2(x, y, z)}(dy^2 + dz^2), \qquad F(x, y, z) > 0, \qquad F(x, y, z) \in C^\infty(\mathcal{S}). \tag{A5}$$

The coordinates $\{x, y, z\}$ cover the whole manifold $\mathcal{S}$. We make no assumptions at this stage about the topology of $\mathcal{S}$. The coordinate x is adapted to P_i in such a way that $P_idx^i = \gamma dx$. The aim is to find for which function F the initial data $(\mathcal{S}, h_{ij}, K_{ij})$ is a time-symmetric Schwarzschild initial data set.

Following our discussion in subsection VI B 1, we set up an orthonormal frame to carry out the calculations. The elements of the orthonormal frame are defined by

$$\frac{P^i\partial_i}{\gamma} = (e_1)^i\partial_i = \partial_x, \qquad (e_2)^i\partial_i = F\partial_y, \qquad (e_3)^i\partial_i = F\partial_z. \tag{A6}$$

Note that, since the vector fields $(e_2)^i$ and $(e_3)^i$ are orthogonal to the integrable 1-form P_idx^i, they give rise to an integrable 2-dimensional distribution. The coordinates (y, z) parametrize the integral submanifolds of the distribution spanned by $(e_2)^i$ and $(e_3)^i$. Also, according to (50) any tangent vector to any of these submanifolds is an eigenvector of $E^i{}_j$ with eigenvalue ρ which means that the basis vectors $(e_2)^i$ and $(e_3)^i$ span the eigenspace associated to the repeated eigenvalue ρ. Thus from (50) we deduce

$$E_{ij} = -2\rho(e^1)_i(e^1)_j + \rho(e^2)_i(e^2)_j + \rho(e^3)_i(e^3)_j. \tag{A7}$$

The equation $P_i = D_i\rho$ implies the relations $\partial_y\rho = \partial_z\rho = 0$, $\partial_x\rho = \gamma$ to be used later. These relations mean that in our adapted coordinates both ρ and γ are functions of x only. Now, if we denote $\partial_x\rho = \rho'$, $\partial_x\gamma = \gamma'$ then using (A3) we get

$$Y = \rho^{-4/3}\gamma = \rho^{-4/3}\rho'. \tag{A8}$$

Next we show that (A4) is indeed an identity when expressed in the adapted frame constructed previously. To that end we need to work out the Ricci rotation coefficients associated to this frame and from them the Riemann tensor and the Ricci tensor. Our conventions for the Ricci rotation coefficients, $\omega^{\boldsymbol{a}}{}_{\boldsymbol{bc}}$, commutation coefficients $c^{\boldsymbol{a}}{}_{\boldsymbol{bc}}$ and Riemann tensor components $r^{\boldsymbol{a}}{}_{\boldsymbol{bcd}}$ are

$$(e_{\boldsymbol{b}})^i\nabla_i(e_{\boldsymbol{c}})^j = \omega^{\boldsymbol{d}}{}_{\boldsymbol{bc}}(e_{\boldsymbol{d}})^j, \qquad c^{\boldsymbol{a}}{}_{\boldsymbol{bc}} = \omega^{\boldsymbol{a}}{}_{\boldsymbol{bc}} - \omega^{\boldsymbol{a}}{}_{\boldsymbol{cb}}, \tag{A9}$$

$$r^{\boldsymbol{c}}{}_{\boldsymbol{fab}} = \omega^{\boldsymbol{c}}{}_{\boldsymbol{ad}}\omega^{\boldsymbol{d}}{}_{\boldsymbol{bf}} - \omega^{\boldsymbol{c}}{}_{\boldsymbol{bd}}\omega^{\boldsymbol{d}}{}_{\boldsymbol{af}} + \Delta_{\boldsymbol{a}}\omega^{\boldsymbol{c}}{}_{\boldsymbol{bf}} - \Delta_{\boldsymbol{b}}\omega^{\boldsymbol{c}}{}_{\boldsymbol{af}} - \omega^{\boldsymbol{c}}{}_{\boldsymbol{df}}c^{\boldsymbol{d}}{}_{\boldsymbol{ab}}, \tag{A10}$$

where $\Delta_{\boldsymbol{a}}$ is the directional derivative associated to the vector field $(e_{\boldsymbol{a}})^i$. In addition, we have the metric condition which in an orthonormal frame takes the form

$$\omega^{\boldsymbol{a}}{}_{\boldsymbol{bc}} - \omega^{\boldsymbol{c}}{}_{\boldsymbol{ba}} = 0. \tag{A11}$$

Evaluating the commutators of the directional derivatives on the coordinates one finds that the only nonvanishing Ricci rotation coefficients are

$$\omega^1{}_{33} = \omega^1{}_{22} = \frac{\partial_xF}{F}, \qquad \omega^2{}_{33} = \partial_yF, \qquad \omega^2{}_{23} = -\partial_zF. \tag{A12}$$

With this information we can work out the components of the Ricci tensor in our orthonormal frame. Using the relation $r_{ij} = E_{ij}$ and (A7) we get

$$-\frac{2(\partial_xF)^2}{F^2} + \frac{\partial_x^2F}{F} = -\rho, \qquad -\frac{\partial_yF\partial_xF}{F} + \partial_{xy}^2F = 0, \qquad -\frac{\partial_zF\partial_xF}{F} + \partial_{xz}^2F = 0, \tag{A13}$$

$$-(\partial_zF)^2 + F\partial_z^2F - (\partial_yF)^2 + F\partial_y^2F - \frac{3(\partial_xF)^2}{F^2} + \frac{\partial_x^2F}{F} = \rho, \tag{A14}$$

and the Hamiltonian constraint $r = 0$ becomes

$$-(\partial_yF)^2 - (\partial_zF)^2 + F(\partial_y^2F + \partial_z^2F) + \frac{2\partial_x^2F}{F} - \frac{5(\partial_xF)^2}{F^2} = 0. \tag{A15}$$

Later we will show how (A13)–(A15) can be solved. Before doing that let us prove that (A4) is an identity for our initial data. In our adapted frame (A4) takes the form

$$(e^2)_i(e^2)_j\Big(9\gamma\rho^3 - \frac{12\gamma^2\rho\partial_x F}{F} + \frac{9\alpha'\rho^2\partial_x F}{F}\Big) + (e^3)_i(e^3)_j\Big(9\gamma\rho^3 - \frac{12\gamma^2\rho\partial_x F}{F} + \frac{9\alpha'\rho^2\partial_x F}{F}\Big) + (e^1)_i(e^1)_j(-28\gamma^3 - 18\gamma\rho^3 + 36\rho\gamma\gamma' - 9\rho^2\gamma'') = 0. \tag{A16}$$

We must show that all the expressions in brackets are zero if (A13) and (A14) hold. Our strategy will be to find relations for γ, γ', γ'' and then plug these relations into (A16). All these derivatives can be calculated by differentiating (A13) and (A14) and using $\rho' = \gamma$ but in order to avoid long and messy calculations we will follow an alternative procedure. We start with the second Bianchi identity which in our case takes the simpler form

$$D^i E_{ij} = 0. \tag{A17}$$

Replacing the electric part by its expression in terms of the adapted frame, Eq. (A7), we find

$$2\rho' - 3\rho({\omega^1}_{22} + {\omega^1}_{33}) = 0, \tag{A18}$$

from which, replacing the rotation coefficients

$$\rho' = \frac{3\rho\partial_x F}{F} = \gamma. \tag{A19}$$

These equations enable us to calculate γ' and γ'' by just differentiating. Note that in Eq. (A16) there are no partial derivatives of F of order greater than one. Therefore, in order to keep our expressions for γ' and γ'' with partial derivatives of F of at most degree 1, we use in each step of the differentiation the first equation of (A13) to replace $\partial_x^2 F$. The final expressions for γ' and γ'' are

$$\gamma' = 3\rho\Big(-\rho + \frac{4(\partial_x F)^2}{F^2}\Big), \tag{A20a}$$

$$\gamma'' = \frac{6\rho(7\rho F^2 - 10(\partial_x F)^2)\partial_x F}{F^3}. \tag{A20b}$$

Putting back the expressions just found for γ, γ' and γ'' in the left-hand side of (A16) we can check explicitly that it vanishes identically. Thus, the initial data set under consideration possesses a KID with the required properties.

To complete our study of the initial data set (A5) we need to solve the partial differential equations (A13)–(A15) or at least show that they have nontrivial solutions. The conditions

$$-\frac{\partial_y F\partial_x F}{F} + \partial_{xy}^2 F = 0, \qquad -\frac{\partial_z F\partial_x F}{F} + \partial_{xz}^2 F = 0, \tag{A21}$$

entail $F(x, y, z) = \Phi(x)G(y, z)$. Inserting this in (A15) we obtain

$$\frac{5\Phi'^2}{\Phi^4} - \frac{2\Phi''}{\Phi^3} = -(\partial_y G)^2 - (\partial_z G)^2 + G(\partial_y^2 G + \partial_z^2 G). \tag{A22}$$

Note that this same equation can be obtained if we eliminate ρ in the first of (A13) and (A14) and insert $F(x, y, z) = \Phi(x)G(y, z)$ in the resulting expression. We have reduced the problem to solving an ordinary differential equation and a partial differential equation

$$\frac{5\Phi'^2}{\Phi^4} - \frac{2\Phi''}{\Phi^3} = k, \qquad -(\partial_y G)^2 - (\partial_z G)^2 + G(\partial_y^2 G + \partial_z^2 G) = k, \tag{A23}$$

where k is a constant. The general solution of these differential equations remains unknown but we can check that the standard time-symmetric slice of Schwarzschild spacetime is among its solutions. These initial data are given in standard spherical coordinates by the expression

$$h_{ij}dx^i dx^j = \Big(1 + \frac{m}{2r}\Big)^4(dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\varphi^2), \tag{A24}$$

We concentrate in a region $\mathcal{U} \subset \mathcal{S}$ defined by $r > m/2$. The topology of $\mathcal{U}$ is $\mathbb{R}^3 \setminus O$ where O represents an open ball. The development of this initial data is one of the static regions of Schwarzschild spacetime. The proof of this statement can now be obtained as a consequence of theorem 3 if we show that (A24) solves (A23). To that end we need to write the latter in the same coordinate system as the former. This is achieved through the coordinate change

$$x = r + m\log r - \frac{m^2}{4r}, \tag{A25}$$

$$y = 2\cot(\theta/2)\cos\varphi, \qquad z = 2\cot(\theta/2)\sin\varphi. \tag{A26}$$

The resulting differential equations are

$$(-5r(m + 2r)\Phi'^2 + 4m\Phi\Phi' + 2r(m + 2r)\Phi\Phi'') = -k\frac{(m + 2r)^5\Phi^4}{16r^3}, \tag{A27}$$

$$\tfrac{1}{4}\tan^2(\theta/2)((\partial_\phi G)^2 - G(\partial_\phi^2 G) - G\sin\theta\cos\theta(\partial_\theta G) - \sin\theta(\partial_\theta G)^2 + G(\partial_\theta^2 G)\sin\theta) = -k, \tag{A28}$$

where $\Phi = \Phi(r)$, $G = G(\theta, \phi)$ and a prime means derivative with respect to r. The choice

$$\Phi = \frac{1}{r(1+\frac{m}{2r})^2}, \qquad G = \frac{1}{\sin^2(\theta/2)}, \tag{A29}$$

brings (A5) into (A24) and solves (A27) and (A28) for $k = -1$. Also this choice entails

$$\rho = \frac{64mr^3}{(m+2r)^6} \Rightarrow \gamma = \frac{768m(m-2r)r^4}{(m+2r)^9}, \tag{A30}$$

$$Y = \frac{3(m-2r)}{m^{1/3}(m+2r)} < 0, \quad \text{if } r > m/2, \tag{A31}$$

which means that (39g) and (39h) hold.

[1] R. Beig and N. O. Murchadha, Phys. Rev. D **57**, 4728 (1998).
[2] D. Brill, J. M. Cavallo, and J. A. Isenberg, J. Math. Phys. (N.Y.) **21**, 2789 (1980).
[3] N. O'Murchadha and E. Malec, Phys. Rev. D **68**, 124019 (2003).
[4] M. J. Pareja and J. Frauendiener, Phys. Rev. D **74**, 044026 (2006).
[5] B. L. Reinhart, J. Math. Phys. (N.Y.) **14**, 719 (1973).
[6] M. A. Scheel, T. W. Baumgarte, G. B. Cook, S. L. Shapiro, and S. A. Teukolsky, Phys. Rev. D **58**, 044020 (1998).
[7] B. G. Schmidt, in *The Conformal Structure of Space-Time. Geometry, Analysis, Numerics*, edited by J. Frauendiener and H. Friedrich (Springer, New York, 2002).
[8] J. A. Valiente Kroon, Classical Quantum Gravity **21**, 3237 (2004).
[9] J. A. Valiente Kroon, Phys. Rev. D **72**, 084003 (2005).
[10] M. D. Kruskal, Phys. Rev. **119**, 1743 (1960).
[11] Y. Choquet-Bruhat and R. Geroch, Commun. Math. Phys. **14**, 329 (1969).
[12] N. O'Murchadha and K. Roszkowski, Classical Quantum Gravity **23**, 539 (2006).
[13] V. D. Zakharov, *Gravitational Waves in Einstein's Theory of Gravitation* (Nauka, Moscow, 1972).
[14] J. J. Ferrando and J. A. Sáez, Classical Quantum Gravity **15**, 1323 (1998).
[15] J. M. Martín-García, see http://metric.iem.csic.es/Martin-Garcia/xAct/.
[16] These definitions of stem from the theory of 2-forms. The natural metric in the space of forms is given by the rank 4 tensor $\frac{1}{8}(g \wedge g)_{\mu\nu\lambda\psi}$. Thus, in terms of this tensor, one has that

$$(U \star V)_{\mu\nu\lambda\psi} = \tfrac{1}{8}(g \wedge g)_{\kappa\pi\sigma\tau} U_{\mu\nu}{}^{\kappa\pi} V_{\lambda\psi}{}^{\sigma\tau} = \tfrac{1}{2} U_{\mu\nu\kappa\pi} V^{\kappa\pi}{}_{\lambda\psi}$$

Similarly, one takes traces according to

$$\mathrm{tr}\, U = \tfrac{1}{8}(g \wedge g)_{\mu\nu\lambda\psi} U^{\mu\nu\lambda\psi} = \tfrac{1}{2} U^{\mu\nu}{}_{\mu\nu}.$$

[17] J. Ehlers and W. Kundt, in *Gravitation: an Introduction to Current Research*, edited by L. Witten (Wiley, New York, 1962).
[18] T. Klösch and T. Strobl, Classical Quantum Gravity **13**, 1191 (1996).
[19] H. Friedrich, Classical Quantum Gravity **13**, 1451 (1996).
[20] C. B. G. McIntosh, R. Arianrhod, S. T. Wade, and C. Hoenselaers, Classical Quantum Gravity **11**, 1555 (1994).
[21] E. Zakhary and C. B. G. McIntosh, Gen. Relativ. Gravit. **29**, 539 (1997).
[22] S. Brian Edgar and A Höglund, J. Math. Phys. (N.Y.) **43**, 659 (2002).
[23] R. Beig and P. T. Chruściel, Classical Quantum Gravity **14**, A83 (1997).
[24] B. Coll, J. Math. Phys. (N.Y.) **18**, 1918 (1977).
[25] V. Moncrief, J. Math. Phys. (N.Y.) **16**, 493 (1975).
[26] M. Mars, Classical Quantum Gravity **16**, 2507 (1999).
[27] M. Mars, Classical Quantum Gravity **17**, 3353 (2000).
[28] W. Simon, Gen. Relativ. Gravit. **16**, 465 (1984).