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(Pseudo)issue of the conformal frame revisited

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The issue of the equivalence between Jordan and Einstein conformal frames in scalar-tensor gravity is revisited, with the emphasis on implementing running units in the latter. The lack of affine parametrization for timelike worldlines and the cosmological constant problem in the Einstein frame are clarified, and a paradox in the literature about cosmological singularities appearing only in one frame is solved. While, classically, the two conformal frames are physically equivalent, they seem to be inequivalent at the quantum level.

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I. INTRODUCTION

Conformal (or Weyl) transformations are widely used in scalar-tensor theories of gravity [1], the theory of a scalar field coupled nonminimally to the Ricci curvature R, and in modified gravity theories in which terms nonlinear in R are added to the Einstein-Hilbert action (due perhaps to quantum corrections [2]). The present acceleration of the Universe discovered with the study of supernovae of type Ia [3] calls either for an exotic form of dark energy (in Einstein gravity or in scalar-tensor theories), or for modifications of gravity described by terms nonlinear in R in the Lagrangian [4], or the addition of terms containing the invariants of the Riemann tensor $R_{ab}R^{ab}$ and $R_{abcd}R^{abcd}$ [5]. To fix the ideas and the terminology, consider a scalar-tensor theory of gravity, described in the Jordan frame by the action [6]

$$S = \int d^4x\sqrt{-g}\left[\frac{f(\phi)R}{2} - \frac{\omega(\phi)}{2}g^{ab}\nabla_a\phi\nabla_b\phi - V(\phi)\right] + \alpha_m\mathcal{L}^{(m)}[g_{ab}, \psi_m], \tag{1.1}$$

where $f > 0$, $S^{(m)} = \int d^4x\sqrt{-g}\mathcal{L}^{(m)}$, and $\mathcal{L}^{(m)}$ is the Lagrangian density describing "ordinary" matter (as opposed to the gravitational scalar field ϕ, which effectively plays the role of a form of nonconventional matter in the field equations. Here g_{ab} is the metric tensor with determinant g, $f(\phi)$ and $\omega(\phi)$ are arbitrary coupling functions, ϕ is the Brans-Dicke-like scalar field with potential $V(\phi)$, and ψ_m collectively denotes the matter fields. The Jordan frame in which the theory (1.1) is formulated is the set of dynamical variables (g_{ab}, ϕ) describing the gravitational field. The effective gravitational coupling is

$$G_{\rm eff} = \frac{1}{8\pi f(\phi)}, \tag{1.2}$$

as can be immediately deduced from inspection of the action (1.1). However, in a Cavendish experiment the effective coupling is instead [7,8]

$$G_{\rm eff}^{(*)} = \frac{2\omega f + 2(\frac{df}{d\phi})^2}{8\pi f[2\omega f + 3(\frac{df}{d\phi})^2]}. \tag{1.3}$$

This expression can also be derived from cosmological perturbation theory [9]. Note that in the Jordan frame description the Lagrangian density $\mathcal{L}^{(m)}[g_{ab}, \psi_m]$ only depends on the metric g_{ab} and the ordinary matter fields ψ_m. As a consequence, this matter is described by the stress-energy tensor

$$T_{ab}^{(m)} = \frac{-2}{\sqrt{-g}}\frac{\delta S^{(m)}}{\delta g^{ab}}, \tag{1.4}$$

and the invariance of $S^{(m)}$ under diffeomorphisms leads to the covariant conservation of $T_{ab}^{(m)}$ [10]

$$\nabla^b T_{ab}^{(m)} = 0. \tag{1.5}$$

As a consequence, test particles in the Jordan frame follow geodesics, the weak equivalence principle [8] is satisfied, and the theory (1.1) is metric. In this frame the kinetic energy term of the scalar ϕ, i.e., $-\omega(\phi)\nabla^a\phi\nabla_a\phi/2$, is noncanonical and has indefinite sign. The experimental constraint $|\omega(\phi_0)| > 40\,000$, where ϕ_0 is the present value of the scalar field, applies [11], unless a potential $V(\phi)$ gives the field a very short range.

Let us consider now the conformal transformation

$$g_{ab} \rightarrow \tilde{g}_{ab} = \Omega^2 g_{ab}, \qquad \Omega = \sqrt{f(\phi)}, \tag{1.6}$$

and the scalar field redefinition

$$\phi \rightarrow \tilde{\phi} = \int \frac{d\phi}{f(\phi)}\sqrt{f(\phi) + \frac{3}{2}\left(\frac{df}{d\phi}\right)^2}. \tag{1.7}$$

This transformation brings the theory into the Einstein conformal frame, i.e., to the set of variables $(\tilde{g}_{ab}, \tilde{\phi})$ in which the action (1.1) takes the form

$$S = \int d^4x\sqrt{-\tilde{g}}\left\{\frac{\tilde{R}}{16\pi} - \frac{1}{2}\tilde{g}^{ab}\tilde{\nabla}_a\phi\tilde{\nabla}_b\phi - \tilde{U}(\tilde{\phi}) + (G\phi)^{-2}\mathcal{L}^{(m)}[\tilde{g}_{ab}, \psi_m]\right\}, \quad (1.8)$$

where

$$\tilde{U}(\tilde{\phi}) = \frac{V[\phi(\tilde{\phi})]}{\Omega^2}, \quad (1.9)$$

and $\tilde{\nabla}_a$ is the covariant derivative operator of the metric $\tilde{g}_{ab}$. Note that the "new" scalar field $\tilde{\phi}$ exhibits a canonical kinetic energy term and it couples minimally to the Ricci curvature $\tilde{R}$ of the new metric $\tilde{g}_{ab}$. However, the action (1.8) does not describe simply general relativity with an extra scalar field $\tilde{\phi}$, because $\tilde{\phi}$ couples explicitly to matter via the prefactor $[G\phi(\tilde{\phi})]^{-2}$ in front of the matter Lagrangian $\mathcal{L}^{(m)}$. The exceptions are forms of conformal matter which obey equations invariant under the conformal transformation (1.6), such as the Maxwell field, a radiation fluid, or a scalar field ψ conformally coupled to R and with zero or quartic potential. The explicit coupling of $\tilde{\phi}$ to all forms of nonconformal matter spoils the equivalence principle in the Einstein frame. In this frame all massive particles deviate from geodesics due to the force, proportional to $\tilde{\nabla}_a\tilde{\phi}$, exerted by $\tilde{\phi}$. By contrast, zero mass particles still move along null geodesics. (This can be realized by noting that the conformal transformation (1.6) does not change the conformally invariant Maxwell equations in four dimensions, which reduce to geometric optics in the high frequency limit.)

The issue has been raised of "which conformal frame is physical," i.e., should one regard the Jordan frame metric g_{ab}, or the Einstein frame metric $\tilde{g}_{ab}$ as physical? This issue has been the subject of much debate and is still contentious due to incorrect formulations of this question. In fact, the question is answered, to a large extent, by Dicke's paper [12] which originally introduced the conformal transformation for Brans-Dicke theory [13], the prototype of scalar-tensor gravity theories. The answer of Ref. [12] is that the two frames are equivalent, provided that the units of mass, length, time, and quantities derived there from scale with appropriate powers of the conformal factor Ω in the Einstein frame. However, Dicke's treatment is valid only at the classical level, while in modern cosmology and in gravitational theories alternative to Einstein gravity, quantum fields in curved space play a significant role and the equivalence of the conformal frames is not clear at all—indeed there are certain indications that the equivalence breaks down at the quantum level. Of course, nothing is known about this equivalence in quantum gravity due to the lack of a definitive theory of quantum gravity.

In view of Dicke's paper, many authors consider the issue of which conformal frame is physical a pseudo problem, and we agree with them to a large extent, apart from the two problems mentioned above. However, while the answer to the question of the physical equivalence of conformal frames may be clear in principle, its *application* to practical situations is a completely different matter. The scaling of units in the Einstein frame is usually forgotten or not taken into account, producing results that range from nonsensical to marginally incorrect, to correct but it is not easy to understand if the conformal transformation (1.6) and (1.7) is applied correctly. (It is worth noting that Dicke himself applied the conformal transformation and the scaling of units incorrectly in the simpler context of Einstein gravity [14]). Misinterpretations of the conformal transformation abound in the literature and fuel the debate on the issue (or pseudoissue) of the conformal frame, while other authors consider the problem a closed one and sharply state that the Einstein frame is physical while the Jordan frame should not be considered at all. The argument for this choice is the positivity of the kinetic energy and the existence of a ground state, but this argument is usually not explored in detail for the specific theories considered. The existing review papers on the subject [15,16] fail to clarify this issue because they do not explicitly state the assumptions made. It appears that they refer to a version of scalar-tensor gravity in the Einstein frame in which the units of mass, length, time, etc. *do not scale* with powers of Ω. This version of the theory has nothing to do with the original Jordan frame and it is physically inequivalent to it, but it has come to be implicitly accepted as a valid theory, which adds to the confusion. It is our opinion that the issue deserves some clarification and that the open problems (Cauchy problem, extension to quantum matter) should be clearly formulated and addressed. In this paper we state as clearly as possible what the problems are, and we show how the divergence of opinions between different authors is due to the fact that two physically different theories in the Einstein frame (with or without scaling of units, respectively) are considered by different authors without realizing, or explicitly stating, which one is the version under examination. As a consequence, much of the existing debate becomes meaningless, the two opposite viewpoints are both correct but they really refer to physically different theories (one of which is not as well motivated as the other), while they are erroneously reported as pertaining to the same physical theory. From a more conservative point of view, instead, only the Einstein frame version of the theory incorporating scaling units is physically motivated. Even accepting this point of view, however, it is not always obvious how to incorporate this scaling of units in a calculation, for example, computing the spectrum of inflationary perturbations in scalar-tensor gravity, and this issue deserves some attention.

Many (perhaps most) researchers in gravitation and cosmology are unaware of the importance of scaling units in the Einstein frame, which is neglected. This issue is discussed in Sec. II, where it is shown that the scaling of units is related to the "anomalous" coupling of the scalar

$\tilde{\phi}$ to matter in the Einstein frame, and to the subsequent violation of the equivalence principle. In Sec. III we examine some consequences of allowing or not the units of fundamental quantities to scale in the Einstein frame, and we resolve an apparent paradox in the literature regarding energy conditions and singularity theorems in the two conformal frames in Sec. IV. The cosmological constant problem and the Cauchy problem are discussed in Sec. V, while Sec. VI contains the conclusions.

II. CONFORMAL TRANSFORMATIONS, JORDAN FRAME, AND EINSTEIN FRAME

Here we recall the basic properties of the conformal transformation to the Einstein frame, the transformation properties of various geometrical quantities, and the conservation equations for the matter stress-energy tensor. The reader is referred to Refs. [10,15–17] for further details.

Consider a spacetime (M, g_{ab}) where M is a smooth manifold with dimension $n > 1$ and g_{ab} is a Lorentzian or Riemannian metric on M. The conformal transformation

$$g_{ab} \rightarrow \tilde{g}_{ab} = \Omega^2 g_{ab}, \tag{2.1}$$

where Ω is a smooth, nowhere vanishing, function of the spacetime point is a point-dependent rescaling of the metric. It changes the length of timelike and spacelike intervals and vectors, but it preserves their timelike or spacelike character. Similarly, null intervals and null vectors according to the "old" metric g_{ab} remain null according to the new metric $\tilde{g}_{ab}$. The light cones are not changed by the conformal transformation (2.1) and the spacetimes (M, g_{ab}) and $(M, \tilde{g}_{ab})$ have the same causal structure; the converse is also true [10]. The inverse metric g^{ab}, the metric determinant g, and the Christoffel symbols transform according to [10,17]

$$\tilde{g}^{ab} = \Omega^{-2} g^{ab}, \qquad \tilde{g} = \Omega^{2n} g, \tag{2.2}$$

$$\tilde{\Gamma}^a_{bc} = \Gamma^a_{bc} + \frac{1}{\Omega}(\delta^a_b \nabla_c \Omega + \delta^a_c \nabla_b \Omega - g_{bc} \nabla^a \Omega), \tag{2.3}$$

while the Riemann and Ricci tensor obey

$$\begin{aligned} \tilde{R}_{abc}{}^d = {} & R_{abc}{}^d + 2\delta^d_{[a} \nabla_{b]} \nabla_c (\ln\Omega) - 2g^{de} g_{c[} \nabla_{b]} \nabla_e (\ln\Omega) \\ & + 2\nabla_{[a}(\ln\Omega) \delta^d_{b]} \nabla_c (\ln\Omega) \\ & - 2\nabla_{[a}(\ln\Omega) g_{b]c} g^{de} \nabla_e (\ln\Omega) \\ & - 2g_{c[a} \delta^d_{b]} g^{ef} \nabla_e (\ln\Omega) \nabla_f (\ln\Omega), \end{aligned} \tag{2.4}$$

$$\begin{aligned} \tilde{R}_{ab} = {} & R_{ab} - (n-2)\nabla_a \nabla_b (\ln\Omega) - g_{ab} g^{ef} \nabla_f \nabla_e (\ln\Omega) \\ & + (n-2)\nabla_a (\ln\Omega) \nabla_b (\ln\Omega) \\ & - (n-2) g_{ab} g^{ef} \nabla_f (\ln\Omega) \nabla_e (\ln\Omega). \end{aligned} \tag{2.5}$$

For the Ricci curvature,

$$\begin{aligned} \tilde{R} &= \tilde{g}^{ab} \tilde{R}_{ab} \\ &= \frac{1}{\Omega^2}\bigg[R - 2(n-1)\Box(\ln\Omega) - (n-1)(n-2) \\ &\quad \times \frac{g^{ab} \nabla_a \Omega \nabla_b \Omega}{\Omega^2}\bigg]. \end{aligned} \tag{2.6}$$

In $n = 4$ dimensions it is

$$\begin{aligned} \tilde{R} &= \frac{1}{\Omega^2}\left(R - \frac{6\Box\Omega}{\Omega}\right) \\ &= \frac{1}{\Omega^2}\left[R - \frac{12\Box(\sqrt{\Omega})}{\sqrt{\Omega}} + 3\frac{g^{ab} \nabla_a \Omega \nabla_b \Omega}{\Omega^2}\right]. \end{aligned} \tag{2.7}$$

The Weyl tensor $C_{abc}{}^d$ with the last index raised is conformally invariant,

$$\tilde{C}_{abc}{}^d = C_{abc}{}^d. \tag{2.8}$$

However, the same tensor with the other indices raised or lowered is not conformally invariant. Note that in the conformally rescaled world the conformal factor Ω plays the role of a form of matter. In fact, if the original metric is Ricci-flat ($R_{ab} = 0$), the new metric is not ($\tilde{R}_{ab} \neq 0$).

If the Weyl tensor of g_{ab} vanishes, also the Weyl tensor of $\tilde{g}_{ab}$ in the conformally related frame vanishes (and vice versa), conformally flat metrics are mapped into conformally flat metrics.

Let us consider covariant conservation for the matter energy-momentum tensor $T^{(m)}_{ab}$. In the Jordan frame it is

$$\nabla^b T^{(m)}_{ab} = 0; \tag{2.9}$$

this equation is not conformally invariant and T_{ab} scales as [10]

$$\tilde{T}^{ab}_{(m)} = \Omega^s T^{ab}_{(m)}, \qquad \tilde{T}^{(m)}_{ab} = \Omega^{s+4} T^{(m)}_{ab}, \tag{2.10}$$

where s is an appropriate conformal weight. As a consequence, the conservation equation in the conformally rescaled world is

$$\begin{aligned} \tilde{\nabla}_a(\Omega^s T^{ab}_{(m)}) = {} & \Omega^s \nabla_a T^{ab}_{(m)} + (s+6)\Omega^{s-1} T^{ab}_{(m)} \nabla_a \Omega \\ & - \Omega^{s-1} g^{ab} T^{(m)} \nabla_a \Omega, \end{aligned} \tag{2.11}$$

in four spacetime dimensions [10]. By conveniently choosing $s = -6$ one obtains

$$\tilde{\nabla}_a \tilde{T}^{ab}_{(m)} = -\tilde{T}^{(m)} \tilde{g}^{ab} \tilde{\nabla}_a (\ln\Omega) \tag{2.12}$$

and

$$\tilde{T}^{(m)} \equiv \tilde{g}^{ab} \tilde{T}^{(m)}_{ab} = \Omega^{-4} T^{(m)}. \tag{2.13}$$

Hence, in the new conformal frame, the stress-energy tensor $\tilde{T}^{(m)}_{ab}$ is not covariantly conserved unless it describes conformally invariant matter with vanishing trace $T^{(m)} = 0$, in which case also $\tilde{T}^{(m)} = 0$ and $\tilde{\nabla}^b \tilde{T}^{(m)}_{ab} = 0$.

It is well known that null geodesics of the Jordan metric g_{ab} are mapped into null geodesics of the Einstein frame metric $\tilde{g}_{ab}$ [10]. Timelike geodesics will be considered in the next section.

III. JORDAN FRAME, EINSTEIN FRAME WITH RUNNING UNITS, AND EINSTEIN FRAME WITH FIXED UNITS

In this section only classical physics of spacetime and matter is considered. Quantum matter will be discussed in Sec. VI.

A viewpoint shared by many authors (see [15,16] for references) states that the Einstein and Jordan conformal frames are physically equivalent. This viewpoint is generally correct as shown below, and it is in open conflict with the viewpoint that the Jordan frame should be abandoned in favor of the Einstein frame because of the presence of negative energy.

A. Einstein frame with running units

The argument of the physical equivalence between the Jordan and Einstein frames dates back to Dicke's 1962 paper introducing the conformal transformation technique for Brans-Dicke theory [12], a paper often forgotten or misread. The basic idea is that the two conformal frames are physically equivalent *provided* that in the Einstein frame the units of time, length, mass, and derived quantities are allowed to scale with appropriate powers of the conformal factor Ω. Physics must be invariant under a choice of the units—this includes not only transformations of units by factors which are the same everywhere in spacetime ("rigid" changes of units or "dilatations"), but also changes of units that depend on the spacetime point. A rescaling of the units of length and time (and, on dimensional grounds, also of mass) is a conformal transformation. Since physics is invariant under a change of units, it is invariant under a conformal transformation provided that the units of length, time, and mass l_u, t_u, and m_u are scaled. The novelty of Dicke's approach consists in allowing these units to be rescaled by different factors at different spacetime points, with the change in each unit being a smooth, nowhere vanishing, function of the spacetime point. Instead of a system of units rigidly attached to the spacetime manifold, the Einstein frame contains a system of units that change with the spacetime location. If one accepts this point of view, the symmetry group of classical physics is enlarged to include conformal transformations *with the associated rescaling of units*.

It is shown in Ref. [12] that g_{ab} scales with the dimensions of a time squared, and since $\tilde{g}_{ab} = \Omega^2 g_{ab}$, it follows that times and lengths scale with Ω, so that

$$dt \rightarrow \tilde{d}t = \Omega dt, \qquad dx^i \rightarrow \tilde{d}x^i = \Omega dx^i \quad (i = 1, 2, 3), \tag{3.1}$$

while for masses

$$m \rightarrow \tilde{m} = \Omega^{-1} m, \tag{3.2}$$

on dimensional grounds.

Since the speed of light in vacuum c is a ratio of space and time, it is invariant and local Lorentz invariance is preserved. The Planck constant, which has dimensions $[h] = [ML^2T^{-1}]$ is left unchanged, while energy with dimensions $[Mc^2]$ scales like a mass. In the Jordan frame of scalar-tensor gravity the effective coupling (1.2) varies, while h, c, the masses of elementary particles, and the coupling constants of physics are true constants, together with the units. The weak equivalence principle holds and the theory is metric. On the contrary, in the Einstein frame the gravitational coupling G is constant and so are h and c, while the masses of elementary particles and the coupling "constants" of nongravitational physics vary with time together with the units of time, length, and mass $\tilde{t}_u$, $\tilde{l}_u$, and $\tilde{m}_u$. In Dicke's viewpoint the Jordan and Einstein frames are merely two equivalent representations of the same physics. One can consider, for example, the proton mass, which has a constant value m_p in the Jordan frame. In the Einstein frame the proton mass depends on Ω (or ϕ) and is $\tilde{m}_p = \Omega^{-1} m_p$. However, in an experiment one measures the *ratio* $\tilde{m}_p/\tilde{m}_u$ between the proton mass and an arbitrarily chosen mass unit $\tilde{m}_u$. Hence, in the Einstein frame it is not $\tilde{m}_p$ that matters, but the ratio

$$\frac{\tilde{m}_p}{\tilde{m}_u} = \frac{\Omega^{-1} m_p}{\Omega^{-1} m_u} = \frac{m_p}{m_u} \tag{3.3}$$

(in the Jordan frame as well, it is only m_p/m_u that is measured). A measurement of the proton mass with respect to the chosen mass unit therefore yields the same value in the Jordan and the Einstein frame. No preferred frame is selected by such a measurement. The outcome of an experiment is the same when analyzed in the Jordan or the Einstein frame. In this context, the problem of which frame is physical is void of content.

In the Einstein frame not only the masses of elementary particles and the mass units, but also the coupling constants of nongravitational physics vary with ϕ. To understand this variation it is useful to consider the following example, to which we will return later [18].

1. Examples

Consider as an example Brans-Dicke theory [13] with a massive Klein-Gordon field ψ as the only form of matter, as described in the Jordan frame by the action

$$S = S^{(BD)} + S^{(KG)} = \int d^4x\sqrt{-g}(\mathcal{L}^{(BD)} + \alpha_{KG}\mathcal{L}^{(KG)}) = \int d^4x\sqrt{-g}\left(\phi R - \frac{\omega}{\phi}g^{ab}\nabla_a\phi\nabla_b\phi\right) - \frac{\alpha_{KG}}{2}\int d^4x\sqrt{-g}(g^{ab}\nabla_a\psi\nabla_b\psi + m^2\psi^2), \tag{3.4}$$

where $\alpha_{KG} = 16\pi G$ is the Klein-Gordon coupling constant. The conformal transformation (1.6) and the scalar field redefinition (1.7) yield

$$\sqrt{-g}(\mathcal{L}^{(BD)} + \alpha_{KG}\mathcal{L}^{(KG)}) = \sqrt{-\tilde{g}}\{\tilde{\mathcal{L}}^{(GR)} + \tilde{\alpha}_{KG}(\phi)\tilde{\mathcal{L}}^{(KG)}[\tilde{g}_{ab}, \psi]\}, \tag{3.5}$$

where $\tilde{\mathcal{L}}^{(GR)} = \tilde{R} - \frac{1}{2}\tilde{g}^{ab}\tilde{\nabla}_a\tilde{\phi}\tilde{\nabla}_b\tilde{\phi}$ is the Einstein-Hilbert Lagrangian density with a canonical scalar field $\tilde{\phi}$,

$$\tilde{\alpha}_{KG}(\tilde{\phi}) = \Omega^{-2}\alpha_{KG} = 32\pi G\exp\left(-8\sqrt{\frac{\pi G}{2\omega+3}}\tilde{\phi}\right), \tag{3.6}$$

$$\tilde{\mathcal{L}}^{(KG)}[\tilde{g}_{ab}, \phi] = \frac{1}{2}\tilde{g}^{ab}\tilde{\nabla}_a\psi\tilde{\nabla}_b\psi + \frac{\tilde{m}^2}{2}\psi^2, \tag{3.7}$$

and

$$\tilde{m}(\tilde{\phi}) = \frac{m}{\Omega} = m\exp\left(-4\sqrt{\frac{\pi G}{2\omega+3}}\tilde{\phi}\right), \tag{3.8}$$

in accordance with Eq. (3.2). In the Einstein frame the mass $\tilde{m}$ of the Klein-Gordon field ψ and its coupling constant $\tilde{\alpha}_{KG}$ acquire a dependence from the Brans-Dicke scalar. This holds true for all forms of matter except conformally invariant matter, which satisfies conformally invariant equations. As an example of such matter, consider the Maxwell field in four spacetime dimensions, described by the matter action

$$S^{(em)} = \int d^4x\sqrt{-g}\alpha_{em}\mathcal{L}^{(em)} = -\int d^4x\sqrt{-g}F_{ab}F^{ab}, \tag{3.9}$$

with $\alpha_{em} = 4$ and $\mathcal{L}^{(em)} = -\frac{1}{4}F_{ab}F^{ab}$, where F_{ab} is the antisymmetric Maxwell tensor. The conformal invariance can be directly verified by computing

$$\sqrt{-g}\mathcal{L}^{(em)} = -\tfrac{1}{4}\sqrt{-g}g^{ac}g^{bd}F_{ab}F_{cd} = -\tfrac{1}{4}(\Omega^{-4}\sqrt{-\tilde{g}})(\Omega^2\tilde{g}^{ac})(\Omega^2\tilde{g}^{bd})F_{ab}F_{cd} = -\tfrac{1}{4}\sqrt{-\tilde{g}}\tilde{g}^{ac}\tilde{g}^{bd}\tilde{F}_{ab}\tilde{F}_{cd}, \tag{3.10}$$

where $\tilde{F}_{ab} = F_{ab}$.

2. Terminology

Implementing the idea that physics should be conformally invariant when units are rescaled leads to conflict with current terminology. Consider, for example, a conformally coupled Klein-Gordon field ψ with a nonzero mass, obeying the equation

$$\Box\psi - \frac{R}{6}\psi - m^2\psi = 0. \tag{3.11}$$

According to standard terminology, the introduction of the mass m breaks the conformal invariance that is present when $m = 0$. One can, however, generalize the notion of conformal invariance by allowing the mass to vary with the scalar ϕ. Upon the use of the relation

$$g^{ab}\nabla_a\nabla_b\psi - \frac{R}{6}\psi = \Omega^3\left[\tilde{g}^{ab}\tilde{\nabla}_a\tilde{\nabla}_b\tilde{\psi} - \frac{\tilde{R}}{6}\tilde{\psi}\right], \tag{3.12}$$

where $\tilde{\psi} \equiv \Omega^{-1}\psi$, one obtains from Eq. (3.11)

$$\tilde{\Box}\tilde{\psi} - \frac{\tilde{R}}{6}\tilde{\psi} - \tilde{m}^2\tilde{\psi} = 0, \tag{3.13}$$

where now $\tilde{m} \equiv \Omega^{-1}m$, in agreement with Eq. (3.2). Hence, the Klein-Gordon equation (3.11) is invariant in form if the current definition of conformal transformation is enlarged to include the notion that masses scale with Eq. (3.2). However, Eq. (3.11) is not conformally invariant according to standard terminology.

B. The equation of motion of massive particles in the Einstein frame with running units

In the Einstein frame the equation of timelike geodesics receives corrections and, as a result, massive particles do not follow geodesics. First, we want to find the transformation property of the four-velocity $u^a = dx^a/d\lambda$ of a massive particle, where λ is a parameter along the geodesic, which is usually not discussed in the literature. The Jordan frame normalization is $u^a u_a = -1$; by assuming that $\tilde{u}_a = \Omega^w u_a$, where w is an appropriate conformal weight, and by imposing the Einstein frame normalization $\tilde{g}^{ab}\tilde{u}_a\tilde{u}_b = -1$, one obtains $w = -1$, or

$$\tilde{u}^a = \Omega^{-1}u^a, \qquad \tilde{u}_a = \Omega u_a. \tag{3.14}$$

These relations can be used to find the relation between the parameters λ and $\tilde{\lambda}$ along the geodesic in the two conformal frames. Since $u^a = dx^a/d\lambda$, $\tilde{u}^a = d\tilde{x}^a/d\tilde{\lambda}$, and lengths scale as $d\tilde{x}^a = \Omega dx^a$, by setting $d\tilde{\lambda} = \Omega^\alpha d\lambda$ one obtains $\tilde{u}^a = \Omega^{1-\alpha}u^a$ which, compared with Eq. (3.14) yields $\alpha = 2$, or

$$d\tilde{\lambda} = \Omega^2 d\lambda, \tag{3.15}$$

which agrees with Eq. (D.6) of Ref. [10]. This relation can also be obtained from the fact that, in terms of proper times $d\tau$ and $d\tilde{\tau}$, we have

$$d\tilde{s}^2 = -d\tilde{\tau}^2 = \tilde{g}_{00}d\tilde{t}^2 = (\Omega^2 g_{00})(\Omega^2 dt^2) = -\Omega^4 d\tau^2 = \Omega^4 ds^2, \tag{3.16}$$

which yields again $d\tilde{s} = \Omega^2 ds$ for the parameter along the timelike curve.

We are now ready to write the equation of motion of massive particles in the Einstein frame. Under the conformal transformation (2.1) the Jordan frame geodesic equation $u^a\nabla_a u^b = 0$ is mapped to [10]

$$u^a\tilde{\nabla}_a u^b = 2u^b \frac{u^c\nabla_c\Omega}{\Omega} + \frac{g^{bd}\tilde{\nabla}_d\Omega}{\Omega}. \tag{3.17}$$

By rewriting this equation in terms of tilded quantities we have

$$\tilde{u}^a\tilde{\nabla}_a\tilde{u}^b = \left(\frac{\tilde{u}^c\tilde{\nabla}_c\Omega}{\Omega}\right)\tilde{u}^b + \frac{\tilde{g}^{bd}\tilde{\nabla}_d\Omega}{\Omega}. \tag{3.18}$$

The first term on the right-hand side appears because the equation is not expressed using an affine parameter, while the second term proportional to the gradient $\tilde{\nabla}_a(\ln\Omega)$ describes the direct coupling of the field ϕ to nonconformal matter in the Einstein frame; it has been likened to a fifth force violating the equivalence principle and making scalar-tensor theory in the Einstein frame nonmetric. It is impossible to achieve an affine parametrization of this timelike curve and thus remove the first term on the right-hand side of Eq. (3.18). In fact, if this could be achieved, the result would be incompatible with the normalization $\tilde{u}^a\tilde{u}_a = -1$. To prove this statement, note that the normalization implies that the four-acceleration $\tilde{a}^b \equiv \tilde{u}^a\tilde{\nabla}_a\tilde{u}^b$ is orthogonal to the four-velocity ($\tilde{u}^b\tilde{a}_b = 0$), a well-known fact [10,19]. Then,

$$0 = \tilde{u}^b\tilde{a}_b \equiv \tilde{u}^b\tilde{u}^a\tilde{\nabla}_a\tilde{u}_b = \tilde{u}^b\tilde{\nabla}_b\Omega, \tag{3.19}$$

implying that the gradient of the conformal factor must be orthogonal to $\tilde{u}^a$ for *any* possible choice of $\tilde{u}^a$: this is clearly absurd. For example, in scalar-tensor cosmology where $\Omega = \Omega(t)$, t being the comoving time of a Friedmann-Lemaitre-Robertson-Walker metric (FLRW), and by choosing $\tilde{u}^a$ as the four-velocity of comoving observers, it follows that $\partial\Omega/\partial t = 0$, which is impossible [20]. Therefore, the term $[\tilde{u}^c\tilde{\nabla}_c(\ln\Omega)]\tilde{u}^b$ in Eq. (3.18) cannot be eliminated or, in other words, affine parametrization cannot be achieved. Equation (3.18) can be rewritten using Eq. (3.2) as

$$\tilde{u}^a\tilde{\nabla}_a\tilde{u}^b = -\left(\frac{\tilde{u}^c\tilde{\nabla}_c\tilde{m}}{\tilde{m}}\right)\tilde{u}^b - \frac{\tilde{g}^{bd}\tilde{\nabla}_d\tilde{m}}{\tilde{m}}. \tag{3.20}$$

Equation (3.20) suggests the interpretation that massive particles deviate from geodesics because their mass is a function of the spacetime point, and this deviation is proportional to the mass gradient. The impossibility of using an affine parametrization is then traced back to the impossibility of eliminating the variation $\tilde{u}^c\tilde{\nabla}_c\tilde{m}$ of the mass $\tilde{m}$ along the direction of motion of the particle. By introducing the three-dimensional metric on the 3-space orthogonal to the four-velocity $\tilde{u}^a$ of the particle,

$$\tilde{h}_{ab} \equiv \tilde{g}_{ab} + \tilde{u}_a\tilde{u}_b, \tag{3.21}$$

where h^a_b is the projection operator on the 3-space of the observer u^a (i.e., $h^a{}_b u^b = h_a{}^b u^a = 0$), Eq. (3.20) is rewritten as

$$\tilde{u}^a\tilde{\nabla}_a\tilde{u}^b = -\frac{\tilde{h}^{bd}\tilde{\nabla}_d\tilde{m}}{\tilde{m}}, \tag{3.22}$$

which shows explicitly that the correction to the equation of motion is given entirely by the variation of the particle mass $\tilde{m}$ in the 3-space of an observer moving with the particle. Removing the term $-(\tilde{u}^c\tilde{\nabla}_c\tilde{m}/\tilde{m})\tilde{u}^b$ from Eq. (3.20) by means of affinely parametrizing the curve would mean introducing corrections to the right-hand side of Eq. (3.22) which are proportional to the derivative of $\tilde{m}$ in the direction of motion $\tilde{u}^a$, and this is impossible. It would mean that the right-hand side of Eq. (3.20) could not be written explicitly as a purely spatial vector, as is instead done in Eq. (3.22), and therefore it could not be the four-acceleration $a^b = u^a\nabla_a u^b$, which satisfies $u^a a_a = 0$.

Equation (3.22) has consequences for cosmology. In the FLRW metric

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-Kr^2} + r^2(d\theta^2 + \sin^2\theta d\varphi^2)\right], \tag{3.23}$$

let u^a be the four-velocity of comoving observers. Since in FLRW scalar-tensor cosmology the scalar field depends only on the comoving time in order to preserve spatial homogeneity, it is $\phi = \phi(t)$, $\Omega = \Omega(t)$, and $\tilde{m} = \tilde{m}(t)$, which implies that the spatial gradient $\tilde{h}^{bd}\tilde{\nabla}_d\tilde{m}$ vanishes identically, and the equation of motion of comoving observers, which is the equation of timelike geodesics when a dust fluid with pressure $P = 0$ fills the Universe, receives no correction in the Einstein frame. Similarly, when there is pressure, the timelike geodesic equation gets corrected by an extra term in P, but no "fifth force" corrections $-\tilde{h}^{bd}\tilde{\nabla}_d(\ln\tilde{m})$ appear. The equivalence between Jordan and Einstein frames with respect to redshift, Boltzmann equation, and particle physics reaction rates in the early Universe is discussed in Ref. [21].

The trajectories of particles with zero mass $\tilde{m} = m = 0$ do not receive corrections when going to the Einstein frame.

C. Einstein frame with fixed units

By now it is clear that if one performs the conformal transformation (1.6) and (1.7) but does not allow the units of length, time, and mass to scale with Ω in the Einstein frame ("fixed units"), one obtains a different physical theory altogether. In this case, the conformal transformation is merely a mathematical device relating the two conformal frames, and the Jordan and Einstein frame are physically inequivalent. If the Jordan frame and the Einstein frame *with fixed units* were physically equivalent, it would mean that the entire realm of (classical) physics is conformally invariant, according to current terminology. But, to quote an example, the Klein-Gordon field obeying Eq. (3.11) with $m \neq 0$ is not conformally invariant in this sense. As another example, consider conformally related metrics which are physically inequivalent such as the Minkowski metric η_{ab} and the FLRW metric given by the line element

$$ds^2 = g_{ab}dx^a dx^b = a^2(\eta)(-d\eta^2 + dx^2 + dy^2 + dz^2), \tag{3.24}$$

where η is conformal time and g_{ab} is manifestly conformally flat: $g_{ab} = \Omega^2\eta_{ab}$ with $\Omega(\eta) = a$. When $da/d\eta > 0$, g_{ab} describes an expanding universe with spacetime curvature, cosmological redshift, possibly a big bang and/or other singularities, and matter. By contrast, the conformally related metric η_{ab} cannot be associated to any of these spacetime features. The two metrics g_{ab} and η_{ab} are physically equivalent only when the fundamental units are allowed to scale with $a(\eta)$ in the spirit of Refs. [12,14]. Then, the Universe (3.24) appears flat when the units of time and length scale as $dt = a(\eta)d\eta$, $d\tilde{x}^i = a(\eta)dx^i$, giving (see [16] for references)

$$ds^2 = -d\tilde{t}^2 + d\tilde{x}^2 + d\tilde{y}^2 + d\tilde{z}^2. \tag{3.25}$$

The recurring debate on the issue of which conformal frame is physical arises from the fact that many authors refer to the Einstein frame by keeping the fundamental units fixed in this frame. The result is a theory, which we shall call ‘‘Einstein frame with fixed units’’ version, which is physically inequivalent to the Jordan frame version of scalar-tensor gravity. These same authors often claim that the Jordan frame version and the Einstein frame with fixed units version are equivalent, forgetting about the scaling of units and Dicke’s paper. The Einstein frame with fixed units version does not share the physical motivations that lead to its Jordan frame cousin. It can even be said that the former arises from a mistake, but given the number of works devoted to this ‘‘wrong’’ theory, we are perhaps facing an (unintentional) new theory of gravity. We leave to the reader the judgment of whether there is enough physical motivation to pursue Einstein frame with fixed units versions of gravitational theories, and we content ourselves to clarify the issue [22].

Let us return for a moment to the Einstein frame with running units: in this frame $\tilde{m}(\phi) = \Omega^{-1}m$ and the ratio of the mass of a particle to the variable mass unit is constant,

$$\frac{\tilde{m}(\phi)}{\tilde{m}_u(\phi)} = \text{constant}. \tag{3.26}$$

This implies that

$$\frac{\tilde{\nabla}_c\tilde{m}}{\tilde{m}} = \frac{\tilde{\nabla}_c\tilde{m}_u}{\tilde{m}_u}, \tag{3.27}$$

and therefore the equation of motion of massive particles (3.22) in the Einstein frame with running units can be written as

$$\tilde{u}^a\tilde{\nabla}_a\tilde{u}^b = -\frac{\tilde{h}^{bd}\tilde{\nabla}_d\tilde{m}_u}{\tilde{m}_u}; \tag{3.28}$$

in other words, the correction to the equation of timelike geodesics and the violation of the equivalence principle can be seen as arising completely from the variation of the mass unit. Therefore, in the Einstein frame with fixed units this correction vanishes and the equivalence principle is satisfied unless, of course, one reintroduces these violations by hand into the theory, but the latter now bears no relation to the original Jordan frame one.

The running of fundamental units can also be seen as the fact that there is an anomalous coupling of $\tilde{\phi}$ to the matter sector in the Einstein frame with running units. This will be clear at the end of this section. We now want to make contact with a different notation that appeared recently in Ref. [23]. The author E. Flanagan parametrizes different conformal frames of a scalar-tensor theory using three different functions of the scalar field $A(\phi)$, $B(\phi)$, and $\alpha(\phi)$. The action is written as

$$S = \int d^4x\sqrt{-g}\left[\frac{A(\phi)}{16\pi G}R - \frac{B(\phi)}{2}g^{ab}\nabla_a\phi\nabla_b\phi - V(\phi)\right] + S^{(m)}[e^{2\alpha(\phi)}g_{ab}, \psi^{(m)}]. \tag{3.29}$$

A conformal transformation is described by

$$g_{ab} \rightarrow \tilde{g}_{ab} = e^{-2\gamma(\phi)}g_{ab}, \tag{3.30}$$

$$\Phi \rightarrow \tilde{\Phi} = h^{-1}(\phi), \quad \text{or} \quad \Phi = h(\tilde{\phi}), \tag{3.31}$$

where γ and h are regular functions with $h' > 0$. The action can be rewritten as

$$S = \int d^4x\sqrt{-g}\left[\frac{\tilde{A}(\tilde{\phi})}{16\pi G}\tilde{R} - \frac{\tilde{B}(\tilde{\phi})}{2}\tilde{g}^{ab}\tilde{\nabla}_a\tilde{\phi}\tilde{\nabla}_b\tilde{\phi} - \tilde{V}(\tilde{\phi})\right] + S^{(m)}[e^{2\tilde{\alpha}(\tilde{\phi})}\tilde{g}_{ab}, \psi^{(m)}], \tag{3.32}$$

where

$$\tilde{\alpha}(\tilde{\phi}) = \alpha[h(\tilde{\phi})] + \gamma(\tilde{\phi}), \tag{3.33}$$

$$\tilde{V}(\tilde{\phi}) = e^{4\gamma(\tilde{\phi})}V[h(\tilde{\phi})], \tag{3.34}$$

$$\tilde{A}(\tilde{\phi}) = e^{2\gamma(\tilde{\phi})}A[h(\tilde{\phi})], \tag{3.35}$$

$$\tilde{B}(\tilde{\phi}) = e^{2\gamma(\tilde{\phi})}\Big\{h'(\tilde{\phi})B[h(\tilde{\phi})] - \frac{3}{4\pi G}h'(\tilde{\phi})\gamma'(\tilde{\phi})A'[h(\tilde{\phi})] - \frac{3}{4\pi G}[\gamma'(\tilde{\phi})]^2 A[h(\tilde{\phi})]\Big\}. \tag{3.36}$$

(i) In these notations the *Jordan frame* corresponds to the choice

$$\alpha = 0, \qquad B = 1, \tag{3.37}$$

and to the free functions $A(\phi)$ and $V(\phi)$. In our notations this corresponds to identifying A with f and (3.29) with the Jordan frame action

$$S = \int d^4x\sqrt{-g}\Big[\frac{f(\phi)}{16\pi G}R - \frac{1}{2}\nabla^c\phi\nabla_c\phi - V(\phi)\Big] + S^{(m)}[g_{ab}, \psi^{(m)}]. \tag{3.38}$$

In fact, Flanagan's Jordan frame action can be generalized to arbitrary $B(\phi)$, which corresponds to our $\omega(\phi)$, obtaining the Jordan frame action

$$S = \int d^4x\sqrt{-g}\Big[\frac{f(\phi)R}{16\pi G} - \frac{\omega(\phi)}{2}\nabla^c\phi\nabla_c\phi - V(\phi)\Big] + S^{(m)}[g_{ab}, \psi^{(m)}]. \tag{3.39}$$

(ii) The *Einstein frame with running units* corresponds to the choice

$$A = 1, \qquad B = 1, \tag{3.40}$$

and to the free functions $\alpha(\phi)$ and $V(\phi)$. In our notations with a tilde denoting Einstein frame quantities, $\tilde{V} = \tilde{U}$ and $e^{2\tilde{\alpha}} = \Omega^{-2}$ and the action (3.29) corresponds to

$$S = \int d^4x\sqrt{-g}\Big\{\frac{\tilde{R}}{16\pi G} - \frac{1}{2}\tilde{g}^{ab}\tilde{\nabla}_a\tilde{\phi}\tilde{\nabla}_b\tilde{\phi} - \tilde{U}(\tilde{\phi}) + \Omega^{-2}\mathcal{L}^{(m)}[\tilde{g}_{ab}, \psi^{(m)}]\Big\}. \tag{3.41}$$

(iii) The *Einstein frame with fixed units* corresponds to the choice

$$A = 1, \qquad B = 1, \qquad \alpha = 0, \tag{3.42}$$

and, in our notations, to the action

$$S = \int d^4x\sqrt{-g}\Big\{\frac{\tilde{R}}{16\pi G} - \frac{1}{2}\tilde{g}^{cd}\tilde{\nabla}_c\tilde{\phi}\tilde{\nabla}_d\tilde{\phi} - \tilde{U}(\tilde{\phi}) + \mathcal{L}^{(m)}[\tilde{g}_{ab}, \psi^{(m)}]\Big\}, \tag{3.43}$$

in which there is no anomalous coupling of the scalar $\tilde{\phi}$ to matter ($\alpha = 0$). The difference between Einstein frame with running units and with fixed units is in the choice of the function α. It could be said that in the Einstein frame with fixed units the function α is not correctly transformed according to Eq. (3.33), while the functions A and B are transformed according to Eqs. (3.34), (3.35), and (3.36). Keeping the units fixed in the Einstein frame causes the masses to remain constant. In our first example of Sec. III A 2, this would correspond to replacing Eq. (3.13) with

$$\tilde{\Box}\tilde{\psi} - \frac{\tilde{R}}{6}\tilde{\psi} - m^2\tilde{\psi} = 0, \tag{3.44}$$

with a constant mass m introduced by hand. Of course, one can *postulate* this equation, which is debatable, but it should at least be made clear that it cannot be derived from Eq. (3.11) by using Dicke's spacetime-dependent rescaling of units. In other words, the first two terms in Eq. (3.44) are obtained with a conformal transformation while the third one is arbitrarily replaced by $-m^2\tilde{\psi}$.

There are situations in cosmology in which the scalar field ϕ is assumed to dominate the dynamics of the Universe and ordinary matter is ignored, setting $S^{(m)} = 0$. In these situations (corresponding to $m = 0$ in the example of Eqs. (3.11) and (3.44)) the running of units does not matter as long as only cosmological dynamics is studied. However, the issue will resurface whenever massive test particles or test fields, or nonconformal matter are introduced into this picture, or when redshift or reaction rates are considered [21].

There is a way that is, in principle, consistent to obtain the Einstein frame with fixed units: if one introduces in the Jordan frame a factor that exactly compensates for the Ω^{-2} factor in front of the matter Lagrangian density when conformally transforming to the Einstein frame, the scalar $\tilde{\phi}$ will couple minimally to matter in this frame. This is done, e.g., in Ref. [24]. However, the price to pay is the nonminimal coupling of ϕ to matter, and the violation of the equivalence principle, in the Jordan frame, which is alien from the spirit of Brans-Dicke and other scalar-tensor theories.

IV. ENERGY CONDITIONS AND SINGULARITY THEOREMS

We now want to discuss the energy conditions in the Jordan and Einstein frame. Let us consider, for the sake of illustration, Brans-Dicke theory represented by the action

$$S = \int d^4x\sqrt{-g}\Big[\phi R - \frac{\omega}{\phi}\nabla^c\phi\nabla_c\phi - V(\phi)\Big] + S^{(m)} \tag{4.1}$$

(the arguments proposed apply, however, to general scalar-tensor theories). The field equations can be written as

$$G_{ab} = \frac{8\pi}{\phi} T^{(m)}_{ab} + T_{ab}[\phi], \tag{4.2}$$

$$\Box\phi = \frac{1}{2\omega + 3}\left[8\pi T^{(m)} + \phi\frac{dV}{d\phi} - 2V\right], \tag{4.3}$$

where

$$T_{ab}[\phi] = \frac{\omega}{\phi^2}\left(\nabla_a\phi\nabla_b\phi - \frac{1}{2}g_{ab}\nabla^c\phi\nabla_c\phi\right) - \frac{V}{2\phi}g_{ab} + \frac{1}{\phi}(\nabla_a\nabla_b\phi - g_{ab}\Box\phi), \tag{4.4}$$

is often identified with an effective stress-energy tensor of the scalar ϕ. There are three possible ways of identifying an effective stress-energy tensor for ϕ [25,26] and the choice of Eq. (4.4) has sometimes been criticized in the literature [26–28]. If the choice (4.4) is accepted, as is common in the literature, it is easy to see that the strong, weak, and dominant energy conditions of general relativity [10] can all be violated by the scalar ϕ regarded as an effective form of matter. This is due to the noncanonical form of T_{ab}; the last term in Eq. (4.4) is linear in the second derivatives of ϕ instead of being quadratic in the first derivatives, and it makes the sign of $T_{00}[\phi]$ indefinite, even causing negative energy densities. The possibility of negative energy is regarded by certain authors as a criterion to discard the Jordan frame *a priori* as unphysical (see [15,16] for references). Since we know that Jordan frame and Einstein frame with running units are physically equivalent, we argue that these authors are left with the Einstein frame with fixed units version of the theory, which is physically ill-motivated. Moreover, it is not a negative kinetic energy that is worrisome, but rather an energy that is unbounded from below, so that the system can decay to lower and lower energy states *ad infinitum* (the electron in the hydrogen atom has negative total energy but there is a ground state of the Hamiltonian which corresponds to a minimum for the spectrum of energy eigenvalues). Hence, the mere possibility of negative energies is not, by itself, an argument to rule out the Jordan frame. As pointed out in Ref. [23], the energy conditions differ in the two frames but there is no physical observable corresponding to the sign of $G_{ab}u^a u^b$ for all timelike vectors u^a, hence there is no measurable inconsistency between the two frames. Furthermore, a positive energy theorem has been shown to hold for special scalar-tensor theories in the Jordan frame [29].

The validity of the energy conditions for the Einstein frame scalar $\tilde{\phi}$ has been emphasized in relation with the Hawking-Penrose singularity theorems [10,30]. If the strong and dominant energy conditions hold for $\tilde{\phi}$ and for ordinary matter in the Einstein frame, the singularity theorems apply, even though the same energy conditions are violated in the Jordan frame. This situation is seen by some as the possibility to circumvent the singularity theorems. In the cosmological context this would imply that it is possible to find solutions that are free of big bang singularities just by going to the Jordan frame. This is clearly impossible if these two conformal frames are physically equivalent: the absence of singularities in one frame and their occurrence in the conformally rescaled theory has thus lead to an apparent paradox [31,32]. If, following Dicke [12], the Jordan and Einstein frames are equivalent, singularities occur in the Einstein frame if and only if they occur in the Jordan frame. The puzzle is quickly resolved as follows (see also Ref. [26]): consider the FLRW metric in the Jordan frame

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1 - Kr^2} + r^2(d\theta^2 + \sin^2\theta d\varphi^2)\right], \tag{4.5}$$

and its Einstein frame cousin

$$d\tilde{s}^2 = -d\tilde{t}^2 + \tilde{a}^2(\tilde{t})\left[\frac{dr^2}{1 - Kr^2} + r^2(d\theta^2 + \sin^2\theta d\varphi^2)\right], \tag{4.6}$$

with $d\tilde{t} = \Omega dt$, $\tilde{a} = \Omega a$, and proper length $d\tilde{l} = \tilde{a}|d\underline{x}| = \Omega a|d\underline{x}| = \Omega dl$. To ascertain whether there is a big bang or other singularity in the Einstein frame with running units it is not sufficient to examine the behavior of the scale factor $\tilde{a}(\tilde{t})$ as $\tilde{t} \rightarrow 0$. One must instead study the ratio of a typical physical (proper) length $\tilde{a}(\tilde{t})|d\tilde{x}|$ to the unit of length $\tilde{l}_u(\tilde{\phi}) = \Omega l_u$, where l_u is the fixed length unit in the Jordan frame and $|d\tilde{x}|$ is the (comoving) coordinate distance in the Einstein frame. This ratio is

$$\frac{\tilde{a}(\tilde{t})|d\tilde{x}|}{\tilde{l}_u(\tilde{\phi})} = \frac{\Omega a(t)|d\underline{x}|}{\Omega l_u} = \frac{a(t)|d\underline{x}|}{l_u}. \tag{4.7}$$

Therefore, $\frac{\tilde{a}|d\tilde{x}|}{\tilde{l}_u(\tilde{\phi})} \rightarrow 0$ if and only if $\frac{a|d\underline{x}|}{l_u} \rightarrow 0$, or a singularity occurs in the Einstein frame if and only if it is present in the Jordan frame. The argument is not yet complete, because one has to make sure that the finite time at which the singularity occurs ("initial time" for a big bang singularity) is not mapped into an infinite time in the other frame. This is easily accomplished by examining the ratio of $\tilde{t}$ to the varying unit of time $\tilde{t}_u(\tilde{\phi}) = \Omega t_u$ in the Einstein frame, where t_u is the fixed unit of time in the Jordan frame. This ratio is

$$\frac{\tilde{t}}{\tilde{t}_u} = \frac{\int_0^t \Omega(\phi)dt'}{\Omega(\phi)t_u} \approx \frac{t}{t_u}, \tag{4.8}$$

as $t \rightarrow 0$ for an initial big bang singularity. Therefore, $\tilde{t} \rightarrow 0$ in the Einstein frame is equivalent to $t \rightarrow 0$ in the Jordan frame.

One can also check whether a singularity in the matter energy density occurs in both frames. The energy density of the cosmic fluid transforms as $\tilde{\rho} = \Omega^{-4}\rho$ on dimensional grounds (for a formal derivation see, e.g., Ref. [26]). The Einstein frame unit of energy is

$$\tilde{\rho}(\tilde{\phi}) \approx \frac{\tilde{m}}{\tilde{l}_u^3} = \frac{\Omega^{-1} m_u}{\Omega^3 l_u^3} = \Omega^{-4}\rho_u, \tag{4.9}$$

where $\rho_u \approx m/l_u^3$ is the (constant) unit of energy density in the Jordan frame. In a big bang singularity, however, it is the ratio $\tilde{\rho}/\tilde{\rho}_u$ that matters, not $\tilde{\rho}$. We have

$$\frac{\tilde{\rho}}{\tilde{\rho}_u} = \frac{\Omega^{-4}\rho}{\Omega^{-4}\rho_u} = \frac{\rho}{\rho_u}, \tag{4.10}$$

hence $\frac{\tilde{\rho}}{\tilde{\rho}_u} \to \infty$ if and only if $\frac{\rho}{\rho_u} \to \infty$, establishing once again the equivalence of the two frames. If one were to consider merely $\tilde{\rho}$ instead of $\tilde{\rho}/\tilde{\rho}_u$, one would erroneously conclude that singularities occur in one conformal frame but not in the other. This happens if the Einstein frame with fixed units is considered, which is not physically equivalent to the Jordan frame (if one wishes to regard it as a physical theory).

V. THE Λ PROBLEM AND THE CAUCHY PROBLEM

In this section we briefly discuss other issues in the realm of classical physics in which the Jordan and the Einstein frames (with running units) prove to be physically equivalent, in spite of claims to the contrary. These are the cosmological constant problem and the Cauchy problem.

It has been claimed that the issue of the conformal frame has implications for the notorious cosmological constant problem [33] of why the cosmological constant energy density is 120 orders of magnitude smaller than what can be calculated with simple quantum mechanics. The stress-energy tensor associated with a cosmological constant $T_{ab}^{(\Lambda)} = \Lambda g_{ab}/(8\pi G)$ provides a Jordan frame energy density $\rho_\Lambda = \Lambda/(8\pi G)$ which is constant, and a conformal cousin $\tilde{\rho}_\Lambda = \Omega^{-4}\rho_\Lambda = e^{-\alpha\tilde{\phi}}\Lambda$ in the Einstein frame, where $\alpha > 0$ is an appropriate constant. Thus, $\tilde{\rho}_\Lambda$ represents a decaying cosmological "constant"; the opinion is often voiced that the exponential factor $e^{-\alpha\tilde{\phi}}$ multiplying Λ in the Einstein frame helps alleviating, if not outright solving, the cosmological constant problem (see, e.g., Sec. 4.22 of Ref. [34]). Again, this would mean that the two conformal frames are physically inequivalent in contrast with the spirit of Dicke's paper [35].

It is easy to see that this argument fails to ease off the cosmological constant problem. Again, what matters in the Einstein frame is not the form (or numerical value) of $\tilde{\rho}_\Lambda$, but the ratio $\tilde{\rho}_\Lambda/\tilde{\rho}_u$, where $\tilde{\rho}_u = \Omega^{-4}\rho_u$ is the unit of energy density in the Einstein frame, and ρ_u is the corresponding Jordan frame unit. The ratio

$$\frac{\tilde{\rho}_\Lambda}{\tilde{\rho}_u} = \frac{\Lambda e^{-\alpha\tilde{\phi}}}{8\pi G\rho_u e^{-\alpha\tilde{\phi}}} = \frac{\Lambda}{8\pi G\rho_u}, \tag{5.1}$$

is the same in the Jordan and Einstein frames and, barring unforeseen complications at the quantum level, the cosmological constant problem is not alleviated a bit by choosing the Einstein frame with running units.

Of course, one could then state that the Einstein frame with fixed units solves the problem because then one would consider $\tilde{\rho}_\Lambda/\rho_u \propto e^{-\alpha\tilde{\phi}}$ instead of $\tilde{\rho}_\Lambda/\tilde{\rho}_u =$ constant; this would be nonsense because the cosmological constant cannot be calculated in one theory (where it is huge) and then mapped into the Einstein frame with fixed units which bears no physical relation with the original Jordan frame. Λ should be calculated directly in the Einstein frame with fixed units and it is still huge. The argument presented here applies also to situations in which the cosmological constant term changes [36].

Finally, we want to comment on the Cauchy problem for scalar-tensor gravity and its implications for the equivalence of the two conformal frames. The folklore about the Cauchy problem is that the mixing of the spin two and spin zero degrees of freedom g_{ab} and ϕ in the Jordan frame makes these variables an inconvenient set for formulating the initial value problem, which is not well posed in the Jordan frame: on the other hand, the Einstein frame variables $(\tilde{g}_{ab}, \tilde{\phi})$ admit a well-posed Cauchy problem completely similar to that of general relativity (see, e.g., the influential paper [37]). Were this true, it would appear that the Jordan and Einstein frame are physically inequivalent in this respect. This position toward the Cauchy problem, however, ignores two older references showing that the Cauchy problem is well posed *in the Jordan frame* for two specific scalar-tensor theories: Brans-Dicke theory with a free scalar ϕ [38] and the theory of a scalar field conformally coupled to the Ricci curvature [39]. The task of studying the Jordan frame Cauchy problem for *general* scalar-tensor theories has been taken on in a recent paper [40] in which it is shown, using generalized harmonic coordinates, that the Cauchy problem is well posed, although further study is necessary for implementing a full 3 + 1 formulation à la York [41] in practical (numerical) applications [40]. This shows that, contrary to the common lore, the Jordan and the Einstein frames are physically equivalent also with respect to the initial value problem. The issue of mapping the details of the Jordan frame Cauchy problem into details of the corresponding Einstein frame problem and, in particular, clarifying the role played by running units, will be discussed elsewhere. It is clear that the equivalence between the two conformal frames breaks down when the conformal transformation breaks down, i.e., when $f(\phi) = 0$ or $f_1 \equiv 2f + 3(df/d\phi)^2 = 0$ (cf. Eqs. (1.6) and (1.7)). However, the Jordan frame initial value problem may not be well posed as well when $f = 0$, and requiring $f > 0$ eliminates also the singularities $f_1 = 0$ (see Refs. [42,43] for a discussion of these singularities and Ref. [44] for conformal continuation past these points).

VI. CONCLUSIONS

It appears that, at the classical level, the Jordan and Einstein frames are physically equivalent when the units of fundamental and derived quantities are allowed to scale appropriately with the conformal factor Ω in the Einstein frame. Previous doubts on the physical equivalence with respect to the Cauchy problem [37] seem to dissipate in the light of recent work [40], although a more comprehensive picture is desirable. The arguments against the equivalence of the two frames raised in the past regard positivity of the energy in the Einstein frame and the indefiniteness of its sign in the Jordan frame ([15,16] and references therein). This is particularly relevant at the quantum level: negative energies do not allow a stable ground state and the system would decay to a lower and lower energy states *ad infinitum*. However, as was pointed out in [23], there is no physical observable corresponding to the sign of $T_{ab}u^a u^b$ or $G_{ab}u^a u^b$, where u^a is a timelike four-vector, and specific examples of scalar-tensor theories that are stable in the Jordan frame have been found [29]. The relevant question to ask, at least at the classical level, is not what the sign of the energy is, but rather whether the energy is bounded from below, which may well occur in scalar-tensor gravity (see, e.g., [45]).

At the quantum level, the issue of the ground state becomes more delicate, as there are more decay channels than at the classical level. Although the conformal equivalence seems to hold to some extent at the semiclassical level, in which the matter fields are quantized while the variables (g_{ab}, ϕ) are classical (see Ref. [23] for a brief discussion and references), this equivalence definitely breaks down when ϕ is quantized [16]. When also g_{ab} is quantized in full quantum gravity, inequivalent quantum theories have been found [23,46–48]. This is not surprising because the conformal transformation can be seen as a Legendre transformation [15]. A similar Legendre transformation is used in the classical mechanics of particles to switch from the canonical coordinates q of the Lagrangian description to the variables (q, p) of the Hamiltonian formalism, and this Legendre map is an example of a canonical transformation [49]. Now, it is well known that Hamiltonians that are classically equivalent become inequivalent when quantized: they exhibit different energy spectra and scattering amplitudes [50]. Therefore, we expect the analogous ''canonical transformations'' between different conformal frames not to be unitary and to yield physically inequivalent theories at the quantum level [51]. A common objection to this statement arising among particle physicists is based on the equivalence theorem of Lagrangian field theory, which states that the S-matrix is invariant under local (nonlinear) field redefinitions [52]. Since the conformal transformation (2.2) and (2.3) is a local nonlinear redefinition of the fields g_{ab} and ϕ, it would seem that quantum physics is invariant under change of the conformal frame. However, this is not true in general because the field theory approach in which the equivalence theorem is derived applies to gravity only in the perturbative regime in which the fields describe small deviations from Minkowski space. In this regime, tree-level quantities can be calculated in any conformal frame with the same results. However, when the metric tensor is allowed full dynamical freedom and is not restricted to be a small perturbation of a fixed background, the field theory approach and the equivalence theorem do not apply. It is plausible that the equivalence theorem can be proved also for fixed backgrounds that are curved and do not coincide with the Minkowski space of effective field theory. However, we are not aware of such a generalization in the literature on quantum field theory on curved space (a proof of a generalized equivalence theorem will be pursued elsewhere). Thus, it is clear that the equivalence theorem fails in the nonperturbative regime; nevertheless, one can consider semiclassical situations in which the metric is classical and the full scalar ϕ is quantized, and it is quite possible that the conformal transformation leaves the quantum physics of ϕ unaffected—after all, the physics of the classical metric is invariant under change of conformal frame and quantization of a scalar field in a fixed background metric poses no problems [53]. Indeed, there are examples in which such semiclassical theories related by a conformal transformation seem to be equivalent [54]. A precise and detailed understanding of the conformal (in)-equivalence at the quantum level, however, requires further work.

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Note added in proof.—After finishing this work we learned about Ref. [58] which agrees with our results and traces the origin of the conformal frame issue to a 1956 paper by Fierz.

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