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Slowly rotating fluid balls of Petrov type D

Michael Bradley,[1,*] Daniel Eriksson,[1,†] Gyula Fodor,[2,‡] and István Rácz[2,§]

[1]*Department of Physics, Umeå University, SE-901 87 Umeå, Sweden*

[2]*KFKI Research Institute for Particle and Nuclear Physics, H-1525, Budapest 114, P.O.B. 49, Hungary*

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The second order perturbative field equations for slowly and rigidly rotating perfect fluid balls of Petrov type D are solved numerically. It is found that all the slowly and rigidly rotating perfect fluid balls up to second order, irrespective of Petrov type, may be matched to a possibly nonasymptotically flat stationary axisymmetric vacuum exterior. The Petrov type D interior solutions are characterized by five integration constants, corresponding to density and pressure of the zeroth order configuration, the magnitude of the vorticity, one more second order constant, and an independent spherically symmetric second order small perturbation of the central pressure. A four-dimensional subspace of this five-dimensional parameter space is identified for which the solutions can be matched to an asymptotically flat exterior vacuum region. Hence these solutions are completely determined by the spherical configuration and the magnitude of the vorticity. The physical properties, like equation of state, shape, and speed of sound, are determined for a number of solutions.

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I. INTRODUCTION

In [1] a second order formalism for slowly and rigidly rotating stars was developed by Hartle. This formalism was applied in [2] to rotating white dwarfs and neutron stars using the Harrison-Wheeler and Tsuruta Cameron V_γ equations of state, and in [3] to the case with constant energy density. In [4] the second order formalism is compared with numerical solutions of the full Einstein equations. For a review of relativistic rotating stars see [5]. In [6] global models for slowly rotating bodies in the post-Minkowskian approximation are treated. In a recent paper [7] second order perturbation theory for the matching of general stationary axisymmetric bodies to an asymptotically flat vacuum has been put on a more solid mathematical ground and the exterior metric is determined to second order.

In this paper we use the Hartle formalism to study perfect fluids of Petrov type D. This condition will be used instead of an equation of state. It was shown in [8] that physically realistic rotating fluid balls cannot be of Petrov types II, III, N, or 0, so the only possible cases are of Petrov types D or I. Hence, it is of interest to closer study the properties of Petrov type D solutions, being the only possible algebraically special solutions. Also, since in the nonrotating spherically symmetric case all interior solutions are of Petrov type D or 0, one might hope to find physically interesting interior solutions of Petrov type D also in the axisymmetric case, at least for slow rotation. However, the quadrupole moment of the rotating configuration will typically deviate from that given by the Kerr metric and hence its exterior metric cannot be Kerr [4]. It is easily verified that such an exterior metric is not of Petrov type D.

The field equations to second order in the small rotational parameter Ω will be solved numerically using fourth order Runge-Kutta. The system reduces to a closed subsystem of six first order differential equations. There are also two more differential equations for two further dependent variables which do not appear in this closed subsystem. Assuming regularity at the center, the solutions of this closed subsystem depend on four constants of integration, corresponding to zeroth order central density and pressure, the magnitude of the angular velocity, and one more second order small constant. Because of scaling invariances we need only consider a two-dimensional subspace of the solution space. The solutions are then matched to a second order axisymmetric vacuum solution using the Darmois-Israel procedure [9,10]. This metric includes the general second order asymptotically flat stationary axisymmetric vacuum solution as a special case. The interior solutions that can be matched to this vacuum form a three-dimensional subspace of the space of solutions. One more freely specifiable parameter, associated with an independent spherically symmetric second order small change of the central pressure, is obtained from the solution of the two remaining equations. Hence the rotating configuration for the asymptotically flat subclass is determined by the spherically symmetric configuration (including a possible second order change of the central pressure) and the magnitude of the angular velocity.

The paper is organized as follows: In Sec. II the method is briefly described and the field equations are presented, along with the Petrov type D condition. Finally, the second order vacuum metric is given. The matching procedure is described in Sec. III and the integration constants for the vacuum solution are solved for in terms of the values of the

*Electronic address: michael.bradley@physics.umu.se
†Electronic address: daniel.eriksson@physics.umu.se
‡Electronic address: gfodor@rmki.kfki.hu
§Electronic address: iracz@sunserv.kfki.hu

interior solution on the matching surface. In Sec. IV the equations are rewritten in a form suitable for numerical integration. The results of the numerical runs are given in Sec. V. First the program is checked against the exact Wahlquist solution, and then the subset for which the solutions are asymptotically flat is determined. Properties like shape, equation of state, and speed of sound are then determined for a number of solutions.

II. PRELIMINARIES

To second order the metric of a slowly rotating axisymmetric object, both in the interior fluid region and the outside vacuum region, can be written as

$$ds^2 = (1+2h)A^2dt^2 - (1+2m)\frac{1}{B^2}dr^2 - (1+2k)r^2[d\theta^2 + \sin^2\theta(d\varphi - \omega dt)^2], \quad (1)$$

where ω is first order and h, m, and k are second order in the rotational parameter [1]. The requirements of regularity at the center and asymptotic flatness imply that the first order function ω depends on r only. The second order functions h, m, k can be given as

$$h = h_0 + h_2P_2(\cos\theta), \qquad m = m_0 + m_2P_2(\cos\theta), \qquad k = k_2P_2(\cos\theta), \quad (2)$$

where h_0, m_0 and h_2, m_2, k_2 are functions of r only, and $P_2(x) = \frac{1}{2}(3x^2 - 1)$ is the second order Legendre polynomial. This result follows from reflection symmetry in the equatorial plane, from the fact that the equations for h, m, and k separate with the ansätze $h = \sum_{i=0}^{\infty} h_i(r)P_i(\cos\theta)$ etc., and from the fact that there are no inhomogeneous terms containing ω in the equations for h_i, k_i, and m_i for $i > 2$. For more details see [1].

The matching of the two spacetime regions happens via the application of a coordinate transformation $\varphi \to \varphi + \Omega t$ in the fluid region. In addition to this, we can also rescale the interior time coordinate first by a constant c_4 while matching the spherical zeroth order solutions, and then later by a second order small constant when doing the matching of the corresponding rotating spacetimes. These then yield the coordinate transformation $t \to c_4(1 + c_3)t$. The first of these coordinate transformations says that the inner fluid region rotates with respect to the distant stationary observers with angular velocity Ω. This parameter Ω is considered to be the small expansion parameter with respect to which ω is first order and the other corrections h, m, k are second order.

The matter content of the interior is modeled by a perfect fluid,

$$T_{ab} = (\rho + p)u_au_b - pg_{ab}.$$

The coordinate system used in (1) is assumed to be comoving with the fluid, i.e. the 4-velocity is assumed to possess the form

$$u^a = (1/\sqrt{g_{00}}, 0, 0, 0) = ((1-h)/A, 0, 0, 0)$$

which also implies that the shear of the fluid is zero so it rotates rigidly.

A. The field equations

In this subsection we list the field equations relevant to various orders. A similar system of equations with a slightly different choice of variables was given in [11]. If no equation of state is specified then the only equation one gets to zeroth order of the rotational parameter is the pressure isotropy condition $G^1{}_1 = G^2{}_2$ which reads

$$B\frac{d^2A}{dr^2} + \frac{d(rA)}{dr}\frac{d(B/r)}{dr} + \frac{A}{r^2B} = 0. \quad (3)$$

Making use of $G^0{}_0 = T^0{}_0$ and $G^1{}_1 = T^1{}_1$, the energy density and pressure of the nonrotating configuration reads

$$\rho_0 = \frac{1}{r^2}\left[1 - \frac{d(rB^2)}{dr}\right], \quad (4)$$

$$p_0 = \frac{1}{r^2}\left[\frac{B^2}{A^2}\frac{d(rA^2)}{dr} - 1\right]. \quad (5)$$

To first order in the rotation parameter, the only relation follows from $G^3{}_0 = 0$,

$$\frac{d}{dr}\left(r^4\frac{B}{A}\frac{d\omega}{dr}\right) + 4r^3\omega\frac{d}{dr}\left(\frac{B}{A}\right) = 0. \quad (6)$$

The second order Einstein equations yield the following four conditions. From $G^1{}_2 = 0$, one gets

$$r\frac{d}{dr}(h_2 + k_2) + r(h_2 - m_2)\frac{1}{A}\frac{dA}{dr} - h_2 - m_2 = 0. \quad (7)$$

The pressure isotropy condition in the angular directions, $G^2{}_2 = G^3{}_3$, gives

$$6(h_2 + m_2) - r^4\frac{B^2}{A^2}\left(\frac{d\omega}{dr}\right)^2 + 4r^3\omega^2\frac{B}{A}\frac{d}{dr}\left(\frac{B}{A}\right) = 0. \quad (8)$$

The equality of the pressure in the angular and radial directions, i.e. $G^1{}_1 = G^2{}_2$, gives two equations. After eliminating the derivative of h_2 using (7), one obtains from the $P_2(\cos\theta)$ part

$$2r\frac{B^2}{A}\frac{dA}{dr}\left(r\frac{dk_2}{dr} - m_2\right) - 2r^2Bh_2\frac{d}{dr}\left(\frac{B}{r}\right) + m_2 - 4k_2 - 5h_2 - \frac{1}{3}r^4\frac{B^2}{A^2}\left(\frac{d\omega}{dr}\right)^2 = 0, \quad (9)$$

while the θ-independent part takes the form

$$6r^3B\frac{d}{dr}\left(\frac{1}{r}A^2B\frac{dh_0}{dr}\right) - 3B^2\frac{d(r^2A^2)}{dr}\frac{dm_0}{dr} + 12A^2m_0 - 3r^4B^2\left(\frac{d\omega}{dr}\right)^2 + 4r^3\omega^2AB\frac{d}{dr}\left(\frac{B}{A}\right) = 0. \quad (10)$$

The energy density function can be decomposed as $\rho = \rho_0 + \rho_2$, where $\rho_2 = \rho_{20} + \rho_{22}P_2(\cos\theta)$ and ρ_{20} and ρ_{22} are second order small functions of the coordinate r given as

$$\rho_{20} = \frac{B}{6r^2A}\left[8r^3\omega^2\frac{d}{dr}\left(\frac{B}{A}\right) + 12\frac{A}{B}\frac{d}{dr}(rB^2m_0) - r^4\frac{B}{A}\left(\frac{d\omega}{dr}\right)^2\right] \quad (11)$$

and

$$\rho_{22} = -\frac{2(3A^2h_2 + r^2\omega^2)}{3r^3A\frac{dA}{dr}}\left[1 - B^2 + r^2\frac{d}{dr}\left(B\frac{dB}{dr}\right)\right]. \quad (12)$$

The analogous decomposition of the pressure is defined by $p = p_0 + p_2 = p_0 + p_{20} + p_{22}P_2(\cos\theta)$, where

$$p_{20} = \frac{B^2}{6r^2A^2}\left[12rA^2\frac{dh_0}{dr} - 12m_0\frac{d}{dr}(rA^2) + r^4\left(\frac{d\omega}{dr}\right)^2\right] \quad (13)$$

and

$$p_{22} = \frac{2B}{3rA}(3A^2h_2 + r^2\omega^2)\frac{d}{dr}\left(\frac{B}{A}\right). \quad (14)$$

The existence of a barotropic equation of state $\rho = \rho(p)$ is equivalent to

$$\frac{\partial\rho}{\partial\theta}\frac{\partial p}{\partial r} - \frac{\partial p}{\partial\theta}\frac{\partial\rho}{\partial r} = 0, \quad (15)$$

which is a geometric condition ensuring the coincidence of the constant pressure and density surfaces. Substituting the decompositions $\rho = \rho_0 + \rho_{20} + \rho_{22}P_2(\cos\theta)$ and $p = p_0 + p_{20} + p_{22}P_2(\cos\theta)$ into this relation yields, up to second order,

$$\rho_{22}\frac{dp_0}{dr} = p_{22}\frac{d\rho_0}{dr}, \quad (16)$$

which is identically satisfied in virtue of the above field equations.

It seems to be plausible to require the equation of state to be independent of the angular velocity. This condition reads

$$\frac{\rho_2}{p_2} = \frac{d\rho_0}{dp_0}. \quad (17)$$

The θ-dependent part of this relation is equivalent to (16), while the spherically symmetric part gives the relation

$$\rho_{20}\frac{\partial p_0}{\partial r} = p_{20}\frac{\partial\rho_0}{\partial r}. \quad (18)$$

Then, by the substitution of the expressions ρ_{20} and p_{20}, given by (11) and (13), together with ρ_0 and p_0 for the zeroth order pressure and density, given by (5), one gets

$$24rA^4\left[r^2\frac{d^2B^2}{dr^2} + 2(1 - B^2)\right]\frac{dh_0}{dr} + 24rAm_0\frac{dA}{dr}\left[4A^2 \times (B^2 - 1) - 4r^2AB\frac{dA}{dr}\frac{dB}{dr} - 4r^2A^2B\frac{d^2B}{dr^2}\right] + 8r^3A^5\frac{dA}{dr}\frac{d}{dr}\left(\frac{B^2}{A^2}\right)\left[3\frac{dm_0}{dr} + \frac{r^2\omega^2}{B^2}\frac{d}{dr}\left(\frac{B^2}{A^2}\right)\right] + A^2\left[2r^2\frac{d^2B^2}{dr^2} - 2r^2A\frac{dA}{dr}\frac{d}{dr}\left(\frac{B^2}{A^2}\right) + 4(1 - B^2)\right] \times\left[r^4\left(\frac{d\omega}{dr}\right)^2 - 12A^2m_0\right] = 0. \quad (19)$$

B. The Petrov type of slowly rotating fluids

The spherically symmetric field equation (3) is usually complemented by a choice of an equation of state for the fluid. Since spherically symmetric static spacetimes are always algebraically special, we do not assume any special equation of state for the nonrotating base solution. As we will see shortly, the deviation from algebraically special cases can arise first when considering the second order terms in the rotational parameter. Here we require the interior solution to remain Petrov type D for slow rotation, thereby completing the system of field equations (3), (6)–(10), and (19) by a further condition, which in some sense plays the role of an equation of state.

In order to calculate the Petrov type, we need a suitable null tetrad. Up to second order, an orthonormal tetrad can be given as

$$e_0^\mu = \left(\frac{1}{A}\left(1 + \frac{\omega^2r^2}{2A^2}\sin^2\theta - h\right), 0, 0, 0\right),$$

$$e_1^\mu = (0, B(1 - m), 0, 0), \qquad e_2^\mu = \left(0, 0, \frac{1}{r}(1 - k), 0\right),$$

$$e_3^\mu = \left(\frac{r\omega}{A^2}\sin\theta, 0, 0, \frac{1}{r\sin\theta}\left(-1 + \frac{\omega^2r^2}{2A^2}\sin^2\theta + k\right)\right).$$

From this we form the null tetrad by the relations

$$\sqrt{2}l^\mu = e_0^\mu + e_3^\mu, \qquad \sqrt{2}k^\mu = e_0^\mu - e_3^\mu,$$
$$\sqrt{2}m^\mu = e_1^\mu + ie_2^\mu.$$

Then the components of the Weyl spinor are given as

$$\Psi_0 = C_{abcd}k^am^bk^cm^d, \qquad \Psi_3 = C_{abcd}k^al^b\bar{m}^cl^d,$$
$$\Psi_1 = C_{abcd}k^al^bk^cm^d, \qquad \Psi_4 = C_{abcd}\bar{m}^al^b\bar{m}^cl^d,$$
$$\Psi_2 = C_{abcd}k^am^b\bar{m}^cl^d.$$

Since $\Psi_1 = 0$ and $\Psi_3 = 0$ (even in the case of fast rotation), the Petrov type is determined by the multiplicities of the roots of the algebraic equation for the complex number a,

$$\Psi_0 + 6\Psi_2a^2 + \Psi_4a^4 = 0. \quad (20)$$

The Petrov type is D if there are two double roots, i.e.

$$9\Psi_2^2 = \Psi_0\Psi_4. \tag{21}$$

We note that the Petrov type can also be D if $\Psi_0 = \Psi_4 = 0$ and $\Psi_2 \neq 0$ but, since then the equation of state can be shown to be $\rho = -p$ (see [8]), we only deal here with the more general case (21). Considering the other possible algebraically special types, the Petrov type cannot be III because of $\Psi_0 = \Psi_4 = 0$, and the Petrov II and N cases also have the nonphysical equation of state $\rho = -p$. Finally, due to a theorem by Collinson [12], the conformally flat case is also excluded.

Hence, the only algebraic special solutions of physical interest one might hope to find are of Petrov type D. Up to second order in the rotational parameter, Eq. (21) gives only one real condition. By substitution of the zeroth and first order field equations (3) and (6) into (21), the *Petrov type D* condition gives the relation [11]

$$\left(rB\frac{dB}{dr} + 1 - B^2\right)(h_2 - m_2) = \frac{r^4A^2}{6}\left[\frac{d}{dr}\left(\frac{B^2\omega}{A^2}\right)\right]^2. \tag{22}$$

Note that m_0 and h_0 do not appear in Eqs. (3), (6)–(9), and (22) and hence this subsystem for A, B, ω, m_2, k_2, and h_2 decouples. Notice that these equations contain m_2 only algebraically. In Sec. V the system will be reformulated as a coupled system of six first order ordinary differential equations. Because of the requirement of a regular center, the solutions to this subsystem will only depend on four constants of integration.

C. Vacuum metric

In the exterior vacuum region, we will use a frame adapted to the asymptotically nonrotating observer. Solving the field equations detailed in Sec. II A by imposing $p = \rho = 0$, the metric functions for the vacuum region are given as follows [1,13]:

$$A^2 = B^2 = 1 - 2M/r, \tag{23}$$

$$\omega = \frac{2aM}{r^3}, \tag{24}$$

$$\begin{gathered} h_0 = \frac{1}{r-2M}\left(\frac{a^2M^2}{r^3} + \frac{r}{2M}c_2\right), \qquad m_0 = \frac{1}{2M-r}\left(\frac{a^2M^2}{r^3} + c_2\right), \\ h_2 = 3c_1 r(2M-r)\log\left(1-\frac{2M}{r}\right) + a^2\frac{M}{r^4}(M+r) + 2c_1\frac{M}{r}(3r^2-6Mr-2M^2)\frac{r-M}{2M-r} + \left(1-\frac{2M}{r}\right)r^2 q_1, \\ k_2 = 3c_1(r^2-2M^2)\log\left(1-\frac{2M}{r}\right) - a^2\frac{M}{r^4}(2M+r) - 2c_1\frac{M}{r}(2M^2-3Mr-3r^2) + (2M^2-r^2)q_1, \\ m_2 = 6a^2\frac{M^2}{r^4} - h_2. \end{gathered} \tag{25}$$

In this approximation, the slowly rotating solution is characterized by the mass M, the first order small rotation parameter a, and the second order small constants c_1, c_2, and q_1. When q_1 takes the value zero, the metric is known to be the general asymptotically flat stationary and axisymmetric vacuum metric to second order (see e.g. [14]). It can be checked by plugging the vacuum quantities into the Petrov type D condition (22) that the solution is of Petrov type D only if both c_1 and q_1 are zero. The metric is then equivalent to the Kerr metric to second order with mass $M \to M - c_2$.

When $q_1 \neq 0$ the metric cannot be asymptotically flat. It is important to keep in mind, however, that without the inclusion of this constant the matching conditions on the zero pressure surface are overdetermined in general [13,15].

III. MATCHING

The matching of the fluid ball to a suitable exterior vacuum region happens at the zero pressure surface. Before matching these two spacetime regions, it is informative to investigate first the structure of the constant pressure surfaces.

A. The constant pressure surfaces

The pressure in the rotating fluid configuration is given by the function $p(r, \theta) = p_0(r) + p_2(r, \theta)$. The surfaces of constant pressure, $\mathcal{S}_{\bar{r}}$, may be labeled by the function $\bar{r}$ defined by the relation

$$p(r, \theta) = \bar{p}_0(\bar{r}) \equiv p_0(\bar{r}) + \delta p(\bar{r}), \tag{26}$$

where p_0 is the corresponding pressure for the nonrotating configuration and $\delta p(\bar{r})$ is a second order small shift of the pressure that changes monotonously from the center, where it takes the value $p_{20}(0)$, to the zero pressure surface, where it becomes zero. The value of the central pressure to second order follows by assuming regularity at the center and by making use of the field equations (10) and (19), along with the relation for p_{20} (13) (see Sec. IV B). It turns out that it will depend, among others, on one freely specifiable constant corresponding to a spherically symmetric perturbation that produces a second order small change of the central pressure (and density). If we choose to consider only rotational perturbations with $p_{20}(0) = 0$, $\delta p(\bar{r})$ may be chosen to be identically zero.

The radial displacement ξ is defined by

$$r = \bar{r} + \xi.$$

To second order one has

$$\begin{aligned} p(r,\theta) &= p_0(r) + p_2(r,\theta) \\ &= p_0(r) + p_{20}(r) + p_{22}(r)P_2(\cos\theta) \\ &= p_0(\bar{r}) + \xi\frac{dp_0}{dr}\bigg|_{\bar{r}} + p_{20}(\bar{r}) + p_{22}(\bar{r})P_2(\cos\theta) \\ &\equiv p_0(\bar{r}) + \delta p(\bar{r}), \end{aligned}$$

implying that ξ possesses the form $\xi = \xi_0 + \xi_2 P_2(\cos\theta)$, where ξ_0 and ξ_2 are given as

$$\xi_0 = -[p_{20}(\bar{r}) - \delta p(\bar{r})]/(dp_0/dr|_{\bar{r}})$$

and

$$\xi_2 = -p_{22}(\bar{r})/(dp_0/dr|_{\bar{r}}). \tag{27}$$

Note that there will be a certain arbitrariness in ξ_0, the average shift of the radius, unless $\delta p(\bar{r})$ is specified. At the origin $\xi_0 = 0$ and on the zero pressure surface r_1, the expressions (5) and (13) together with $\delta p(r_1) = 0$ give

$$\xi_0 = \frac{1}{12rB\frac{dA}{dr}\frac{d}{dr}(\frac{A}{B})}\bigg[12rA^2\frac{dh_0}{dr} - 12m_0\frac{d}{dr}(rA^2) + r^4\bigg(\frac{d\omega}{dr}\bigg)^2\bigg]\bigg|_{r=r_1} \tag{28}$$

for ξ_0. If we choose $\delta p(\bar{r}) \equiv 0$ this expression, with r_1 substituted with $\bar{r}$, holds for any $\bar{r}$ in the interval $[0, r_1]$. From (5) and (14) $\xi_2(\bar{r})$ is given by

$$\xi_2 = -\frac{(3A^2h_2 + r^2\omega^2)}{3A\frac{dA}{dr}}\bigg|_{r=\bar{r}}. \tag{29}$$

The circumference of the intersection of a constant pressure surface $\mathcal{S}_{\bar{r}}$ and the equatorial plane $\theta = \pi/2$, which is in fact a circle, is obtained from

$$\begin{aligned} dl^2 &= (1+2k)r^2\sin^2\theta d\varphi^2 \\ &= (1+2k_0-k_2)\bigg(\bar{r}+\xi_0-\frac{\xi_2}{2}\bigg)^2 d\varphi^2, \end{aligned}$$

giving

$$l_1 = 2\pi\bar{r}\bigg(1 + k_0 + \frac{\xi_0}{\bar{r}} - \frac{k_2}{2} - \frac{\xi_2}{2\bar{r}}\bigg).$$

The length of the curve γ, yielded by the intersection of $\mathcal{S}_{\bar{r}}$ and a plane including the axis of rotational symmetry, is given as

$$l_2 = 2\pi\bar{r}\bigg(1 + k_0 + \frac{\xi_0}{\bar{r}} + \frac{k_2}{4} + \frac{\xi_2}{4\bar{r}}\bigg),$$

where we have used the relation

$$\begin{aligned} dl^2 &= (1+2k)r^2d\theta^2 \\ &= (1+2k_0+2k_2P_2(\cos\theta)) \\ &\quad\times(\bar{r}+\xi_0+\xi_2P_2(\cos\theta))^2d\theta^2, \end{aligned}$$

along with the fact that the term obtained by substituting $dr = -3\xi_2\cos\theta\sin\theta d\theta$ into the line element (1) is of fourth order and is hence dropped. The constant pressure surfaces are oblate iff $l_1 > l_2$, i.e. whenever

$$k_2 + \frac{\xi_2}{\bar{r}} < 0. \tag{30}$$

Another way of determining the oblateness of the constant pressure surfaces is possible by comparing the radial distance from the origin to the curve γ and the analogous distance to second order in the eccentricity parameter ϵ, $r = a(1 - \frac{1}{2}\epsilon^2\cos^2\theta)$, for an ellipse in $\mathbb{R}^2$ with semimajor axis a. The curve γ is then found to be an ellipse up to second order, with ϵ^2 given as

$$\epsilon^2 = -3\bigg[\frac{\xi_2}{B} + \int_0^{\bar{r}}\frac{m_2}{B}d\bar{r}\bigg]\bigg/\int_0^{\bar{r}}\frac{d\bar{r}}{B}. \tag{31}$$

For infinitesimally small values of $\bar{r}$, (31) reduces to

$$\epsilon^2 = -3\bigg(m_2 + \frac{\xi_2}{\bar{r}}\bigg),$$

according to which the constant pressure surfaces are oblate iff

$$m_2 + \frac{\xi_2}{\bar{r}} < 0. \tag{32}$$

Notice that this inequality, along with the relations (61) and (62) in Sec. IV B, also justifies that the two different characterizations of oblateness are compatible.

Note, finally, that $d\bar{r}$ is a form field orthogonal to the constant pressure surfaces $\mathcal{S}_{\bar{r}}$. Up to second order, the corresponding normalized field is

$$n_a = (0, (1+m)/B, 3\xi_2\sin\theta\cos\theta/B, 0). \tag{33}$$

B. The matching

In this section we match the interior rotating fluid solution to an exterior vacuum region at the zero pressure surface $\mathcal{S}_{\bar{r}=r_1}$.

In the vacuum exterior region, suitable hypersurfaces for matching are determined by the condition [16]

$$\tilde{\Omega}^2 g_{\varphi\varphi} + 2\tilde{\Omega}g_{\varphi t} + g_{tt} = 1 - \tilde{C}, \tag{34}$$

where $\tilde{\Omega}$ and $\tilde{C}$ are constants. To second order such a surface can be given as

$$r = r_1 + \chi = r_1 + \chi_0 + \chi_2 P_2(\cos\theta), \qquad (35)$$

where χ_0 and χ_2 are second order small constants. The unit normal to this surface is given by

$$n_a^{(v)} = (0, (1 + m^{(v)})/B^{(v)}, 3\chi_2 \sin\theta \cos\theta / B^{(v)}, 0), \qquad (36)$$

where the uppercase index $^{(v)}$ here and after refers to vacuum quantities.

To adjust the coordinates in the two regions, we apply a rigid rotation in the interior by the transformation $\varphi \to \varphi + \Omega t$. Also, we can rescale the interior time coordinate by a constant c_4 while matching the spherical basis solutions, and then later by a second order small constant when doing the matching of the corresponding rotating configuration $t \to c_4(1 + c_3)t$. We do not have such freedom in choosing the time coordinate and applying rotation in the exterior region since we want a coordinate system adapted to asymptotically nonrotating stationary observers.

Together with the values of the other parameters, the location of the matching surface $\mathcal{S}_{r_1}$ is determined by the Darmois-Israel conditions [9,10]. In particular, these conditions pick out the zero pressure surface as the matching surface.

The Darmois-Israel conditions require that the induced metrics agree on the matching surface $\mathcal{S}_{r_1}$,

$$ds^2|_{\mathcal{S}_{r_1}} = ds^2_{(v)}|_{\mathcal{S}_{r_1}}, \qquad (37)$$

as well as the induced second fundamental forms,

$$K|_{\mathcal{S}_{r_1}} = K^{(v)}|_{\mathcal{S}_{r_1}}, \qquad (38)$$

where K is defined as

$$K \equiv K_{ab} dx^a dx^b \equiv h_a{}^c h_b{}^d n_{(c;d)} dx^a dx^b,$$

with

$$h_a{}^b = n_a n^b + \delta_a{}^b$$

being the projection operator onto the hypersurface orthogonal to the normal vector n_a.

Writing g_{ab} and K_{ab} as

$$g_{ab} = g_{ab}^{(0)} + g_{ab}^{(1)} + g_{ab}^{(2)}, \qquad K_{ab} = K_{ab}^{(0)} + K_{ab}^{(1)} + K_{ab}^{(2)},$$

where the superscripts $^{(0)}$, $^{(1)}$, $^{(2)}$ indicate zeroth, first, and second order terms, respectively, one obtains to second order

$$\begin{aligned} g_{ab}(r) &= g_{ab}^{(0)}(r) + g_{ab}^{(1)}(r) + g_{ab}^{(2)}(r) \\ &= g_{ab}^{(0)}(r_1 + \xi) + g_{ab}^{(1)}(r_1) + g_{ab}^{(2)}(r_1) \\ &= g_{ab}^{(0)}(r_1) + \frac{\partial g_{ab}^{(0)}(r)}{\partial r}\bigg|_{r=r_1} \xi + g_{ab}^{(1)}(r_1) + g_{ab}^{(2)}(r_1), \end{aligned} \qquad (39)$$

and similarly for K_{ab}, on the matching surface given by $r = r_1 + \xi$. An analogous result holds in the outer region. From (37)–(39) we then obtain the following equations to order by order.

Zeroth order:

$$A^{(v)} = c_4 A, \qquad B^{(v)} = B, \qquad A^{(v)}{}_{,r} = c_4 A_{,r}. \qquad (40)$$

First order:

$$\omega^{(v)} = c_4(\omega - \Omega), \qquad \omega^{(v)}{}_{,r} = c_4 \omega_{,r}. \qquad (41)$$

Second order:

$$h_2^{(v)} = h_2, \qquad k_2^{(v)} = k_2, \qquad h_0^{(v)} = h_0 + c_3, \qquad (42)$$

$$c_4 A(h_0^{(v)} - h_{0,r}) + \xi_0 (A^{(v)}{}_{,rr} - c_4 A_{,rr}) = 0, \qquad (43)$$

$$\begin{aligned} c_4 A(h_2^{(v)} - h_{2,r}) &+ \xi_2 (A^{(v)}{}_{,rr} - c_4 A_{,rr}) \\ &- c_4 r A_{,r} (k_2^{(v)} - k_{2,r}) = 0, \end{aligned} \qquad (44)$$

$$B(m_0^{(v)} - m_0) = \xi_0 (B^{(v)}{}_{,r} - B_{,r}), \qquad (45)$$

$$B(m_2^{(v)} - m_2) = \xi_2 (B^{(v)}{}_{,r} - B_{,r}) + rB(k_2^{(v)} - k_{2,r}), \qquad (46)$$

$$\chi_0 = \xi_0, \qquad \chi_2 = \xi_2. \qquad (47)$$

All quantities here are evaluated at $r = r_1$, i.e. at the zeroth order radius. From the zeroth order equations, we solve for M and c_4 as

$$M = \frac{1}{2} r_1 (1 - B^2), \qquad c_4 = \frac{B}{A}. \qquad (48)$$

The third equation is equivalent to the zero pressure condition and also gives the radius r_1 implicitly from

$$r_1 = \frac{A}{2B^2 \frac{dA}{dr}} (1 - B^2). \qquad (49)$$

To first order we solve for a and Ω

$$a = \frac{B r_1^3}{3A(B^2 - 1)} \frac{d\omega}{dr}, \qquad \Omega = \frac{r_1}{3} \frac{d\omega}{dr} + \omega. \qquad (50)$$

From six of the nine second order equations, we can solve for c_1, c_2, c_3, q_1, χ_0, and χ_2 as

$$c_1 = \frac{B^2[r_1^4 B^2(B^4-3)(\frac{d\omega}{dr})^2 + 36A^2h_2(1-B^4) + 72A^2B^2(h_2+k_2)]}{9(B^2-1)^6A^2r_1^2}, \tag{51}$$

$$c_2 = \frac{\xi_0}{2}\left(B^2 - 1 + 2r_1B\frac{dB}{dr}\right) - r_1B^2m_0 - \frac{r_1^5}{36}\frac{B^2}{A^2}\left(\frac{d\omega}{dr}\right)^2, \tag{52}$$

$$c_3 = \frac{r_1^4}{36A^2}\left(\frac{d\omega}{dr}\right)^2 + \frac{c_2}{r_1B^2(1-B^2)} - h_0, \tag{53}$$

$$\begin{aligned} q_1 = \frac{1}{9r_1^2A^2(B^2-1)^6}\bigg[& 18k_2A^2(B^4-1)(B^4-8B^2+1) \\ & + 216A^2B^2\ln B(2B^2(h_2+k_2) + (1-B^4)h_2) \\ & + 36h_2A^2B^2(B^2-1)(B^4+B^2-8) \\ & + r_1^4\left(\frac{d\omega}{dr}\right)^2 B^2((B^2-1)(2+11B^2-7B^4) \\ & + 6B^2(B^4-3)\ln B)\bigg], \end{aligned} \tag{54}$$

$$\chi_0 = \xi_0 \quad \text{and} \quad \chi_2 = \xi_2, \tag{55}$$

where ξ_0 and ξ_2 are obtained from (28) and (29). The remaining three equations turn out to be identically satisfied due to the other matching equations and the field equations. Note that we did *not* assume the Petrov type D condition (22) when calculating the matching conditions. Hence an appropriate matching can be done, i.e. the vacuum metric in Sec. II C is general enough for describing the exterior of any axisymmetric rigidly rotating perfect fluid ball up to second order.

IV. NUMERICAL INTEGRATION

In this section we provide a reformulation of the field equations which is more suitable for numerical integration. By doing this we can get higher precision at the origin where apparent singularities arise; moreover, the freely specifiable constants are identified more easily this way.

A. Integrating the zeroth order field equation

In order to simplify (3) it is convenient to redefine the functions A, h, m, and k in terms of the function ν as

$$A = e^{\nu}, \qquad h = \tilde{h}e^{-2\nu}, \qquad m = \tilde{m}e^{-2\nu}, \qquad k = \tilde{k}e^{-2\nu}, \tag{56}$$

giving

$$\begin{aligned} ds^2 = {} & e^{2\nu}(1+2\tilde{h}e^{-2\nu})dt^2 - (1+2\tilde{m}e^{-2\nu})\frac{1}{B^2}dr^2 \\ & - (1+2\tilde{k}e^{-2\nu})r^2[d\theta^2 + \sin^2\theta(d\varphi - \omega dt)^2]. \end{aligned}$$

The equations simplify considerably due to the fact that only the derivative of ν will appear. Hence we introduce the function z by

$$\frac{z}{B} = r\frac{d\nu}{dr} + 1. \tag{57}$$

Then the zeroth order equation (3) becomes first order in z and algebraic in B [17],

$$Br\frac{dz}{dr} + 2B^2 + z^2 - 4Bz + 1 = 0. \tag{58}$$

Furthermore, the pressure of the nonrotating configuration (5) takes the form

$$p_0 = \frac{1}{r^2}(2Bz - B^2 - 1). \tag{59}$$

B. Series expansion around a regular center

For sufficiently regular configurations close to the center, the metric coefficients can be given as power series in r. Assuming that the central pressure and density are finite, it follows that $B(0) = z(0) = 1$. The assumption of smoothness of the configurations at the symmetry center, in the spacetime sense, implies that the odd coefficients in the expansions of the basic variables are zero. Although the smoothness implies the vanishing of the odd coefficients without the use of the field equations, the requirement of smoothness of central density and pressure, together with the field equations, also implies the smoothness of the metric functions.

The vanishing of the odd coefficients can be shown in the generic case by plugging power series expansions of the dependent variables into the field equations. In these cases the smoothness of the density and pressure results from these considerations. If odd powers of r are included in the expansion of the metric variables, then it can be shown that the field equations and the Petrov D condition, together with the assumption of finite and positive central pressure and density, imply the vanishing of the coefficients of all odd terms. However, odd terms may appear in special cases, and then the metric ceases to be smooth at the origin. For example, consider the spherically symmetric case when the only field equation is given by (58). If r^3 terms are included in the expansion of B, then one also obtains a nonsmooth density gradient $\frac{d\rho_0}{dr}|_{r=0} \neq 0$. However, since the field equation implies that the pressure gradient $\frac{dp_0}{dr}|_{r=0}$ is always zero, the squared speed of sound is vanishing at the origin, i.e. $v_s^2 = \frac{dp_0}{d\rho_0}|_{r=0} = 0$. It is of interest to find the minimal requirements needed to guarantee that the solution is regular to any order. A conjecture is that this is the case when the central pressure and density are finite and $\frac{dp}{d\rho}|_{r=0} \neq 0$ [18]. If the Petrov D condition is used instead of an equation of state to specify the configuration, then it can be shown that $\frac{dp}{d\rho}|_{r=0} = 0$ can occur only if either the central pressure or density is negative.

Hence, assuming a smooth center in the spacetime sense, the odd powers will be omitted hereafter. Plugging the expressions

$$\begin{aligned} B &= 1 + b_1 r^2 + b_2 r^4 + \ldots, \\ z &= 1 + z_1 r^2 + z_2 r^4 + \ldots, \\ \omega &= \omega_0 + \omega_1 r^2 + \omega_2 r^4 + \ldots, \\ \tilde{h}_2 &= h_2^{(0)} + h_2^{(1)} r^2 + h_2^{(2)} r^4 \ldots, \\ \tilde{m}_2 &= m_2^{(0)} + m_2^{(1)} r^2 + m_2^{(2)} r^4 \ldots, \\ \tilde{k}_2 &= k_2^{(0)} + k_2^{(1)} r^2 + k_2^{(2)} r^4 \ldots \end{aligned} \tag{60}$$

into the field equations then justifies that all coefficients can be given in terms of b_1, z_1, ω_0, and $h_1 \equiv h_2^{(1)}$. To zeroth order one obtains

$$h_2^{(0)} = m_2^{(0)} = k_2^{(0)} = 0, \tag{61}$$

then to second order

$$m_2^{(1)} = k_2^{(1)} = -h_2^{(1)} \equiv -h_1, \qquad \omega_1 = \tfrac{2}{5}\omega_0(z_1 - 3b_1), \tag{62}$$

and finally to fourth order

$$\begin{aligned} b_2 &= -\frac{b_1^2}{2} + \frac{3\omega_0^2}{50h_1}(z_1 - 3b_1)^2, \qquad z_2 = b_1 z_1 - \frac{z_1^2}{2} - b_1^2, \\ h_2^{(2)} &= \frac{h_1(3z_1 - 13b_1)}{14} - \frac{\omega_0^2(z_1 - 3b_1)[2h_1(22z_1 - 31b_1) + 3\omega_0^2(z_1 - 3b_1)]}{210h_1(z_1 - b_1)}, \\ k_2^{(2)} &= \frac{\omega_0^2}{6}(z_1 - 3b_1) + \frac{h_1}{2}(b_1 - z_1) - h_2^{(2)}, \qquad m_2^{(2)} = \frac{2\omega_0^2}{3}(z_1 - 3b_1) - h_2^{(2)}, \\ \omega_2 &= \frac{\omega_0(z_1 - 3b_1)}{70h_1}[h_1(z_1 - 33b_1) - 3\omega_0^2(z_1 - 3b_1)]. \end{aligned} \tag{63}$$

The expansion of the density and pressure of the nonrotating configuration can be written as

$$\rho_0 = \rho_{0c} - \frac{3\omega_0^2}{20h_1}(p_{0c} + \rho_{0c})^2 r^2 + O(r^4), \tag{64}$$

$$p_0 = p_{0c} - \tfrac{1}{12}(3p_{0c}^2 + 4\rho_{0c}p_{0c} + \rho_{0c}^2)r^2 + O(r^4), \tag{65}$$

where the central density and pressure are given by

$$\rho_{0c} = -6b_1, \qquad p_{0c} = 2z_1. \tag{66}$$

This shows that for realistic configurations $b_1 < 0$ and $z_1 > 0$; consequently, the $z_1 - b_1$ term in the denominator of $h_2^{(2)}$ is nonvanishing. Also, the existence of a local maximum of the density at the center implies $h_1 > 0$. The pressure always has a local maximum at $r = 0$ if the central values are positive.

Assuming $h_1 = 0$ implies that the higher coefficients in the expansion of h_2, m_2, and k_2 are zero. We conjecture that $h_2 = m_2 = k_2 \equiv 0$, which is also supported by a numerical calculation. From this it follows that $\omega = \omega_0 =$ constant and that $A = B = \sqrt{1 + Cr^2}$. But this simply gives the de Sitter or anti de Sitter solutions, depending on the sign of the integration constant C, in a rotating frame.

Plugging the expansions of the original (no tilde) second order spherical perturbation quantities

$$\begin{aligned} h_0 &= h_0^{(0)} + h_0^{(1)} r^2 + h_0^{(2)} r^4 + \ldots, \\ m_0 &= m_0^{(0)} + m_0^{(1)} r^2 + m_0^{(2)} r^4 + \ldots \end{aligned} \tag{67}$$

together with (60) into the two remaining equations (10) and (19) gives that two constants, e.g. $h_0^{(0)}$ and $m_0^{(1)}$, are freely specifiable, whereas the other coefficients can be expressed in terms of these two. Of these, only $m_0^{(1)}$ is essential since the constant $h_0^{(0)}$ can be absorbed by a second order rescaling of the time coordinate. To second order one gets

$$m_0^{(0)} = 0 \quad \text{and} \quad h_0^{(1)} = \frac{5m_0^{(1)}h_1(z_1 - b_1)}{2\omega_0^2(z_1 - 3b_1)} + \frac{m_0^{(1)}}{2} \tag{68}$$

unless $z_1 = 3b_1$, corresponding to $p_{0c} = -\rho_{0c}$. From this result it follows that p_{20}, as given by (13), at the origin is

$$p_{20}(r = 0) = \frac{10m_0^{(1)}h_1(b_1 - z_1)}{\omega_0^2(3b_1 - z_1)}. \tag{69}$$

Hence the central pressure will be unchanged if $m_0^{(1)} = 0$. However, the higher order coefficients in the expansions (67) are still nonzero, i.e. the expansions of h_0 and m_0 start with r^4 terms. We note that, in general, $\rho_{20}(r = 0) = 6m_0^{(1)}$ and $p_{20}(r = 0) = 4h_0^{(1)} - 2m_0^{(1)}$.

Purely spherically symmetric perturbations, corresponding to a small change of central pressure but unchanged equation of state, of the type (67) can be obtained. These kinds of perturbations are possible even when there is no rotation at all. By choosing $\omega_0 = h_1 = 0$ and $m_0^{(1)} \neq 0$, a spherically symmetric perturbation, with a second order shift of the central pressure given by

$$p_{20}(r=0) = \frac{6m_0^{(1)}(b_1 - z_1)(3b_1 - z_1)}{5b_1^2 + 10b_2}, \qquad (70)$$

is produced. This expression remains valid in general, even when the Petrov D condition is not assumed.

C. System of differential equations

Motivated by the results of the previous section, it is advantageous to define the new dependent variables β, ζ, $\tilde{y}$, $\hat{h}$, $\hat{k}$, and $\hat{m}$ through

$$B = 1 + r^2\beta, \qquad z = 1 + r^2\zeta, \qquad \omega_{,r} = 2r\tilde{y},$$
$$\tilde{h}_2 = r^2\hat{h}, \qquad \tilde{k}_2 = r^2(r^2\hat{k} - \hat{h}), \qquad \tilde{m}_2 = r^2(r^2\hat{m} - \hat{h}). \qquad (71)$$

The closed subsystem of equations (3), (6)–(9), and (22) then takes the form

$$\frac{d\zeta}{dr} = -\frac{r(2\beta(\beta-\zeta)+\zeta^2)}{\beta r^2+1}, \qquad \frac{d\beta}{dr} = \frac{2\omega^2(\zeta-3\beta)-3\hat{m}}{2\omega^2 r(\beta r^2+1)} + \frac{\tilde{y}^2 r}{\omega^2}(\beta r^2+1) + \frac{r\beta(\zeta-3\beta)}{(\beta r^2+1)}, \qquad \frac{d\omega}{dr} = 2\tilde{y}r,$$

$$\frac{d\tilde{y}}{dr} = \frac{3\hat{m}-5\omega\tilde{y}}{r\omega(\beta r^2+1)^2} - \frac{r\tilde{y}^2}{\omega^2}(2\omega+\tilde{y}r^2) + \frac{r\tilde{y}(3\hat{m}-10\beta\omega^2(2+\beta r^2))}{2\omega^2(\beta r^2+1)^2},$$

$$\frac{d\hat{h}}{dr} = \frac{[\hat{m}(3\hat{h}+\omega^2) - 4\omega^2(\zeta\hat{h}+\hat{k}-2\beta\hat{h})]}{2r\omega^2(\zeta-\beta)(\beta r^2+1)} - \frac{r\tilde{y}^2(2\omega^2+3\hat{h})(\beta r^2+1)}{3\omega^2(\zeta-\beta)} + \frac{r[r^2\hat{m}(\zeta-\beta)^2 - \beta\hat{h}(2\zeta-3\beta)]}{(\zeta-\beta)(\beta r^2+1)}, \qquad (72)$$

$$\frac{d\hat{k}}{dr} = \frac{\hat{m}-4\hat{k}+2\hat{h}(\beta-\zeta)}{r(\beta r^2+1)} + \frac{r(2\hat{k}(\zeta-3\beta)+\hat{m}\zeta)}{\beta r^2+1},$$

while $\hat{m}$ can be solved for algebraically as

$$\hat{m} = \frac{2}{3}\omega^2(\zeta-3\beta) + \frac{2}{3}r^2\tilde{y}^2(\beta r^2+1)^2 + \frac{2r^2\omega^2}{3}\left[\frac{\tilde{y}(\beta r^2+1)^2[2\omega(\zeta-3\beta) - \tilde{y}(1+\beta r^4(\zeta-\beta)+2\beta r^2)]}{r^2(\beta r^2+1)[\omega^2(\zeta-2\beta)-\tilde{y}(\beta r^2+1)(r^2\tilde{y}+2\omega)] + 3\hat{h} - \beta r^2\omega^2}\right]$$
$$+ \frac{2r^2\omega^2}{3}\left[\frac{\beta(\zeta-2\beta)(3\hat{h}-r^2\omega^2(\zeta-3\beta)) - \omega^2(\zeta-3\beta)^2}{r^2(\beta r^2+1)[\omega^2(\zeta-2\beta)-\tilde{y}(\beta r^2+1)(r^2\tilde{y}+2\omega)] + 3\hat{h} - \beta r^2\omega^2}\right].$$

Boundary conditions at $r = 0$ are given as

$$\beta(0) = b_1, \qquad \zeta(0) = z_1, \qquad \omega(0) = \omega_0,$$
$$\tilde{y}(0) = \omega_1 = \frac{2}{5}\omega_0(z_1 - 3b_1), \qquad \hat{h}(0) = h_1,$$
$$\hat{k}(0) = k_2^{(2)} + h_2^{(2)} = \frac{\omega_0^2}{6}(z_1 - 3b_1) + \frac{h_1}{2}(b_1 - z_1).$$

The relation

$$\hat{m}(0) = m_2^{(2)} + h_2^{(2)} = \tfrac{2}{3}\omega_0^2(z_1 - 3b_1)$$

is then satisfied identically. As we have seen, there are four freely specifiable constants: b_1, z_1, ω_0, and h_1.

The system of equations (72) possesses two types of scale invariances. The first one is associated with the rescaling of the r coordinate, $r \to \alpha r$, under which transformation the dependent variables scale as

$$\beta, \zeta, \hat{m}, \hat{k}, \tilde{y} \to \frac{\beta}{\alpha^2}, \frac{\zeta}{\alpha^2}, \frac{\hat{m}}{\alpha^2}, \frac{\hat{k}}{\alpha^2}, \frac{\tilde{y}}{\alpha^2}, \qquad \omega, \hat{h} \to \omega, \hat{h}. \qquad (73)$$

There is also a rescaling associated to the rescaling of the rotational parameter ω_0, following the rule $\omega_0 \to \gamma\omega_0$, which induces the transformation

$$\omega, \tilde{y} \to \gamma\omega, \gamma\tilde{y}, \qquad \hat{h}, \hat{m}, \hat{k} \to \gamma^2\hat{h}, \gamma^2\hat{m}, \gamma^2\hat{k},$$
$$\beta, \zeta, r \to \beta, \zeta, r. \qquad (74)$$

It is interesting that a combination of the above two rescalings with $\gamma = 1/\alpha$ yields a similarity transformation of the investigated system.

Because of these scale invariances of the equations, two of the constants, e.g., ω_0 and b_1, can be fixed. All other configurations can be obtained by rescaling. Note also that b_1 and z_1 can be expressed in terms of the zeroth order central density and pressure as

$$b_1 = -\tfrac{1}{6}\rho_{0c}, \qquad z_1 = \tfrac{1}{2}p_{0c}.$$

Some of the above equations contain terms of the type "F/r;" thereby they are apparently singular at the origin, so we collected them to the beginning of the right-hand sides. It can be checked case by case that all of the corresponding numerators vanish at the origin. Nevertheless, in determining the values of the corresponding ratios numerically, it turned out to be advantageous to use as the fundamental variables the differences ζ_Δ, β_Δ, ω_Δ, $\tilde{y}_\Delta$, $\hat{h}_\Delta$, $\hat{m}_\Delta$, $\hat{k}_\Delta$ between the variables ζ, β, ω, $\tilde{y}$, $\hat{h}$, $\hat{m}$, $\hat{k}$ and their exact values ζ_0, β_0, ω_0, $\tilde{y}_0$, $\hat{h}_0$, $\hat{m}_0$, $\hat{k}_0$ at the origin. Because of the cancellation of the terms involving the exact values at the origin, what remains from the numerators will be proportional to r^2.

V. NUMERICAL SOLUTIONS

The system (72), when rewritten in terms of the variables ζ_Δ, β_Δ, ω_Δ, $\tilde{y}_\Delta$, $\hat{h}_\Delta$, $\hat{m}_\Delta$, $\hat{k}_\Delta$, was solved using fourth order Runge-Kutta. A check of the convergence factor

$$C_n = \frac{f_n - f_{2n}}{f_{2n} - f_{4n}},$$

where n is the number of points in a given r interval, was performed for various quantities f. We found that the errors decreased according to the expectations, i.e. the value of C_n was found to be close to 16.

We also checked whether the scale invariance properties of the field equations were reproduced properly by our numerical code. When we rescaled the freely specifiable boundary data according to (73) or (74), then all the dependent variables scaled in the appropriate way.

In scanning the four-dimensional parameter space, due to the scaling invariances, without loss of generality, we fixed $b_1 = -1$ and $\omega_0 = 0.1$ while we varied the central values z_1 and h_1. Notice that, to have positive central densities and pressures, the relations $b_1 < 0$ and $z_1 > 0$ also have to be satisfied. The integrations were carried out until the zero pressure surface was reached.

A. Check of Wahlquist

The code was checked for the Wahlquist solution [19], which is of Petrov type D. To second order it is given by [13]

$$ds^2 = f_0(1+2h)dt^2 - 2\frac{1+2m}{\mu_0\kappa^2 f_0}dx^2 - \frac{2}{\mu_0\kappa^2}\sin^2 x \times (1+2k)[d\theta^2 + \sin^2\theta(d\varphi - \omega dt)^2] \quad (75)$$

with

$$f_0 = 1 + \frac{1}{\kappa^2}(1 - x\cot x) \quad (76)$$

and

$$\omega = \frac{\mu_0 r_0}{2\sin^2 x}(1 - x\cot x). \quad (77)$$

The transformation to the Hartle variables used in (1) is given by

$$r = \sqrt{\frac{2}{\mu_0\kappa^2}}\sin x.$$

For the functions h, m, and k, see [13]. For the Wahlquist solution the four starting values are given by

$$b_1 = \frac{\mu_0}{12}(1 - 3\kappa^2), \qquad z_1 = \frac{\mu_0}{4}(1-\kappa^2), \qquad \omega_0 = \frac{\mu_0 r_0}{6},$$

and

$$h_1 = -\frac{\mu_0^2 r_0^2 \kappa^2}{60} \quad (78)$$

in terms of the three integration constants μ_0, κ, and r_0. Solving for h_1 gives

$$h_1 = \frac{(z_1 - 3b_1)\omega_0^2}{5(b_1 - z_1)}. \quad (79)$$

Note that $h_1 < 0$ for positive central density and pressure, implying, together with Eq. (64), the well-known fact that the density of the Wahlquist solution has a minimum at the center.

In Fig. 1 the relative error between the analytical solution for the rotational function ω, as given by (77), and the numerical solution is plotted for various resolutions.

As it was already found [13,14], the shape of the Wahlquist fluid ball is always prolate which is also in accordance with the positive sign of the quantity $k_2 + \xi_2/r_1$. The quantity q_1 is not zero for the Wahlquist solution which is equivalent to the fact that it cannot be matched to an asymptotically flat exterior solution to second order [13,14,20]. The value of c_1 was also found to be negative for all the tested Wahlquist configurations. This is also verified by analyzing the analytical expression for c_1 given in [14]. It would be interesting to know whether the sign of this quantity is related to the shape in general.

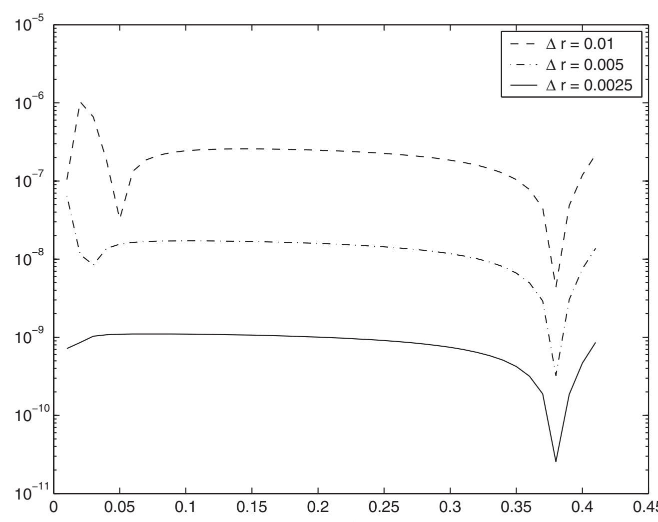

FIG. 1. The relative error $|\omega_{\rm numerical} - \omega_{\rm analytic}|/\omega_{\rm analytic}$ is shown for the resolutions $\Delta r = 0.01$, $\Delta r = 0.005$, and $\Delta r = 0.0025$. Starting values are $z_1 = -b_1 = 1$ and $\omega_0 = 0.1$, giving $h_1 = -0.004$.

B. Asymptotically flat solutions

A solution to the field equations (72) is asymptotically flat iff $q_1 = 0$. In Fig. 2 the points of the dashed curve represent configurations with value $q_1 = 0$ in the $z_1 h_1$ plane, while in Fig. 3 a section of the same curve for small z_1, corresponding to small central pressures, is given. Naturally, the asymptotically flat solutions, represented by points belonging to these curves, always have finite radii, but increasing z_1 further, corresponding to the increase of the central pressure, we get into a region where the density becomes negative before a zero pressure surface is reached. The limiting curve where the pressure and density become zero at the same radius is shown in Figs. 2 and 3 by the solid lines. Because of the negative density, the pressure ceases to be a monotonic function of r for configurations above the $\rho_{r_1} = 0$ curve.

Since for the asymptotically flat solutions h_1 may be seen as a function of z_1, we see that h_2, m_2, and k_2 for these solutions are determined by b_1, z_1, and ω_0. The analysis done in Sec. IV B for the remaining field equations showed that, essentially, one constant of integration is freely specifiable for the functions h_0 and m_0. This implies then that the second order configuration is completely determined by the zeroth order spherical configuration, a second order spherically symmetric perturbation, and the magnitude of the rotation.

C. Are there Kerr-like solutions?

To our knowledge the only known source for the Kerr metric is the thin rotating disk of dust with $a = m$ found by Neugebauer and Meinel [21]. It is tempting to investigate whether there can exist a fluid ball belonging to the class

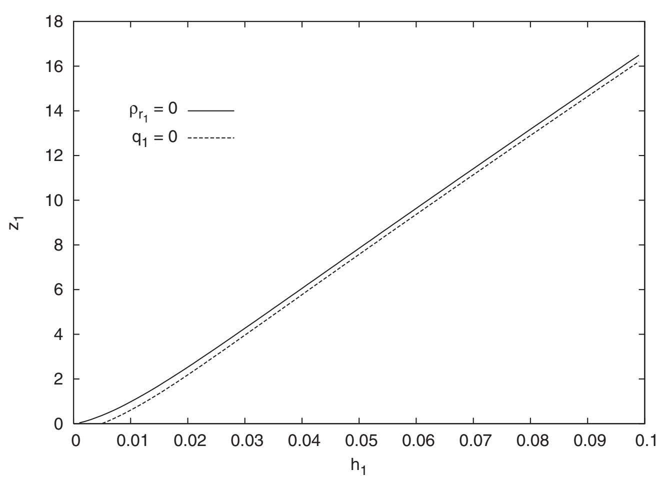

FIG. 2. Along the solid curve the density and pressure become zero for the same value of the radius $r = r_1$. Below this curve those configurations can be found which can be matched to an exterior vacuum region. The dashed curve represents those configurations in the $z_1 h_1$ plane for which the exterior vacuum region is asymptotically flat.

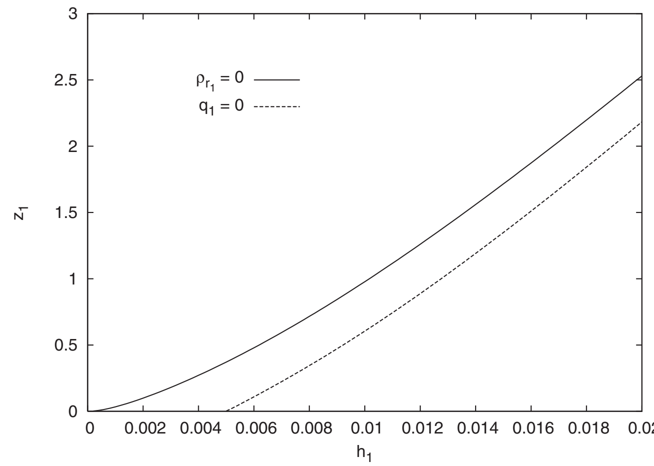

FIG. 3. The same curves as in Fig. 2 are shown for small z_1, corresponding to low central pressures.

investigated in this paper that could be matched to the Kerr solution to second order. To settle this issue note first that the metric of the exterior region becomes the Kerr metric with mass parameter $M - c_2$ iff $q_1 = c_1 = 0$. However, the numerical runs indicated that either $c_1 > 0$ for $h_1 > 0$ or $c_1 < 0$ for $h_1 < 0$ occurs, in general, i.e. the desired matching seems not to be supported. Note that these numerical findings are in accordance with some earlier results, see e.g. [4,22], telling us that typically the exterior metric deviates from the Kerr metric due to the ellipsoidal shape of the rotating fluid ball. To this end it is illuminating to consider an expansion of the exterior metric for large r, which gives the following leading terms of g_{00} (with $q_1 = 0$):

$$g_{00} = 1 - \frac{2M(1 - \frac{c_2}{M})}{r} + \frac{2MP_2(\cos\theta)(a^2 + \frac{16}{5}M^4 c_1)}{r^3}, \tag{80}$$

i.e., the associated quadrupole moment reads (cf., e.g. [23])

$$Q_{11} = Q_{22} = -Q_{33}/2 = -2M(a^2 + \tfrac{16}{5}M^4 c_1)$$

in an asymptotically Cartesian system with the 3-axis along the axis of rotation. In Fig. 4 the value of c_1 as a function of the central pressure $p_{0c} = 2z_1$ along the $q_1 = 0$ curve is shown.

D. Some asymptotically flat solutions with a reasonable equation of state

In this subsection we present some properties, like equation of state and speed of sound, for some of the physically interesting inner fluid ball configurations which can be matched to a suitable asymptotically flat exterior vacuum region up to second order, i.e. those solutions for which q_1 vanishes. According to Table I the value of h_1 has to be smaller than around 0.012 for these solutions to have subluminal speed of sound, $v_s^2 = dp/d\rho < 1$. For all con-

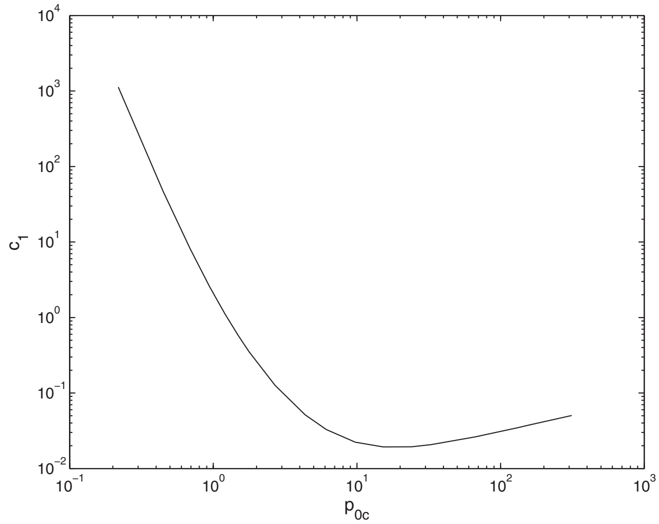

FIG. 4. The constant c_1 as a function of zeroth order central pressure, p_{0c}, along the curve $q_1 = 0$.

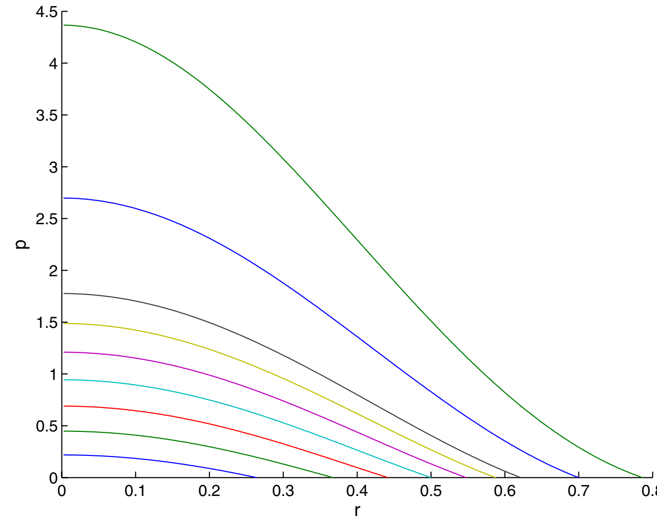

FIG. 5 (color online). The zeroth order pressure p_0 is shown as a function of r for various central pressures along the curve $q_1 = 0$. From top to bottom the central pressures are given by 4.3666, 2.6978, 1.7762, 1.4872, 1.21, 0.944, 0.6896, 0.448, and 0.2184, respectively, whereas the central density is $\rho_{0c} = 6 = -6b_1$.

figurations in this interval the speed of sound also increases when approaching the center, i.e. the fluid becomes stiffer as would be expected on physical grounds.

In Figs. 5 and 6 the zeroth order pressure and density, p_0 and ρ_0, are shown as functions of r for some configurations with central pressure p_{0c} between 0.218 and 4.366, while in Figs. 7 and 8 the equation of state, i.e. p as a function of ρ, and the square of the speed of sound, $v_s^2 = \frac{dp}{d\rho}$, as a function of r are depicted, respectively, for the same family of solutions. Notice that the value of v_s^2 at the surface of the fluid ball seems to be independent of the values of the free parameters at the center. However, a closer look shows that the value slowly changes from 0.272 to 0.261 with increasing central pressure for the configurations plotted in Fig. 8.

In general, the equations of state cannot be polytropic since the density does not tend to zero while approaching the matching surface. However, the equations of state can be approximated with a polytropic one close to the center. The last two columns in Table I provide the central values of the adiabatic index

$$\kappa \equiv \frac{n}{p}\frac{dp}{dn} = \frac{p+\rho}{p}\frac{dp}{d\rho}, \tag{81}$$

where n is the baryon number density, and the Newtonian adiabatic index

TABLE I. The central pressure $p_{0c} = 2z_1$, the radius of the zero pressure surface r_1, the shape of the zero pressure surface ($a \equiv k_2 + \xi_2/r_1$ is negative for an oblate configuration), the shape of the constant pressure surfaces close to the center given by $b \equiv (k_2 + \xi_2/\bar{r})|_{\bar{r}=0}$, the value of the constant c_1, the maximal speed of sound v_s^2, the zeroth order energy density at the matching surface, and, finally, the central values of the adiabatic indices $\kappa = \frac{p+\rho}{p}\frac{dp}{d\rho}$ and $\kappa_N = \frac{\rho}{p}\frac{dp}{d\rho}$ are shown for some configurations which can be matched to an asymptotically flat exterior (with $q_1 = 0$). Although for all listed configurations the central density $\rho_{0c} = 6 = -6b_1$, any central density can be obtained using the rescaling freedom (73).

h_1	$p_{0c} = 2z_1$	r_1	a	b	c_1	v_s^2	$\rho_0(r_1)$	κ	κ_N
0.006	0.2184	0.264	−0.0086	−0.00841	$1.13 \cdot 10^3$	0.36	5.3021	10.3	9.9
0.007	0.4480	0.367	−0.0089	−0.00844	46.675	0.44	4.7244	6.3	5.9
0.008	0.6896	0.441	−0.0091	−0.00843	8.165	0.54	4.2416	5.2	4.7
0.009	0.9440	0.500	−0.0093	−0.00839	2.556	0.64	3.8318	4.7	4.1
0.01	1.210	0.548	−0.0095	−0.00831	1.105	0.74	3.485	4.4	3.7
0.011	1.4872	0.587	−0.0098	−0.00822	0.584	0.85	3.1898	4.3	3.4
0.012	1.7762	0.621	−0.0100	−0.00812	0.352	0.97	2.9324	4.2	3.3
0.015	2.6978	0.700	−0.0109	−0.007805	0.1259	1.35	2.3441	4.4	3.0
0.02	4.3666	0.786	−0.0126	−0.00733	0.0514	2.05	1.7335	4.9	2.8
0.05	15.2	1.029	−0.0254	−0.00620	0.0193	6.76	0.664 28	9.4	2.7
0.1	32.8	1.190	−0.0449	−0.00594	0.0210	14.95	0.421 64	17.7	2.7
1	312.8	1.636	−0.2297	−0.00637	0.0503	164.58	0.213 98	168	3.2

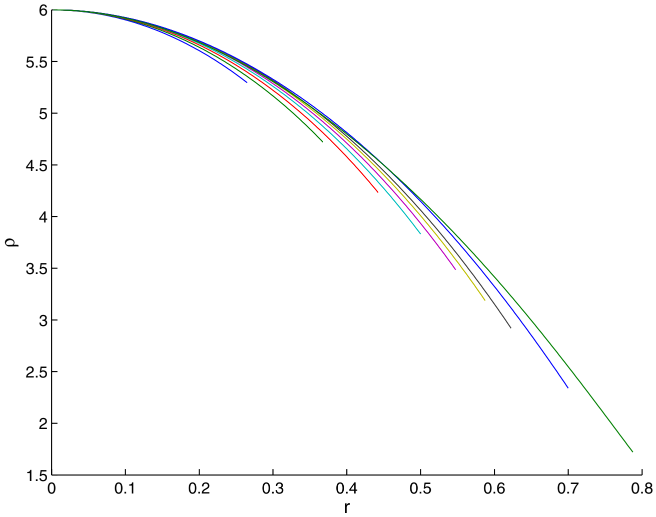

FIG. 6 (color online). The zeroth order density ρ_0 is shown as a function of r for various central pressures along the curve $q_1 = 0$. From top to bottom the central pressures are as given in Fig. 5.

$$\kappa_N \equiv \frac{\rho}{p}\frac{dp}{d\rho}, \tag{82}$$

which approximates κ for low pressures, respectively. Unfortunately, the value of κ is not in the preferred range $4/3 - 5/3$ that is considered to be physically acceptable in the case of compact neutron stars or white dwarfs.

As we have seen in Sec. III A, the surfaces of constant pressure, $\mathcal{S}_r$, are determined by the relation $r = \bar{r} + \xi_0 + \xi_2 P_2(\cos\theta)$, where ξ_2 is given by (29). These surfaces are oblate iff $k_2(\bar{r}) + \xi_2/\bar{r} < 0$. In terms of the functions ν, β, ζ, $\hat{h}$, $\hat{k}$, and ω, the expression $k_2(\bar{r}) + \xi_2/\bar{r}$ can be written as

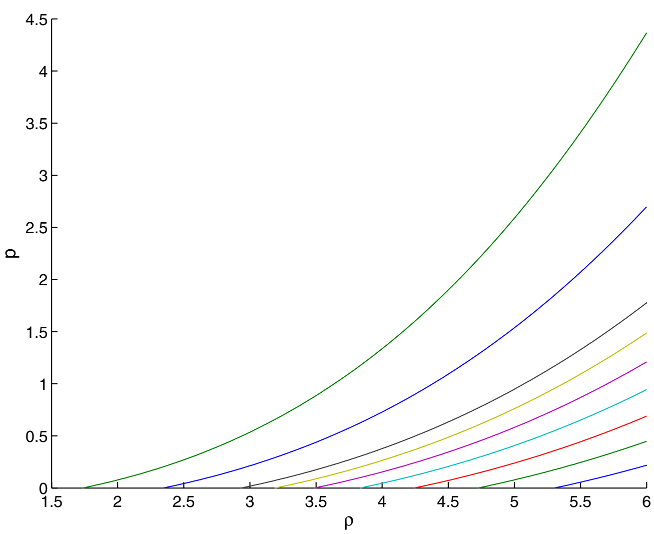

FIG. 7 (color online). The equation of state $p = p(\rho)$ is shown for the same configurations as before.

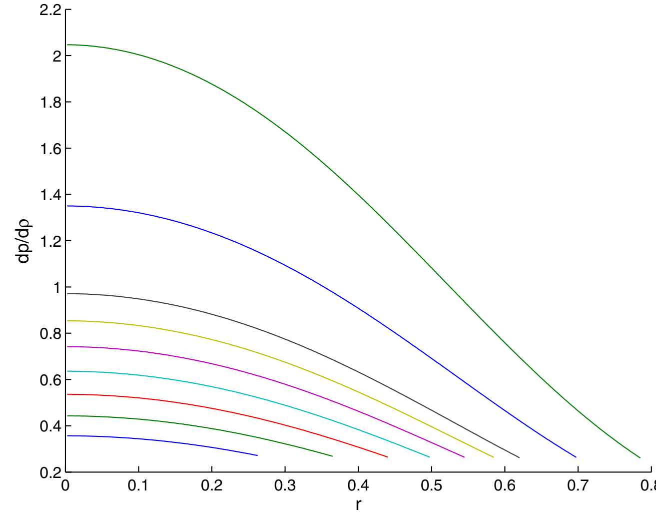

FIG. 8 (color online). The square of the speed of sound, $v_s^2 = \frac{dp}{d\rho}$, is shown for the same configurations as in the previous figures.

$$k_2(\bar{r}) + \xi_2/\bar{r} = e^{-2\nu}\left(r^4\hat{k} - r^2\hat{h} - \frac{(3\hat{h} + \omega^2)(1 + \beta r^2)}{3(\zeta - \beta)}\right). \tag{83}$$

Using the Taylor expansion around the center, obtained in Sec. IV B, one gets in the limit $\bar{r} \to 0$

$$\lim_{\bar{r}\to 0} k_2(\bar{r}) + \xi_2/\bar{r} = -\tfrac{1}{3}e^{-2\nu(0)}(\omega_0^2 + 3h_1)/(z_1 - b_1). \tag{84}$$

Note that $e^{\nu(0)}$ may be fixed to 1 since its value only corresponds to a rescaling of the time coordinate. Since $z_1 > 0$ and $b_1 < 0$ for realistic configurations we see that close to the center the surfaces of constant pressure are oblate iff $h_1 > -\omega_0^2/3$. In Table I the values of $a \equiv (k_2(\bar{r}) + \xi_2/\bar{r})|_{\bar{r}=r_1}$ and $b \equiv (k_2(\bar{r}) + \xi_2/\bar{r})|_{\bar{r}=0}$ are also indicated for several configurations. In virtue of the negative signs of these parameters, the constant pressure surfaces are all oblate for these configurations.

In Table I the central pressure $p_{0c} = 2z_1$, the radius of the zero pressure surface r_1, the shape at the zero pressure surface, the shape close to the center, the value of c_1, the maximal speed of sound v_s^2, the zeroth order density at the zero pressure surface, and the adiabatic index are given for a sequence of solutions with $q_1 = 0$.

VI. CONCLUSIONS

The most important finding of this paper is that a subclass of slowly rotating perfect fluid balls of Petrov type D can be matched to asymptotically flat vacuum spacetimes and also that, in general, slowly rotating perfect fluid balls can be matched to nonasymptotically flat vacuum exteriors determined by Eqs. (23)–(25). Our numerical results support the conclusion that neither of the Petrov type D inner fluid solutions can be matched to second order to the Kerr metric, which is in accordance with the general expectation that the ellipsoidal shape of the rotating fluid ball produces an extra contribution to the quadrupole moment which should also be present in the corresponding quadrupole moment of the external field [4,22]. It was also found that there is a range in parameter space for which the value of the central pressure is relatively low and the speed of sound is also subluminal. The equation of state was also determined for various solutions belonging to the investigated class. It is clear that the equation of state cannot be polytropic since, in general, the energy density does not vanish at the zero pressure surface. Nevertheless, the equation of state can be approximated close to the center by a polytropic one. Unfortunately, the corresponding adiabatic index κ was found to take values out of the physically preferred range.

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