Loop quantum cosmology and the $k = -1$ Robertson-Walker model

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The loop quantization of the negatively curved $k = -1$ Robertson-Walker model poses several technical challenges. We show that the issues can be overcome and a successful quantization is possible that extends the results of the $k = 0, +1$ models in a natural fashion. We discuss the resulting dynamics and show that for a universe consisting of a massless scalar field, a bounce is predicted in the backward evolution in accordance with the results of the $k = 0, +1$ models. We also show that the model predicts a vacuum repulsion in the high curvature regime that would lead to a bounce even for matter with vanishing energy density. We finally comment on the inverse volume modifications of loop quantum cosmology and show that, as in the $k = 0$ model, the modifications depend sensitively on the introduction of a length scale which *a priori* is independent of the curvature scale or a matter energy scale.

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I. INTRODUCTION

One of the major cosmological parameters yet to be determined precisely pertains to the spatial curvature of the universe. While current observations indicate that the universe is very nearly flat, they do not yet provide irrefutable evidence as to whether on very large scales the curvature is positive, negative, or exactly zero (the $k = +1, -1, 0$ Robertson-Walker (RW) models, respectively). The current observations merely provide evidence of a prediction of inflation; namely, the curvature scale is sufficiently large such as to appear very nearly flat to an observer. Such a feature is explained by inflation through a period of accelerated expansion in the early universe that inflates the curvature scale to very large values. Thus most work on structure formation has assumed that the universe is exactly flat with $k = 0$ which is a good approximation for the post inflationary epoch. However, it is the period of the early universe where the curvature can play an important role and thus should not be neglected.

It is also the high energy regime of the early universe where quantum gravity is expected to be a requirement for a complete description. While no complete and fully accepted quantum theory of gravity exists, a leading candidate exists which is known as loop quantum gravity (LQG) [1–3]. The application of LQG techniques to the cosmological setting, loop quantum cosmology (LQC), has so far been restricted to the $k = 0, +1$ models (see [4] for a review). One of the major successes of the models of LQC so far is the resolution of the classical singularity predicted in the $k = 0, +1$ models [5] which can result in a repulsive gravitational force at high energies that leads to a big bounce of the universe [6–9]. Thus an open question remains as to whether these results hold in the negatively curved $k = -1$ model and whether a loop quantization even exists.

The $k = -1$ model has not been constructed in LQC due to technical issues that inhibit a successful quantization. The $k = -1$ model can be derived as the isotropic limit of the homogeneous Bianchi V model which lacks a correct Hamiltonian framework [10]. The Hamiltonian framework is essential to the canonical quantization scheme of both LQG and LQC and thus this failure presents a roadblock to quantization. Notwithstanding this issue, as we shall show the $k = -1$ model also leads to subtle features in the choice of dynamical variables in LQC that require careful attention when attempting a quantization.

In this paper we will show that these issues can be successfully overcome leading to a loop quantization of the model. We will show that the Hamiltonian framework can be constructed specifically for the isotropic Bianchi V model and that the theory can be quantized incorporating techniques similar to those used in the loop quantization of spherically symmetric models. The resulting quantum theory is in a form that is similar to the $k = 0, +1$ LQC models and thus shares many of the same features. We show directly that the model predicts a big bounce in the backward evolution of the universe sourced by a massless scalar field. We describe this behavior in terms of an effective Friedmann equation that is quadratic in the matter energy density. Furthermore the effective Friedmann equation predicts a vacuum repulsion in the Planckian curvature regime, whereby a bounce would be triggered even with vanishing matter density. Finally, we comment on the inverse volume effects predicted by LQC and show that they are dependent on the introduction of a scale into the model which is not determined from the curvature scale or any matter energy scale. We discuss the phenomenological implications of this.

II. CLASSICAL FRAMEWORK

We begin with the classical framework that will form the basis of the loop quantization for the $k = -1$ model. Loop

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quantum gravity (and hence loop quantum cosmology) is based on a Hamiltonian framework using connection-triad variables as the gravitational field variables. The goal of this section is to consider the connection-triad variables which are invariant under the symmetries of the Bianchi V group (which leads to the $k = -1$ model), and then construct the Hamiltonian in terms of the reduced variables, and finally show that the equations of motion derived from the Hamiltonian give back the usual cosmological equations of motion for the open model.

The starting point for the homogeneous cosmological model we consider are the Bianchi models. The homogeneous metric is given by

$$ds^2 = -N(t)^2 dt^2 + \alpha_{ij}(t)\, {}^o\omega_a^i\, {}^o\omega_b^j dx^a dx^b, \tag{1}$$

where $\alpha_{ij}(t)$ are the dynamical components of the metric, $N(t)$ is known as the lapse and represents the rescaling freedom of the time coordinate, and ${}^o\omega_a^i$ are a basis of left-invariant one-forms determined by the group structure of the Bianchi model being considered. The left-invariant one-forms satisfy

$$d\, {}^o\omega^i = -\tfrac{1}{2} C^i{}_{jk}\, {}^o\omega^j \wedge {}^o\omega^k, \tag{2}$$

where $C^i{}_{jk}$ are the structure constants of the isometry group and thus characterize the Bianchi model. For the open $k = -1$ model, we consider the Bianchi V model with structure constants that can be taken of the form

$$C^i{}_{jk} = \delta^i_k \delta_{j1} - \delta^i_j \delta_{k1}. \tag{3}$$

The structure constants satisfy $C^i{}_{ij} \neq 0$ which in the language of [11] implies that the Bianchi V model is class B. This fact will be important in what we consider later.

In a particular choice of coordinates, Eqs (2) can be solved explicitly to give the left-invariant one-forms as

$${}^o\omega^1 = dx, \tag{4}$$

$${}^o\omega^2 = e^{-x} dy, \tag{5}$$

$${}^o\omega^3 = e^{-x} dz, \tag{6}$$

where the coordinates x, y, z are valued on the real line representative of the fact that we are considering the spatially noncompact $k = -1$ model with topology homeomorphic to $\mathbb{R}^3$. Thus we have not chosen a particular compactification of the $k = -1$ model and work in the usual model with infinite spatial extent. If we consider the isotropic limit of this Bianchi model with $\alpha_{ij}(t) = a^2(t)\delta_{ij}$ with $a(t)$ representing the scale factor, and fix the lapse to be equal to one, then the metric

$$\begin{aligned} ds^2 &= -dt^2 + a^2 \delta_{ij}\, {}^o\omega_a^i\, {}^o\omega_b^j dx^a dx^b \\ &= -dt^2 + a^2(dx^2 + e^{-2x} dy^2 + e^{-2x} dz^2) \end{aligned} \tag{7}$$

can be shown to have constant negative spatial curvature and hence corresponds to the open $k = -1$ model. The usual hyperbolic metric in hyperbolic coordinates $ds^2 = -dt^2 + a^2(d\psi^2 + \sinh^2\psi(d\theta^2 + \sin^2\theta d\phi^2))$ can be recovered with the following transformation

$$x = -\ln(\cosh\psi - \sinh\psi \cos\theta), \qquad y = \frac{\sin\theta \cos\phi}{\coth\psi - \cos\theta},$$
$$z = \frac{\sin\theta \sin\phi}{\coth\psi - \cos\theta}.$$

With the form of the metric (7), Einstein's equations lead to a set of differential equations satisfied by the scale factor $a(t)$ given by the Friedmann equation

$$\left(\frac{\dot a}{a}\right)^2 = \frac{\kappa}{3}\rho_M + \frac{1}{a^2} \tag{8}$$

and the acceleration equation

$$\frac{\ddot a}{a} = -\frac{\kappa}{6}(\rho_M + 3p_M) \tag{9}$$

with ρ_M and p_M being the matter density and pressure, respectively. Here $\kappa = 8\pi G$ with G being Newton's constant.

In addition to the left-invariant one-forms, for what follows we will also need a basis of vector fields ${}^o e_i^a$ which also are left invariant. The left-invariant vector fields have commutators which provide a representation of the Lie algebra under consideration

$$[{}^o e_i, {}^o e_j] = C^k{}_{ij}\, {}^o e_k \tag{10}$$

and are also dual to ${}^o\omega_a^i$ thus satisfying

$${}^o e_i^a\, {}^o\omega_a^j = \delta_i^j. \tag{11}$$

In the chosen coordinates for the Bianchi V model, ${}^o e_i^a$ are given explicitly as

$${}^o e_1 = \partial_x, \qquad {}^o e_2 = e^x \partial_y, \qquad {}^o e_3 = e^x \partial_z \tag{12}$$

whence it is simple to satisfy Eq. (10).

The classical framework of loop quantum cosmology (LQC) diverges from the standard framework in two important ways. The first is that the equations of motion are derived from a Hamiltonian framework which allows for a canonical quantization of the theory. Second, the variables that form the basis for quantization are not the usual metric ones (i.e., the scale factor). This framework follows directly from that used in the full theory of loop quantum gravity and it is these changes that allow for a rigorous quantization of gravity. The canonical set of variables consists of an orthonormal triad E_i^a (of density weight one) which encodes the information of spatial geometry, and an SU(2) valued connection A_a^i which is canonically conjugate to E_i^a. The starting point of LQC is to reduce these variables to the symmetry of the cosmological model. We can use the basis provided by the left-invariant one-forms and vector fields to accomplish this.

Starting with the triad E^a_i, we expand using the basis vector fields as

$$E^a_i = \sqrt{{}^o q}\tilde{p}(t){}^o e^a_i, \tag{13}$$

where $\tilde{p}(t)$ represents the dynamical component of the triad. The factor $\sqrt{{}^o q} = e^{-2x}$ is a density weight provided by the hyperbolic metric ${}^o q_{ab} = {}^o\omega^i_a {}^o\omega^i_b$ which gives the triad E^a_i its density weight. E^a_i encodes the spatial geometry in a specific fashion being that it is related to the spatial three-metric q_{ab} through

$$E^a_i E^{bi} = |q| q^{ab}. \tag{14}$$

Using this relation, we find that $\tilde{p}$ is related to the scale factor as

$$|\tilde{p}| = a^2, \tag{15}$$

where the absolute value indicates that we are allowing $\tilde{p}$ to take on positive and negative values in contrast to the scale factor which is usually assumed to be strictly non-negative. A change in sign of $\tilde{p}$ corresponds to a change in orientation of the triad E^a_i leaving the metric q_{ab} invariant.

The first nontriviality of the $k = -1$ arises when we consider a symmetric connection A^i_a. From the $k = 0, +1$ models, we expect that an isotropic connection can be decomposed using the left-invariant one-forms as $A^i_a = \tilde{c}(t){}^o\omega^i_a$ [12,13] with $\tilde{c}$ being the *only* dynamical component. In this form, the connection is diagonal in the basis of left-invariant one-forms. However, this form must be consistent with the fact that on the half-shell (after solving Hamilton's equations for $\dot{E}^a_i$), A^i_a is determined from the dynamics of the spatial metric as

$$A^i_a = \gamma K^i_a + \Gamma^i_a, \tag{16}$$

where K^i_a is the extrinsic curvature, γ is known as the Barbero-Immirzi parameter (a real valued ambiguity parameter of loop quantum gravity), and Γ^i_a is the spin connection. Upon symmetry reduction, the extrinsic curvature can be shown to be of diagonal form[1] $K^i_a = \mathrm{sgn}(\tilde{p})\dot{a}{}^o\omega^i_a$ which is consistent with the connection being diagonal. However, this is not the case with the spin connection Γ^i_a. The formula for the spin connection is given by

$$\Gamma^i_a = -\tfrac{1}{2}\epsilon^{ijk} e^b_j(\partial_a e^k_b - \partial_b e^k_a + e^c_k e^l_a \partial_c e^m_b \delta_{lm}), \tag{17}$$

where e^i_a is the physical triad satisfying

$$e^i_a e^i_b = q_{ab}. \tag{18}$$

The physical triad e^i_a is related to E^a_i through

$$e^a_i = \frac{1}{\sqrt{|q|}} E^a_i. \tag{19}$$

Using the symmetric form of E^a_i (13) and evaluating (17), one finds that the spin connection is given by

$$\Gamma^i_a = \Gamma^i_j {}^o\omega^j_a \tag{20}$$

with

$$\Gamma^i_j = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \tag{21}$$

whence it is clear that the spin connection is nondiagonal, and the assumption that the connection is diagonal is not consistent. We must therefore take the connection to be of nondiagonal form

$$A^i_a = A^i_j(t){}^o\omega^j_a, \qquad A^i_j = \begin{pmatrix} \tilde{c}(t) & 0 & 0 \\ 0 & \tilde{c}(t) & -\tilde{c}_2(t) \\ 0 & \tilde{c}_2(t) & \tilde{c}(t) \end{pmatrix}. \tag{22}$$

In this form, the connection has two dynamical components $\tilde{c}$ and $\tilde{c}_2$, where on the half-shell $\tilde{c} = \mathrm{sgn}(\tilde{p})\gamma\dot{a}$ is determined from the extrinsic curvature and $\tilde{c}_2 = 1$. This is in contrast to the $k = 0, +1$ models where the connection can safely be assumed to be diagonal and only has one dynamical component.

With the symmetry reduced connection-triad variables, the next step is to show that the Hamiltonian formulation leads to the correct classical equations of motion. Yet here another problem arises. The Bianchi V model is of class B type where, as first shown in [10], the ADM Hamiltonian formulation in the general homogeneous case fails. The main issue is that the equations of motion derived from the symmetry reduced Hamiltonian do not agree with Einsteins' equations after symmetry reduction. In other words, the symmetry reduction and Hamiltonian formulation do not commute in the class B models. While this may seem a fatal issue for the $k = -1$ model, it was shown in [10] that the Hamiltonian formulation does not fail for the isotropic limit of the Bianchi V model which is precisely the case in which we are interested. This failure does hinder the extension of the results presented here to the anisotropic Bianchi V model, but we will now show explicitly that the Hamiltonian formulation of the isotropic model using the connection-triad variables leads to the correct equations of motion. Another avenue worth exploration is whether the analysis of [10] holds in general for the Hamiltonian formulation based on the connection-triad variables used in loop quantum gravity which is manifestly different than the ADM Hamiltonian formulation and thus may not suffer from the same issues. We do not attempt to address this possibility here.

Therefore, our aim now is to plug in the symmetry reduced connection (22) and triad (13) into the Hamil-

[1]The $\mathrm{sgn}(p)$ arises because the extrinsic curvature one-form carries the signature of the triad which is evident from the definition $K^i_a = e^{ai}K_{ab}$ where K_{ab} is the usual extrinsic curvature of the Arnowitt-Deser-Misner (ADM) formulation which does not carry information about the orientation.

tonian of the full theory and show that we get back the correct equations of motion (8) and (9). The action written in terms of the connection-triad variables[2] is given as [1]

$$S_{GR}[E, A, \lambda^i, N^a, N] = \int dt \int d^3x \frac{1}{\kappa\gamma} E^a_i \mathcal{L}_t A^i_a - [\lambda^i G_i + N^a \mathcal{C}_a + N\mathcal{C}_{GR}], \quad (23)$$

whence the Hamiltonian is a sum of constraints: G_i is the Gauss constraint, $\mathcal{C}_a$ is the diffeomorphism constraint, and $\mathcal{C}_{GR}$ is the Hamiltonian constraint. The parameters λ^i, N^a, N are Lagrange multipliers which enforce the vanishing of the constraints. The first term of the action indicates that the connection and triad are canonically conjugate with Poisson brackets

$$\{A^i_a(x), E^b_j(y)\} = \kappa\gamma\delta^i_j\delta^b_a\delta(x - y). \quad (24)$$

Hamilton's equations for the connection A^i_a and triad E^b_j can then be shown to be equivalent to Einstein's equations.

When inserting the symmetry reduced connection and triad into the action, the first issue we face is that the spatial integration in the action diverges since we are considering the noncompact $k = -1$ model. This same issue arises in the noncompact $k = 0$ model and would arise in any cosmological quantization scheme based on a Hamiltonian or action framework. To overcome this, we choose to follow the technique used in the $k = 0$ model for LQC [15]; namely, we restrict the spatial integration to a finite sized fiducial cell with a fixed background volume

$$V_0 = \int d^3x \sqrt{{}^o q}. \quad (25)$$

Note that the extent of the fiducial cell is fixed on the manifold or in other words has fixed comoving coordinates. Thus, as the universe expands for instance, so would the physical size of the fiducial cell. The choice in the fiducial cell remains a quantum ambiguity and we will be interested in determining whether the resulting quantum theory makes predictions dependent on V_0. As we now show, the choice in fiducial cell has no effect classically, but that is not true in the quantum case which we will discuss later.

Now with the understanding that we are limiting the spatial integrations in the action to the fiducial cell we can insert the symmetry reduced connection and triad (13) and (22). The canonical term is given by

$$\int dt \int d^3x \frac{1}{\kappa\gamma} E^a_i \mathcal{L}_t A^i_a = \int dt \frac{3V_0}{\kappa\gamma} \tilde{p}\, \dot{\tilde{c}} \quad (26)$$

which indicates that $\tilde{c}$ and $\tilde{p}$ are canonically conjugate with Poisson brackets

$$\{\tilde{c}, \tilde{p}\} = \frac{\kappa\gamma}{3V_0}. \quad (27)$$

The Gauss constraint in terms of the reduced variables is given by

$$G_i \equiv \partial_a E^a_i + \epsilon^k_{ij} A^j_a E^a_k = \frac{2V_0}{\kappa\gamma} \tilde{p}(\tilde{c}_2 - 1)\delta_{i1} \quad (28)$$

and thus is nonvanishing. This is in contrast to the $k = 0$, $+1$ models where the Gauss constraint vanishes indicative of the fact that a complete gauge fixing of the Gauss constraint was performed in those models. This suggests that we should gauge fix the Gauss constraint by setting $\tilde{c}_2$ to be identically equal to one. With this, the Gauss constraint vanishes and additionally the diffeomorphism constraint C_a can be shown to vanish. With this gauge fixing the connection is now of the form

$$A^i_j = \begin{pmatrix} \tilde{c} & 0 & 0 \\ 0 & \tilde{c} & -1 \\ 0 & 1 & \tilde{c} \end{pmatrix} \quad (29)$$

and we are now left with two dynamical phase-space variables $\tilde{p}$ and $\tilde{c}$ and one surviving constraint, the Hamiltonian constraint. This is exactly the situation in the $k = 0$, $+1$ models.

The dynamics of the model is now entirely encoded in the Hamiltonian constraint which is given by

$$\mathcal{C}_{GR} = -\frac{6V_0}{\gamma^2}\sqrt{|\tilde{p}|}(\tilde{c}^2 - \gamma^2) \quad (30)$$

and the entire gravitational action becomes[3]

$$S_{GR}[\tilde{p}, \tilde{c}, N] = \int dt \frac{3V_0}{\kappa\gamma} \tilde{p}\, \dot{\tilde{c}} - \frac{N}{2\kappa}\left[-\frac{6V_0}{\gamma^2}\sqrt{|\tilde{p}|}(\tilde{c}^2 - \gamma^2)\right], \quad (31)$$

whence the total Hamiltonian including matter is given by

$$\mathcal{H} = -\frac{3V_0 N}{\kappa\gamma^2}\sqrt{|\tilde{p}|}(\tilde{c}^2 - \gamma^2) + \mathcal{H}_M \quad (32)$$

with $\mathcal{H}_M$ denoting the matter Hamiltonian.

With the Hamiltonian and Poisson structure we can now derive the classical equations of motion. We first have Hamilton's equations $\dot{x} = \{x, \mathcal{H}\}$ for any phase-space variable x, and further the Hamiltonian itself must vanish since it is proportional to the Hamiltonian constraint. Starting with Hamilton's equations for $\tilde{p}$ we find

[2]The action can also be derived from a Legendre transform of the covariant Holst action written in terms of a four dimensional so (1) and (3) connection and a cotetrad [14].

[3]The extra factor of 2κ appearing below the lapse N appears because the Hamiltonian constraint used in previous works of LQC differs from the Hamiltonian constraint in the full theory given in [1] by the factor of 2κ. A constant factor multiplying the Hamiltonian constraint does not affect any physical results.

$$\dot{\tilde{p}} = \{\tilde{p}, \mathcal{H}\} = -\frac{\kappa\gamma}{3V_0}\frac{\partial\mathcal{H}}{\partial\tilde{c}} = \frac{2\sqrt{|\tilde{p}|}}{\gamma}\tilde{c}, \qquad (33)$$

where for the equations of motion we have fixed the lapse $N = 1$. Notice that the factors of V_0 cancel appearing both in the numerator of the Hamiltonian (32) and in the denominator of the Poisson brackets (27). Furthermore, we have assumed that the matter Hamiltonian only couples to the spatial geometry i.e., is only a function of $\tilde{p}$ and not $\tilde{c}$. This assumption is true for scalar fields and perfect fluids, though it is not true for fermions for instance which we do not consider (see [16] for discussions for the inclusion of fermions in LQG with physical effects dependent on the Barbero-Immirzi parameter γ). Using $|\tilde{p}| = a^2$ we can write the left-hand side of the Friedmann equation as

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{\tilde{c}^2}{\gamma^2|\tilde{p}|}. \qquad (34)$$

Now we can use the vanishing of the constraint to relate the right-hand side to the matter density. Using $\mathcal{H} = 0$ we find

$$H^2 = \frac{\kappa}{3}\frac{\mathcal{H}_M}{V_0\tilde{p}^{3/2}} + \frac{1}{|\tilde{p}|} = \frac{\kappa}{3}\rho_M + \frac{1}{a^2}, \qquad (35)$$

where we have used $\mathcal{H}_M/(V_0\tilde{p}^{3/2}) = \rho_M$. Thus the reduced Hamiltonian gives back the correct Friedmann equation. Similarly the acceleration equation can be derived by considering Hamilton's equation for $\dot{\tilde{c}}$ once the matter Hamiltonian is explicitly specified.

This derivation demonstrates explicitly that the Hamiltonian framework presented here leads to Einsteins equations for the open $k = -1$ model. This also demonstrates that the equations of motion classically are insensitive to the choice in fiducial cell V_0 which was introduced to regulate the divergent spatial integrals in the action and resulting Hamiltonian. Furthermore, the Hamiltonian is similar in form to the $k = 0$, $+1$ models and thus is indicative that a successful loop quantization is possible.

Before turning to the quantization, we would like to make a closer connection to the LQC work of the $k = 0$, $+1$ models. There we can define untilded variables by rescaling $\tilde{p}$ and $\tilde{c}$ by a factor dependent on the fiducial cell as

$$p \equiv V_0^{2/3}\tilde{p}, \qquad c \equiv V_0^{1/3}\tilde{c}. \qquad (36)$$

In terms of the untilded variables, the Poisson bracket is now independent of V_0

$$\{c, p\} = \frac{\kappa\gamma}{3} \qquad (37)$$

and the Hamiltonian constraint becomes

$$\mathcal{C}_{GR} = -\frac{3}{\kappa\gamma^2}\sqrt{|p|}(c^2 - V_0^{2/3}\gamma^2). \qquad (38)$$

The relation between the rescaled triad and the scale factor is given by

$$|p| = V_0^{2/3}a^2 \qquad (39)$$

as well as the half-shell relation

$$c = \mathrm{sgn}(p)V_0^{1/3}\gamma\dot{a}. \qquad (40)$$

We will use the Hamiltonian based on the untilded variables as the starting point for quantization. Notice that V_0 appears explicitly in the Hamiltonian constraint (38) which is in contrast to the $k = 0$ model where V_0 drops out of the constraint.

It will be important in the interpretation of the quantum theory to understand the physical meaning of the variable p. From the relation (39), we find

$$|p|^{3/2} = V_0a^3 = V_{\mathrm{cell}} \qquad (41)$$

which one recognizes as representing the *physical* volume of the fiducial cell. Note that in order to physically measure the value of p, one would need to prescribe the size of the fiducial cell. For instance, if today the fiducial cell is taken to have Planckian physical volume: $V\mathrm{cell} = l_{\mathrm{P}}^3$, then p is similarly Planckian: $p = l_{\mathrm{P}}^2$. This can be so despite the fact that, assuming we live in an open $k = -1$ universe, the value of the scale factor a is astronomically large. Thus there is no direct correlation between the value of the scale factor a, and the value of p. Again the value of p is highly dependent on the size of the fiducial cell.

Let us conclude this section with a further note about the fiducial cell. Since we will be interested in whether the quantum predictions are sensitive to this choice, we would like to know how the classical variables transform under a change in its size. For instance, let us consider that the fiducial cell is resized as

$$V_0 \rightarrow V_0' = \xi^3 V_0 \qquad (42)$$

then from their definition (36), the untilded variables transform as

$$p \rightarrow p' = \xi^2 p, \qquad (43)$$

$$c \rightarrow c' = \xi c. \qquad (44)$$

Note that the scale factor a (and therefore $\tilde{p}$ and $\tilde{c}$) do not make reference to the fiducial cell and therefore do not rescale under this change. The scaling of p can be understood from (41) by noting that the value of p is determined from the physical volume of the cell, and thus if the cell is enlarged, we expect the value of p to be larger. The untilded variables therefore do not classically have direct physical meaning as they can be freely rescaled under this transformation. What is physical classically are *changes* in p and c where for instance the Hubble rate $H = \frac{1}{2}(\frac{\dot{p}}{p})$ is invariant under a resizing of the fiducial cell.

III. QUANTIZATION

With the classical framework completed, we can now turn to the loop quantization of the model. To achieve this in the canonical quantization scheme involves the following steps. First one chooses a set of basic variables and finds a quantum representation of their algebra in order to construct what is known as the kinematical Hilbert space. The next step is to construct an operator corresponding to the Hamiltonian constraint that is self-adjoint in the kinematical Hilbert space. Finally, the physical Hilbert space consists of wave functions that are annihilated by the constraint operator and that have finite norm in a suitable physical inner product (which typically is not equivalent to the kinematical inner product). One then interprets the theory by evaluating expectation values of observables on physical wave functions. By considering the $k = -1$ model sourced with a massless scalar field, this program can be carried out to completion. The construction of the $k = -1$ model presented here follows closely that of the $k = 0$ model presented in [6], and thus we will omit many of the technical details and refer the reader to that article for a complete description.

A. Kinematical Hilbert space

To construct the kinematical Hilbert space we first must consider the elementary variables that will form the basis for quantization. From the full theory of LQG, one does not take the bare connection A_a^i and triad E_i^a as the basic variables. Rather, in the case of the connection, one integrates A_a^i along edges and then exponentiates the quantity leading to a holonomy. The holonomy variables are then taken as the basic configuration variables. The momentum variables are fluxes which are constructed by integrating of the triad over a two-surface. In the cosmological setting, fluxes are simply proportional to p which therefore forms an elementary variable. On the other hand, the holonomies amount to exponentials of the connection c and it is this fact that becomes the departure point of LQC from previous versions of quantum cosmology based on a Schrodinger-type quantization of the Hamiltonian.

Thus let us consider the holonomies in detail. In the $k = 0$ model they consist of integrating the connection along edges generated by the left-invariant vector fields and assume the form $h_i = \cos(\frac{\bar{\mu}c}{2}) + 2\sin(\frac{\bar{\mu}c}{2})\tau_i$ where $\bar{\mu}$ is equal to the fiducial length of the edge divided by $V_0^{1/3}$, and τ_i are the generators of SU(2) satisfying $\{\tau_i, \tau_j\} = \epsilon_{ij}{}^k\tau_k$. With holonomies of this form, the algebra generated is that of the almost periodic functions (which look like exponentials of the connection $e^{i\bar{\mu}c}$) and the kinematical Hilbert space assumes a simple form [15]. However, when we consider holonomies of the connection in the $k = -1$ model considered here, they take on a more complicated form where for instance the holonomy along the edge generated by ${}^oe_2^a$ is given by

$$h_2(\bar{\mu}) = \cos\frac{\bar{\mu}\sqrt{c^2 + V_0^{2/3}}}{2} + \frac{2[c\tau_2 - V_0^{1/3}\tau_3]}{\sqrt{c^2 + V_0^{2/3}}} \times \sin\frac{\bar{\mu}\sqrt{c^2 + V_0^{2/3}}}{2} \tag{45}$$

and thus the algebra generated is no longer simply that of the almost periodic functions. Finding a representation of the algebra would be difficult.

However, we can exploit a technique used in the loop quantization of other models such as the spherically symmetric models of LQG [17] as well as the quantization of the Schwarzschild horizon interior [18]. The complicated form of the holonomies of the connection arises because of the nondiagonal form of the connection (29). If we consider instead holonomies of the connection minus the spin-connection (essentially holonomies of the extrinsic curvature) as done in [17,18], then the holonomies are of a form equivalent to the $k = 0, +1$ models

$$h_i = \cos\left(\frac{\bar{\mu}c}{2}\right) + 2\sin\left(\frac{\bar{\mu}c}{2}\right)\tau_i, \tag{46}$$

where again c refers to the diagonal component of the connection in (29). In the full theory, the extrinsic curvature is not a connection and hence its holonomies are not defined. However, in the reduced setting we have performed a complete SU(2) gauge fixing to arrive at symmetric connections and thus it is possible to regard the extrinsic curvature as a connection. The resulting quantization will be a slight departure from that predicted by the full theory and thus care must be taken when interpreting the results. In Sec. V, we will comment on the regime where we expect the differences to occur.

Additionally we shall follow the prescription of [6] leading to improved dynamics for LQC. Namely, in contrast to the original literature of LQC, we assume that the parameter $\bar{\mu}$ appearing in the holonomies is a function of p and not a constant. The motivation for this can be seen as twofold. First let us consider the issue of the fiducial cell dependence. The quantity $\bar{\mu}c$ appears in the holonomies (46) and we have shown that under a resizing of the fiducial cell, the connection c scales according to Eq. (44). Quantum corrections can arise when $\bar{\mu}c$ becomes on the order of 1 [19] and thus we can generate arbitrarily large quantum corrections by choosing a larger fiducial cell as long as $\bar{\mu}$ is a fixed constant. However, if $\bar{\mu}$ scales as

$$\bar{\mu} \propto \frac{1}{\sqrt{|p|}} \tag{47}$$

then we find that the quantity $\bar{\mu}c$ is invariant under a resizing of the fiducial cell. A direct result of this in the $k = 0$ model is that the bounce occurs when the matter energy density is on the order of Planckian [6] which is to be expected on physical grounds. On the other hand, if $\bar{\mu}$ is a fixed constant, the bounce can occur even at largely sub-Planckian densities and even a cosmological constant can trigger a future recollapse of the universe [7,8,20]. The second motivation for this scaling comes from the method proposed in [15] to constrain the value of $\bar{\mu}$ based on using the minimum area eigenvalue of LQG in constructing the Hamiltonian constraint operator. We will discuss this in more detail when we construct the constraint operator. For now let us assume that $\bar{\mu}$ is given as

$$\bar{\mu} = \sqrt{\frac{\Delta}{|p|}}, \tag{48}$$

where Δ is a constant to be fixed later.

The kinematical Hilbert space can then be constructed and a basis is given by eigenstates of the $\hat{p}$ operator labeled by a real parameter v with eigenvalues

$$\hat{p}|v\rangle = \frac{\kappa\gamma\hbar}{6}\left(\frac{|v|}{K}\right)^{2/3}|v\rangle, \tag{49}$$

where the constant K is given by

$$K = \frac{2}{3}\sqrt{\frac{\kappa\gamma\hbar}{6\Delta}}. \tag{50}$$

Similarly the states $|v\rangle$ are eigenstates of the fiducial cell volume operator

$$\hat{V}_{\rm cell}|v\rangle = \left(\frac{\kappa\gamma\hbar}{6}\right)^{2/3}\frac{|v|}{K}|v\rangle. \tag{51}$$

The parameter v runs over the entire real line, but the spectrum is discrete in the sense that the states $|v\rangle$ are normalizable satisfying

$$\langle v'|v\rangle = \delta_{v'v}. \tag{52}$$

A general quantum state is a continuous sum over the basis states $|v\rangle$ as well as any matter degrees of freedom. We will interest ourselves in the inclusion of a scalar field degree of freedom whence a general quantum state is given by

$$|\Psi\rangle = \int d\phi \sum_{v} \Psi(v, \phi)|v, \phi\rangle \tag{53}$$

with the kinematical inner product between two states given by

$$\langle\Psi_1|\Psi_2\rangle_{\rm kin} = \int d\phi \sum_{v} \bar{\Psi}_1(v, \phi)\Psi_2(v, \phi). \tag{54}$$

A quantum state which lies in the kinematical Hilbert space has a finite kinematical norm which implies

$$\int d\phi \sum_{v} \bar{\Psi}(v, \phi)\Psi(v, \phi) < \infty. \tag{55}$$

This constitutes the kinematical Hilbert space as well as the action of the basic flux operator $\hat{p}$. Additional basic operators are required in the form of holonomy operators which can be built using the formula (46) and the basic exponential operators

$$\hat{h}_{\pm} = \exp(\mp i\widehat{\bar{\mu}c}/2). \tag{56}$$

The basis $|v\rangle$ has been chosen such that the exponential operators act simply as shift operators

$$\hat{h}_{\pm}\Psi(v) = \Psi(v \pm 1). \tag{57}$$

An important feature of the quantization is that since holonomies form the basic configuration variables, there is no basic operator corresponding to the connection $\hat{c}$. In order to construct such an operator, one has to approximate it using the basic holonomy operators. An example of this is given by the Hamiltonian constraint to which we turn now.

B. Quantum difference equation

The next step in quantization is to construct a Hamiltonian constraint operator that is self-adjoint on the kinematical Hilbert space. The classical expression for the gravitational part of the constraint is again given by

$$\mathcal{C}_{GR} = -\frac{6}{\gamma^2}\sqrt{|p|}(c^2 - \gamma^2 V_0^{2/3}) \tag{58}$$

which is equivalent to the $k = 0$ model up to the $\gamma^2 V_0^{2/3}$ term in the parentheses. The main complication in constructing the gravitational part of the Hamiltonian constraint operator is the lack of an operator for the bare connection. Thus the c^2 term must be quantized using holonomies. Following the results from the $k = 0$ model, the following classical reexpression

$$\mathcal{C}_{GR} = -\frac{4}{\kappa\hbar\gamma^3\bar{\mu}^3}\sum_{ijk}\epsilon^{ijk}\,{\rm tr}[(h_i h_j h_i^{-1}h_j^{-1} - 2\bar{\mu}^2\gamma^2 V_0^{2/3}\tau_i\tau_j)h_k\{h_k^{-1}, V\}] \tag{59}$$

can be shown to give back the classical expression (58) in the limit as $\bar{\mu}$ is taken to zero. This expression is now readily quantizable with the major nontriviality being that we cannot take the limit as $\bar{\mu}$ goes to zero as that would require a $\hat{c}$ operator. Thus in the quantum constraint operator we do not take the limit, instead leaving $\bar{\mu}$ to be a finite parameter given by the expression (48).

In order to constrain the parameter Δ in the definition of $\bar{\mu}$ (or equivalently the parameter K in (50)), in the $k = 0$ model one can connect to the full theory of LQG by shrinking $\bar{\mu}$ until the closed loop spanned by the edges of the holonomies $h_i h_j h_i^{-1}h_j^{-1}$ has the minimum physical area eigenvalue of LQG. This fixes the value of Δ to be equal to the minimum area eigenvalue of LQG $\Delta = 2\sqrt{3}\pi\gamma l_{\rm P}^2$ which implies that K is given by $K = \frac{2\sqrt{2}}{3\sqrt{3\sqrt{3}}}$ [6]. However, in the $k = -1$ model this interpretation does not hold since the edges do not close. Thus we cannot *a priori* make the same assignment of K. We can however turn to the $k = +1$ model for guidance. There, the quantization has been performed using holonomies of the extrinsic curvature where the loop similarly does not close [13,21]. The quantization of the $k = +1$ involving holonomies of the connection and using a closed loop for the constraint operator appears in [9] and there the value of K is constrained to the same value as the $k = 0$ model using the same procedure. Furthermore, the quantization using holonomies of the connection is quantitatively similar to the one using holonomies of the extrinsic curvature in the $v \gg 1$ regime which can be taken as evidence that the same value of K should be used in both quantizations. With this in mind, we will leave this issue open and assume that the parameter K is on the order of 1 without explicitly fixing its value.

With the caveats mentioned, the construction of the constraint operator follows that of the $k = 0$ model (see [6] for details). The action on the operator is given by

$$\hat{C}_{GR}\Psi(v) = f_+(v)\Psi(v+1) + f_0(v)\Psi(v) + f_-(v)\Psi(v-1) \tag{60}$$

with the functions f given by

$$f_+(v) = \frac{27}{16}\sqrt{\frac{\kappa\gamma\hbar}{6}}\frac{K}{\gamma^2}|v+2||\,|v+1| - |v+3|\,|, \tag{61}$$

$$f_-(v) = f_+(v-4), \tag{62}$$

$$f_0(v) = -f_+(v) - f_-(v) + g(v), \tag{63}$$

and the function $g(v)$ representing the modification coming from the $k = -1$ model given explicitly as

$$g(v) = \frac{3V_0^{2/3}}{K^{1/3}}\sqrt{\frac{\kappa\gamma\hbar}{6}}|v|^{1/3}|\,|v+1| - |v-1|\,|. \tag{64}$$

Thus the contribution from the $k = -1$ model amounts to the addition of a term $g(v)$ that acts diagonally on the basis states $|v\rangle$.

To discuss dynamics and interpret the difference equation we can add matter in the form of a massless scalar field as done in [6]. Since the difference equation will be of similar form as the $k = 0$ model most of the results remain valid. With a massless scalar field, the full constraint is given by

$$\hat{C} = \hat{C}_{GR} + \kappa\hat{p}^{-3/2}\hat{P}_\phi^2, \tag{65}$$

where P_ϕ is the canonical momentum to the scalar field. Since the Hamiltonian is independent of the scalar field ϕ, the conjugate momentum P_ϕ is a constant of motion classically. The classical Friedmann equation is given by

$$H^2 = \frac{\kappa}{6}\frac{P_\phi^2}{V_0^2 a^6} + \frac{1}{a^2} \tag{66}$$

which can be solved explicitly in terms of conformal time $d\eta = adt$ giving

$$a^2(\eta) = \sqrt{\frac{\kappa P_\phi^2}{6V_0^2}}\sinh(2\eta) \tag{67}$$

and similarly the scalar field evolves as

$$\phi(\eta) = \frac{1}{2}\sqrt{\frac{6}{\kappa}}\ln(\tanh\eta) + \phi_0. \tag{68}$$

Both are monotonic functions and thus can play the role of emergent time. If we choose the scalar field to play the role of emergent time the evolution of a is given by

$$a^2(\phi) = \sqrt{\frac{\kappa P_\phi}{6}}\mathrm{csch}\sqrt{2\kappa/3}(\phi - \phi_0). \tag{69}$$

For the quantization of the matter part of the constraint, the operator for P_ϕ acts simply as $\hat{P}_\phi = -i\hbar\partial/\partial\phi$. The inverse volume operator $\hat{p}^{-3/2}$ requires careful treatment as the naive inverse of the $\hat{p}$ operator does not lead to a densely defined self-adjoint operator owing to the fact the *normalizable* state $|v = 0\rangle$ lies in the spectrum of the $\hat{p}$ operator. Using techniques from the full theory [22,23], the application to LQC leads to a bounded self-adjoint operator [24] with eigenvalues given by [6]

$$\hat{p}^{-3/2}\Psi(v) = \left(\frac{6}{\kappa\gamma\hbar}\right)^{3/2}B(v)\Psi(v) \tag{70}$$

with the function $B(v)$ given by

$$B(v) = \left(\frac{3}{2}\right)^3 K|v||\,|v+1|^{1/3} - |v-1|^{1/3}|^3. \tag{71}$$

This inverse volume operator represents one choice among many possible choices of the types that have been explored in [24]. In particular there is freedom to use a particular spin J SU(2) representation to define the holonomies. The operator shown here corresponds to using the fundamental representation ($J = 1/2$) in accordance with arguments indicating that the theory should be quantized using that value [25,26]. The behavior of $B(v)$ changes for $v < 1$ and $v > 1$. For $v < 1$, $B(v)$ behaves polynomially and increases for large values of v while it vanishes at the singularity $v = 0$. For $v > 1$, $B(v)$ approaches the classical expression $B(v) \approx (\frac{\kappa\gamma\hbar}{6})^{3/2}p^{-3/2}$. The physical meaning of $v < 1$ is dependent on the choice of fiducial cell and we will discuss this in more detail in Sec. V.

With the matter constraint operator, the difference equation can be rearranged into the form

$$\frac{\partial^2\Psi(v,\phi)}{\partial\phi^2} = -\hat{\Theta}\Psi(v,\phi) \equiv -(\hat{\Theta}_0 + \hat{\Theta}_{-1})\Psi(v,\phi) \tag{72}$$

with the $\hat{\Theta}_0$ operator equivalent to the $\hat{\Theta}$ operator in the

$k = 0$ model [6]

$$\hat{\Theta}_0\Psi(v) = -B(v)^{-1}[C^+(v)\Psi(v+4) + C^0(v)\Psi(v) + C^-(v)\Psi(v)], \tag{73}$$

$$C^+(v) = \frac{3\kappa K}{64}|v+2|||v+1| - |v+3||, \tag{74}$$

$$C^-(v) = C^+(v-4), \tag{75}$$

$$C^0(v) = -C^+(v) - C^-(v). \tag{76}$$

The $k = -1$ model contributes the $\hat{\Theta}_{-1}$ operator which acts diagonally on $\Psi(v)$ as

$$\hat{\Theta}_{-1}\Psi(v) = -B(v)^{-1}\frac{\kappa\gamma^2 V_0^{2/3}}{12K^{1/3}}|v|^{1/3}||v+1| - |v-1||\Psi(v). \tag{77}$$

The combined operator $\hat{\Theta}$ is self-adjoint[4] but not positive definite. If we restrict ourselves to the positive part of the spectrum of $\hat{\Theta}$ then the physical inner product can be constructed in a simple fashion. Namely we restrict to eigenstates $e_\omega(v)$ of the $\hat{\Theta}$ operator: $\hat{\Theta}e_\omega(v) = \omega^2 e_k(v)$. Since $\hat{\Theta}$ is self-adjoint and we are restricting to the positive part of the spectrum, by spectral analysis we can construct an operator corresponding to the square root $\sqrt{\hat{\Theta}}$. Solutions to the difference equation then split into positive and negative frequency solutions satisfying a first order Schrodinger-like equation

$$\mp i\frac{\partial\Psi(v,\phi)}{\partial\phi} = \sqrt{\hat{\Theta}}\Psi(v,\phi). \tag{78}$$

In this form, the difference equation is like a standard evolution equation in terms of the scalar field ϕ. We can restrict to the positive frequency solution space when considering physical wave functions whence the physical inner product is given in analogy with the Schrodinger inner product of quantum mechanics as

$$\langle\Psi_1|\Psi_2\rangle_{\rm phys} = \sum_v B(v)\bar{\Psi}_1(v,\phi_0)\Psi_2(v,\phi_0). \tag{79}$$

As in quantum mechanics, the physical inner product can be evaluated at any "time" ϕ_0, and the difference Eq. (78) guarantees that the result is independent of ϕ_0.

Finally, to interpret the physical wave functions requires the evaluation of expectation values of observables. Technically, we require Dirac observables which correspond to quantum operators which commute with the constraint operator so as to lead to unambiguous gauge invariant observables. Following the $k = 0$ model [6] the scalar field momentum $\hat{P}_\phi = -i\hbar\partial/\partial\phi$ is an observable whose operator trivially commutes with the constraint operator. An additional observable is the value of v at a given instant in time ϕ_0 labeled $v|_{\phi_0}$. The expectation value of this observable is given by

$$\langle\Psi|\hat{v}|_{\phi_0}|\Psi\rangle = \frac{\sum_v B(v)v\bar{\Psi}(v,\phi_0)\Psi(v,\phi_0)}{\langle\Psi|\Psi\rangle_{\rm phys}}. \tag{80}$$

IV. DYNAMICS

With the inclusion of the massless scalar field, the resulting dynamics and interpretation of the theory can be understood by constructing suitable semiclassical states [6,8,27]. The dynamics of the theory is most easily understood by choosing the scalar field ϕ to play the role of the internal clock. Following the procedure set forth in [6,8], eigenfunctions of the $\hat{\Theta}$ operator are calculated and then Fourier transformed to get physical wave function solutions to the quantum difference equation. Thus given the eigenfunctions $e_\omega(v)$ we choose a Gaussian profile $e^{-(\omega-\omega_*)^2/2\sigma^2}e^{i\omega\phi_*}$ peaked around a large value of the scalar field momentum $P_\phi = \hbar\omega_*$ with spread σ and peaked around a value of the scalar field ϕ_*. Physical wave functions are constructed through the Fourier transform

$$\Psi(v,\phi) = \int_{-\infty}^{\infty} d\omega e^{-(\omega-\omega_*)^2/2\sigma^2}e^{i\omega\phi_*}e_\omega(v)e^{i\omega\phi} \tag{81}$$

which are thus by construction solutions to the difference equation. Numerically, the procedure is to first calculate the eigenstates $e_\omega(v)$. Typically, there is a twofold degeneracy in the eigenstates, and this is removed by choosing the eigenstate that matches the positive frequency Wheeler-DeWitt solution[5] for large v. Once the eigenstates are calculated, the Fourier transform (81) is calculated using the Fast Fourier transform algorithm.

An example of a numerical simulation is shown in Fig. 1. The state is initially peaked around a large value of v and evolves towards the singularity while remaining sharply peaked. Instead of plunging into the singularity as expected from the classical dynamics, the state bounces leading to an expanding universe. The results of the quantum dynamics are qualitatively similar to the $k = 0$, $+1$ models [6,9] and the bounce occurs when the energy density of the scalar field is Planckian.

The behavior of the dynamics can be understood in terms of an effective classical description. This amounts

[4]Technically $\hat{\Theta}$ is self-adjoint on the Hilbert space $L^2(\mathbb{R}_{\rm Bohr}, B(v)d\mu_{\rm Bohr})$ where $\mathbb{R}_{\rm Bohr}$ refers to the almost periodic functions. The extra factor of $B(v)$ is due to the fact the $\hat{C}_{GR}$ is self-adjoint in the kinematical Hilbert space $L^2(\mathbb{R}_{\rm Bohr}, d\mu_{\rm Bohr})$ and that $\hat{\Theta} \propto B^{-1}(v)\hat{C}_{GR}$

[5]Wheeler-DeWitt solutions are eigenstates of the operator $\hat{\Theta}$ which is the continuous differential operator that approximates the difference operator $\hat{\Theta}$ in the large v limit. See [6,8] for more details.

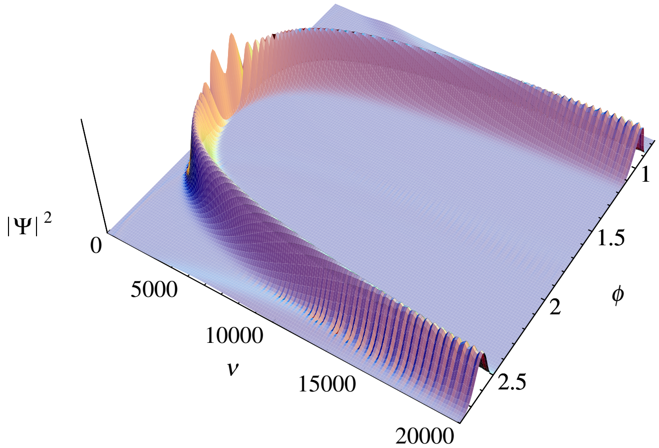

FIG. 1 (color online). Evolution of the semiclassical state initially peaked at a large value of v. The state remains sharply peaked and bounces before reaching the singularity $v = 0$. After the bounce, the state continues to remain sharply peaked and leads to an expanding universe. The values of the numerical parameters used in the simulation were $\omega_* = 700$, $\sigma = 20$, $V_0 = 1$, and $K = 1/2$.

to considering an effective modified Hamiltonian constraint through which effective classical equations of motion are calculated. Note that by nature this sort of effective description cannot completely encode the predictions from the quantum theory and care must be taken when applying the effective theory in more general settings. In particular if the wave function becomes nonsharply peaked, then additional modifications to the dynamics are expected to become appreciable [28]. In the numerical simulations performed for this work, the wave function remains sharply peaked throughout the evolution, and the effective description provides an accurate description which we show explicitly now.

The effective Hamiltonian is given by (see [19,20,25,28–30] for various discussions on the issue)

$$\mathcal{H}_{\rm eff} = -\frac{3\sqrt{|p|}}{\kappa\gamma^2\bar{\mu}^2}\sin^2(\bar{\mu}c) + \frac{3\sqrt{|p|}V_0^{2/3}}{\kappa} + |p|^{3/2}\rho_M, \tag{82}$$

where again $\bar{\mu}$ is a function of p given by

$$\bar{\mu} = \sqrt{\frac{\Delta}{|p|}}. \tag{83}$$

Note that in this effective Hamiltonian, we are implicitly assuming the $v \gg 1$ limit. In particular, in this limit the $B(v)$ eigenvalues that would appear in the matter part of the Hamiltonian are approximated by the classical expression; namely

$$B(v) = \frac{K}{v} + (v^{-3}) \tag{84}$$

and thus the matter density takes on its classical form

$$\rho_M = \frac{P_\phi^2}{2p^3} + \mathcal{O}(p^{-9/2}). \tag{85}$$

In this effective Hamiltonian we are therefore ignoring the inverse volume corrections to the matter Hamiltonian and will show that this is a good approximation by comparison with the quantum dynamics.

With this effective Hamiltonian we can derive an effective Friedmann equation. To do this first we note that the left-hand side of the Friedmann equation involving the Hubble rate squared can be written as

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{1}{4}\left(\frac{\dot{p}}{p}\right)^2 \tag{86}$$

which is a simple consequence from the fact that $p \propto a^2$ from Eq. (36). The time derivative $\dot{p}$ is calculated from Hamilton's equation $\dot{p} = \{p, \mathcal{H}_{\rm eff}\}$ giving

$$H^2 = \frac{1}{4}\left(\frac{\dot{p}}{p}\right)^2 = \frac{1}{\gamma^2\bar{\mu}^2|p|}\sin^2\bar{\mu}c\cos^2\bar{\mu}c$$
$$= \frac{1}{\gamma^2\bar{\mu}^2|p|}\sin^2\bar{\mu}c(1 - \sin^2\bar{\mu}c). \tag{87}$$

Finally, we can use the vanishing of the Hamiltonian to relate $\sin^2\bar{\mu}c$ to ρ_M which gives

$$\sin^2\bar{\mu}c = \gamma^2\bar{\mu}^2V_0^{2/3} + \frac{\kappa\gamma^2\bar{\mu}^2|p|}{3}\rho_M. \tag{88}$$

Putting these together and writing in terms of the scale factor $|p| = V_0^{2/3}a^2$ we get for the effective Friedmann equation

$$H^2 = \left(\frac{\kappa}{3}\rho_M + \frac{1}{a^2}\right)\left(1 - \frac{\gamma^2\Delta}{a^2} - \frac{\kappa\gamma^2\Delta}{3}\rho_M\right). \tag{89}$$

The first term in parentheses is the classical right-hand side of the Friedmann equation and thus the second term in parentheses represents the quantum modifications. The bounce can be understood as arising when the second term vanishes; namely, when the matter density reaches a maximum

$$\rho_{\rm max} = \frac{3}{\kappa\gamma^2\Delta} - \frac{3}{\kappa a^2}, \tag{90}$$

where the first term is precisely the same form as the critical density $\rho_c = \frac{3}{\kappa\gamma^2\Delta}$ arising in the $k = 0$ model and the second term forms an additional contribution from the $k = -1$ model. Notice that the actual value of the matter density at the bounce point depends on the value of the scale factor at the bounce point. To determine the bounce scale factor a_c and the value of the bounce energy density for the massless scalar field, we can solve for when the matter density equals the maximum value

$$\frac{P_\phi^2}{2V_0^2a_c^6} = \frac{3}{\kappa\gamma\Delta} - \frac{3}{\kappa a_c^2}. \tag{91}$$

If the scalar field momentum is sufficiently large, then a_c is sufficiently large so that the second term is negligible and we find that the bounce energy density agrees with the form of the $k = 0$ critical density $\rho_{\max} \approx \rho_c$. The actual value of ρ_c is dependent explicitly on the value of Δ which by (50) depends on the value of K. If K is on the order of 1, then (50) implies that Δ is on the order of the Planck length squared, and one finds that ρ_c is on the order of the Planck density. From the arguments of [6], the critical density in the $k = 0$ model is valued at $\rho_c = .82\rho_p$.

It is evident from the effective Friedmann equation (89) and from the form of the maximum energy density (90) that arbitrary matter with positive energy density will trigger a bounce. Furthermore, the effective Friedmann equation predicts a minimum scale factor $a_{\min}$ that the open universe can reach. Namely, even in the vacuum energy density case the right-hand side of the effective Friedmann equation is negative and thus forbidden for values of the scale factor below

$$a_{\min} = \gamma\sqrt{\Delta} \tag{92}$$

which again is on the order of the Planck length if Δ is on the order of the Planck length squared. Thus the open model constructed here predicts a vacuum repulsion in the high curvature regime.

We can compare the predictions of the effective Friedmann equation with the quantum dynamics as a method of testing the validity of the effective theory. In Fig. 2, the expectation value of observable $\langle \hat{v}|_{\phi_0}\rangle$ is plotted along with the spread $\langle \widehat{\Delta v}|_{\phi_0}\rangle$. The solid line is the trajectory predicted from the effective Friedmann equation (89) which agrees quite well with the expectation values. We see that the effective Friedmann equation accounts for the bounce at the right moment and agrees very well in the post bounce regime. This testifies as to the validity of the effective theory in the massless scalar field model considered. Furthermore, we have ignored the inverse volume corrections to the matter part of the effective Hamiltonian and thus the quantum dynamics are not sensitive to those effects. The reason for this is that the bounce occurs at a value of v much larger than 1. In particular for the values of the parameters chosen in Fig. 2, the bounce value of v is 228.015. In order to probe the small v regime, one would need a semiclassical state with a small value of P_ϕ yet such states behave nonsemiclassically with large spread and thus the effective description would not be valid and the quantum state would not be a good description of our universe.

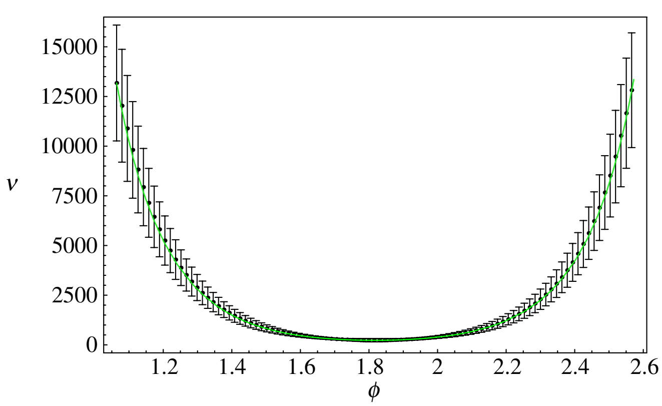

FIG. 2 (color online). Expectation value (dots) of $v|_\phi$ observable with the error bars representing the dispersion. The expectation values are approximated well by the predicted values (solid line) from the effective Friedmann equation (89).

However, as we mentioned one should keep in mind that in more complicated models, the effective theory shown here can in principle deviate from the quantum dynamics with deviations that may depend of the quantum state. Thus it is an open issue to understand better in what regimes the deviations occur and whether or not the deviations can be accounted for in a more complete effective picture. An effective theory that takes into account the quantum degrees of freedom (such as the spread of the wave function) can be found in [28], and thus merits testing with the quantum dynamics in more complicated scenarios.

V. DISCUSSION

We have shown explicitly that a successful loop quantization of the $k = -1$ model exists with the correct semiclassical limit. In this quantization the results of the $k = 0$, $+1$ models are extended and the classical singularity can be resolved even leading to a big bounce with a massless scalar field. This is further testament to robustness of the predictions of LQC.

Several caveats of the model require discussion. First is that our model was constructed using holonomies of the extrinsic curvature as opposed to holonomies of the connection as done in the full theory. The reason for using this quantization is that the holonomies of the connection (an example of which is given in formula (45)) are not almost periodic functions thus rendering a loop quantization difficult. As stated, this technique has been utilized in the loop quantization of the spherically symmetric models as well as in the inhomogeneous cosmological model of [31]. An important question is therefore what are the implications of the quantization using holonomies of the extrinsic curvature.

We can turn to the closed $k = +1$ model where both quantizations have been performed, with holonomies of the extrinsic curvature being used in the earlier work [13] while holonomies of the connection comprising the quantization are in the more recent work of [9]. The two quantizations can be shown to be in agreement in the $v \gg 1$ limit, with the differences restricted to the small volume $v \ll 1$ regime. We can understand the reason for this behavior in the following heuristic way. The holonomies of the connection consist of exponentials of $\bar{\mu}$ times the connection; i.e., $\bar{\mu}(\gamma K^i_a + \Gamma^i_a)$ where for the nonflat models the spin-connection components Γ_a^i are constant valued. Since we are taking $\bar{\mu}$ to scale as $p^{-1/2}$ (equivalently $v^{-1/3}$), then for large values of v the quantity $\bar{\mu}\Gamma_a^i$ is guaranteed to be small. Thus, the difference of holonomies of the connection and extrinsic curvature are expected to be negligible in the large v limit. This is precisely what is observed in the $k = +1$ model.

Therefore, the results of the $k = +1$ model indicate that for the $k = -1$ quantization presented here, the results are expected to be valid in the $v \gg 1$ regime. This does not affect any of the results presented here provided that the semiclassical state does not approach the $v < 1$ regime. As we have mentioned, the bounce occurs at a value of $v \gg 1$ for universes which behave semiclassically. Furthermore, in the $k = +1$ model similarly the bounce occurs at $v \gg 1$ for universes which reach macroscopic size before recollapsing [9]. Thus we expect that the physical results presented in this paper, such as the quantum bounce, are largely insensitive to whether the quantization is performed using holonomies of the connection or extrinsic curvature.

Additionally there is the issue of the dependence of the quantum results on the size of the fiducial cell. First we can ask if the effective Friedmann equation (89) is dependent on the fiducial cell and therefore the prediction of the quantum bounce. Classical quantities such as the scale factor a and matter energy density do not make reference to the fiducial cell and thus do not rescale. This implies that the effective Friedmann equation (89) is invariant under a change in fiducial cell. Note that the result crucially depends on the fact that $\bar{\mu}$ is not taken to be a constant, but scales as $p^{-1/2}$. Thus the prediction of the bounce does not make reference to the fiducial cell.

The same statement cannot be made about the inverse volume corrections appearing in the quantum matter density of the scalar field. The eigenvalues $B(v)$ give back the classical behavior for $v \ll 1$ but in general behave as

$$B(v) \propto \begin{cases} v^4 & v \ll 1 \\ v^{-1} & v \gg 1 \end{cases}. \tag{93}$$

The parameter v is proportional to the physical volume of the fiducial cell, and thus must scale if we resize the fiducial cell. The exact scaling under a resizing of the fiducial cell $V_0 \rightarrow V_0' = \xi^3 V_0$, is given as

$$v \rightarrow v' = \xi^3 v. \tag{94}$$

For a given value of the scale factor, a larger fiducial cell implies a larger value of v. In terms of the scale factor, v is related as

$$v = V_0 K\left(\frac{6}{8\pi\gamma}\right)^{3/2} \frac{a^3}{l_P^3} \tag{95}$$

which makes evident that the value of v depends explicitly on the fiducial cell volume V_0 for a *fixed* value of the scale factor. If we enlarge the fiducial cell, then the value of v should also increase which in turn *reduces* the effects of the inverse volume eigenvalues. Vice versa, a smaller fiducial cell implies stronger inverse volume effects.

Thus, when considering phenomenological applications involving the inverse volume modifications, one must specify the scale at which the inverse volume effects are non-negligible. In other words, the critical scale separating the quantum regime from the classical regime corresponds to $v = 1$ which in terms of a critical scale factor a_* gives

$$a_* = \sqrt{\frac{8\pi\gamma}{6}} \frac{l_P}{K^{1/3}} V_0^{-1/3} \tag{96}$$

which indicates the explicit dependence on the fiducial cell. Again, a larger value of V_0 implies a smaller a_* which pushes the quantum effects into the higher curvature regime and vice versa. If the fiducial cell volume V_0 and K are on the order of 1, then a_* is on the order of the Planck length, but note that the critical scale is not necessarily Planckian.

The issue of the scale dependence of the inverse volume modifications occurs additionally in the $k = 0$ model where again a fiducial cell is required to quantize the spatially infinite model (see discussions in [6,32]). The preceding arguments remain valid for this model and a scale must be introduced. On the other hand, the compact $k = +1$ model does not require a fiducial cell since the spatial integrations do not diverge. There, inverse volume modifications occur when the physical volume of the entire universe is Planckian. In other words, the scale at which the quantum effects occur is provided by physical volume of the universe. For the closed model this is equivalent to the high curvature Planckian regime.

Since the scale at which the inverse volume effects occur is given by the physical size of the fiducial cell in the $k = 0, -1$ models, an important issue is to determine what sets the scale in loop quantum cosmology. The fiducial cell was introduced in order to regulate the infinite spatial integrations appearing in the action and Hamiltonian and thus is not expected to be physically relevant. One possibility is that the scale is provided in an *inhomogeneous* treatment of loop quantum cosmology. An inhomogeneous model of loop quantum cosmology has been developed in [31] based on a fixed lattice quantization. In that model, the scale corresponds to the physical size of the lattice links. Yet, the inhomogeneous model does not provide a prescription to determine the size of the scale, which must be specified by hand and is not necessarily tied to matter degrees of freedom or the curvature scale. The naive expectation would be that the lattice spacing should be Planckian in size, but if the model describes the current universe then we would expect to see inverse volume modifications occurring today, a prediction which is clearly ruled out by observations.

Whatever determines the scale inherent in LQC models, one is faced with constraining the predictions with observations. As mentioned, if the scale is too small, then inverse volume corrections might be predicted in the near past which would alter the Friedmann dynamics and be observationally detectable. If the lattice links of an inhomogeneous model provide the scale, the links must be sufficiently larger than the Planck scale in the recent history of the universe, but presumably not too large to spoil particle physics. If the scale provided by the lattice links expands with the growing universe (i.e. the lattice links grow with the universe), then ensuring that they are not too large today, while being not too small in the earlier universe could be challenging and might require fine tuning. The inhomogeneous model of [31] has the behavior that the lattice links expand with the universe and thus would face this constraint. However, as mentioned in [31], one possibility is that in a more systematically derived inhomogeneous lattice model of loop quantum cosmology, the scale provided by the lattice links would dynamically change and thus might not grow with the expanding universe. This type of behavior is mimicked in the homogeneous setting when $\bar{\mu}$ scales as a function of $p^{-1/2}$, a quantization feature which was first proposed in [6] and has been utilized in this paper. With this scaling behavior, the holonomy edges defining the Hamiltonian constraint operator decrease in physical length with the expanding universe. The results of the improved quantization appear better grounded on a physical basis and thus this behavior would seem to be a requirement for constructing inhomogeneous models.

Furthermore, we have shown that the inverse volume modifications play no important role in the quantum dynamics for universes which behave semiclassically since the bounce occurs for $v \gg 1$. Additionally, in the $k = +1$ model, for universes which grow to macroscopic size, again the inverse volume modifications play no role [9]. We these indications, along with the arguments that the ambiguity parameter j should be its lowest value $1/2$ [25,26], these results give evidence that the inverse volume modifications may not play a significant role in the evolution of the universe.

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