

Static wormhole solution for higher-dimensional gravity in vacuumGustavo Dotti,¹ Julio Oliva,^{2,3} and Ricardo Troncoso³¹*Facultad de Matemática, Astronomía y Física, Universidad Nacional de Córdoba, Ciudad Universitaria, (5000) Córdoba, Argentina*²*Departamento de Física, Universidad de Concepción, Casilla, 160-C, Concepción, Chile*³*Centro de Estudios Científicos (CECS), Casilla 1469, Valdivia, Chile*

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A static wormhole solution for gravity in vacuum is found for odd dimensions greater than four. In five dimensions the gravitational theory considered is described by the Einstein-Gauss-Bonnet action where the coupling of the quadratic term is fixed in terms of the cosmological constant. In higher dimensions $d = 2n + 1$, the theory corresponds to a particular case of the Lovelock action containing higher powers of the curvature, so that in general, it can be written as a Chern-Simons form for the AdS group. The wormhole connects two asymptotically locally AdS spacetimes each with a geometry at the boundary locally given by $\mathbb{R} \times S^1 \times H_{d-3}$. Gravity pulls towards a fixed hypersurface located at some arbitrary proper distance parallel to the neck. The causal structure shows that both asymptotic regions are connected by light signals in a finite time. The Euclidean continuation of the wormhole is smooth independently of the Euclidean time period, and it can be seen as instanton with vanishing Euclidean action. The mass can also be obtained from a surface integral and it is shown to vanish.

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The quest for exact wormhole solutions in general relativity, which are handles in the spacetime topology, has appeared repeatedly in theoretical physics within different subjects, ranging from the attempt of describing physics as pure geometry, as in the ancient Einstein-Rosen bridge model of a particle [1], to the concept of “charge without charge” [2], as well as in several issues concerning the Euclidean approach to quantum gravity (see, e.g., [3]). More recently, during the 80’s, motivated by the possibility of quick interstellar travel, Morris, Thorne and Yurtsever pushed forward the study of wormholes from the point of view of “reverse engineering”, i.e., devising a suitable geometry that allows this possibility, and making use of the Einstein field equations in order to find the corresponding stress-energy tensor that supports it as an exact solution [4]. However, one of the obstacles to circumvent, for practical affairs, is the need of exotic forms of matter, since it is known that the required stress-energy tensor does not satisfy the standard energy conditions (see, e.g., [5]). Besides, the pursuit of a consistent framework for a unifying theory of matter and interactions has led to a consensus in the high energy community that it should be formulated in dimensions higher than four. However, for general relativity in higher dimensions, the obstacle aforementioned concerning the stress-energy tensor persists. Nonetheless, in higher dimensions, the straightforward dimensional continuation of Einstein’s theory is not the only option to describe gravity. Indeed, even from a conservative point of view, and following the same basic principles of general relativity, the most general theory of gravity in higher dimensions that leads to second order field equations for the metric is described by the Lovelock action [6], which is nonlinear in the curvature. In this vein, for the simplest extension, being quadratic in the curvature, it has been found that the so-called Einstein-Gauss-Bonnet theory,

admits wormhole solutions that would not violate the weak energy condition provided the Gauss-Bonnet coupling constant is negative and bounded according to the shape of the solution [7].

Here it is shown that in five dimensions, allowing a cosmological (volume) term in the Einstein-Gauss-Bonnet action, and choosing the coupling constant of the quadratic term such that the theory admits a single anti-de Sitter (AdS) vacuum, allows the existence of an exact static wormhole solution in vacuum. As explained below, the solution turns out to have “mass without mass” and connects two asymptotically locally AdS spacetimes each with a geometry at the boundary that is not spherically symmetric. It is worth to remark that no energy conditions can be violated since the whole spacetime is devoid of any kind of stress-energy tensor. In what follows, the five-dimensional case is worked out in detail, and next we explain how the results extend to higher odd dimensions for a special class of theories among the Lovelock class, which are also selected by demanding the existence of a unique AdS vacuum.

Static wormhole in five dimensions.—The action for the Einstein-Gauss-Bonnet theory with a volume term can be written as

$$I_5 = \kappa \int \epsilon_{abcde} \left(R^{ab} R^{cd} + \frac{2}{3l^2} R^{ab} e^c e^d + \frac{1}{5l^4} e^a e^b e^c e^d \right) e^e,$$

where $R^{ab} = d\omega^{ab} + \omega_f^a \omega^{fb}$ is the curvature 2-form for the spin connection ω^{ab} , and e^a is the vielbein. The coupling of the Gauss-Bonnet term has been fixed so that the theory possesses a unique AdS vacuum of radius l . In the absence of torsion, the field equations can be simply writ-

ten as

$$\mathcal{E}_a := \epsilon_{abcde} \bar{R}^{bc} \bar{R}^{de} = 0, \quad (1)$$

where $\bar{R}^{bc} := R^{bc} + \frac{1}{l^2} e^b e^c$. These equations are solved by the following metric

$$ds_5^2 = l^2[-\cosh^2(\rho - \rho_0) dt^2 + d\rho^2 + \cosh^2(\rho) d\Sigma_3^2], \quad (2)$$

where ρ_0 is an integration constant and $d\Sigma_3^2$ stands for the metric of the base manifold which can be chosen to be locally of the form $\Sigma_3 = S^1 \times H_2$. The radius of the hyperbolic manifold H_2 turns out to be $3^{-1/2}$, so that the Ricci scalar of Σ_3 has the value of -6 , as required by the field equations. The metric (2) describes a static wormhole with a neck of radius l , located at the minimum of the warp factor of the base manifold, at $\rho = 0$. Since $-\infty < \rho < \infty$, the wormhole connects two asymptotically locally AdS spacetimes so that the geometry at the boundary is locally given by $\mathbb{R} \times S^1 \times H_2$. Actually, it is simple to check that the field equations are solved provided the base manifold Σ_3 has a negative constant Ricci scalar. Indeed, for a metric of the form (2) the vielbeins can be chosen as

$$e^0 = l \cosh(\rho - \rho_0) dt; \quad e^1 = l d\rho; \\ e^m = l \cosh(\rho) \tilde{e}^m,$$

where \tilde{e}^m is the dreibein of Σ_3 , so that the curvature two form is such that the only nonvanishing components of \bar{R}^{ab} read

$$\bar{R}^{0m} = \cosh(\rho_0) dt \wedge \tilde{e}^m; \quad \bar{R}^{mn} = \tilde{R}^{mn} + \tilde{e}^m \tilde{e}^n. \quad (3)$$

Replacing Eqs. (3) in the field Eqs. (1) it turns out that the components \mathcal{E}_0 and \mathcal{E}_m are identically satisfied. The remaining field equation, $\mathcal{E}_1 = 0$, reads

$$\cosh(\rho_0) dt \wedge \epsilon_{mnp} (\tilde{R}^{mn} \tilde{e}^p + \tilde{e}^m \tilde{e}^n \tilde{e}^p) = 0,$$

which implies that Σ_3 must be a manifold with a constant Ricci scalar satisfying

$$\tilde{R} = -6. \quad (4)$$

One can notice that the field Eqs. (1) are deterministic unless Σ_3 were chosen as having negative constant curvature, i.e., being locally isomorphic to H_3 . If so, the field equations would degenerate in such a way that the component g_{tt} of the metric becomes an arbitrary function of ρ . This degeneracy is a known feature of the class of theories considered here [8], and is overcome by choosing a base manifold satisfying (4) but not being of constant curvature. A simple example of a compact smooth three-dimensional manifold fulfilling these conditions is given by $\Sigma_3 = S^1 \times H_2/\Gamma$, where H_2 has radius $\frac{1}{\sqrt{3}}$, and Γ is a freely acting discrete subgroup of $O(2, 1)$. It worth pointing out that Σ_3 is not an Einstein manifold, and that any nontrivial solution

of the corresponding Yamabe problem (see e.g. [9]) provides a suitable choice for Σ_3 .

The causal structure of the wormhole is depicted in Fig. 1, where the dotted vertical line shows the position of the neck, and the solid bold lines correspond to the asymptotic regions located at $\rho = \pm\infty$, each of them resembling an AdS spacetime but with a different base manifold since the usual sphere S^3 must be replaced by Σ_3 . The line at the center stands for $\rho = \rho_0$. It is apparent from the diagram that null and timelike curves can go forth and back from the neck. Furthermore, note that radial null geodesics are able to connect both asymptotic regions in finite time. Indeed, one can see from (2) that the coordinate time that a photon takes to travel radially from one asymptotic region, $\rho = -\infty$, to the other at $\rho = +\infty$ is given by

$$\Delta t = \int_{-\infty}^{+\infty} \frac{d\rho}{\cosh(\rho - \rho_0)} = [2 \arctan(e^{\rho - \rho_0})]_{-\infty}^{+\infty} = \pi,$$

which does not depend on ρ_0 . Thus, any static observer located at $\rho = \rho_0$ says that this occurred in a proper time given by πl . Note also that this observer actually lives on a static timelike geodesic, and it is easy to see that a small perturbation along ρ makes him to oscillate around $\rho = \rho_0$. This means that gravity is pulling towards the fixed hypersurface defined by $\rho = \rho_0$ which is parallel to the neck. Hence, the constant ρ_0 corresponds to a modulus parametrizing the proper distance between this hypersurface and the neck. Actually, one can explicitly check that radial timelike geodesics are always confined since they satisfy

$$\frac{1}{2} \dot{\rho}^2 - \frac{E^2}{2 \cosh^2(\rho - \rho_0)} = C_0,$$

$$\dot{t} - \frac{E}{\cosh^2(\rho - \rho_0)} = 0,$$

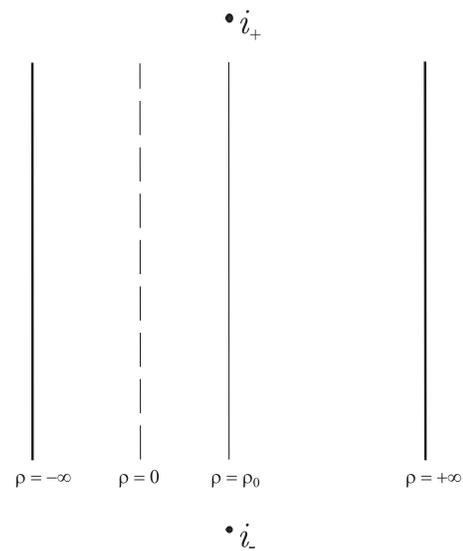


FIG. 1. Penrose diagram for the wormhole.

where the dot stand for derivatives with respect to the proper time τ , and the velocity is normalized as $u_\mu u^\mu = 2l^2 C_0$. Thus, one concludes that the position of a radial geodesic, $\rho(\tau)$, in proper time behaves as a particle in a Pöschl-Teller potential. Therefore, as it can also be seen from the Penrose diagram, null and spacelike radial geodesics connect both asymptotic regions in finite time. Furthermore, timelike geodesics for which $2l^2 C_0 = -1$ are shown to be confined.

Euclidean continuation and a finite action principle.—The Euclidean continuation of the wormhole metric (2) is smooth independently of the Euclidean time period, so that the wormhole could be in thermal equilibrium with a heat bath of arbitrary temperature. It is then useful to evaluate the Euclidean action for this configuration. It has been shown in [10] that the action I_5 can be regularized by adding a suitable boundary term in a background independent way, which can be written just in terms of the extrinsic curvature and the geometry at the boundary. The total action then reads $I_T = I_5 - B_4$, where the boundary term reads

$$B_4 = \kappa \int_{\partial M} \epsilon_{abcde} \theta^{ab} e^c \left(R^{de} - \frac{1}{2} \theta^d{}_f \theta^{fe} + \frac{1}{6l^2} e^d e^e \right). \quad (5)$$

For the wormhole solution, the boundary of M is of the form, $\partial M = \partial M_+ \cup \partial M_-$, where ∂M_- has a reversed orientation with respect to that of ∂M_+ . Using the fact that the only non vanishing components of the second fundamental form θ^{ab} for the wormhole (2), for each boundary, are given by

$$\theta^{01} = \sinh(\rho - \rho_0) d\tau; \quad \theta^{m1} = \sinh(\rho) \tilde{e}^m, \quad (6)$$

where τ now stands for the Euclidean time, it is simple to verify that the action principle I_T attains an extremum for the wormhole solution.

Let us evaluate the action I_T for the wormhole (2) with a base manifold of the form $\Sigma_3 = S^1 \times H_2/\Gamma$. Assuming that the boundaries, are located at $\rho = \rho_+$ and $\rho = \rho_-$, respectively, one obtains that

$$\begin{aligned} I_5 &= B_4 \\ &= 2\kappa l \beta \sigma [3 \sinh(\rho_0) + 8 \cosh^3(\rho) \sinh(\rho - \rho_0)] \rho^\pm, \end{aligned}$$

where β is the Euclidean time period, and $\sigma = \frac{8\pi^2}{3} R_0 (g - 1)$ is the volume of the base manifold Σ_3 in terms of the radius of S^1 and the genus of H_2/Γ , given by R_0 and g , respectively.

Therefore, remarkably, the regularized Euclidean action I_T does not depend on the integration constant ρ_0 , and it vanishes for each boundary regardless their position. This means that the Euclidean continuation of the wormhole can be seen as an instanton with vanishing Euclidean action.

Consequently, the total mass of the wormhole is found to vanish since $M = -\frac{\partial I_T}{\partial \beta} = 0$. The same results extend to any base manifold with a Ricci scalar satisfying Eq. (4).

It is worth pointing out that the value of the regularized action for the wormhole is lower than the one for AdS spacetime, which turns out to be $I_T(\text{AdS}) = 6\Omega_3 \kappa \beta$, where Ω_3 is the volume of S^3 . However, AdS spacetime has a negative “vacuum energy” given by $M_{\text{AdS}} = -6\Omega_3 \kappa$.

Mass from a surface integral.—The fact that the action principle I_T has an extremum for the wormhole solution, also allows to compute the mass from the following surface integral [10]

$$Q(\xi) = \kappa \int_{\partial \Sigma} \epsilon \left(I_\xi \theta \frac{e}{l} + \theta I_\xi \frac{e}{l} \right) \left(\tilde{R} + \frac{1}{2} \theta^2 + \frac{1}{2l^2} e^2 \right), \quad (7)$$

which is obtained by the straightforward application of Noether’s theorem [11]. The mass is obtained evaluating (7) for the timelike Killing vector $\xi = \partial_t$, and one then confirms that the mass, $M = Q(\partial_t)$, vanishes for the Lorenzian solution. We would like to remark that, following this procedure, one obtains that the contribution to the total mass coming from each boundary reads

$$Q_\pm(\partial_t) = \pm 6\sigma \kappa \sinh(\rho_0), \quad (8)$$

where $Q_\pm(\partial_t)$ is the value of (7) at $\partial \Sigma_\pm$, which again does not depend on ρ_+ and ρ_- . This means that for a positive value of ρ_0 , the mass of the wormhole appears to be positive for observers located at ρ_+ , and negative for the ones at ρ_- , such that the total mass always vanishes. This provides a concrete example of what Wheeler dubbed “mass without mass”. Hence, the integration constant ρ_0 could also be regarded as a parameter for the apparent mass at each side of the wormhole, which vanishes only when the solution acquires reflection symmetry, i.e., for $\rho_0 = 0$.

The wormhole in higher odd dimensions.—The five-dimensional static wormhole solution in vacuum, given by Eq. (2), can be extended as an exact solution for a very special class of gravity theories among the Lovelock family in higher odd dimensions $d = 2n + 1$. In analogy with the procedure in five dimensions, the theory can be constructed so that the relative couplings between each Lovelock term are chosen so that the action has the highest possible power in the curvature and possesses a unique AdS vacuum of radius l . The field equations then read

$$\mathcal{E}_A := \epsilon_{ab_1 \dots b_{2n}} \bar{R}^{b_1 b_2} \dots \bar{R}^{b_{2n-1} b_{2n}} = 0, \quad (9)$$

which are solved by the straightforward extension of (2) to higher dimensions

$$ds^2 = l^2 [-\cosh^2(\rho - \rho_0) dt^2 + d\rho^2 + \cosh^2(\rho) d\Sigma_{2n-1}^2],$$

where ρ_0 is an integration constant, and $d\Sigma_{2n-1}^2$ stands for the metric of the base manifold. In the generic case, the base manifold must solve the following equation [12]

$$\epsilon_{m_1 \dots m_{2n-1}} \bar{R}^{m_1 m_2} \dots \bar{R}^{m_{2n-3} m_{2n-2}} \tilde{\epsilon}^{m_{2n-1}} = 0, \quad (10)$$

where $\tilde{\epsilon}^m$ is the vielbein of Σ_{2n-1} . Note that this is a single scalar equation.

As in the five-dimensional case, it is worth to remark that the field Eqs. (9) are deterministic unless Σ_{2n-1} solves the field equations for the same theory in $2n-1$ dimensions with unit AdS radius. If so, the field equations would degenerate and the metric component g_{tt} would be an arbitrary function of ρ . In particular, the hyperbolic space H_{2n-1} falls within this degenerate class. Therefore, in order to circumvent this degeneracy, the base manifold must fulfill Eq. (10), but without solving simultaneously the field equations for the same theory in $2n-1$ dimensions with unit AdS radius.

A simple example of a compact smooth $(2n-1)$ -dimensional manifold fulfilling the latter conditions is given by $\Sigma_{2n-1} = S^1 \times H_{2n-2}/\Gamma$, where H_{2n-2} has radius $(2n-1)^{-1/2}$, and Γ is a freely acting discrete subgroup of $O(2n-2, 1)$. Note that Σ_{2n-1} is not an Einstein manifold.

The metric in higher odd dimensions then describes a static wormhole with a neck of radius l connecting two asymptotic regions which are locally AdS spacetimes, so that the geometry at the boundary is given by $\mathbb{R} \times S^1 \times H_{2n-2}/\Gamma$. The wormhole in higher dimensions shares the features described in the five-dimensional case, including the meaning of the parameter ρ_0 , and its causal structure is depicted in Fig. 1.

As in the five-dimensional case, the Euclidean continuation of the wormhole metric is smooth and it has an arbitrary Euclidean time period. The Euclidean action can be regularized in higher odd dimensions in a background independent way as in Ref. [10], by the addition of a suitable boundary term which is the analogue of (5), and can also be written in terms of the extrinsic curvature and the geometry at the boundary. The nonvanishing components of the second fundamental form θ^{ab} acquire the same form as in Eq. (6) for higher dimensions, so that it is easy to check that the regularized action has an extremum for the wormhole solution. As in the five-dimensional case, the Euclidean continuation of the wormhole can be seen as an instanton with a regularized action that vanishes independently of the position of the boundaries, so that its mass is also found to vanish. This means that AdS spacetime has a greater action than the wormhole, but a lower ‘‘vacuum energy’’.

The wormhole mass for the Lorenzian solution can also be shown to vanish making use of a surface integral which is the extension of (7) to higher odd dimensions [10]. The contribution to the total mass coming from each boundary

does not depend on the location of the boundaries and is given by

$$Q_{\pm}(\partial_t) = \pm \alpha_n \sigma \kappa \sinh(\rho_0),$$

so that for a nonvanishing integration constant ρ_0 , the wormhole appears to have ‘‘mass without mass’’. Here $\alpha_n := [(1-2n)^{n-1} - 2^n(1-n)^{n-1}](2n-1)!$.

It is simple to show that for different base manifolds, the Euclidean action also vanishes, and the surface integrals for the mass possess a similar behavior.

Final remarks.— The existence of interesting solutions in vacuum could be regarded as a criterion to discriminate among the different possible gravity theories that arise only in dimensions greater than four. Indeed, it has been shown that, among the Lovelock family, selecting the theories as having a unique maximally symmetric vacuum solution guarantees the existence of well-behaved black hole solutions [13,14]. In turn, it has been shown that demanding the existence of simple compactifications describing exact black p -brane solutions, selects the same class of theories [15] (see also [16]). In this sense, the theory possessing the highest possible power in the curvature with a unique AdS vacuum is particularly interesting and it is singled out for diverse reasons. It is worth to mention that in this case, the Lagrangian can be written as a Chern-Simons gauge theory for the AdS group [17], so that the local symmetry is enlarged from Lorentz to AdS. One can see that the wormhole solution found here is not only a vacuum solution for these theories, but also for their locally supersymmetric extension in five [18] and higher odd dimensions [19]. The compactification of the wormhole solution is straightforward since it has been shown that it always admits a base manifold with a S^1 factor. This means that in one dimension below, the geometrical and causal behavior is similar to the one described here, but in this case the base manifold is allowed to be locally a hyperbolic space without producing a degeneracy of the field equations. Note that the dimensionally reduced solution is supported by a nontrivial dilaton field with a nonvanishing stress-energy tensor.

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