

General relativity from a gauged Wess-Zumino-Witten term

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In this paper two things are done. First it is shown how a four-dimensional gauged Wess-Zumino-Witten term arises from the five-dimensional Einstein-Hilbert plus Gauss-Bonnet Lagrangian with a special choice of the coefficients. Second, the way in which the equations of motion of four-dimensional General Relativity arise is exhibited.

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I. INTRODUCTION

Since the proof of power counting non renormalizability [1] of General Relativity (GR), the scientific community has increasingly accepted the idea that Einstein's theory is an effective field theory [2]. Although some candidates for the high energy limit of GR have emerged, no one has managed to give a power counting renormalizable theory that also reproduces the dynamical behavior of GR in four dimensions.

The most general gravitational action in five dimensions has one additional free parameter besides the cosmological constant [3]. These two parameters can be chosen so that the Lagrangian becomes a Chern-Simons form, acquiring some of the features that make $D = 3$ gravity a theory with zero beta function [4]. This particular $D = 5$ gravitation theory is a gauge theory where the spin connection ω and the vielbein e are parts of a single connection for the Lie algebras $so(4, 2)$, $so(5, 1)$ or $iso(4, 1)$. Gravitation theories that have these properties exist in all odd dimensions. They have been studied in [5,6] and a review of them can be found in Ref. [7]. Another compelling reason to consider these gauge theories for the $SO(D - 1, 2)$ group, they admit an immediate supersymmetric extension [8,9]. Moreover, the local supersymmetry in those theories is realized off shell and without invoking auxiliary fields or ad-hoc constraints.

In this work, the topological sector -in the sense that no metric is needed to construct it-, of the gauged Wess-Zumino-Witten (gWZW) Lagrangian, is shown to arise from higher-dimensional gravity. In the case where the connection is valued in the Lie algebra $su_L(3) \times su_R(3)$, the gWZW term plus a kinetic piece for the Goldstone fields, arises as the effective Lagrangian of QCD [10]. In our case, the gauge group will not be $SU_L(3) \times SU_R(3)$, but the anti-de-Sitter group in five dimensions, $SO(4, 2)$.

II. FIVE-DIMENSIONAL GRAVITY AND GWZW TERMS

The most general five-dimensional, ghost-free [11,12], gravitational action is given by [13]

$$S(\omega, e) = \int_M \varepsilon_{abcde} (\alpha_2 R^{ab} R^{cd} e^e + \alpha_1 R^{ab} e^c e^d e^e + \alpha_0 e^a e^b e^c e^d e^e) \quad (1)$$

where the curvature twoform is written in terms of the Lorentz (spin) connection ω , as

$$R^{ab} = d\omega^{ab} + \omega^a{}_c \omega^{cb} = \frac{1}{2} R^{ab}{}_{\mu\nu} dx^\mu dx^\nu \quad (2)$$

The vielbein e^a is related to the spacetime metric through $g_{\mu\nu} = e^a_\mu e^b_\nu \eta_{ab}$, and $\eta_{ab} = \text{diag}(-, +, +, +, +)$ is the Lorentz-invariant metric. The vielbein and the Lorentz connection are regarded as independent fields. The field equations associated with the variations of ω are satisfied if the torsion, $De^a \equiv de^a + \omega^a{}_b e^b$, is set equal to zero. In the sector of the theory where the torsion is zero and the vielbein is invertible, ω is a function of the vielbein, and the usual second order equations for the metric are recovered from the field equations obtained from the variation with respect to e .

An interesting accident occurs when the constants in the action are in the ratio $\alpha_2:\alpha_1:\alpha_0 = 1:2/3:1/5$. In that case, the action can be rewritten as a Chern-Simons theory [5,6,8,9],

$$S(\mathcal{A}) = \kappa \int_M \left\langle \mathcal{A} d\mathcal{A} d\mathcal{A} + \frac{3}{2} \mathcal{A}^3 d\mathcal{A} + \frac{3}{5} \mathcal{A}^5 \right\rangle = \kappa \int_M \text{CS}(\mathcal{A}), \quad (3)$$

where

$$\mathcal{A} = \frac{1}{2} \omega^{ab} J_{ab} + e^a J_{a5}, \quad \langle J_{ab} J_{cd} J_{e5} \rangle = \varepsilon_{abcde}, \quad (4)$$

κ is dimensionless, $[J_{AB}, J_{CD}] = -J_{AC} \eta_{BD} + J_{BC} \eta_{AD} -$

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($A \leftrightarrow B$), and $\langle \dots \rangle$ stands for an invariant symmetric trace in the algebra.

In this way, the action acquires an enlarged gauge symmetry. If $\eta_{55} > 0$ the gauge group is $SO(5, 1)$ and the action has positive cosmological constant. For $\eta_{55} < 0$ the gauge group is $SO(4, 2)$ and the cosmological constant is negative. In the latter case, localized deformations of the geometry give rise to asymptotically locally anti-de Sitter geometries.

Strictly speaking, under a gauge transformation the action (3) is not gauge invariant but changes by a closed form plus a boundary term. This quasi invariance is a source of ambiguities in an asymptotically AdS spacetime, where the boundary terms that arise by gauge transformations change the action and modify the conserved charges, producing even divergent values for them. This problem can be circumvented if the action principle is modified by the addition of some new terms that do not modify the field equations but render the action truly gauge invariant [14,15]. The trick is to replace the Lagrangian in (3) by a transgression form,

$$S(\mathcal{A}, \bar{\mathcal{A}}) = \kappa \int_M \text{CS}(\mathcal{A}) - \text{CS}(\bar{\mathcal{A}}) + \kappa \int_{\partial M} B(\mathcal{A}, \bar{\mathcal{A}}), \quad (5)$$

where

$$B(\mathcal{A}, \bar{\mathcal{A}}) = - \left\langle \mathcal{A} \bar{\mathcal{A}} \left(\mathcal{F} + \bar{\mathcal{F}} - \frac{1}{2} \mathcal{A}^2 - \frac{1}{2} \bar{\mathcal{A}}^2 + \frac{1}{2} \mathcal{A} \bar{\mathcal{A}} \right) \right\rangle. \quad (6)$$

The transgression form is the object which appears in the Chern-Weil theorem, that states that the pullback of invariant polynomials of the curvature $P(\mathcal{F})$ are members of cohomology groups of the manifold where they are defined [16],

$$dP(\mathcal{F}) = 0, \quad P(\mathcal{F}) - P(\bar{\mathcal{F}}) = dTP(\mathcal{A}, \bar{\mathcal{A}}), \quad (7)$$

where $TP(\mathcal{A}, \bar{\mathcal{A}})$ is defined by Eq. (7) up to a closed form. The gauge invariant, globally-defined expression for $TP(\mathcal{A}, \bar{\mathcal{A}})$ stands for the transgression form. In $2n - 1$ dimensions, the transgression takes the form

$$TP_{2n-1}(\mathcal{A}, \bar{\mathcal{A}}) = n \int_0^1 dt \langle (\mathcal{A} - \bar{\mathcal{A}}) \mathcal{F}_t^{n-1} \rangle, \quad (8)$$

where $\mathcal{F}_t = d\mathcal{A}_t + \mathcal{A}_t \mathcal{A}_t$, $\mathcal{A}_t = \mathcal{A}(1-t) + \bar{\mathcal{A}}t$. Thus, the boundary term $B(\mathcal{A}, \bar{\mathcal{A}})$ is uniquely determined by the Chern-Weil theorem.

The field equations for \mathcal{A} are the same, whether the action principle is defined by (5) or by (3). However, there is a problem interpreting the physical meaning of the field $\bar{\mathcal{A}}$. An interpretation was proposed in [15], where the Lagrangian $\text{CS}(\bar{\mathcal{A}})$ was considered as defined in a manifold with opposite orientation to the one for $\text{CS}(\mathcal{A})$. An

alternative is to regard $\bar{\mathcal{A}}$ not as a dynamical field but as a means of constructing the boundary term which makes the action finite [17]. A different philosophy will be followed here, that is to regard $\bar{\mathcal{A}}$ and \mathcal{A} as two connections defining the same non trivial, principal bundle. That is, they are related by a gauge transformation.

If a non trivial bundle is considered, the integrand does not exist globally either in (3) or in (5). This non existence problem will be treated in [18], where a detailed study of the definition of an action principle in a manifold divided into patches, for a non trivial principal bundle, will be presented [19]. In the case of a non trivial bundle, however, the action (5) can be treated formally provided more than one chart is used. In this case, it is necessary to introduce connection one-forms defined on each chart, such that, in the overlap of two charts the connections are related by a gauge transformation, $\bar{\mathcal{A}} = h^{-1} \mathcal{A} h + h^{-1} dh \equiv \mathcal{A}^h$, where h is a transition function which determines the non triviality of the bundle.

Replacing $\bar{\mathcal{A}} = \mathcal{A}^h$ in (5), it is straightforward to check that the action takes the form of a gWZW term,

$$\begin{aligned} S(h, \mathcal{A}) = & - \frac{\kappa}{10} \int_{M^5} \left\langle h^{-1} dh (h^{-1} dh)^2 (h^{-1} dh)^2 \right\rangle \\ & + \kappa \int_{M^4} \left\langle dhh^{-1} \mathcal{A} \left(d\mathcal{A} + \frac{1}{2} \mathcal{A}^2 \right) \right\rangle \\ & - \frac{\kappa}{2} \int_{M^4} \left\langle dhh^{-1} \mathcal{A} \{ (dhh^{-1})^2 + \mathcal{A} dhh^{-1} \} \right\rangle \\ & - \kappa \int_{M^4} \left\langle \mathcal{A} \mathcal{A}^h \left(\mathcal{F} + \mathcal{F}^h - \frac{1}{2} \mathcal{A}^2 \right. \right. \\ & \left. \left. - \frac{1}{2} (\mathcal{A}^h)^2 + \frac{1}{2} \mathcal{A} \mathcal{A}^h \right) \right\rangle. \quad (9) \end{aligned}$$

where the curvature is $\mathcal{F} = d\mathcal{A} + \mathcal{A} \mathcal{A}$ and $\mathcal{F}^h = h^{-1} \mathcal{F} h$. This action is invariant under the adjoint action of the gauge group, namely,

$$\mathcal{A} \rightarrow g^{-1} \mathcal{A} g + g^{-1} dg, \quad h \rightarrow g^{-1} h g. \quad (10)$$

As has been shown, the principle of gauge invariance, through the mathematical structure of the theory of principal bundles, provides a compactification mechanism. Beginning with a five-dimensional gauge theory that has no metric in it, a $D = 4$ gauge invariant theory has been obtained. This is a compactification mechanism alternative to Kaluza-Klein. The relation between three-dimensional Chern-Simons theories and two-dimensional gWZW models was early realized in [20]. Originally, gWZW terms were obtained in [10] by trial and error, and it was later shown that they can be obtained systematically see, e.g., [21–23].

Other attempts to relate the $D = 4$ gWZW Lagrangians to $D = 5$ Chern-Simons theory can be found in the literature (see for instance, Refs. [24,25]). However, asymptotic conditions for the metric were always assumed in order to reproduce the kinetic term for the Goldstone fields, not

present in (9). As can be seen from the previous discussion, such a strong assumption is not required here, the kinetic term does not appear, but the gWZW term arises naturally from the transgression form.

The action (9), usually supplemented with a kinetic term for the so-called Goldstone fields, requires the introduction of the Hodge dual, which in turn requires the existence of a metric in the manifold. The point of view followed here is that the metric arises from the components of a gauge connection in a broken phase of the theory, but it is not assumed to be defined *a priori*. In the next section the action principle (9) for a particular class of h will be studied further.

III. THE GWZW TERM AS A GRAVITATIONAL ACTION

The action (9) describes a theory with $SO(4, 2) \times SO(4, 2)$ gauge symmetry spontaneously broken to its diagonal subgroup $SO(4, 2)$ [10,26]. The gauge invariance is manifest since the connection takes its values in the diagonal subalgebra only [22]. However, in order to describe the known low energy gravitational behavior, it is necessary to reduce the symmetry to the usual $SO(3, 1)$ local symmetry present in the Einstein-Hilbert action. A way to do this is by replacing the gauge group $SO(4, 2)$ by the coset $\frac{SO(4,2)}{\mathbb{R}}$, where R stands for the group of transformations generated by $e^{\phi J_{45}}$ and an element of the coset, h , is a representative of the equivalence class $[h] = \{h \sim h' \Leftrightarrow h' = e^{\phi J_{45}} h, h, h' \in SO(4, 2)\}$. Using the adjoint action, the stability group of this coset is $SO(3, 1) \times R$ and it corresponds to the residual gauge invariance present in the theory, that is, $h \in [h] \Rightarrow g^{-1} h g \in [h] \Leftrightarrow g \in SO(3, 1) \times R$.

Fixing the Goldstone field associated to J_{45} in the action (9), corresponds to reducing the gauge symmetry down to $SO(3, 1) \times R$. There is an interesting geometrical interpretation of this. Suppose we have a six-dimensional manifold M^6 and delete a four-dimensional submanifold M^4 (Fig. 1).

Then, by considering the integral of a characteristic class on $M^6 - M^4$, the action induced on M^4 is (9) [27],

$$\int_{M^6 - M^4} \langle \mathcal{F} \mathcal{F} \mathcal{F} \rangle = S(h, \mathcal{A}). \quad (11)$$

The field ϕ can be interpreted in terms of the six-dimensional pseudo-Riemannian geometry, as a deficit

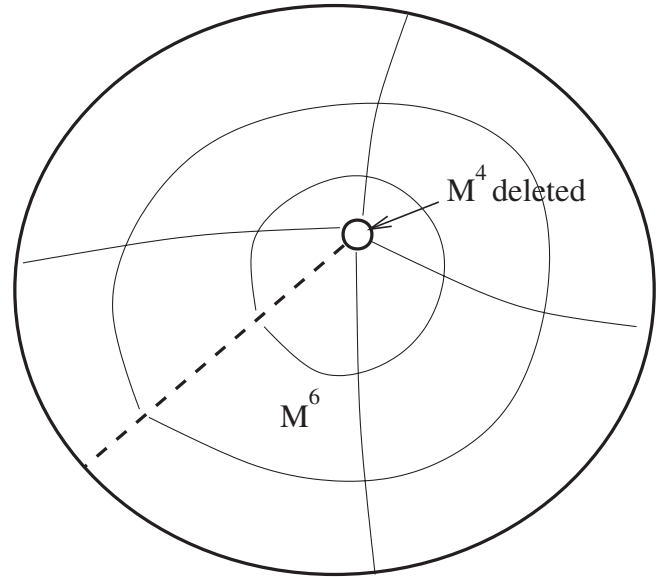


FIG. 1. A four-dimensional defect in a six-dimensional manifold. The submanifold M^4 has been deleted, as indicated by the infinitesimal loop in the center of the diagram.

angle around the four-dimensional defect M^4 which, as shown here, is related to the four-dimensional cosmological constant. Here we have assumed that ϕ is a constant, thus breaking part of the gauge symmetry “by hand”. In terms of this geometrical picture, the defect is assumed to have a fixed deficit angle.

Now, in order to write the field equations associated with the gWZW term (9) for the coset $\frac{SO(4,2)}{\mathbb{R}}$, it is helpful to decompose the connection and the curvature in a way that reflects the $SO(3, 1)$ symmetry,

$$\mathcal{A} = \frac{1}{2} \omega^{ab} J_{ab} + b^a J_{a4} + e^a J_{a5} + \Phi J_{45}, \quad (12)$$

$$\begin{aligned} \mathcal{F} = & \frac{1}{2} (R^{ab} + e^a e^b - b^a b^b) J_{ab} + [D b^a + e^a \Phi] J_{a4} \\ & + [D e^a + b^a \Phi] J_{a5} + [d\Phi - b_a e^a] J_{45}, \end{aligned} \quad (13)$$

where $D e^a = d e^a + \omega^a_b e^b$. The field equations associated to the variation of the Goldstone fields in the coset $\frac{SO(4,2)}{\mathbb{R}}$ are

$$\begin{aligned} \kappa \int_{M^4} \left\langle h^{-1} \delta h \left\{ (\mathcal{F}^h)^2 + \mathcal{F}^2 + \mathcal{F}^h \mathcal{F} - \frac{3}{4} [\mathcal{A}^h - \mathcal{A}, \mathcal{A}^h - \mathcal{A}] (\mathcal{F}^h + \mathcal{F}) + \frac{1}{8} [\mathcal{A}^h - \mathcal{A}, \mathcal{A}^h - \mathcal{A}]^2 \right. \right. \\ \left. \left. + \frac{1}{2} (\mathcal{A}^h - \mathcal{A}) [\mathcal{F}^h + \mathcal{F}, \mathcal{A}^h - \mathcal{A}] \right\} \right\rangle = 0, \end{aligned} \quad (14)$$

while the 15 equations of motion that arise from the variation of the connection, are

$$\kappa \int_{M^4} \left\langle \delta \mathcal{A} (\mathcal{A}^h - \mathcal{A}) \left(\mathcal{F}^h + 2\mathcal{F} - \frac{1}{4} [\mathcal{A}^h - \mathcal{A}, \mathcal{A}^h - \mathcal{A}] \right) \right\rangle - (h \leftrightarrow h^{-1}) = 0. \quad (15)$$

In order to obtain the equations of motion in an explicit form, it is necessary to pick a representative of the coset. In any open set, it can be parametrized by 14 coordinates π , as $h = e^{\phi J_{45}} e^{\pi^{a4} J_{a4}} e^{\pi^{a5} J_{a5}} e^{(1/2)\pi^{ab} J_{ab}}$, where ϕ is an arbitrary, real, constant. The purely gravitational sector of the theory, that is the one in which only the vielbein and the spin connection are present, corresponds to setting b^a , Φ and the 14 Goldstone fields equal to zero. In this dynamical sector the set of Eqs. (15) reduces to

$$\varepsilon_{abcd} e^b (R^{cd} + \mu e^c e^d) \sinh \phi = 0 \quad (16)$$

$$\varepsilon_{abcd} e^c D e^d \sinh \phi = 0. \quad (17)$$

Here the constant μ is given by $(1 + 2 \cosh \phi)/3$. Excluding the trivial case $\phi = 0$ implies $\mu > 1$, and Eq. (17) implies that the torsion is zero. Solving the torsion for the spin connection and replacing it back in (16), the Einstein's equations in standard form are obtained. It is reassuring to check that these field configurations also satisfy the 14 field equations obtained from (14).

In order to write Einstein's equations it is necessary to assume that the vielbein e^a_μ is invertible, and to make contact with a metric theory, it is also necessary to rescale the vielbein with a parameter with dimensions of length, $e^a = \bar{e}^a/l$. In this way, the metric is $g_{\mu\nu} = \bar{e}^a_\mu \bar{e}^b_\nu \eta_{ab}$, the effective cosmological constant acquires its usual units $\Lambda = (1 + 2 \cosh \phi)l^{-2}$, and (16) can be rewritten as

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - \Lambda g_{\mu\nu} = 0. \quad (18)$$

However, as was shown in Ref. [4], assuming the invertibility of the vielbein spoils the possibility of making a sensible, quantum mechanical, perturbative expansion around $\mathcal{A} = 0$. The rescaling of e^a is also unhelpful in the sense that, if no such rescaling is done, all the parameters of the theory are dimensionless, which would suggest the possibility of power counting renormalizability of the theory.

IV. DISCUSSION AND OUTLOOK

Here we have shown that General Relativity is a dynamical sector of a gWZW theory for the coset $\frac{SO(4,2)}{\mathbb{R}}$. It can be checked that the same phenomenon occurs if in the $SO(4, 2)$ gWZW, a representative of the group is taken as $h = e^{\phi J_{45}} e^{\pi^{a4} J_{a4}} e^{\pi^{a5} J_{a5}} e^{\pi^{ab} J_{ab}} = e^{\phi J_{45}} \bar{h}$, and ϕ is kept fixed in the action reducing the symmetry to $SO(3, 1) \times R$. The four-dimensional spacetime arises in the sector of solution space characterized by $\bar{h} = 1$, $\det e \neq 0$, $b = 0 = \mathcal{A}^{45}$.

Having obtained the equations of General Relativity, a more detailed analysis of the dynamical structure of the theory (9) is necessary. Generically, as it happens with all higher-dimensional Chern-Simons theories, the system will possess degenerate dynamical sectors [28–30]. A deeper understanding of this problem would be required prior to any study of the quantum properties of the action (9).

The theory changes dramatically if the ϕ field is regarded as dynamical. In that case, there is no purely gravitational sector with only e^a and ω^{ab} nonzero. However there are solutions of gravity coupled to the other fields, including an interesting class of gravitational solitons, that is, Lorentzian, everywhere regular, classical solutions. A two-parameter family of these solitons will be presented in [31].

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