# CMB anisotropies due to cosmological magnetosonic waves

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We study scalar mode perturbations (magnetosonic waves) induced by a helical stochastic cosmological magnetic field and derive analytically the corresponding cosmic microwave background (CMB) temperature and polarization anisotropy angular power spectra. We show that the presence of a stochastic magnetic field, or an homogeneous magnetic field, influences the acoustic oscillation pattern of the CMB anisotropy power spectrum, effectively acting as a reduction of the baryon fraction. We find that the scalar magnetic energy density perturbation contribution to the CMB temperature anisotropy is small compared to the contribution to the CMB $E$-polarization anisotropy.

## I. INTRODUCTION

A promising explanation for observed uniform magnetic fields in galaxies is that they are the amplified remnants of a seed cosmological magnetic field (for reviews see Refs. [1]) generated in the early Universe [2,3]. A seed magnetic field may have an helical part [4]. Magnetic helicity plays an important role in magnetohydrodynamical processes in the primordial plasma as well as in cosmological perturbation dynamics; In particular, magnetic helicity influences the inverse cascade mechanism—when energy is transferred from small to large scales—[5], and as a result affects large-scale magnetic field formation [6].

The average energy density and helicity of the magnetic field must be small to be consistent with the observed large-scale spatial isotropy of the Universe. In this case the linear theory of gravitational instability can be used to study perturbation dynamics [7–10]. A cosmological magnetic field induces scalar, vector and tensor perturbations [9]. At linear order each mode evolves separately. (At second order the modes are coupled and this results in non-Gaussian effects [11]).

The vector (vorticity) and tensor (gravitational waves) perturbation modes induced by a cosmological magnetic field have attracted a lot of interest [12–20]. This is partially because they give rise to a $B$-polarization CMB anisotropy signal, which vanishes for density (scalar mode) perturbations at linear order. Any cosmological signature of a primordial magnetic field is a potential probe (for a short review see Ref. [21]). For example, the limit on a chemical-potential-like distortion of the CMB Planck spectrum leads to a limit on a cosmological magnetic field of order $10^{-8}$–$10^{-9}$ Gauss on 1–500 kpc lengthscales [22]. Similar limits on a cosmological magnetic field generated during inflation [3] are obtained from CMB temperature and polarization anisotropy and non-Gaussianity data from vector and tensor perturbation modes [23].

On the other hand, the scalar mode of magnetically driven perturbations also has a significant effect on CMB fluctuations: fast magnetosonic waves shift the CMB temperature anisotropy power spectrum acoustic peaks [24–26]. In this paper we present a systematic treatment of scalar magnetized perturbations that complements earlier work [10,19,24–30]. Using the total angular momentum formalism [31] and analytical approximation techniques described in Ref. [14] we obtain analytical expressions for CMB temperature and polarization anisotropies.

Reference [27] presents numerical computations of scalar CMB temperature and polarization anisotropies in the case of a stochastic cosmological magnetic field with magnetic field power spectral indices $n_B = 1, 2, 3$. Here we consider a general cosmological magnetic field and contrary to Ref. [27] we account for the Lorentz force term in the Euler equation for baryons, in accord with the analyses of Refs. [9,10,19]. The main new results are approximate analytical expressions for the CMB fluctuations. This analysis allows us to identify two different effects arising from the magnetic field: (i) a rescaling of the photon-baryon fluid sound speed (that is responsible for the shift of the CMB acoustic peaks); and, (ii) effects from nonzero magnetic anisotropic stress (that is responsible for the additional CMB $E$-polarization anisotropy signal).

The rest of this paper is organized as follows. In the next Section we describe the magnetic field, including the power spectrum, the anisotropic stress tensor, and its connection with the scalar (longitudial) part of the Lorentz force. We present analytical expressions for two-point correlations functions of the magnetic field energy density, the scalar part of the Lorentz force, and the magnetic field anisotropic stress. In Sec. III we derive the equations that govern scalar magnetic perturbations (the Einstein and matter conservation equations) and discuss general solutions and the initial conditions we adopt. In Secs. IV, V, and VI we use semianalytical approximations to compute CMB

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temperature and polarization anisotropies (as well as cross-correlations between temperature and $E$-polarization anisotropies). We conclude in Sec. VII.

## II. MAGNETIC FIELD STATISTICAL PROPERTIES

### A. Power spectrum

We assume the presence of a Gaussianly-distributed stochastic helical cosmological magnetic field generated during or prior to the radiation-dominated epoch, with the energy density of the field a first-order perturbation to the Friedmann-Lemaître-Robertson-Walker (FLRW) homogeneous cosmological spacetime model. We neglect fluid back-reaction onto the magnetic field, therefore the spatial and temporal dependence of the field separates, $\mathbf{B}(t, \mathbf{x}) = \mathbf{B}(\mathbf{x})/a^2$. Here $a(t)$ is the cosmological scale factor, normalized to unity at the present time $t_0$, and $\mathbf{B}(\mathbf{x})$ is the magnetic field at the present time. Since the magnetic field energy density is first order, the magnetic field is $1/2$ order.

Smoothing on a comoving length $\lambda$ with a Gaussian smoothing kernel $\propto \exp[-x^2/\lambda^2]$, we obtain the smoothed magnetic field with mean squared magnetic field $B_\lambda{}^2 = \langle \mathbf{B}(\mathbf{x}) \cdot \mathbf{B}(\mathbf{x}) \rangle|_\lambda$ and mean squared magnetic helicity $H_\lambda{}^2 = \lambda |\langle \mathbf{B}(\mathbf{x}) \cdot [\nabla \times \mathbf{B}(\mathbf{x})] \rangle|_\lambda$ [16,18]. Corresponding to the smoothing length $\lambda$ is the smoothing wavenumber $k_\lambda = 2\pi/\lambda$. We use

$$B_j(\mathbf{k}) = \int d^3x e^{i\mathbf{k} \cdot \mathbf{x}} B_j(\mathbf{x}),$$
$$B_j(\mathbf{x}) = \int \frac{d^3k}{(2\pi)^3} e^{-i\mathbf{k} \cdot \mathbf{x}} B_j(\mathbf{k}), \qquad (1)$$

when Fourier transforming between position and wavenumber spaces. We assume flat spatial hypersurfaces (consistent with current observational indications, e.g., Ref. [32]).

We also assume that the primordial plasma is a perfect conductor on all scales larger than the Silk damping wavelength $\lambda_S$ (the thickness of the last-scattering surface) set by photon and neutrino diffusion. On much smaller scales we model magnetic field damping by an ultraviolet cutoff wavenumber $k_D = 2\pi/\lambda_D$ that is due to the damping of Alfvén waves from photon viscosity [29,30] (see Eq. (1) of Ref. [18]); here $\lambda_D \ll \lambda_S$.

Under these assumptions the magnetic field two-point correlation function in wavenumber space is

$$\langle B_i^\star(\mathbf{k}) B_j(\mathbf{k}') \rangle = (2\pi)^3 \delta^{(3)}(\mathbf{k} - \mathbf{k}')[P_{ij}(\hat{\mathbf{k}}) P_B(k)$$
$$+ i\epsilon_{ijl}\hat{k}_l P_H(k)]. \qquad (2)$$

Here $i$ and $j$ are spatial indices, $i, j \in (1, 2, 3)$, $\hat{k}_i = k_i/k$ a unit wavevector, $P_{ij}(\hat{\mathbf{k}}) = \delta_{ij} - \hat{k}_i\hat{k}_j$ the transverse plane projector, $\epsilon_{ijl}$ the antisymmetric symbol, and $\delta^{(3)}(\mathbf{k} - \mathbf{k}')$ the Dirac delta function. The power spectra of the symmetric and helical parts of the magnetic field, $P_B(k)$ and $P_H(k)$, are assumed to be simple power laws on large scales

$$P_B(k) = P_{B0}k^{n_B} = \frac{2\pi^2\lambda^3 B_\lambda^2}{\Gamma(n_B/2 + 3/2)}(\lambda k)^{n_B},$$

$$P_H(k) = P_{H0}k^{n_H} = \frac{2\pi^2\lambda^3 H_\lambda^2}{\Gamma(n_H/2 + 2)}(\lambda k)^{n_H}, \qquad k < k_D,$$
$$\qquad (3)$$

and vanish on small scales where $k > k_D$. The spectral indexes $n_B$ and $n_H$ are constrained by the requirement of finiteness of mean magnetic field energy density ($n_B > -3$) and mean magnetic helicity ($n_H > -4$) in the infrared region at small $k$. In addition, causality requires $P_B(k) \geq |P_H(k)|$ (the Schwartz inequality), [33].

### B. Anisotropic stress-energy tensor

We consider the effects of a cosmological magnetic field on the CMB at high redshift when the Universe is hot and the plasma a good conductor. As a result the magnetic field lines are dragged by the matter fluid and this generates a weak "frozen" electric field $\mathbf{E} = -\mathbf{v} \times \mathbf{B}$, where $\mathbf{v}$ is the perturbed (first order) 3-velocity of the fluid. We neglect this weak electric field in what follows since the energy density of this electric field contributes at third order in the perturbation expansion. At the current time, the space-space part of Maxwell stress-energy tensor for the magnetic field is

$$\tau_{ij}^{(B)}(\mathbf{x}, \eta_0) = \frac{1}{4\pi}\left[ B_i(\mathbf{x}) B_j(\mathbf{x}) - \frac{1}{2}\delta_{ij} B^2(\mathbf{x}) \right], \quad (4)$$

where $\eta_0$ is the current value of conformal time $\eta = \int^\eta dt/a(t)$. The energy density and the anisotropic trace-free part of the space-space components of the stress-energy tensor of the magnetic field at the current time are

$$\rho_B(\mathbf{x}, \eta_0) = \frac{1}{8\pi}B^2(\mathbf{x}) = -\tau^{\text{tr}}(\mathbf{x}, \eta_0), \qquad (5)$$

$$\tau_{ij}^{(A)}(\mathbf{x}, \eta_0) = \frac{1}{4\pi}\left[ B_i(\mathbf{x}) B_j(\mathbf{x}) - \frac{1}{3}\delta_{ij} B^2(\mathbf{x}) \right]$$
$$= \tau_{ij}^{(B)}(\mathbf{x}, \eta_0) - \frac{1}{3}\delta_{ij}\tau^{\text{tr}}(\mathbf{x}, \eta_0), \qquad (6)$$

where $\tau^{\text{tr}} = \delta_{ij}\tau_{ij}^{(B)}$. Both $\rho_B(\mathbf{x}, \eta_0) = \rho_B(\mathbf{x}, \eta)a^4$ and $\tau_{ij}^{(A)}(\mathbf{x}, \eta_0) = \tau_{ij}^{(A)}(\mathbf{x}, \eta)a^4$ are quadratic in the magnetic field, and their wavenumber-space transforms are convolutions of the magnetic field

$$\rho_B(\mathbf{k}, \eta_0) = -\tau^{\text{tr}}(\mathbf{k}, \eta_0) = \frac{1}{8\pi}\int \frac{d^3p}{(2\pi)^3}B_l(\mathbf{p})B_l(\mathbf{k} - \mathbf{p}), \qquad (7)$$

$$\tau_{ij}^{(A)}(\mathbf{k}, \eta_0) = \frac{1}{4\pi} \int \frac{d^3 p}{(2\pi)^3} \left[ B_i(\mathbf{p}) B_j(\mathbf{k} - \mathbf{p}) \right.$$
$$\left. - \frac{1}{3} \delta_{ij} B_l(\mathbf{p}) B_l(\mathbf{k} - \mathbf{p}) \right]. \qquad (8)$$

We assume that Eq. (6) is a first-order perturbation and decompose it into scalar, vector, and tensor parts, $\tau_{ij}^{(A)}(\mathbf{x}) = \Pi_{ij}^{(S)}(\mathbf{x}) + \Pi_{ij}^{(V)}(\mathbf{x}) + \Pi_{ij}^{(T)}(\mathbf{x})$ [7,28]. This decomposition is more conveniently done in wavenumber space. We use projection operators and find $\Pi_{ij}^{(V)}(\mathbf{k}) = [P_{ib}(\hat{\mathbf{k}})\hat{k}_j + P_{jb}(\hat{\mathbf{k}})\hat{k}_i]\hat{k}_a \tau_{ab}(\mathbf{k})$ (for the vector part) [14,18] and $\Pi_{ij}^{(T)}(\mathbf{k}, \eta_0) = [P_{ia}(\hat{\mathbf{k}})P_{jb}(\hat{\mathbf{k}}) - P_{ij}(\hat{\mathbf{k}})P_{ab}(\hat{\mathbf{k}})/2]\tau_{ab}(\mathbf{k}, \eta_0)$ (for the tensor part) [14,16]. The scalar parts of $\rho_B$ and $\tau_{ij}^{(B)}$ determine the scalar part of the magnetic source. The scalar part $\Pi_{ij}^{(S)}(\mathbf{k})$ has to be proportional to $\hat{k}_i \hat{k}_j - \delta_{ij}/3$ [7], so we define

$$\Pi_{ij}^{(S)}(\mathbf{k}) = \frac{3}{2}\left(\hat{k}_i \hat{k}_j - \frac{1}{3}\delta_{ij}\right)\Pi^{(S)}(\mathbf{k}). \qquad (9)$$

Here the scalar $\Pi^{(S)}(\mathbf{k}, \eta) = \Pi^{(S)}(\mathbf{k}, \eta_0)/a^4$ is associated with the anisotropic stress of the magnetic field. (At leading order the isotropic pressure has the same time dependence, $p_B(\mathbf{x}, \eta) = p_B(\mathbf{x}, \eta_0)/a^4$, and is related to the magnetic field energy density by $p_B(\mathbf{x}, \eta) = \rho_B(\mathbf{x}, \eta)/3$.) It is straightforward to determine $\Pi^{(S)}(\mathbf{k})$ by applying $\hat{k}_n \hat{k}_m - \delta_{nm}/3$ on $\tau_{nm}^{(A)}(\mathbf{k})$, i.e.,

$$\Pi^{(S)}(\mathbf{k}) = \hat{k}_n \hat{k}_m \tau_{nm}^{(A)}(\mathbf{k}), \qquad (10)$$

where we use $\delta_{nm}\tau_{nm}^{(A)} = 0$. Our $\Pi^{(S)}(\mathbf{k}, \eta_0)$ is related to the $\tau^{(S)}(\mathbf{k}, \eta_0)$ of Ref. [11] through $\tau^{(S)}(\mathbf{k}, \eta_0) = 3\Pi^{(S)}(\mathbf{k}, \eta_0)/2$.

The scalar $\Pi^{(S)}$ is related to the scalar part of the Lorentz force $\mathbf{L}(\mathbf{x}, \eta_0) = -[\mathbf{B}(\mathbf{x}) \times (\nabla \times \mathbf{B}(\mathbf{x}))]/(4\pi)$ and the isotropic pressure $p_B(\mathbf{x}, \eta_0)$. We introduce a scalar $L^{(S)}$ defined by $L_i^{(S)}(\mathbf{x}, \eta_0) = \nabla_i L^{(S)}(\mathbf{x}, \eta_0)$, where $L_i^{(S)}$ is the scalar part of the Lorentz force. Using the Maxwell equation $\nabla \cdot \mathbf{B} = 0$, the Lorentz force is

$$L_i(\mathbf{x}, \eta_0) = \frac{1}{4\pi}\left[ B_j(\mathbf{x})\nabla_j B_i(\mathbf{x}) - \frac{1}{2}\nabla_i B^2(\mathbf{x}) \right], \qquad (11)$$

and the corresponding scalar part is derived through the scalar

$$\nabla^2 L^{(S)}(\mathbf{x}, \eta_0) = \nabla_i L_i^{(S)}(\mathbf{x}, \eta_0)$$
$$= \frac{1}{4\pi}\left[ (\nabla_i B_j(\mathbf{x}))\nabla_j B_i(\mathbf{x}) - \frac{1}{2}\nabla^2 B^2(\mathbf{x}) \right], \qquad (12)$$

where $\nabla^2 = \nabla_i \nabla_i$ is the Laplace operator and we have used $\nabla \cdot \mathbf{B} = 0$. In position space Eq. (10) reads

$$\nabla^2 \Pi^{(S)}(\mathbf{x}, \eta_0) = \frac{1}{4\pi}\left[ \nabla_i \nabla_j (B_i(\mathbf{x})B_j(\mathbf{x})) - \frac{1}{3}\nabla^2 B^2(\mathbf{x}) \right], \qquad (13)$$

and comparison with Eq. (12) results in

$$\Pi^{(S)}(\mathbf{x}, \eta_0) = \frac{\rho_B(\mathbf{x}, \eta_0)}{3} + L^{(S)}(\mathbf{x}, \eta_0), \qquad (14)$$

the analog of Eq. (5.44) of Ref. [10].

Since we consider a stochastic magnetic field we present here various correlations and averages of the magnetic source for scalar perturbations. The wavenumber-space scalar two-point correlation function is

$$\langle \Pi^{(S)\star}(\mathbf{k}, \eta_0)\Pi^{(S)}(\mathbf{k}', \eta_0) \rangle = (2\pi)^3 |\Pi^{(S)}(k, \eta_0)|^2$$
$$\times \delta^{(3)}(\mathbf{k} - \mathbf{k}'), \qquad (15)$$

where the power spectrum $|\Pi^{(S)}(k, \eta_0)|^2$ depends only on $k = |\mathbf{k}|$. (The vector and tensor two-point correlation functions are given in Refs. [14,16–18].) Two-point correlation functions of the magnetic field energy density and the scalar part of the Lorentz force are defined in a similar manner,

$$\langle \rho_B^\star(\mathbf{k}, \eta_0)\rho_B(\mathbf{k}', \eta_0) \rangle = (2\pi)^3 |\rho_B(k, \eta_0)|^2 \delta^{(3)}(\mathbf{k} - \mathbf{k}'), \qquad (16)$$

$$\langle L^{(S)\star}(\mathbf{k}, \eta_0)L^{(S)}(\mathbf{k}', \eta_0) \rangle = (2\pi)^3 |L^{(S)}(k, \eta_0)|^2$$
$$\times \delta^{(3)}(\mathbf{k} - \mathbf{k}'). \qquad (17)$$

Note that $L^{(S)}(\mathbf{x}, \eta)$ is related to $\mathcal{F}(\mathbf{x}, \eta)$ of Ref. [9] through $\nabla^2 L^{(S)}(\mathbf{x}, \eta) = \mathcal{F}(\mathbf{x}, \eta)/(4\pi)$, so $k^2 L^{(S)}(\mathbf{k}, \eta) = -\mathcal{F}(\mathbf{k}, \eta)/(4\pi)$.

The scalar power spectra $|\Pi^{(S)}(k)|^2$, $|\rho_B(k)|^2$, and $|L^{(S)}(k)|^2$ are determined by the symmetric part of the magnetic field power spectrum $P_B(k)$ and do not depend on the magnetic helicity spectrum $P_H(k)$ (also see Ref. [11]),

$$|\Pi^{(S)}(k)|^2 = \frac{1}{576\pi^5} \int d^3 p P_B(p)P_B(|\mathbf{k} - \mathbf{p}|)$$
$$\times [9(1 - \gamma^2)(1 - \beta^2) - 6(1 + \gamma\mu\beta - \gamma^2 - \beta^2)$$
$$+ (1 + \mu^2)], \qquad (18)$$

$$|\rho_B(k)|^2 = \frac{1}{256\pi^5} \int d^3 p P_B(p)P_B(|\mathbf{k} - \mathbf{p}|)(1 + \mu^2), \qquad (19)$$

$$|L^{(S)}(k)|^2 = \frac{1}{256\pi^5} \int d^3 p P_B(p)P_B(|\mathbf{k} - \mathbf{p}|)$$
$$\times [4\gamma\beta(\gamma\beta - \mu) + (1 + \mu^2)]. \qquad (20)$$

Here $\gamma = \hat{\mathbf{k}} \cdot \hat{\mathbf{p}}$, $\beta = \mathbf{k} \cdot (\mathbf{k} - \mathbf{p})/(k|\mathbf{k} - \mathbf{p}|)$, and $\mu = \mathbf{p} \cdot (\mathbf{k} - \mathbf{p})/(p|\mathbf{k} - \mathbf{p}|)$. The relations to the two-point correlation functions given in Ref. [11] are

$\langle \tau^\star(\mathbf{k}) \tau(\mathbf{k}') \rangle = \langle \rho_B^\star(\mathbf{k}) \rho_B(\mathbf{k}') \rangle$ and $\langle \tau^{\star S}(\mathbf{k}) \tau^S(\mathbf{k}') \rangle = 9 \langle \Pi^{(S)}(\mathbf{k}) \Pi^{(S)}(\mathbf{k}') \rangle / 4$; $\tau(\mathbf{k})$ and $\tau^S(\mathbf{k})$ are given in Eqs. (2.15) of Ref. [11].

Using the power law magnetic field power spectrum of Eq. (3), we can obtain expressions for the power spectra $|\Pi^{(S)}(k)|^2$, $|\rho_B(k)|^2$, and $|L^{(S)}(k)|^2$ in the semianalytical approximation where we divide the integration range into $p < k$ and $p > k$ parts and consider two limiting ranges $p \ll k$ and $p \gg k$ [14,16,18]. In particular, the magnetic energy density correlation power spectrum is[1]

$$|\rho_B(k, \eta_0)|^2 = \frac{3(k_D \lambda)^{2n_B + 3} \lambda^3 B_\lambda^4}{32(2n_B + 3) \Gamma^2(n_B/2 + 3/2)}$$
$$\times \left[ 1 + \frac{n_B}{n_B + 3} \left( \frac{k}{k_D} \right)^{2n_B + 3} \right]. \quad (25)$$

For $n_B > -3/2$ this expression is dominated by the cutoff scale $k_D$, and for large $k_D$ it does not depend on $k$, while for $n_B < 3/2$ we get $|\rho_B(k, \eta_0)|^2 \propto k^{2n_B + 3}$. It can be shown [14] that $|\rho_B(k, \eta_0)|^2 \simeq 9|L^{(S)}(k, \eta_0)|^2 / 8 \simeq |L^{(S)}(k, \eta_0)|^2$ and $|\rho_B(k, \eta_0)|^2 \simeq 9|\Pi^{(S)}(k, \eta_0)|^2 / 4$.[2]

Increasing magnetic helicity reduces the vector part of Lorentz force two-point correlation function $\langle L_i^{(V)} L_i^{(V)} \rangle$ ([9,18]; this reduces parity-even CMB fluctuations [16,18]), but leaves the scalar part unchanged ($|L^{(S)}|^2$ is independent of magnetic helicity). In contrast, Refs. [9,10,27] neglect the Lorentz force for a maximally helical magnetic field. They argue as follows: since $\langle \mathbf{B} \cdot (\nabla \times \mathbf{B}) \rangle$ is maximal, the average of the Lorentz force $\propto$

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[1]The power spectrum $|\rho_B(k)|^2$ is related to the two-point correlation function of the energy density of the magnetic field. By definition the r.m.s. magnetic field energy density $\rho_B^{\mathrm{rms}}$ is

$$[\rho_B^{\mathrm{rms}}(\eta_0)]^2 = \langle \rho_B(\mathbf{x}) \rho_B(\mathbf{x}) \rangle = \frac{1}{2\pi^2} \int_0^\infty dk k^2 |\rho_B(k, \eta_0)|^2. \quad (21)$$

Replacing the upper limit of integration by the cutoff scale $k_D$ and using Eq. (25) below, we find

$$\rho_B^{\mathrm{rms}}(\eta_0) = \frac{\sqrt{n + 6} B_\lambda^2 (k_D \lambda)^{n_B + 3}}{8\sqrt{2}(n + 3)\pi \Gamma(n_B/2 + 3/2)}. \quad (22)$$

$\rho_B^{\mathrm{rms}}(\eta_0)$ differs from the average magnetic energy density $\bar\rho_B = \langle \rho_B \rangle$ which is determined by the power spectrum $E_B(k) = k^2 P_B(k)/\pi^2$,

$$\bar\rho_B(\eta_0) = \frac{1}{8\pi} \langle B_l(\mathbf{x}) B_l(\mathbf{x}) \rangle = \frac{1}{8\pi} \int_0^\infty dk E_B(k), \quad (23)$$

which, using the cutoff scale $k_D$, gives

$$\bar\rho_B(\eta_0) = \frac{B_\lambda^2 (k_D \lambda)^{n_B + 3}}{4\pi(n_B + 3)\Gamma(n_B/2 + 3/2)}; \quad (24)$$

for $n_B = -3$, $\bar\rho_B(\eta_0) = B_\lambda^2/(8\pi)$. We note, in particular, that the average Lorentz force vanishes but the r.m.s. Lorentz force is not zero.

[2]It may be shown that $L^{(S)} \simeq -2\sqrt{2}\rho_B/3 \simeq -\rho_B$ and $\Pi^{(S)} \simeq -2\rho_B/3$.

$\langle \mathbf{B} \times (\nabla \times \mathbf{B}) \rangle$ for a such field is minimal or even zero—this is a valid approximation for a homogeneous field, and results in the force-free approximation [9]—but this is not applicable for a stochastic field. In the case of a stochastic field the average Lorentz force is zero (as is the average magnetic field itself) but the Lorentz force two-point correlation is nonzero (see footnote 2 above). This affects stochastic peculiar motions (vorticity perturbations) of charged particles [18], and, as we show below, the dynamics of density perturbations also. In the stochastic field case the force-free approximation should be used with caution.

## III. SCALAR MAGNETIC PERTURBATIONS

In this section we study the dynamics of linear magnetic energy density perturbations about a spatially-flat FLRW background with scalar metric fluctuations. The metric tensor can be decomposed into a spatially homogeneous background part and a perturbation part, $g_{\mu\nu} = g_{\mu\nu}^{(0)} + \delta g_{\mu\nu}$; Greek letters are used for spacetime indices, $\mu, \nu \in (0, 1, 2, 3)$. For a spatially-flat model, and working with conformal time, the background FLRW metric $g_{\mu\nu}^{(0)} = a^2 \eta_{\mu\nu}$, where $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$ is the Minkowski metric tensor. Scalar perturbations are gauge dependent because the mapping of coordinates between the perturbed physical manifold and the background is not unique. We work in Newtonian (longitudinal) gauge in which the metric tensor is shear free [8]. Scalar perturbations to the geometry are then described by two scalar gravitational potentials $\Psi$ and $\Phi$ where

$$\delta g_{00}^{(S)} = -2a^2 \Psi, \qquad \delta g_{ij}^{(S)} = 2a^2 \Phi \delta_{ij}. \quad (26)$$

In this gauge, neglecting vector and tensor fluctuations, the line element $ds^2 = a^2(\eta)[-(1 + 2\Psi)d\eta^2 + \delta_{ij}(1 + 2\Phi)dx^i dx^j]$.

Matter perturbations are described by the perturbation of the complete stress-energy tensor, $\delta\tau_\mu^\nu$, which is the sum of the perturbed fluid and electromagnetic stress-energy tensors, $\delta\tau_\mu^\nu = \sum_f \delta\tau_{f\mu}^\nu + \tau_\mu^{(B)\nu}$. The subscript $f$ denotes the three different matter components we consider, photons ($\gamma$), baryons ($b$), or cold dark matter ($c$), which we model as fluids. A magnetic field influences perturbations without changing the background metric which is determined by the photon, baryon, and cold dark matter background densities. For simplicity we ignore neutrinos; Refs. [9,10,23] account for the effects of these relativistic weakly interacting particles. Since we focus on dynamics at large redshift we also ignore a possible cosmological constant or dark energy. We decompose perturbations into plane waves $\propto \exp(i\mathbf{k} \cdot \mathbf{x})$ and in what follows equations are presented in wavenumber space.

The magnetic field source affects the motion of baryons. Before recombination the photon-baryon plasma is dominated by photons and has the relativistic equation of state

$p = \rho/3$. The baryon-photon momentum density ratio $R(\eta) = 3\rho_b(\eta)/(4\rho_\gamma(\eta)) = 3a\Omega_{0b}/(4\Omega_{0\gamma})$, where $\rho_\gamma$ and $\rho_b$ are the energy densities of photons and baryons, $\Omega_{0\gamma}$ and $\Omega_{0b}$ are the photon and baryon density parameters measured today, i.e., $\Omega_{0f} = \rho_f(\eta_0)/\rho_{cr}$, where $\rho_{cr} = 8\pi G/(3H_0^2)$ is the critical Einstein-de Sitter density and $H_0$ is the Hubble constant. At early times $R \ll 1$; at the last-scattering surface when photons and baryons decouple $R_{dec} \simeq 0.35$.

We consider a primordial magnetic field and assume that the stress-energy of the magnetic field is not compensated by anisotropic stress in the fluid. That is, we have nonzero initial gravitational potentials but vanishing initial fluid energy density and velocity perturbations. This is possible for a magnetic field generated during inflation [3,28]. This will limit the spectral index of the magnetic field, $n_B < 0$.[3]

In Secs. III A and III B below we present the linear perturbation theory equations for the metric and matter perturbations. In Sec. III C we discuss the speed of sound rescaling in the presence of a magnetic field, while in Sec. III D we consider initial conditions.

## A. Metric perturbations

In zero-shear (Newtonian) gauge, part of Einstein's equation for scalar metric perturbations become two Poisson equations [7,8], which in wavenumber space are

$$k^2\Phi = 4\pi Ga^2\left[\rho_B + \sum_f \rho_f\delta_f + 3\frac{\dot{a}}{ak}\sum_f(\rho_f + p_f)v_f^{(S)}\right],$$
(27)

$$k^2(\Psi + \Phi) = -4\pi Ga^2\left[2\sum_f p_f\Pi_f^{(S)} + \rho_B + 3L^{(S)}\right].$$
(28)

Here $\delta_f$ is the $f$-th fluid density perturbation, $v_f^{(S)}$ is the scalar part of the $f$-th fluid velocity perturbation $\mathbf{v}_f^{(S)} = \hat{\mathbf{k}}v_f^{(S)}$, $\Pi_f^{(S)}$ is the scalar part of the anisotropic $f$-th fluid stress-energy tensor, $\Pi_f^{(S)} = (\hat{k}_i\hat{k}_j - \delta_{ij}/3)\tau_{f,ij}$, and an overdot represents a conformal time derivative $\partial/\partial\eta$. For economy of notation we do not explicitly show the wave vector ($\mathbf{k}$) or time ($\eta$) dependence of the variables in these and following equations. (For instance, the magnetic field energy density $\rho_B(\eta) = \rho_B(\eta_0)/a^4$, the Lorentz force $L^{(S)}(\eta) = L^{(S)}(\eta_0)/a^4$, and the anisotropic stress of the magnetic field $\Pi^{(S)}(\eta) = \Pi^{(S)}(\eta_0)/a^4$.) Note that the combination on the r.h.s. of Eq. (27),

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[3]This limit on the magnetic field spectral index was obtained for a magnetic field generated through coupling of the inflation and hypercharge during inflation [3]. Ref. [34] argues for the same result while Ref. [35] relaxes the limit to $n_B < 2$.

$$\mathcal{D}_f = \rho_f\delta_f + 3\frac{\dot{a}}{ak}(\rho_f + p_f)v_f^{(S)}$$

$$= \rho_f\left[\delta_f + 3\frac{\dot{a}}{a}(1 + \omega_f)\frac{v_f^{(S)}}{k}\right],$$
(29)

(where $\omega_f = p_f/\rho_f$ is the equation of state parameter for the $f$-th fluid) corresponds to the gauge-invariant total (magnetic-field-induced) $f$-th fluid energy density perturbation [36].

To derive Eq. (27) we use the Einstein equation for $\delta\tau_0^i$, which is

$$-\dot{\Phi} + \frac{\dot{a}}{a}\Psi = 4\pi Ga^2\sum_f(\rho_f + p_f)v_f^{(S)}/k.$$
(30)

As a consequence of the high conductivity, $\sigma \gg 1$, of the primordial plasma, the $0i$ component of the magnetic field stress-energy tensor, $\propto \mathbf{E} \cdot \mathbf{B}$, being suppressed by $1/\sigma$, does not contribute to the r.h.s. of Eq. (30). The energy density of the electric field, $\propto E^2$, is suppressed by $1/\sigma^2$ and does not contribute to the r.h.s. of Eq. (27).

The trace of the space-space part of the Einstein equation gives an additional constraint equation. Since this equation is for the trace $\tau_i^i$ [9,27], it does not have a contribution from magnetic anisotropic stress (but isotropic pressure does contribute),

$$\ddot{\Phi} + \frac{\dot{a}}{a}(2\dot{\Phi} - \dot{\Psi}) + \left[\left(\frac{\dot{a}}{a}\right)^2 - 2\frac{\ddot{a}}{a}\right]\Psi + \frac{k^2}{3}(\Psi + \Phi)$$

$$= -4\pi Ga^2\left(\sum_f c_{S,f}^2\rho_f\delta_f + \frac{\rho_B}{3}\right).$$
(31)

Here $c_{S,f}^2 = dp_f/d\rho_f$ is the square of the speed of sound in the $f$-th fluid.

Eqs. (27), (28), and (31) govern the evolution of the scalar metric perturbations $\Phi$ and $\Psi$, if the density, velocity, and anisotropic stress perturbations for each $f$-th component are known. For unmagnetized perturbations, the energy density and anisotropic stress of the magnetic field, $\rho_B(\mathbf{k}, \eta)$ and $\Pi^{(S)}(\mathbf{k}, \eta)$, vanish. In this case Eq. (28) results in the simple relation $\Phi = -\Psi$. The presence of collisionless particles, such as neutrinos, induces anisotropic stress [23,31]. Even through we neglect neutrinos, in our case a stochastic magnetic field induces anisotropic stress and so violates the condition $\Phi = -\Psi$.

Using Eq. (28), setting $p_b = 0 = p_c$, and neglecting neutrinos, Eq. (31) becomes

$$\ddot{\Phi} + \frac{\dot{a}}{a}(2\dot{\Phi} - \dot{\Psi}) + \left[\left(\frac{\dot{a}}{a}\right)^2 - 2\frac{\ddot{a}}{a}\right]\Psi$$

$$= -\frac{4\pi Ga^2\rho_\gamma}{3}\left(\delta_\gamma - \frac{3L^{(S)}}{\rho_\gamma}\right) = -\frac{\dot{a}^2}{2a^2}\left(\delta_\gamma - \frac{3L^{(S)}}{\rho_\gamma}\right),$$
(32)

where the last step uses the zeroth-order Friedmann equation in the radiation-dominated model. Eqs. (27) and (28)

may be combined together,

$$k^2(2\Phi + \Psi) = 4\pi G a^2 \left[ \rho_\gamma \left( \delta_\gamma - \frac{3L^{(S)}}{\rho_\gamma} \right) + \rho_b \delta_b + \rho_c \delta_c \right.$$

$$\left. + 3\frac{\dot{a}}{ak} \sum_f (\rho_f + p_f) v_f^{(S)} \right]. \qquad (33)$$

We show below that the combination $\delta_\gamma \rho_\gamma - 3L^{(S)}$ on the r.h.s. of this equation also appears in the equations for matter perturbations and reflects an effective temperature rescaling (induced by the presence of the Lorentz force).

Using Eq. (14), Eqs. (27) and (28) may be rewritten as

$$k^2 \Phi = 4\pi G a^2 \left[ 3\Pi^{(S)} + \sum_f \rho_f \delta_f - 3L^S \right.$$

$$\left. + 3\frac{\dot{a}}{ak} \sum_f (\rho_f + p_f) v_f^{(S)} \right], \qquad (34)$$

$$k^2(\Psi + \Phi) = -12\pi G a^2 \left[ \frac{2}{3} \sum_f p_f \Pi_f^{(S)} + \Pi^{(S)} \right]. \qquad (35)$$

These equations show that we should take note of two effects, one a consequence of anisotropic stress, the other arising from nonzero energy density and peculiar velocity perturbations.

### B. Matter perturbations

The first-order energy conservation equations for photons, baryons, and the CDM fluid are

$$\dot{\delta}_\gamma + \frac{4}{3} k v_\gamma^{(S)} + 4\dot{\Phi} = 0, \qquad (36)$$

$$\dot{\delta}_b + k v_b^{(S)} + 3\dot{\Phi} = 0, \qquad (37)$$

$$\dot{\delta}_c + k v_c^{(S)} + 3\dot{\Phi} = 0. \qquad (38)$$

The Lorentz force directly modifies only the Euler equation for the baryons, since only baryons are charged. The Euler equation for the CDM fluid is identical to that in a model without a magnetic field. Prior to decoupling, photons are tightly coupled to baryons and they move together. Since the Lorentz force affects the motion of baryons, the presence of a cosmological magnetic field also influences the evolution of photons. The first-order Euler or momentum conservation equations in the tight coupling regime (prior to last scattering) for photons, baryons, and CDM are [9,24]

$$\dot{v}_\gamma^{(S)} - \frac{k}{4} \delta_\gamma - k\Psi + \dot{\tau}(v_\gamma^{(S)} - v_b^{(S)}) + \frac{2k}{5} \Theta_2^{(S)} = 0, \quad (39)$$

$$\dot{v}_b^{(S)} + \frac{\dot{a}}{a} v_b^{(S)} - k\Psi - \frac{\dot{\tau}}{R}(v_\gamma^{(S)} - v_b^{(S)}) + \frac{kL^{(S)}}{\rho_b} = 0, \quad (40)$$

$$\dot{v}_c^{(S)} + \frac{\dot{a}}{a} v_c^{(S)} - k\Psi = 0. \qquad (41)$$

In Eq. (39) $\Theta_2^{(S)} = 5\Pi_\gamma^{(S)}/12$ is the quadrupole moment of the photon temperature fluctuation and reflects the anisotropic nature of Thomson-Compton scattering. The photon density fluctuation is related to the temperature monopole moment, $\delta_\gamma = 4\Theta_0^{(S)}$, and the perturbed photon velocity is the dipole term $v_\gamma^{(S)} = \Theta_1^{(S)}$. In Eqs. (39) and (40) $\dot{\tau}$ is the differential visibility function; $\dot{\tau} = n_e x_e \sigma_T a$ where $n_e(z)$ is the charged particle number density, $x_e(z)$ is the plasma ionization fraction, and $\sigma_T$ is the Thompson cross section. In Eq. (40) the term $\propto \dot{\tau}$ (the so-called "baryon drag force" term) reflects the coupling between photons and baryons, and so determines the velocity difference between the photon and baryon fluids. In the lowest order of the tight coupling approximation the $\dot{\tau}(v_\gamma^{(S)} - v_b^{(S)})$ terms in Eqs. (39) and (40) vanish since $v_b^{(S)} \approx v_\gamma^{(S)}$.

At early time the hydrodynamical description for photons is a reasonable approximation due to their strong interaction with baryons and short mean free path. In this case $\Pi_\gamma^{(S)} = 0$ and so $\Theta_2^{(S)} = 0$. Subtracting Eq. (40) from Eq. (39), multiplying by $R$, and using Eq. (37), the velocity difference between the photon and baryon fluids, $\Delta v_{\gamma b} = v_\gamma^{(S)} - v_b^{(S)}$, obeys

$$R\Delta \dot{v}_{\gamma b} + (1 + R)\dot{\tau}\Delta v_{\gamma b} = \frac{k}{4}\left( \frac{3L^{(S)}}{\rho_\gamma} + R\delta_\gamma \right)$$

$$- \frac{R\dot{a}}{ka}(\dot{\delta}_b + 3\dot{\Phi}), \qquad (42)$$

where $\Delta \dot{v}_{\gamma b} = \dot{v}_\gamma^{(S)} - \dot{v}_b^{(S)}$. $R \propto a$, so at early times it is very small, and at these times Eq. (42) results in[4]

$$\Delta v_{\gamma b} \simeq \frac{3kL^{(S)}}{4\dot{\tau}\rho_\gamma}. \qquad (44)$$

For modes with wavelengths larger than the Hubble radius, the difference between photon and baryon fluid velocities is significantly suppressed by the $k/\dot{\tau}$ factor in this equation.

To derive an equation that describes the photon-baryon fluid in the tight coupling approximation, where $v_b^{(S)} \simeq v_\gamma^{(S)} \equiv v^{(S)}$, we multiply Eq. (40) by $R$ and add Eq. (39) to get

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[4]For a model without a magnetic field $\Delta v_{\gamma b}$ obeys

$$R\Delta \dot{v}_{\gamma b} + (1 + R)\dot{\tau}\Delta v_{\gamma b} = R\left[ \frac{k\delta_\gamma}{4} - \frac{\dot{a}}{ka}(\dot{\delta}_b + 3\dot{\Phi}) \right], \quad (43)$$

and so for $R \ll 1$ $\Delta v_{\gamma b}$ vanishes. In the presence of a magnetic field the Lorentz force term is responsible for a nonzero velocity difference at early times.

$$\frac{\partial}{\partial \eta}[(1 + R)v^{(S)}] = \frac{k}{4}\left(\delta_\gamma - \frac{3L^{(S)}}{\rho_\gamma}\right) + (1 + R)k\Psi. \quad (45)$$

Here we have made use of the relation $\dot{a}/a = \dot{R}/R$. Equation (45) is valid over a limited range of times prior to decoupling. After decoupling photon evolution must be described by the Boltzmann transport equation and $\dot{\tau}$ is not large enough to ensure the equality of the photon and baryon fluid velocities, i.e., $v_\gamma^{(S)} \neq v_b^{(S)}$, so Eq. (45) is not valid. Also, after decoupling it is no longer possible to ignore the photon anisotropic stress term (i.e., the temperature quadrupole moment $\Pi_\gamma^{(S)} = 12\Theta_2^{(S)}/5$) that appears in Eq. (39).

Using $\delta_\gamma = 4\Theta_0^{(S)}$ and $v_\gamma^{(S)} = \Theta_1^{(S)}$, Eqs. (36) and (45) may be expressed in terms of temperature multipoles,

$$\dot{\Theta}_0^{(S)} = -\frac{k}{3}\Theta_1^{(S)} - \dot{\Phi}, \quad (46)$$

$$\frac{\partial}{\partial \eta}[(1 + R)\Theta_1^{(S)}] = k\left[\Theta_0^{(S)} - \frac{3L^{(S)}}{4\rho_\gamma} + (1 + R)\Psi\right]. \quad (47)$$

The quantity $\Theta_0^{(S)} + \Psi$ is usually called the effective temperature, and in a model with a cosmological magnetic field it obeys the second order differential equation

$$\frac{\partial}{\partial \eta}[(1 + R)(\dot{\Theta}_0^{(S)} + \dot{\Psi})] + \frac{k^2}{3}(\Theta_0^{(S)} + \Psi)$$
$$= -\frac{Rk^2}{3}\Psi + \frac{\partial}{\partial \eta}[(1 + R)(\dot{\Psi} - \dot{\Phi})] + \frac{k^2 L^{(S)}}{4\rho_\gamma}. \quad (48)$$

This equation is derived from Eqs. (46) and (47). In addition to the usual baryon drag force term $\propto Rk^2\Psi/3$ and gravitational potential time derivative difference term $\propto (\dot{\Psi} - \dot{\Phi})$, there is a term on the r.h.s. of this equation, $\propto k^2 L^{(S)}/\rho_\gamma$, which directly reflects the presence of the cosmological magnetic field.

Since $L^{(S)}(\eta)/\rho_\gamma(\eta)$ is time independent, we may use $\Theta_0^{(S)} + \Psi - 3L^{(S)}/(4\rho_\gamma)$ as the generalized effective temperature in the case when a cosmological magnetic field is present. Equation (48), rewritten in terms of $\bar{\Theta}_0^{(S)}$, where $4\bar{\Theta}_0^{(S)} = 4\Theta_0^{(S)} - 3L^{(S)}/\rho_\gamma = \delta_\gamma - 3L^{(S)}/\rho_\gamma$, is

$$\frac{\partial}{\partial \eta}[(1 + R)(\dot{\bar{\Theta}}_0^{(S)} + \dot{\Psi})] + \frac{k^2}{3}(\bar{\Theta}_0^{(S)} + \Psi)$$
$$= -\frac{Rk^2}{3}\Psi + \frac{\partial}{\partial \eta}[(1 + R)(\dot{\Psi} - \dot{\Phi})]. \quad (49)$$

$\bar{\Theta}_0^{(S)}$ reflects the rescaling of the photon fluid energy density perturbation in the presence of a magnetic field.

In a model without a cosmological magnetic field, defining $m_{\rm eff} = 1 + R$, Eq. (48) can be rewritten as (also see Eq. (83) of Ref. [31]),

$$\frac{\partial}{\partial \eta}(m_{\rm eff}\dot{\Theta}_0^{(S)}) + \frac{k^2}{3}\Theta_0^{(S)} = -\frac{k^2}{3}m_{\rm eff}\Psi - \frac{\partial}{\partial \eta}(m_{\rm eff}\dot{\Phi}). \quad (50)$$

Equation (50) is the second order differential equation that governs the dynamics of photon density perturbations ($\delta_\gamma = 4\Theta_0^{(S)}$). In the absence of gravitational potentials the r.h.s. of Eq. (50) vanishes. The l.h.s. of Eq. (50) differs from the equation for an undriven simple harmonic oscillator only by the time-dependent factor $m_{\rm eff}$. Just like the case for a harmonic oscillator, Eq. (50) has two independent solutions—sine and cosine modes—that depend on initial conditions. Defining the photon-baryon fluid sound speed $c_S = 1/\sqrt{3m_{\rm eff}}$, for $\dot{m}_{\rm eff}/m_{\rm eff} \ll \omega$ where $\omega = c_S k$ is the oscillation frequency, the JWKB solutions of Eqs. (46) and (47) are [31]

$$\Theta_0^{(S)} = A_1 m_{\rm eff}^{-1/4}\cos(ks + \phi), \quad (51)$$

$$\Theta_1^{(S)} = A_1\sqrt{3}m_{\rm eff}^{-3/4}\sin(ks + \phi). \quad (52)$$

Here $A_1$ is the amplitude, $s = \int c_S d\eta$ is the acoustic Hubble radius, and $\phi$ is the phase. The constants $A_1$ and $\phi$ depend on initial conditions.

We note that baryon pressure has been neglected, $p_b = 0$, in the baryon Euler Eq. (40). Consequently Eq. (48) also assumes that the baryon pressure vanishes. We discuss this assumption in the following subsection C. Here we assume vanishing baryon pressure and study acoustic oscillations driven by a weak Lorentz force. We again neglect gravitational potentials but now retain the last, Lorentz force, term on the r.h.s. of Eq. (48). In this case the leading JWKB terms in the equation are (again under the assumption that $\dot{m}_{\rm eff}/m_{\rm eff} \ll \omega$),

$$\ddot{\Theta}_0^{(S)} + k^2 c_S^2\Theta_0^{(S)} = \frac{k^2 L^{(S)}}{4\rho_\gamma m_{\rm eff}} = \frac{3}{4}c_S^2 k^2\frac{L^{(S)}}{\rho_\gamma}. \quad (53)$$

The solutions of Eq. (53) are of a similar oscillatory form to those in Eqs. (51) and (52), but now there is a constant shift of $\Theta_0^{(S)} \to \Theta_0^{(S)} + 3\rho_B/(4\rho_\gamma)$ (here we have used $L^{(S)} \simeq -\rho_B$), while $\Theta_1^{(S)} = -3\dot{\Theta}_0^{(S)}/k$ remains unchanged.

### C. Acoustic oscillations in the baryon fluid

The propagation of magnetosonic waves and magnetohydrodynamical instabilities in the expanding Universe are discussed in detail in Ref. [37]. The effects on CMB temperature anisotropies of magnetosonic waves in a homogeneous magnetic field are studied in Ref. [24], where it is shown that a homogeneous magnetic field induces three types of MHD waves (fast and slow magnetosonic and Alfvén waves) in an expanding Universe.

Fast magnetosonic waves result in a rescaling of the fluid sound speed, i.e., $c_S \to \sqrt{c_S^2 + v_A^2}$, where the original fluid sound speed $c_S$ is characterized by the fluid pressure $p$ and

energy density $\rho$, the Alfvén speed $v_A = B_0/\sqrt{4\pi(\rho + p)}$, and $B_0$ is the (unperturbed background) homogeneous magnetic field strength [38]. Fast magnetosonic waves require a small inhomogeneous magnetic field $B_1$, so we write the total magnetic field $\mathbf{B} = \mathbf{B}_0 + \mathbf{B}_1$, where $|\mathbf{B}_1| \ll |\mathbf{B}_0|$. The induction law in this case is

$$\frac{\partial \mathbf{B}_1}{\partial t} = \nabla \times [\mathbf{v} \times \mathbf{B}_0], \qquad (54)$$

where we work the leading order in the inhomogeneity. For the case of a homogeneous magnetic field, Adams *et al.* [24] consider a zeroth-order (background) magnetic field $\mathbf{B}_0$ with zeroth-order energy density, $\rho_B \propto B_0^2$, small compared to the energy density of the photon-baryon fluid. In this case $\mathbf{B}_1$ is a first-order perturbation. The fluid 3-velocity perturbation $\mathbf{v}$ is also first order and satisfies the linearized Euler equation, [24,37],

$$\rho \frac{\partial \mathbf{v}}{\partial t} + \nabla p + \frac{1}{4\pi}[\mathbf{B}_0 \times (\nabla \times \mathbf{B}_1)] = 0. \qquad (55)$$

Here the pressure gradient $\nabla p$ is related to the sound speed through $\nabla p = c_S^2 \nabla \rho$. In addition, the perturbed magnetic field obeys the Gauss law $\nabla \cdot \mathbf{B}_1 = 0$.

Assuming tight coupling between photons and baryons, multiplying the baryon Euler equation by $R$ and adding the photon Euler equation,[5] Adams *et al.* [24] obtain the Euler equation for the photon-baryon fluid accounting for a zeroth-order spatially homogeneous background magnetic field (also see Eq. (52) of Ref. [39]),

$$\frac{\partial}{\partial \eta}[(1 + R)v] - c_{S,b}^2 k \delta_b$$
$$= \frac{k}{4} \delta_\gamma + (1 + R)k\Psi$$
$$+ \frac{k}{4\pi(\rho_\gamma + p_\gamma)}[\hat{\mathbf{k}} \cdot \{\mathbf{B}_0 \times (\hat{\mathbf{k}} \times \mathbf{B}_1)\}], \quad (56)$$

where $c_{S,b}$ is the baryon fluid (not the photon-baryon fluid) sound speed.

Compared to the stochastic magnetic field case of Eq. (45) and (56)—for a homogeneous background magnetic field—contains an additional baryon pressure term on the l.h.s., $c_{S,b}^2 k \delta_b$, while the term on the r.h.s. of Eq. (56) $\propto \hat{\mathbf{k}} \cdot \{\mathbf{B}_0 \times (\hat{\mathbf{k}} \times \mathbf{B}_1)\}$ is the analog of the Lorentz force term $\propto L^{(S)}$ on the r.h.s. of Eq. (45). We emphasize that Eq. (45) is valid for a stochastic magnetic field and that here the smoothed amplitude of the magnetic field $B_\lambda$ is $1/2$ order in the perturbation expansion, while $\bar{\rho}_B(\propto B_\lambda^2)$ and $L^{(S)}$ are first order. We argue below that at linear order the additional baryon pressure term ($c_{S,b}^2 k \delta_b$) in Eq. (56) can be neglected. However, we emphasize that even if the sound speed in the uncoupled baryon fluid vanishes, i.e.,

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[5]The baryon and photon Euler equations are Eqs. (11) and (13) of Ref. [24].

$c_{S,b} = 0$, the effective sound speed in the coupled photon-baryon fluid is not the same as the sound speed in the uncoupled photon fluid, Because of the tight coupling between photons and baryons the photon-baryon fluid sound speed depends on the baryon fraction, $c_S = 1/\sqrt{3m_{\text{eff}}}$ [see Eq. (50)] so the coupling between baryons and photons reduces the sound speed from the $1/\sqrt{3}$ value for an uncoupled photon fluid.

The last term on the r.h.s. of Eq. (56) induces fast magnetosonic waves in the presence of a homogeneous magnetic field. These magnetosonic waves change the photon-baryon fluid sound speed, increasing it relative to the case without a magnetic field. In the limit of a weak magnetic field, the effective sound speed is $\bar{c}_S = \sqrt{1/(3m_{\text{eff}}) + v_A^2}$ [24]. The Alfvén speed $v_A$ here is that defined in the photon-baryon fluid, i.e.,

$$v_A^2 = \frac{B_0^2}{4\pi(\rho_B + 4\rho_\gamma/3)} = \frac{3B_0^2}{16\pi(1 + R)\rho_\gamma}. \qquad (57)$$

For the more realistic case of a stochastic magnetic field, the Alfvén speed should be defined in the terms of the smoothed magnetic field $B_\lambda$ [18]. Equation (45) is the analog of Eq. (56) for the case of a stochastic magnetic field. For this stochastic field case we define the Alfvén speed squared as $\bar{v}_A^2 = 3\bar{\rho}_B/(2(1 + R)\rho_\gamma)$. Here we neglect the baryon pressure, $p_b \approx 0$. Note that for the case $n_B = -3$ our definition of the Alfvén speed coincides with that used in Ref. [24] under the assumption that $B_\lambda = B_0$, i.e., $v_A = \bar{v}_A$. The term $\propto L^{(S)}$ on the r.h.s. of Eq. (45) ensures that the effective sound speed is rescaled in a manner similar to that for an homogeneous magnetic field,

$c_S \rightarrow \bar{c}_S = \sqrt{1/(3m_{\text{eff}}) + \bar{v}_A^2}$.

This rescaling of the sound speed may be formally described as a baryon energy density fraction change $R \rightarrow \bar{R} = R - \Delta R$, as follows. We define $\bar{m}_{\text{eff}} = \bar{c}_S^2/3$, then

$$\bar{m}_{\text{eff}} = \frac{m_{\text{eff}}}{1 + 9B_0^2/(16\pi\rho_\gamma)}, \qquad (58)$$

so the rescaling of the sound speed induced by the presence of a magnetic field (a homogeneous or stochastic field) is equivalent to a reduction of the baryon fraction, $\Delta R = 3\bar{v}_A^2 m_{\text{eff}}^2/(1 + 3\bar{v}_A^2 m_{\text{eff}})$. Consequently, this increase of the sound speed (relative to that of a model without a magnetic field) induces shifts of the CMB anisotropy angular power spectrum peaks comparable to shifts resulting from a reduction of the baryon density $\rho_b \rightarrow \rho_b - 3(1 + R)B_0^2/(4\pi)$; here we have used the fact that $\bar{v}_A \ll \bar{c}_S$. This was noted, for a homogeneous magnetic field, from the result of numerical simulations, in Ref. [24]; the analytical results we have derived, for a homogeneous or a stochastic magnetic field, are new. The reduction of the baryon fraction reduces the baryon drag force ($\propto Rk^2\Psi$) on the r.h.s. of Eq. (48). As a result there are two effects: a

shift of the CMB anisotropy angular power spectrum peak positions and a reduction of the peak amplitudes.

We also note that the Lorentz force in the Poisson Eq. (34) can be treated as reducing the photon effective temperature. In particular, photon energy density perturbations ($\delta_\gamma$) can be compensated by the Lorentz force $L^S$ if $\delta_\gamma(\eta) = 3L^{(S)}/\rho_\gamma$ [also see Eq. (53)]. Note that $L^{(S)}/\rho_\gamma$ is time independent.

Ref. [26] (their Eq. 46) considers a modified form of Eq. (56) for the case of a stochastic magnetic field: they discard the Lorentz force term. Ref. [27] also neglects the Lorentz force contribution (their Eq. 13 and App. A). Both references argue that such a force-free approximation is justified by the infinite conductivity of the plasma which results in a vanishing electric field in the metric perturbation equation. We have noted however, at the end of Sec. II, that the force-free approximation cannot be used in this manner. Here we point out that the current, $\propto \nabla \times \mathbf{B}$, does not vanish, and so a Lorentz force term must be present in the baryon Euler equation.

We have shown that the Lorentz force term in the baryon Euler equation results in shifts of the CMB anisotropy angular power spectrum peaks, relative to the case without a magnetic field.[6] This result, obtained for a stochastic magnetic field, generalizes that of Ref. [24] for an homogeneous magnetic field; for $n_B = -3$ our estimate reproduces the acoustic oscillations shown in Fig. 1 of Ref. [24]. On the other hand this result contradicts Fig. 1 of Ref. [27]. This figure is for the case of a stochastic magnetic field and it indicates that the magnetic field effectively increases the baryon fraction (while Ref. [27] considers a magnetic field with positive spectral index, $n_B > 0$, this cannot explain the effective increase of the baryon fraction they find).

We note that Ref. [26] retains the baryon pressure term $\propto c_{S,b}^2 k \delta_b$ (and argues that $c_{S,b}^2$ is related to the gradient of the magnetic pressure, $\nabla p_B$) in the photon-baryon Euler Eq. (56). Ref. [26] claims that their CMB anisotropy angular power spectrum peak shifts are similar to those found in Ref. [24], because of this term. The $c_{S,b}^2 k \delta_b$ term does induce a rescaling of the sound speed. However, this term is second order ($\delta_b$ and $\nabla p_B$ are first order), does not contribute at linear order, and so can be discarded compared to the Lorentz force contribution which they neglect.

### D. Initial conditions

The presence of a cosmological magnetic field modifies the initial conditions for the monopole $\Theta_0^{(S)}$ and gravitational potential, $\Phi$ and $\Psi$, perturbations. A proper treatment in an inflation model requires analysis of quantum mechanical fluctuations during inflation, see, e.g.,

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[6]Below we discuss the peak shifts related to the baryonic pressure effect, which are small compared the shifts discussed here.

Refs. [40] for the case without a magnetic field. Here we adopt a more phenomenological approach.

As discussed in Refs. [9,10], there are three different types of perturbations to consider, depending on conditions in the radiation-dominated epoch: (i) adiabatic, where the initial gravitational potentials are large compared to the magnetic field fraction of the energy density, i.e., $\rho_B/\rho_\gamma \ll \Psi_{\mathrm{in}}$ and $\rho_B/\rho_\gamma \ll \Phi_{\mathrm{in}}$, so the magnetic field energy density may be ignored; (ii) quasiadiabatic, where $\rho_B/\rho_\gamma \lesssim \Psi_{\mathrm{in}}$, $\rho_B/\rho_\gamma \lesssim \Phi_{\mathrm{in}}$; and, (iii) isocurvature, where the magnetic field energy density fraction dominates over the initial gravitational potentials, $\rho_B/\rho_\gamma \gg \Psi_{\mathrm{in}}$ and $\rho_B/\rho_\gamma \gg \Phi_{\mathrm{in}}$.

Since adiabatic and quasiadiabatic CMB perturbations have been studied in some detail (for a review see Ref. [39]), we focus on isocurvature fluctuations induced by a cosmological magnetic field.

We assume (as is conventional in the case of an isocurvature solution) that initial values of all relevant variables are determined by the magnetic field energy density and anisotropic stress. Under such an assumption the initial conditions for the gravitational potentials are obtained through Eqs. (27) and (35) (assuming that initial fluid perturbations are zero),

$$k^2\Phi_{\mathrm{in}} = \frac{4\pi G}{a_{\mathrm{in}}^2} \rho_B(\eta_0),$$

$$k^2(\Phi_{\mathrm{in}} + \Psi_{\mathrm{in}}) = -\frac{12\pi G}{a_{\mathrm{in}}^2} \Pi^{(S)}(\eta_0), \tag{59}$$

where $a_{\mathrm{in}}$ is the value of the scale factor when the initial conditions are applied. Adding Eqs. (59) and using Eq. (14) implies that $k^2(2\Phi_{\mathrm{in}} + \Psi_{\mathrm{in}}) = -12\pi G L^{(S)}(\eta_0)/a_{\mathrm{in}}^2$.

Using these initial conditions for the gravitational potentials and assuming a weak magnetic field, one may obtain solutions for scalar magnetic perturbations with wavelengths larger than the Hubble radius (i.e., the leading terms in an expansion in $k\eta \ll 1$).

### IV. CMB TEMPERATURE ANISOTROPIES

In this section we compute the CMB temperature anisotropies due to the presence of a cosmological magnetic field. We assume the existence of a cosmological magnetic field on scales larger than Hubble radius, thus we assume that a magnetic field has been generated during inflation [3]. Our analysis below holds for a magnetic field with spectral index $n_B$ larger than $-3$.

CMB temperature fluctuations are caused by scalar perturbations due to: i) initial intrinsic inhomogeneities on the last-scattering surface; (ii) the relativistic Doppler effect due to the baryon velocity as the photon propagates to the observer; (iii) the difference in the gravitational potential between the points of photon emission and the observer (the usual Sachs-Wolfe effect, SW); and, (iv) changes in

the gravitational potential as the photon propagates (the so-called integrated Sachs-Wolfe effect, ISW) [31,39,41].

Using the total angular momentum formalism [31], the angular power spectrum of the CMB temperature anisotropy measured today is

$$C_\ell^{\Theta\Theta(S)} = \frac{2}{\pi} \int dk k^2 \frac{\Theta_\ell^{(S)\star}(\eta_0, k)}{2\ell+1} \frac{\Theta_\ell^{(S)}(\eta_0, k)}{2\ell+1}. \quad (60)$$

Here $l$ is the multipole index and $\Theta_\ell^{(S)}(\eta_0, k)$ is the $l$-th multipole moment of the (scalar-sourced) temperature fluctuation and is determined by the integral solution of the Boltzmann transport equation [31],

$$\frac{\Theta_\ell^{(S)}(\eta_0, k)}{2\ell+1} = \int_0^{\eta_0} d\eta e^{-\tau} [\{\dot{\tau}(\Theta_0^{(S)} + \Psi) + \dot{\Psi} - \dot{\Phi}\}$$
$$\times j_\ell^{(S,0)}(k\eta_0 - k\eta) + \dot{\tau} v_b^{(S)} j_\ell^{(S,1)}(k\eta_0 - k\eta)$$
$$+ \dot{\tau} P^{(S)} j_\ell^{(S,2)}(k\eta_0 - k\eta)]. \quad (61)$$

Here $P^{(S)} = (\Theta_2^{(S)} - \sqrt{6}E_2^{(S)})/10$ is the anisotropic (quadrupolar) part of the Compton scattering cross section—which is a source of CMB polarization anisotropies—where $\Theta_2^{(S)}$ and $E_2^{(S)}$ are the temperature and $E$-polarization quadrupole moments,[7] and the radial functions

$$j_\ell^{(S,0)}(x) = j_\ell(x), \qquad j_\ell^{(S,1)}(x) = j_\ell'(x),$$
$$j_\ell^{(S,2)}(x) = \tfrac{1}{2}[3j_\ell'' + j_\ell(x)], \quad (62)$$

where $j_\ell$ is the spherical Bessel function and a prime represents a derivative with respect to $x$.

Equation (61) includes the four effects responsible for the CMB temperature anisotropies mentioned above. The initial photon temperature ($\propto P^{(S)}$), SW ($\propto (\Theta_0^{(S)} + \Psi)$), and baryon velocity Doppler ($\propto v_b^{(S)}$) effects are present on the last-scattering surface and so in Eq. (61) they appear with the factor $e^{-\tau}\dot{\tau}$; the ISW effect, $\propto (\dot{\Psi} - \dot{\Phi})$, contributes from decoupling until today and thus is suppressed by the factor $e^{-\tau}$.

We are interested in the large-scale CMB temperature anisotropy due to a cosmological magnetic field. The contribution from the ISW effect is negligible when compared to the SW effect. Also, compared to the SW effect, the Doppler term $\propto v_b^{(S)}$ plays a secondary role when $1 + R > 1$ [31]. The quadrupole term $P^{(S)} \propto k\Theta_1^{(S)}/\dot{\tau}$ (see Eq. (90) of Ref. [31] and the discussion around Eq. (78) in Sec. V below) is strongly suppressed for $k\eta \ll 1$, and thus we neglect it on large angular scales. On large angular scales the largest contribution arises from the ordinary SW effect so we approximate the temperature integral solution for the

scalar perturbation as

$$\frac{\Theta_\ell^{(S)}(\eta_0, k)}{2\ell+1} \simeq \int_0^{\eta_0} d\eta e^{-\tau}\dot{\tau}(\Theta_0^{(S)} + \Psi)j_\ell(k\eta_0 - k\eta). \quad (63)$$

The visibility function $\dot{\tau}e^{-\tau}$ is sharply peaked at decoupling, so we use the approximation, [41],

$$\frac{\Theta_\ell^{(S)}(\eta_0, k)}{2\ell+1} \simeq [\Theta_0^{(S)}(\eta_{\text{dec}}) + \Psi(\eta_{\text{dec}})]j_\ell(k\eta_0 - k\eta_{\text{dec}})$$
$$\simeq -\frac{1}{3}\Phi(\eta_{\text{dec}})j_\ell(k\eta_0), \quad (64)$$

where $\eta_{\text{dec}}$ is the value of conformal time at decoupling. This is the familiar SW, $\Phi/3$, result. This expression for the temperature anisotropy multipole moment depends on $\Phi(\eta_{\text{dec}})$, so we must solve for $\Phi(\eta)$.

The gravitational potential $\Phi(\eta)$ obeys Eq. (27). To solve this equation we decompose $\Phi$ as

$$\Phi(\eta) = \Phi_1(\eta) + \Phi_2(\eta), \quad (65)$$

where the potential $\Phi_1$ is due to the magnetic field energy density and $\Phi_2$ is related to the energy density and velocity perturbations in the fluids (these perturbations in the fluid are induced by the magnetic field anisotropic stress). From Eq. (27), the potentials $\Phi_1(\eta)$ and $\Phi_2(\eta)$ obey

$$k^2\Phi_1 = 4\pi Ga^2\rho_B, \quad (66)$$

$$k^2\Phi_2 = 4\pi Ga^2\sum_f \mathcal{D}_f, \quad (67)$$

where $\mathcal{D}_f$ is the gauge-invariant energy density perturbation in the $f$-th fluid, Eq. (29).

Mathematically $\mathcal{D}_f$ should be obtained through the solutions for the energy density ($\delta_f$) and velocity ($v_f^{(S)}$) perturbations of the $f$-th fluid. $\delta_f$ and $v_f^{(S)}$ obey fairly complicated second order differential equations (see Sec. III B) that are not straightforward to integrate. However, since we only consider perturbations arising from a magnetic field, it is expected that on scales larger than the Hubble radius the sum of the density and velocity perturbations in the fluids, $\sum_f \mathcal{D}_f$, should be of order the magnetic field energy density, while on small scales radiation pressure prevents perturbations from growing [14]. Thus $\sum_f \mathcal{D}_f \le \rho_B$.[8] Using this, Eqs. (66) and (67) imply

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[7]The zeroth-order term in the expansion of the photon Boltzmann transport equation for $\ell = 2$ relates $\Theta_2^{(S)}$ and $E_2^{(S)}$ as $E_2^{(S)} = -\sqrt{6}\Theta_2^{(S)}/4$, leading to $P^{(S)} = \Theta_2^{(S)}/4$ [31].

[8]We justify this approximation by using the fact that the evolution of the magnetic-field-induced energy density and velocity perturbations on large scales must follow the dynamics of the magnetic source. On the other hand, the homogeneous solutions of Eq. (27) on small scales (in a model without a magnetic field) are sound waves (oscillating energy density perturbations). To obtain solutions of the inhomogeneous equation we must integrate the product of the source and the homogeneous solutions; as a consequence of the oscillation, the resulting integral will be negligibly small.

that the maximal value for $\Phi$ is

$$\Phi(\eta) = \frac{\Phi(\eta_0)}{a^2(\eta)} = \frac{8\pi G\rho_B(\eta_0)}{k^2 a^2}. \quad (68)$$

With this expression for $\Phi(\eta)$, the temperature anisotropy multipole moment, Eq. (64), becomes

$$\frac{\Theta_\ell^{(S)}(\eta_0, k)}{2\ell + 1} \simeq -\frac{8\pi G}{3k^2 a_{\rm dec}^2} \rho_B(\eta_0, k) j_\ell(k\eta_0), \quad (69)$$

where $a_{\rm dec}$ is the value of the scale factor at decoupling. Using this in Eq. (60), the CMB temperature anisotropy angular power spectrum is given by

$$C_\ell^{\Theta\Theta(S)} = \frac{2}{\pi}\left(\frac{8\pi G}{3a_{\rm dec}^2}\right)^2 \int_0^\infty dk \frac{|\rho_B(\eta_0, k)|^2}{k^2} j_\ell^2(k\eta_0). \quad (70)$$

Now $|\rho_B|^2$ is given in Eq. (25), so in terms of the Bessel function $J_{\ell+1/2} = \sqrt{2x/\pi}\,j_\ell(x)$, we find

$$C_\ell^{\Theta\Theta(S)} = \frac{2\pi^2 G^2 B_\lambda^4 \lambda^{2n_B+6}}{3(2n_B + 3)\Gamma^2(n_B/2 + 3/2)a_{\rm dec}^4\eta_0} \int_0^\infty dk \frac{1}{k^3}$$
$$\times \left[k_D^{2n_B+3} + \frac{n_B}{n_B + 3}k^{2n_B+3}\right]J_{\ell+1/2}^2(k\eta_0). \quad (71)$$

The integral in Eq. (71) may be evaluated by using Eq. (6.574.2) of Ref. [42]. For $n_B < -3/2$, when the magnetic source is dominated by the term proportional to $k^{2n_B+3}$, we find

$$\ell^2 C_\ell^{\Theta\Theta(S)} = \frac{2^{2n_B+1} n_B \Gamma(-2n_B)}{(n_B + 3)\Gamma^2(-n_B + 1/2)} A_{\Theta\Theta}^{(S)} \ell^{2n_B+2}, \quad (72)$$

while for $n_B > -3/2$, when the magnetic source is dominated by $k_D^{2n_B+3}$, we have

$$\ell^2 C_\ell^{\Theta\Theta(S)} = \frac{1}{2}(k_D\eta_0)^{2n_B+3} A_{\Theta\Theta}^{(S)} \ell^{-1}. \quad (73)$$

In these expressions

$$A_{\Theta\Theta}^{(S)} = \frac{\pi^2 G^2 B_\lambda^4 \lambda^{2n_B+6}}{3(2n_B + 3)\Gamma^2(n_B/2 + 3/2)a_{\rm dec}^4\eta_0^{2n_B+2}}. \quad (74)$$

Eqs. (72) and (73) describe the angular power spectrum of the CMB temperature anisotropy induced by scalar magnetic perturbations. The maximum growth rate of the power spectrum $\ell^2 C_\ell$ with $\ell$ is $\ell^{-1}$. This occurs for $n_B > -3/2$. So, as expected, the CMB scalar temperature fluctuations due to a stochastic magnetic field are strongly suppressed at small angular scales. The suppression is stronger for an inflation generated magnetic field with $n_B \to -3$ ($\ell^2 C_\ell \propto \ell^{-4}$).

## V. CMB POLARIZATION ANISOTROPY

In this section we compute the scalar magnetic-field-induced CMB $E$-polarization anisotropy angular power spectrum. Scalar (density) perturbations only induce electric type $E$-polarization anisotropies.

In the total angular momentum formalism [31], the $E$-polarization anisotropy angular power spectrum is

$$C_\ell^{EE(S)} = \frac{2}{\pi} \int dk k^2 \frac{E_\ell^{(S)\star}(\eta_0, k)}{2\ell + 1} \frac{E_\ell^{(S)}(\eta_0, k)}{2\ell + 1}, \quad (75)$$

where $E_\ell$ is the $l$-th multipole moment of the $E$-polarization anisotropy. It is determined by the integral solution of the Boltzmann transport equation [31],

$$\frac{E_\ell^{(S)}}{2\ell + 1} = -\sqrt{6} \int_0^{\eta_0} d\eta e^{-\tau}\dot{\tau}P^{(S)}(\eta)e_\ell^{(S)}(k\eta_0 - k\eta). \quad (76)$$

Here $P^{(S)}$ is defined below Eq. (61) and $e_\ell^{(S)}$ are the $E$-polarization radial functions,

$$e_\ell^{(S)}(x) = \sqrt{\frac{3}{8}\frac{(\ell + 2)!}{(\ell - 2)!}}\frac{j_\ell(x)}{x^2}. \quad (77)$$

Polarization anisotropies, being connected with the anisotropic stress, are determined by shear viscosity. They are generated during last scattering when the tight coupling is enhanced by the fast growth of the inverse differential visibility function $1/\dot{\tau}$ [31]. In the lowest order of the $k/\dot{\tau}$ expansion, the set of Boltzmann equation solutions are given in Eq. (90) of Ref. [31], and for the scalar perturbation mode,

$$P^{(S)} = \frac{2k}{9\dot{\tau}}\Theta_1^{(S)}. \quad (78)$$

Here $\Theta_1^{(S)}$ is the dipole moment of the temperature anisotropy and obeys Eq. (47). A similar equation holds for the case of scalar magnetic perturbations, see the discussion in Sec. 3 of the last of Refs. [19]. In this case, in addition to fluid shear viscosity, there is also a contribution from magnetic anisotropic stress $\Pi^{(S)}$. Since we focus only on magnetic-field-induced effects, we consider $\Pi^{(S)}$ to be the dominant source of polarization. Then, according to the technique used in Secs. IV.B–D of Ref. [31], and using footnotes 2 and 7, we have

$$P^{(S)}(\eta) = -\frac{k^2\eta}{3\sqrt{2}\dot{\tau}\rho_{\gamma0}}\Pi^{(S)} = \frac{\sqrt{2}k^2\eta}{9\dot{\tau}\rho_{\gamma0}}\rho_B(\eta_0, k). \quad (79)$$

Using these expressions for $P^{(S)}$ and $e_\ell^{(S)}$, Eqs. (77) and (79), in Eq. (76), we find

$$\frac{E_\ell^{(S)}}{2\ell + 1} = -\frac{\sqrt{(\ell + 2)(\ell + 1)\ell(\ell - 1)}\rho_B(\eta_0, k)}{3\sqrt{2}\rho_{\gamma0}}$$
$$\times \int_0^{\eta_0} d\eta \frac{\eta e^{-\tau}}{(\eta_0 - \eta)^2} j_\ell(k\eta_0 - k\eta). \quad (80)$$

Here we use again the fact that the visibility function $e^{-\tau}\dot{\tau}$ peaks at decoupling and approximate the $E$-polarization

integral solution as

$$\frac{E_\ell^{(S)}}{2\ell + 1} \simeq -\frac{\sqrt{(\ell + 2)(\ell + 1)\ell(\ell - 1)}\rho_B(\eta_0, k)\eta_{\text{dec}}}{3\sqrt{2}\rho_{\gamma 0}\dot{\tau}(\eta_{\text{dec}})\eta_0^2} j_\ell(k\eta_0).$$

(81)

To obtain the angular power spectrum $C_\ell^{EE(S)}$ we need $\dot{\tau}(\eta_{\text{dec}})$. From Eq. (C3) of Ref. [41], and assuming for the current value of the baryon energy density parameter $\Omega_b(\eta_0) = \rho_b/\rho_{\text{cr}} = 0.05$ and that the redshift at decoupling $z_{\text{dec}} = 1100$, we get

$$\dot{\tau}(z_{\text{dec}}) = 8.05\left(\frac{\dot{a}}{a}\right)_{\text{dec}}.$$

(82)

Photon-baryon decoupling occurs during the matter dominated epoch, when $a(\eta) \propto \eta^2$, so $(\dot{a}/a)_{\text{dec}} = 2/\eta_{\text{dec}}$, and $\dot{\tau}(\eta_{\text{dec}}) \simeq 16.1/\eta_{\text{dec}}$.

Using Eqs. (25), (81), and (82), the CMB $E$-polarization angular power spectrum of Eq. (75) is

$$C_\ell^{EE(S)} = \frac{(\ell + 2)(\ell + 1)\ell(\ell - 1)B_\lambda^4 \lambda^{2n_B+6}\eta_{\text{dec}}^4}{3 \times 2^{14}(2n_B + 3)\Gamma^2(n_B/2 + 3/2)\rho_{\gamma 0}^2 \eta_0^5}$$
$$\times \int_0^\infty dk k \left[k_D^{2n_B+3} + \frac{n_B}{n_B + 3} k^{2n_B+3}\right]$$
$$\times J_{\ell+1/2}^2(k\eta_0).$$

(83)

To evaluate this integral we consider two cases: $n_B < -3/2$, when the integral is dominated by the second term ($\propto k^{2n_B+3}$); and, $n_B > -3/2$, when the main contribution to the integral comes from the term $\propto k_D^{2n_B+3}$. In the first case, for $-3 < n_B < -2$, the integral may be exactly evaluated using Eq. (6.574.2) of Ref. [42]),[9]

$$\ell^2 C_\ell^{EE(S)} \simeq \frac{n_B \Gamma(-n_B - 2)}{4\sqrt{\pi}(n_B + 3)\Gamma(-n_B - 3/2)} A_{EE}^{(S)}\ell^{2n_B+10}$$
$$(-3 < n_B < -2).$$

(84)

For $-2 \leq n_B < -3/2$ we may evaluate integral using the semianalytical approximation of the Appendix of Ref. [18]. For $x \gg 1$ $J_{l+1/2}(x) \simeq \sqrt{2/(\pi x)}\cos[x - (l + 1)\pi/2]$, Eq. (9.2.1) of Ref. [43], and replacing the oscillatory function $\cos^2 x$ by its r.m.s. value of $1/2$, we get (see Eq. (B2) of Ref. [18]) for $n_B = -2$,

$$\ell^2 C_\ell^{EE(S)} = -2\ln\left(\frac{k_D \eta_0}{\ell}\right) A_{EE}^{(S)}\ell^6 \qquad (n_B = -2), \quad (85)$$

while for $-2 < n_B < -3/2$,

$$\ell^2 C_\ell^{EE(S)} \simeq \frac{n_B}{2(n_B + 3)(n_B + 2)}(k_D \eta_0)^{2n_B+4} A_{EE}^{(S)}\ell^6$$
$$(-2 < n_B < -3/2).$$

(86)

For the case when $n_B > -3/2$, using $x_D^{2n_B+3} \times \int_0^{x_D} dx x J_{\ell+1/2}^2(x) = x_D/\pi$, we have

$$\ell^2 C_\ell^{EE(S)} \simeq \frac{8}{9\pi}(k_D \eta_0)^{2n_B+4} A_{EE}^{(S)}\ell^6 \qquad (n_B > -3/2).$$

(87)

In these expressions

$$A_{EE}^{(S)} = \frac{B_\lambda^4 \lambda^{2n_B+6}\eta_{\text{dec}}^4}{3 \times 2^{14}(2n_B + 3)\Gamma^2(n_B/2 + 3/2)\eta_0^{2n_B+10}\rho_{\gamma 0}^2}.$$

(88)

Contrary to the temperature anisotropy, the $E$-polarization anisotropy angular power spectrum $\ell^2 C_\ell^{EE}$ grows rapidly with $\ell$ (the fastest growth rate is $\ell^6$). An $\ell^6$ dependence for $E$-polarization is also expected in the absence of a magnetic field [31].

## VI. TEMPERATURE—POLARIZATION CROSS CORRELATIONS

In this section we compute the scalar magnetic-field-induced CMB temperature—$E$-polarization cross-correlation anisotropy angular power spectrum [31],

$$C_\ell^{\Theta E(S)} = \frac{2}{\pi} \int dk k^2 \frac{\Theta_\ell^{(S)\star}(\eta_0, k)}{2\ell + 1} \frac{E_\ell^{(S)}(\eta_0, k)}{2\ell + 1}.$$

(89)

Using Eqs. (25), (69), (81), and (82), we find

$$C_\ell^{\Theta E(S)} \simeq l^2 \frac{\sqrt{2}\pi G B_\lambda^4 \lambda^{2n_B+6}\eta_{\text{dec}}^2}{3 \times 2^{10}(2n_B + 3)\Gamma^2(n_B/2 + 3/2)\eta_0^3 a_{\text{dec}}^2 \rho_{\gamma 0}}$$
$$\times \int \frac{dk}{k}\left[k_D^{2n_B+3} + \frac{n_B}{n_B + 3} k^{2n_B+3}\right] J_{l+1/2}^2(k\eta_0),$$

(90)

under the approximations discussed in the two previous sections.

Evaluating the integral in Eq. (90), we find, for $-3 < n_B < -3/2$,

$$\ell^2 C_\ell^{\Theta E(S)} \simeq \frac{n_B \Gamma(-n_B - 1)}{\sqrt{\pi}(n_B + 3)\Gamma(-n_B - 1/2)} A_{\Theta E}^{(S)}\ell^{2n_B+6}, \quad (91)$$

while for $n_B > -3/2$,

$$\ell^2 C_\ell^{\Theta E(S)} \simeq (k_D \eta_0)^{2n_B+3} A_{\Theta E}^{(S)}\ell^3, \quad (92)$$

where

$$A_{\Theta E}^{(S)} = \frac{\sqrt{2}\pi G B_\lambda^4 \lambda^{2n_B+6}\eta_{\text{dec}}^2}{3 \times 2^{11}(2n_B + 3)\Gamma^2(n_B/2 + 3/2)a_{\text{dec}}^2 \eta_0^{2n_B+6}\rho_{\gamma 0}}.$$

(93)

As in the case of temperature fluctuations, the temperature—$E$-polarization cross-correlation angular power spectrum $\ell^2 C_\ell^{\Theta E(S)}$ has maximum growth rate $\propto \ell^3$ for $n_B > -3/2$.

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[9]The magnetic field source is nonzero up to the damping scale $k_D$. Since the integral is dominated by small wavenumbers we can replace the upper-cutoff scale $k_D$ by $\infty$.

## VII. CONCLUSIONS

We present a systematic discussion of scalar isocurvature magnetic perturbations (magnetosonic cosmological waves) in a Universe with a stochastic primordial magnetic field. We derive the complete set of equations that govern the dynamics of linear magnetic energy density perturbations.

A stochastic magnetic field shifts the acoustic peaks of the CMB temperature anisotropy angular power spectrum, acting in a similar way as a reduction of the baryon fraction. This result extends the work of Adams *et al.* [24] who studied a homogeneous magnetic field. The second important effect that comes from a stochastic magnetic field is a nonzero anisotropic stress which generates a CMB $E$-polarization anisotropy.

We obtain approximate analytical expressions for the CMB anisotropy angular power spectra $C_\ell^{\Theta\Theta(S)}$, $C_\ell^{EE(S)}$, and $C_\ell^{\Theta E(S)}$. Numerical values of these spectra depend on four parameters: the cut-off wavenumber $k_D$; the smoothing length $\lambda$; the amplitude of the smoothed magnetic field $B_\lambda$; and, the magnetic field power spectral index $n_B$.

We find that the scalar CMB temperature anisotropy power spectrum amplitude $C_\ell^{\Theta\Theta(S)}$ rapidly decreases with increasing $\ell$, consequently, the magnetic field energy density contribution to the total CMB temperature anisotropy signal is suppressed at large multipole number. On the other hand, the contribution from the magnetic field aniso-tropic stress to the $E$-polarization anisotropy becomes large on small angular scales and should be accounted for when estimating the magnetic field contribution to the CMB $E$-polarization anisotropy (it must be added to the vector CMB $E$-polarization) [17,18].

Since scalar CMB temperature perturbations induced by a stochastic magnetic field do not depend on magnetic helicity, precise measurements of the CMB temperature anisotropy angular power spectra peak positions and amplitudes, combined with a CMB-independent measurement of the baryon fraction and CMB polarization Faraday rotation data, should provide information on the symmetric part of the magnetic field power spectrum, $P_B$. This determination of $P_B$ together with future CMB $B$-polarization anisotropy data (which is sensitive to both power spectra, $P_B$ and $P_H$), could lead to a constraint on cosmological magnetic helicity.

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