Future gravitational physics tests from ranging to the BepiColombo Mercury planetary orbiter

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Milani *et al.* recently have published careful and fundamental studies of the accuracy with which both gravitational physics information and the solar quadrupole moment can be obtained from Earth-Mercury distance data. To complement these results, a quite different analysis method is used in the present paper. We calculate the first-order corrections to the Keplerian motion of a single planet around the Sun due to the parameterized post-Newtonian theory parameters β, γ, α_1, α_2, and ξ, as well as corrections due to the solar quadrupole moment J_2 and a possible secular change in $GM_\odot$. The Nordtvedt parameter η that is used in tests of the strong equivalence principle also is included in this analysis. The expected accuracies are given for 1 yr, 2 yr, and 8 yr mission durations, assuming that the planet-planet and asteroid-planet perturbations are accurately known. The "modified worst-case" error analysis method that we use is quite different from the usual covariance analysis method based on assumed uncorrelated random errors, plus a bias that is fixed or that changes in a prescribed way. We believe this is appropriate because systematic measurement errors are likely to be the main limitation on the accuracy of the results. Our final estimated uncertainties are one-third of the errors that would result if a 4.5-cm rms systematic error had the most damaging possible variation with time. We discuss the resulting uncertainties for several different subsets of orbital and relativity parameters.

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I. INTRODUCTION

The most accurate method available for determining planetary orbits is based on accurate range measurements from the Earth to a spacecraft orbiting another planet or to a lander placed on the planet's surface. The classic results from range measurements to the Mariner 9 Mars orbiter and to the Viking orbiters and landers provided much improved ephemerides for both the Earth and Mars. In addition, strong tests of the predictions of general relativity were carried out [1–4].

This paper is the fourth in a series of investigations to determine how much more accurately one could test the predictions of present gravitational theory, and also determine the solar quadrupole moment, if high-accuracy range measurements were made during a Mercury Orbiter mission. In the past, range measurement capability frequently was limited because it had been included on various missions for navigation purposes rather than for specific scientific measurements. With a combination of Ka-band and X-band ranging capability, and a system designed specifically for accurate range measurements, much more favorable performance appears feasible.

The situation is more complicated for orbiters than for landers because of the need to convert from the measured Earth-spacecraft distance to the desired Earth-planet distance. This involves determining the orbit of the spacecraft about the planetary center of mass, which requires solving from the tracking data for a number of spatial harmonics of the gravitational field and for radiation pressure and often other parameters. Nongravitational perturbations of an irregular nature also frequently are present, such as the firing of attitude-control jets that give unbalanced forces, unless care is taken in the mission design to avoid them. The orbit determination accuracy of Mariner 9 was affected substantially by such problems and by the fact that the spacecraft was in a 12-hr orbit with low periapsis.

To reduce the orbit determination problem, we concentrate here on what could be achieved with a Mercury Orbiter in a nearly circular orbit and with an altitude roughly equal to the planetary radius, as is the case for the LAGEOS satellites in orbit around the Earth. For those satellites, roughly 1 cm orbit accuracy is achieved routinely based on laser range measurements from the Earth's surface. For ranging from the Earth to a Mercury Orbiter, the geometry would not be as favorable for observations over a short period because of only having measurements from a particular direction. However, over longer times, it appears that the changes in the Earth-Mercury direction with respect to the satellite orbit plane plus Mercury's rotation would permit accurate determination of Mercury's low-degree gravity field, the satellite orbit around Mercury, and the orbits of Mercury and the Earth around the Sun.

In view of being optimistic about the satellite orbit determination part of the problem for a favorable type of orbit, we analyzed what could be done with accurate knowledge of the center-to-center distance between the Earth and Mercury as a function of time assuming random uncorrelated errors in the daily normal point range errors. However, assumptions such as one range measurement per day, with something like 5 or 10 cm accuracy and random errors, led to much too optimistic results, since a substantial part of the error is likely to be systematic, as discussed below. Thus a different type of analysis, based on what is called a ''modified worst-case analysis,'' was done instead. This approach will be described in detail later.

The results from the modified worst-case study were quite encouraging but were not pursued further, partly because the prospects for a Mercury Orbiter mission of any kind seemed very uncertain. The situation now has changed dramatically. NASA's MESSENGER mission was launched to Mercury in August 2004, and will arrive there in 2011. It will have a 12 hr period orbit with 80 deg inclination, 200 km periapsis altitude, and 15 000 km apoapsis altitude. Its X-band Doppler tracking capability will permit valuable new information to be obtained about Mercury's gravity field. But the mission was not planned to carry out tests of gravitational theory and does not have a high-accuracy ranging system.

However, the European Space Agency is planning a dual spacecraft Mercury Orbiter mission called BepiColombo for launch in 2012. It has sensitive tests of relativity as one of its major objectives. It will start its 1 yr official science mission in 2016. Its lower altitude Planetary Orbiter spacecraft will have a polar orbit with 2.3 h period, 400 km periapsis altitude, and 1500 km apoapsis altitude. Its ranging and Doppler systems are planned to give very high-accuracy results.

The tracking system [5] will be a five-link X-band plus Ka-band system of the type needed to correct completely for the interplanetary electron density along the measurement path. In addition, it is planned to include a 20 MHz ranging sidetone in order to give considerably higher resolution than usually can be obtained. The accuracy with which Mercury's gravity field and rotation can be determined from the mission has been investigated in detail by Milani *et al.* [6]. In addition, in a second paper by Milani *et al.*, a detailed study has been made of relativity tests and measurements of related quantities that can be carried out with the ranging system [7].

In the papers by Milani *et al.* [6,7], it was found that the random error in determining the distance from the Earth to the center of mass of Mercury for a typical observing run of about 8 hr would be 4.5 cm rms. However, a considerably larger systematic error due to nonlinear drift in the ranging system was allowed for. The form of this error was assumed to be $50\sin[(\pi/2)\times(t/365)]$ cm, where t is the time in days from the beginning of the 1 yr orbiting phase of the mission. Systematic errors in the on-board accelerometer also were considered. Only 1 yr of observations was assumed. The Nordtvedt effect, involving mainly the effect of Jupiter on the orbits of Mercury and the Earth around the Sun, was included in the study by solving numerically for the orbital perturbations.

After the publication of the careful study by Milani *et al.* [7], we decided that it would be worthwhile to extend our results and to publish them. The reason is that our approach for considering the possible effect of systematic measurement errors is quite different from that of Milani *et al.* [7]. Since it is difficult to know ahead of time just how systematic errors are going to affect the scientific results, we believe that the different studies are complementary to each other. Further studies will be needed in order to understand and evaluate the differences in the results.

Following the preliminary worst-case earth-mercury ranging analysis, three studies of the satellite orbit determination part of the problem for a favorable type of orbit around Mercury were published. One paper [8] investigated the requirements on the transponder satellite in order to achieve 3-cm range accuracy and 1×10^{-14} Doppler accuracy for 10-min integration times. It was believed at the time that this was achievable with a low-power Ka/X-band dual-frequency sidetone ranging system and a 30 cm diameter antenna. However, it was pointed out to us later that a fifth channel was needed to achieve the full accuracy. This involves converting the X-band uplink signal to Ka-band before retransmitting it to the ground [5].

The second paper was a covariance study of the orbit determination problem for a transponder satellite in a polar orbit at 2439 km altitude [9]. Range and Doppler measurements from a single tracking station on the Earth were assumed for 8-hr periods every 2 or 3 days. The 3-cm range uncertainty was considered to be systematic, but the 1×10^{-14} Doppler errors were assumed random. A gravity field for Mercury that is complete through degree and order 10 was solved for from only 40 carefully chosen arcs of data. The conclusion was that the center-to-center Earth-Mercury distance could be determined with 6-cm accuracy whenever tracking is done. In view of the 88-day orbit period for Mercury, the accuracy would not degrade substantially during periods of 2 or 3 days between measurements. The third study [10] included a considerably more complete radiation pressure model for the spacecraft, and the capability of achieving 6-cm accuracy for the Earth-Mercury distance was verified.

In the present paper, we assume that a single ''normal point'' giving the Earth-Mercury distance is determined once each day. The uncertainty is taken to be 4.5 cm, based on the more recent and more complete studies of Milani *et al.* [6,7]. The uncertainty in the scientific information obtained scales directly with the normal point uncertainty that is assumed, so the results can be adjusted for the case of different spacecraft orbits around Mercury or different measurement accuracy.

It is particularly important to emphasize that the normal point errors are unlikely to be random from day to day. Inaccuracies in the orbit determination model are likely to give errors that vary with the orbital and rotational frequencies of the planet, their harmonics, and differences from the orbital frequency of the Earth. Range measurement errors may well vary with the range, which depends on the synodic frequency, with spacecraft temperature, and with the long-term effects of aging in the spacecraft electronics. From estimates of the completely random part of the range error due to signal-to-noise considerations, it appears that this contribution to the uncertainty will be considerably below the level due to systematic errors. For this reason, we believe that a standard covariance analysis of the expected scientific results could give over-optimistic conclusions, unless specific systematic error parameters are introduced that allow for the expected correlations between the errors at different times.

Instead of the usual covariance analysis assuming uncorrelated errors, we will use what we call a modified worst-case analysis. The basis for this approach has been given by Hauser [11] and was used by him in considering the scientific information expected from the Lunar Laser Ranging Experiment. A similar modified worst-case analysis was used by Anderson *et al.* [12] as one of two error analysis methods in considering Earth-Mars ranging to test the strong equivalence principle. And a worst-case analysis was used by Nordtvedt [13] in considering the proposed Close Solar Probe mission. The worst-case analysis approach was developed independently a number of years ago at the Jet Propulsion Laboratory and at other institutions.

The main part of this paper will be devoted to providing the necessary framework for studies, using our approach, of the scientific results achievable for gravitational physics tests and for the determination of the solar quadrupole moment from accurate range measurements to another planet. A major simplifying assumption made throughout most of the paper is to neglect all asteroid-planet perturbations and all planet-planet perturbations, except for those due to Jupiter. This means that the effects on the results of uncertainties in the masses and orbits for the other planets and in the asteroidal masses are not present in the analysis. Including such perturbations unfortunately is beyond the scope of the present paper. However, it is clear how the analysis can be extended in the future to remove this limitation.

In Sec. II we derive, for each parameter of interest, the corrections of lowest order to the unperturbed Newtonian equations of planetary motion, including those for the Nordtvedt effect [7,12,14–17]. The systematic error analysis is presented in Sec. III, and the 19 parameters considered in the analysis are discussed. The manner in which the partial derivative of the range with respect to how each of the parameters is calculated is described in Sec. IV. The application of the analysis to the case of a Mercury Orbiter mission is discussed in Sec. V, and the results for several possible mission lengths and for several different parameter sets are given. The interpretation of the results is discussed and used as a basis for providing rough estimates of the effects of neglecting the mass and orbit uncertainties for the other planets and the asteroids. Finally, the overall results are discussed in Sec. VI.

II. THE PERTURBED EQUATIONS OF MOTION

A. Relativity parameters

Nearly every metric theory of gravity that has been suggested so far can be fit into the generalized 10-parameter parameterized post-Newtonian (PPN) framework [18–21], except for possible cosmological effects on the gravitational constant or for MOND type theories. In this paper we consider six of these parameters, γ, β, α_1, α_2, α_3, and ξ, and their effects on the Earth-Mercury distance. Here γ represents the strength of the correction to the spatial part of the metric and β represents the strength of the nonlinear terms in the component g_{00} of the metric tensor. The parameters α_1, α_2, and α_3 measure effects due to the velocity of the Sun relative to an assumed universal rest frame, and ξ is the coefficient of the Whitehead term in g_{00} [21], which represents an interaction between the solar system and the Galaxy's mass:

$$g_{00} = -1 + 2\xi \int \frac{\rho(x')\rho(x'')(\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3} \cdot \left[\frac{\mathbf{x}' - \mathbf{x}''}{|\mathbf{x} - \mathbf{x}''|} - \frac{\mathbf{x} - \mathbf{x}''}{|\mathbf{x}' - \mathbf{x}''|}\right] d^3x' d^3x'' + \ldots . \quad (2.1)$$

Any theory of gravity that predicts preferred location effects has ξ nonzero [18]. The coefficient of the Newtonian potential term in g_{00} is taken to be α = unity since its deviation from unity has been measured to be $< 1.4 \times 10^{-4}$ by Gravity Probe A [22,23], and because the value $\alpha = 1$ is required in order that planetary orbits be described by Newtonian Physical Laws to lowest order [24]. For theories in which conservation of energy and momentum hold, $\alpha_3 = 0$. For spatially isotropic theories, $\alpha_1 = \alpha_2 = \alpha_3 = \xi = 0$. In general relativity, $\gamma = \beta = 1$ and $\alpha_1 = \alpha_2 = \alpha_3 = \xi = 0$.

We have not considered the parameters ξ_1, ξ_2, ξ_3, or ξ_4, since they entered into the equations either through planet-planet interactions, most of which are not treated in this paper, or through the Nordtvedt effect [14–16]. The Nordtvedt effect gives rise to a term of relatively large amplitude in the Earth-Mercury range, so we have included the Nordtvedt parameter η in the analysis. The Nordtvedt parameter can be checked on by the lunar ranging experiment [25–28]. Also, Sherman [17] (see also [26]) and Anderson *et al.* [12] have shown that a good determination of η can be obtained from accurate range measurements between the Earth and Mars if the perturbations due to Jupiter are considered.

The metric we consider is

$$-ds^2 = \left[1 - 2\frac{m}{r} + 2\beta\frac{m^2}{r^2} + (\alpha_1 - \alpha_2 - \alpha_3)\frac{w^2}{c^2}\frac{m}{r} + \alpha_2\frac{m}{r}\left(\frac{\mathbf{w}\cdot\mathbf{r}}{cr}\right)^2 - 2\xi\frac{GM_G}{c^2R_G^3}\frac{(\mathbf{R}_G\cdot\mathbf{r})^2}{r^2}\frac{m}{r}\right](cdt)^2 + \left[\left(\frac{1}{2}\alpha_1 - \alpha_2\right)\frac{m}{r}\frac{w_j}{c} + \alpha_2\left(\frac{\mathbf{w}\cdot\mathbf{r}}{cr^2}\right)\frac{m}{r}x_j\right](cdt)dx^j - \left(1 + 2\gamma\frac{m}{r}\right)(dx^2 + dy^2 + dy^2). \tag{2.2}$$

Here, $\mathbf{w}$ is the velocity of the Sun relative to a preferred frame, which we shall assume to be the velocity relative to the thermal background radiation detected by Smoot *et al.* [29]: $\mathbf{w} = 3.71 \times 10^{10} \times (-0.970, 0.139, -0.197)$ m/day in ecliptic coordinates. The other quantities in Eq. (2.2) are $m = GM_\odot/c^2 = 1.477$ km for the Sun; M_G is the mass of our Galaxy; $\mathbf{R}_G$ is the vector from the galactic center to the Sun, with $GM_G/c^2R_G = 5 \times 10^{-7}$, and $\mathbf{R}_G/R_G = (-0.0694, -0.9921, -0.105)$ in ecliptic coordinates.

With the exception of the ξ term, the expression given in (2.2) is the metric for a massless test body in the field of a fixed point mass $M_\odot$ at the origin. The parameter ξ measures possible coupling between the Sun's mass and the background Galactic mass.

The geodesic equations of motion for the metric (2.2), when written using coordinate time t as the independent variable, take the form

$$\frac{d^2x^k}{dt^2} = -\Gamma^k_{\alpha\beta}\frac{dx^\alpha}{dt}\frac{dx^\beta}{dt} + \Gamma^0_{\alpha\beta}\frac{dx^\alpha}{dt}\frac{dx^\beta}{dt}\frac{dx^k}{dt}\frac{1}{c}; \tag{2.3}$$

these equations may then be used to express the acceleration of the test body as

$$\frac{d^2\mathbf{r}}{dt^2} = -\frac{GM_\odot}{r^3}\mathbf{r} + \delta\mathbf{a}_{\mathrm{PPN}}, \tag{2.4}$$

where $\delta\mathbf{a}_{\mathrm{PPN}}$ is interpreted as a small non-Newtonian perturbing acceleration modifying the Newtonian acceleration in flat Euclidean space. For the PPN parameters of interest, and with one additional term involving the rotation of the sun added (see later discussion),

$$\delta\mathbf{a}_{\mathrm{PPN}} = \frac{GM_\odot}{r^3}\left\{\left[2(\beta+\gamma)\frac{m}{r} - \gamma\left(\frac{v^2}{c^2}\right)\right]\mathbf{r} + 2(1+\gamma)\frac{(\mathbf{r}\cdot\mathbf{v})\mathbf{v}}{c^2}\right\} + \frac{m}{r^3}\mathbf{r}\left[\frac{1}{2}\alpha_1(\mathbf{w}\cdot\mathbf{v}) + \frac{3}{2}\alpha_2\left(\frac{\mathbf{w}\cdot\mathbf{r}}{r}\right)^2\right] - w\frac{m}{r^3}\left[\frac{1}{2}\alpha_1(\mathbf{r}\cdot\mathbf{v}) + \alpha_2(\mathbf{r}\cdot\mathbf{w})\right] - \frac{1}{3}\alpha_3\left(\frac{\Omega}{M_\odot c^2}\right)(\mathbf{w}\times\lambda) + \xi\left(\frac{GM_G}{c^2R_G^3}\right)GM_\odot\nabla\left[\frac{(\mathbf{R}_G\cdot\mathbf{r})^2}{r^3}\right] + \frac{GM_\odot}{r^3}\mathbf{r}\left[\frac{w^2}{2c^2}(\alpha_1 - \alpha_2 - \alpha_3)\right], \tag{2.5}$$

where λ is the angular spin velocity of rotation of the Sun about its rotation axis ($|\lambda| = 3 \times 10^{-6}$/sec) and $\Omega/M_\odot c^2 = -3.52 \times 10^{-6}$ is the ratio of the Sun's self-gravitational energy to its rest energy. The acceleration given by Eq. (2.5) will be used below to derive the corresponding PPN perturbations in the planetary orbital elements.

Note that $\delta\mathbf{a}_{\mathrm{PPN}}$ contains a term proportional to the unperturbed Newtonian acceleration:

$$\frac{GM_\odot}{r^3}r\left[\frac{w^2}{2c^2}(\alpha_1 - \alpha_2 - \alpha_3)\right].$$

The effect of this term on planetary orbits could be simulated by a change in the Sun's mass, and therefore we have dropped this term. The term involving λ in $\delta\mathbf{a}_{\mathrm{PPN}}$ also deserves some discussion. It has not been derived from the point mass metric, Eq. (2.2), but from a corresponding metric appropriate for extended bodies. This term is due to possible coupling between the Sun's spin, λ, and the preferred frame velocity $\mathbf{w}$, and represents a uniform acceleration $\mathbf{A}$ of the Sun through the Universe given by $\mathbf{A} = (1/3)\alpha_3(\Omega/M_\odot c^2)\mathbf{w}\times\boldsymbol{\lambda} = 2.311 \times 10^4\alpha_3(0.132, 0.938, 0.15)$ m/day^2 in ecliptic coordinates. As a result, there is some ambiguity in what to take for the origin of the coordinate system. The results for $\delta\mathbf{a}_{\mathrm{PPN}}$ above, and the perturbations that follow, take $\mathbf{r}$ to be the vector between the accelerated Sun and the planet.

By integrating the Lagrange planetary perturbation equations [30], we will compute the first-order corrections to the Keplerian orbital elements arising from $\delta\mathbf{a}_{\mathrm{PPN}}$. These, in turn, will be used to find the corresponding effects on the Earth-Mercury distance. The methods used and the results are given in Sec. IV and in the appendix.

B. Solar quadrupole moment

The value of the dimensionless solar quadrupole moment J_2 from solar oscillation data has been reported as $(1.7 \pm 0.4) \times 10^{-7}$ by Duvall *et al.* [31] and as about 10% less than this value by Brown *et al.* [32]. These results are consistent with the value expected for nearly uniform rotation of the Sun. However, the accuracy achievable from Earth-Mercury distance measurements appears to be considerably better. For this reason, and because of the expected correlation of J_2 with β and other parameters, it is necessary to include J_2 in our parameter list. The acceleration of a test body in the field of the Sun must then be modified to read

$$\mathbf{a} = \nabla\left(\frac{GM_\odot}{r}\right) + \delta\mathbf{a}_{\rm PPN} - \frac{GM_\odot}{2} J_2 \nabla\left[\frac{R_\odot^2}{r^3}(3\sin^2\varphi - 1)\right], \tag{2.6}$$

where $R_\odot$ is the radius of the Sun and φ is the latitude of the planet with respect to the solar equator.

To simplify the algebra involved in computations of perturbations arising from the solar quadrupole moment, it is convenient to choose a reference plane that coincides with the plane of the solar equator. The inclination and longitude of node of this plane relative to the ecliptic are $I = 7°15'$, $\Omega = 75°04'$, respectively. Fitzpatrick [30] has published integrated expressions for the changes in orbital elements that have been used in our computations.

C. Evolving gravitational constant

It has been conjectured that the gravitational constant G may be changing with time [33–37]. We define the parameter $\lambda_G = -\dot{G}/G$, where we assume that $\dot{G} = dG/dt$ is constant in the first approximation. The effect of λ_G on a test mass acceleration is to introduce an additional perturbation

$$\delta\mathbf{a} = \frac{GM_\odot \lambda_G t \mathbf{r}}{r^3}. \tag{2.7}$$

The perturbing effect of this acceleration on the orbital elements may easily be integrated using the Lagrange planetary perturbation equations. The results are given in the appendix.

D. Strong equivalence principle violation

The effect of strong equivalence principle (SEP) violation due to the presence of Jupiter can be considered to be a direct planetary perturbation on the orbits of Earth and Mercury. The perturbed equations of motion of a planet orbiting the sun, perturbed by the Nordtvedt effect [14,15], have been derived by Milani *et al.* [7] in heliocentric coordinates and are taken to be

$$\frac{d^2\mathbf{q}_i}{dt^2} = -\frac{G(M_\odot + M_i)\mathbf{q}_i}{q_i^3} + \eta\frac{\Omega}{M_\odot c^2}\frac{GM_j\mathbf{q}_{ij}}{q_{ij}^3}, \tag{2.8}$$

where

$$\mathbf{q}_{ij} = \mathbf{q}_i - \mathbf{q}_j, \tag{2.9}$$

$\mathbf{q}_i$ is the heliocentric position vector of planet i, $\mathbf{q}_j$ is the position of Jupiter, $M_\odot$ is the Sun's mass, M_i is the mass of planet i, M_j is Jupiter's mass, η is the Nordvedt parameter, and $\Omega/M_\odot c^2 = -3.52 \times 10^{-4}$ is the ratio of the Sun's gravitational self-energy to its rest mass energy.

The orbital inclinations and eccentricities of Earth and Jupiter are all very small. In treating the SEP-violating orbital perturbations due to Jupiter, we shall therefore treat the unperturbed orbits as coplanar and circular, with radii a_i and a_j, respectively. We denote the orbital angular velocities by ω_i and ω_j, respectively, and the synodic frequency Ω_i between the planet and Jupiter by S_i, where

$$S_i = (\omega_i - \omega_j)t + \Delta\varphi = \Omega_i t + \Delta\varphi; \tag{2.10}$$

φ is the initial angle between the Earth and Jupiter, subtended by the sun, and Ω_i is the synodic frequency.

The equations of motion for the perturbed orbit can be separated into a part that is radial (parallel to the heliocentric radius) and a part that is tangential (perpendicular to the radius). We denote these perturbations by δr_r and δr_t, respectively. Only terms linear in the perturbations are retained. The reciprocal of the unperturbed distance between planet i and Jupiter (planet j) is

$$\frac{1}{\sqrt{a_i^2 + a_j^2 - 2a_i a_j \cos S_i}} = \sum_{n=0}^{\infty} \frac{a_i^n}{a_j^{n+1}} P_n(\cos S_i), \tag{2.11}$$

where $P_n(x)$ is the Legendre polynomial of order n. Then the equations of motion are

$$\delta\ddot{r}_r - 2\omega_i\delta\dot{r}_t - 3\omega_i^2\delta r_r = -\frac{GM_j\eta\Omega_0}{a_j^2} \times \sum_1^\infty n\left(\frac{a_i}{a_j}\right)^n P_n(\cos S_i), \tag{2.12}$$

$$\delta\ddot{r}_t + 2\omega_i\delta\dot{r}_r = +\frac{GM_j\eta\Omega_0}{a_i(\omega_i - \omega_j)}\frac{d}{dt}\sum_1^\infty \frac{a_i^n}{a_j^{n+1}} P_n(\cos S_i), \tag{2.13}$$

where

$$\omega_i^2 = \frac{G(M_\odot + M_i)}{a_i^3}. \tag{2.14}$$

The Legendre polynomials on the right-hand sides of the equations of motion can be expanded in power series in the variables $\cos S_i$, and a term such as $(\cos S_i)^n$ can be reduced to a Fourier series by repeatedly employing trigonometric identities such as

$$(\cos S)^m = (\cos S)^{m-2}(1 + \cos(2S))/2; \tag{2.15}$$

$$\cos(mS)\cos(nS) = (\cos(mS + nS) + \cos(mS - nS))/2. \tag{2.16}$$

Thus the equations of motion can be written in the form

$$\delta\ddot{r}_r - 2\omega_i\delta\dot{r}_t - 3\omega_i^2\delta r_r = \sum_1^\infty A_n\cos(nS_i), \tag{2.17}$$

$$\delta\ddot{r}_t + 2\omega_i\delta\dot{r}_r = \sum_1^\infty B_n\sin(nS_i). \tag{2.18}$$

Then solutions for the perturbations can be found by assuming

$$\delta r_r = \sum_n \delta r_r^{(n)} \cos(nS), \tag{2.19}$$

$$\delta r_t = \sum_n \delta r_t^{(n)} \sin(nS). \tag{2.20}$$

The equations for perturbations of a given synodic Fourier frequency $(n\Omega_i)$ are then coupled and the solutions are

$$\delta r_r^{(n)} = -\frac{n\Omega_i A_n - 2\omega_i B_n}{n\Omega_i(n^2\Omega_i^2 - \omega_i^2)}, \tag{2.21}$$

$$\delta r_t^{(n)} = \frac{2n\omega_i\Omega_i A_n - (n^2\Omega_i^2 + 3\omega_i^2)B_n}{n^2\Omega_i^2(n^2\Omega_i^2 - \omega_i^2)}. \tag{2.22}$$

This method of solution is similar to that employed by Laplace [38] in computing the Newtonian perturbations of the inner planets due to Jupiter. The details are straightforward but lengthy, and we give only the final results here. For the Earth, the radial and tangential perturbations are (in meters)

$$\begin{aligned} \delta r_r &= 374.83\cos(S_e) - 4.87\cos(2S_e) - 0.35\cos(3S_e) \\ &\quad - 0.04\cos(4S_e), \\ \delta r_t &= -796.20\sin(S_e) + 6.94\sin(2S_e) + 0.43\sin(3S_e) \\ &\quad + 0.04\sin(4S_e). \end{aligned} \tag{2.23}$$

For the planet Mercury, the inclination is not small. However, changing the assumed value of Mercury's inclination in the calculations of the modified worst-case errors has a negligible effect on these errors. Therefore, we have used the analysis given above for the SEP-violating effect on Mercury's orbit and find for Mercury

$$\begin{aligned} \delta r_r &= 81.83\cos(S_m) - 0.09\cos(2S_m), \\ \delta r_t &= -165.92\sin(S_m) + 0.12\sin(2S_m). \end{aligned} \tag{2.24}$$

The partial derivative of the Earth-Mercury range with respect to the parameter η is then

$$\frac{\partial\rho}{\partial\eta} = \hat{N}_{em}\cdot(\delta\mathbf{r}_e - \delta\mathbf{r}_m), \tag{2.25}$$

where $\hat{N}_{em}$ is a unit vector from Mercury to Earth.

Since Mercury's orbit radius is relatively small compared to that of Earth, the unit vector $\hat{N}_{em}$ is primarily in the direction of Earth's radius. Therefore, the tangential perturbation of Earth's orbit contributes only in a minor way in the perturbation of the range, and the net SEP-violating contribution to the range would be only a few hundred meters (if $\eta = 1$).

There are at least two easy ways of incorporating the SEP-violating range perturbation. One way is to treat η as an independent parameter. Results for this case are presented in Table IV. Another way is to make use of the dependence of η on other parameters: $\eta = 4\beta - 3\gamma - 1$, and to therefore add to the partial derivatives with respect to β and γ the contributions

$$\frac{\partial\rho}{\partial\beta}\bigg|_\eta = 4\frac{\partial\rho}{\partial\eta}, \frac{\partial\rho}{\partial\gamma}\bigg|_\eta = -3\frac{\partial\rho}{\partial\eta}. \tag{2.26}$$

Since γ is determined mainly by the time delay, estimates of uncertainty in γ are hardly affected. However, the uncertainty in β is significantly reduced, as is the uncertainty in J_2 due to the high correlation between other perturbations arising from β and J_2. We present here only the results obtained treating η as an independent parameter.

The equations of motion could be numerically integrated; however, initial conditions must then be carefully chosen to eliminate unwanted solutions of the homogeneous equations of motion, that is, Eqs. (2.12) and (2.13) with no perturbing term proportional to the parameter η. There exist two linearly independent solutions of such equations, oscillating with frequencies ω_i, which have nothing to do with SEP violation and which if included would introduce spurious time signatures that would tend to reduce the estimates of uncertainty in η.

E. Time delay

In the above sections we have considered the dynamical effects on a planet's orbit. In addition, there is a nondynamical effect, the gravitational time delay [39] of the tracking signal. For the PPN coordinate system used here, the apparent shift in range is given by

$$\Delta\rho = m(1+\gamma)\ln\left\{\frac{r_E + r_M + \rho}{r_E + r_M - \rho}\right\}, \tag{2.27}$$

where ρ is the Earth-Mercury range, and r_E and r_M are the Sun-Earth and Sun-Mercury distances, respectively. No other PPN parameters contribute to the time delay to this order. (See Refs. [1,18] for further discussion.)

III. SYSTEMATIC ERROR ANALYSIS

A. Modified worst-case systematic error analysis

Since the apparent accelerations (2.5)–(2.8) of both the Earth and Mercury and the time delay (2.27) depend on the parameters γ, β, J_2, α_i, ξ, λ_G, and η, these parameters can be estimated from ranging observations between the two planets. Any realistic estimate for the post fit uncertainties of given parameters must include the effects of nonrandom, systematic errors. For this purpose we have employed a modified worst-case systematic error analysis, which is similar to that of Hauser (1974) mentioned earlier.

Let ρ stand for the calculated distance between the Earth and Mercury. Suppose there are m parameters, d_k $(k = 1, 2, \ldots, m)$, which affect the theoretical calculation of ρ. Then we write

$$\rho = (t, d_1, d_2, \ldots, d_m) = \rho(t, \{d_k\}) = \rho(t) \qquad (3.1)$$

for the calculated range at time t. Let $R(t)$ be the range that is actually observed at time t in the experiment. The data analysis consists of adjusting the parameters $\{d_k\}$, so that the residuals $\Delta(t) = R(t) - \rho(t)$ are as small as possible in the least-squares sense. Thus we try to find the corrections Δd_k to the initial parameter values d_k^0 so that the quantity

$$\sum_i [R(t_i) - \rho(t_i, \{d_k^0 + \Delta d_k\})]^2$$

is minimized. Here, t_i represents the time of the ith observation. The first-order equations for the corrections Δd_k that result from the standard minimization procedure are

$$\sum_i \frac{\partial \rho(t_i)}{\partial d_k} \Delta\rho(t_i, \{d_\ell^0\}) = \sum_{j=1}^{m} \sum_i \left(\frac{\partial \rho(t_i)}{\partial d_k} \frac{\partial \rho(t_i)}{\partial d_j} \right) \Delta d_j, \qquad (3.2)$$

where

$$\Delta\rho(t_i, \{d_\ell^0\}) = R(t_i) - \rho(t_i, \{d_\ell^0\}) \equiv \Delta\rho(t_i) \qquad (3.3)$$

is the observed range residual. Or, defining the matrix **C** with elements

$$C_{kj} = \sum_i \left(\frac{\partial \rho(t_i)}{\partial d_k} \frac{\partial \rho(t_i)}{\partial d_j} \right), \qquad (3.4)$$

Eq. (3.2) becomes

$$\sum_i \frac{\partial \rho(t_i)}{\partial d_k} \Delta\rho(t_i) = \sum_{j=1}^{m} C_{kj} \Delta d_j. \qquad (3.5)$$

The element C_{kj} of the $m \times m$ matrix **C** measures the cross correlation between the time signatures of the parameters d_k and d_j over the data set. If the time signatures corresponding to the parameters $\{d_k\}$, as represented by the partial derivatives $\partial\rho/\partial d_k$, are all linearly independent, then the matrix **C** will be nonsingular. Equation (3.5) can then be inverted and the solutions for the corrections Δd_ℓ to the initial values, d_ℓ^0, of the parameters are then

$$\Delta d_\ell = \sum_{j=1}^{m} B_{\ell j} \sum_i \frac{\partial \rho(t_i)}{\partial d_j} \Delta\rho(t_i), \qquad (3.6)$$

where **B** is the matrix inverse to **C**.

Once these corrections, Δd_ℓ, have been found, there still remains the question of their sensitivity to either experimental or modeling errors. Suppose $\delta\rho(t_i)$ is the error in the residual $\Delta\rho(t_i)$; then because Eq. (3.6) gives a linear relationship between the residuals and the corrections Δd_ℓ to the parameters, there will result an error δd_ℓ in the calculated correction given by

$$\delta d_\ell = \sum_{j=1}^{m} B_{\ell j} \sum_i \frac{\partial \rho(t_i)}{\partial d_j} \delta\rho(t_i). \qquad (3.7)$$

If the errors in the range residuals $\delta\rho(t_i)$ are random and uncorrelated, then upon repeating the range measurements $\rho(t_i)$ many times one would expect

$$\langle \delta\rho(t_i)\delta\rho(t_j) \rangle = \delta_{ij}\sigma^2, \qquad (3.8)$$

where the expectation value represented by the brackets $\langle\rangle$ are considered as an average over an "ensemble" of similar experiments, and σ^2 is the variance in the residuals. On forming the average of the quantity δd_ℓ^2 from Eq. (3.7) and using (3.8), we obtain the usual expression for the rms error in the correction to the lth parameter [40]:

$$\delta d_\ell = \sigma\sqrt{B_{\ell\ell}}. \qquad (3.9)$$

It can then easily be seen that, as the number of data points increases, the random error approaches zero. This implies that for a ranging experiment where there are an enormous number of data points, it is the time-dependent systematic errors that are of most importance.

The actual time dependence of the systematic errors is unknown, of course, but different sorts of estimates can be made. It is possible, for example, to make reasonable estimates of the systematic errors that lie at each of the important frequencies. This estimated error budget could then be substituted into Eq. (3.7) to give the errors δd_l for each l, as was done in one approach used by Hauser [11] for the lunar ranging problem. A similar approach is to use an estimated covariance matrix of the observation errors, as discussed by Kaula [40]. Despite the subjective nature of such estimates, we believe that these methods are among the most reliable for assessing errors in results from data sets that already exist.

For possible future measurements, we start instead from the worst-case approach. This approach assumes only that we know the total rms magnitude of the error. In particular, for each parameter d_ℓ we find the time-dependent systematic error that causes the maximum uncertainty in that parameter, subject only to the condition that

$$\left(\frac{1}{N} \sum_i \delta\rho(t_i)^2 \right)^{1/2} = \sigma. \qquad (3.10)$$

Using the standard method of Lagrange multipliers to maximize the error δd_ℓ given by Eq. (3.7) subject to the constraints (3.10), the worst-case error is found to occur when the residual has the time dependence given by

$$\delta\rho_\ell(t_i) = \frac{1}{S_\ell} \sigma \sum_j B_{\ell j} \frac{\partial \rho(t_i)}{\partial d_j}, \qquad (3.11)$$

where

$$S_\ell = (B_{\ell\ell}/N)^{1/2} \qquad (3.12)$$

is a normalization factor needed to satisfy Eq. (3.10). Substituting Eq. (3.11) back into (3.7) gives the worst-case estimate for δd_l, which works out to be

$$\delta d_\ell = \sigma(N B_{\ell\ell})^{1/2}. \tag{3.13}$$

Hauser (1974) obtained this same result, except with a $\leq$ sign instead of an equality.

Thus the worst-case systematic error in the parameter d_l is larger by a factor of $N^{1/2}$ than for the random error case. Generally speaking, if changes in a parameter do not introduce secular effects in the calculated range, the worst-case error given by Eq. (3.13) approaches a fixed limit as the number of observations becomes very large. If secular effects are introduced by changes in a parameter, then the error given by Eq. (3.13) in the determination of that parameter may continue to decrease as additional measurements are made.

Clearly it is unduly pessimistic to expect that every parameter would have the worst-case error (for that parameter) associated with its determination. Unless particular known time dependences seem likely to dominate the error budget, a substantial part of the systematic error probably will occur as irregular drifts, terms with frequencies that have little effect on the parameters of interest, or as a constant offset. Also, the time dependence of the range error associated with having the worst possible result for one parameter may not produce the worst errors for other parameters of interest. For these reasons, we choose to modify the worst-case results by dividing them by a factor k. In some cases a value as small as 2 or as large as 10 may be justified, but for the present problem we have subjectively chosen to use a value of 3. This seems reasonable in view of the number of parameters involved, the relatively high frequency of Mercury's motion, and the absence of a reason to expect that most of the systematic error will have the worst possible time dependence. We call this the modified worst-case analysis.

B. The parameter set

In this paper, we are mainly concerned with placing limits on the accuracy with which the parameters γ, β, J_2, α_1, α_2, α_3, ξ, λ_G, and η can be determined. Consequently, these parameters must be included in the parameter set $\{d_k\}$. The other parameters considered will be the product $GM_\odot$, together with the standard, unperturbed Keplerian elements for each planet.

Because the Earth-Mercury distance is unaffected by any rotation of the reference coordinate system, three of the Keplerian elements may be eliminated from consideration; hence only nine independent Keplerian elements, which are all taken relative to the ecliptic, are used in this analysis. These parameters are the semimajor axes of the Earth and of Mercury (a_1, a_2); eccentricities of the Earth and of Mercury (e_1, e_2); the differences between perihelion angle and initial longitude ($\tilde{\omega}_1 - L_{10}$, $\tilde{\omega}_2 - L_{20}$); the difference between the position of the line of nodes and initial longitude of Mercury ($\Omega_2 - L_{20}$); the orbital inclination of Mercury (I_2); and the difference of initial longitudes of Mercury and of the Earth ($L_{20} - L_{10}$). The total number of parameters is thus 19. We have also considered three other cases. For the two spatial isotropy cases, in which the parameters α_1, α_2, α_3, and ξ may be assumed to vanish, one with η not included and one including it, there are 14 or 15 parameters in the set. For the case where general relativity is assumed to be correct and G to be constant, so that γ, β, λ_G, and η are not solved for either, there are 11 parameters.

IV. CALCULATION OF PARTIAL DERIVATIVES

A. Range derivatives

The error analysis described in the previous section requires the calculation of partial derivatives of the Earth-Mercury range with respect to each of the parameters: $\partial\rho(t, \{d\})/\partial d_j$ ($j = 1, \ldots, 19$). We first discuss the calculation of derivatives with respect to the parameters γ, β, J_2, α_1, α_2, α_3, ξ, and λ_G. These derivatives are obtained by integration of the Lagrangian perturbation equations [30]. For a small perturbation from any one of the parameters, the corresponding change in range ρ is regarded as arising from first-order time-dependent perturbations of the Keplerian elements of the two planets. For this purpose, the independent Keplerian elements of a planet are taken to be the semimajor axis a, eccentricity e, perihelion longitude $\tilde{\omega}$, position of line of nodes Ω, inclination I, and mean anomaly M. First-order perturbations in the semimajor axis arising from the perturbations in acceleration will be denoted by Δa:

$$\Delta a = \Delta a(t) \equiv a(t) - a(t_0) = \int_{t_0}^{t} \dot{a} dt \tag{4.1}$$

with similar meanings for Δe, $\Delta\tilde{\omega}$, $\Delta\Omega$, ΔI, and ΔM. The time derivatives $\dot{a}$, $\dot{e}$, etc., are obtained from the Lagrangian perturbation equations when the perturbing acceleration is given by Eqs. (2.5)–(2.7).

Algebraic expressions for the perturbations in the Keplerian elements have been obtained by expressing the appropriate time-dependent integrands [such as $\dot{a}$ in Eq. (4.1)] in terms of the true anomaly or the eccentric anomaly and then integrating. Results are given in the appendix, except for the case of the perturbation due to the solar quadrupole moment J_2. For the perturbations from $J_2 \neq 0$ we use instead the results of Fitzpatrick [30] with the epoch chosen so that $\Delta a(t_0) = 0$. When considering the total perturbation, say Δa, from the acceleration (2.5)–(2.7), it is evident that because only first-order corrections are retained Δa will be a sum of terms each of which is linear in one of the parameters γ, β, J_2, α_i, ξ, λ_G, or η. If these parameters are denoted by $d_{11}, d_{12}, \ldots, d_{19}$, respectively, then we have, for example,

$$\Delta a = \sum_{i=11}^{19} \frac{\partial a}{\partial d_i} d_i. \tag{4.2}$$

This linear dependence of Δa, Δe, etc. on the parameters $\{d_i\}$ allows us to consider the effects of each parameter separately on the Earth-Mercury range. (Later in this section and in the appendix where it is clear that perturbations due only to one specific parameter are being considered, we will also use the symbol Δa in place of $\partial a/\partial d_i$, where d_i is tacitly assumed to be unity.)

Suppose, for example, we have found the perturbation in each Keplerian element from the PPN parameter β. Then, the total derivative of the range with respect to β is

$$\frac{\partial\rho}{\partial\beta} = \frac{\partial\rho}{\partial a_1}\frac{\partial a_1}{\partial\beta} + \frac{\partial\rho}{\partial a_2}\frac{\partial a_2}{\partial\beta} + \frac{\partial\rho}{\partial e_1}\frac{\partial e_1}{\partial\beta} + \frac{\partial\rho}{\partial e_2}\frac{\partial e_2}{\partial\beta} + \frac{\partial\rho}{\partial\tilde{\omega}_1}\frac{\partial\tilde{\omega}_1}{\partial\beta} + \frac{\partial\rho}{\partial\tilde{\omega}_2}\frac{\partial\tilde{\omega}_2}{\partial\beta} + \frac{\partial\rho}{\partial\Omega_1}\frac{\partial\Omega_1}{\partial\beta} + \frac{\partial\rho}{\partial\Omega_2}\frac{\partial\Omega_2}{\partial\beta} + \frac{\partial\rho_1}{\partial I_1}\frac{\partial I_1}{\partial\beta} + \frac{\partial\rho}{\partial I_2}\frac{\partial I_2}{\partial\beta} + \frac{\partial\rho}{\partial M_1}\frac{\partial M_1}{\partial\beta} + \frac{\partial\rho}{\partial M_2}\frac{\partial M_2}{\partial\beta}, \tag{4.3}$$

where the subscripts 1 and 2 denote orbital elements for the Earth and Mercury, respectively. Equation (4.3) can also be written in a shorter form by replacing $\partial a_i/\partial\beta$ by Δa_i, etc., understanding that Δa_i is the perturbation in a_i due only to β (with β set equal to unity).

Because we shall express the planetary orbital elements in ecliptic coordinates, in which Ω_1 is ill defined ($I_1 = 0$ in these coordinates), it is convenient to replace the contributions to the perturbation arising through Ω_1 by replacing the term $(\partial\rho\partial\Omega_1)(\partial\Omega_1/\partial d_i)$ by the following limit

$$\frac{\partial\rho}{\partial\Omega_1}\frac{\partial\Omega_1}{\partial d_i} \to \lim_{I_1\to 0}\left\{\frac{1}{\sin I_1}\frac{\partial\rho}{\partial\Omega_1}\right\} \times \lim_{I_1\to 0}\left\{\sin I_1\frac{\partial\Omega_1}{\partial d_i}\right\} = \lim_{I_1\to 0}\left\{\frac{1}{\sin I_1}\frac{\partial\rho}{\partial\Omega_1}(\sin I_1\Delta\Omega_1)\right\}. \tag{4.4}$$

Thus we write, for example,

$$\frac{\partial\rho}{\partial\beta} = \frac{\partial\rho}{\partial a_1}\Delta a_1 + \frac{\partial\rho}{\partial a_2}\Delta a_2 + \frac{\partial\rho}{\partial e_1}\Delta e_1 + \frac{\partial\rho}{\partial e_2}\Delta e_2 + \frac{\partial\rho}{\partial\tilde{\omega}_1}\Delta\tilde{\omega}_1 + \frac{\partial\rho}{\partial\tilde{\omega}_2}\Delta\tilde{\omega}_2 + \lim_{I_1\to 0}\left\{\frac{1}{\sin I_1}\frac{\partial\rho}{\partial\Omega_1}(\sin I_1\Delta\Omega_1)\right\} + \frac{\partial\rho}{\partial\Omega_2}\Delta\Omega_2 + \frac{\partial\rho}{\partial I_1}\Delta I_1 + \frac{\partial\rho}{\partial I_2}\Delta I_2 + \frac{\partial\rho}{\partial M_1}\Delta M_1 + \frac{\partial\rho}{\partial M_2}\Delta M_2, \tag{4.5}$$

where on the right it is understood that in this example Δa_1, Δa_2, etc., refer only to contributions arising from β. Similar results hold for $\partial\rho/\partial d_i$ for $d_i = d_{11}, \ldots, d_{19}$.

Equation (4.5) shows that to find the derivative of the range with respect to $d_{11}, \ldots, d_{19}$, it is necessary to first compute the derivatives with respect to the unperturbed Keplerian elements a_i, e_i, etc. (The derivative with respect to any element evident in (4.5) is computed keeping all other elements fixed.) For this purpose we express the ecliptic coordinates (x, y, z) of a planet in terms of Keplerian elements [30]:

$$x = r[\cos\Omega\cos(f+\omega) - \cos I\sin\Omega\sin(f+\omega)],$$
$$y = r[\sin\Omega\cos(f+\omega) + \cos I\cos\Omega\sin(f+\omega)], \tag{4.6}$$
$$z = r\sin I\sin(f+\omega),$$

where f is the true anomaly, $\omega = \tilde{\omega} - \Omega$ is the argument of perihelion, and r is the radial distance between the planet and the Sun. For the Earth, $I_1 = 0$ and neither ω_1 nor Ω_1 is physically well defined. However, we can assign ω_1 and Ω_1 temporary values consistent with $\omega_1 + \Omega_1 = \tilde{\omega}_1$ (which *is* well defined), noting that no physically meaningful result can ever depend on these values. The radial distance r is given by

$$r = a(1-e^2)/(1+e\cos f). \tag{4.7}$$

The true anomaly and eccentric anomaly E may be related by comparing the above with another equivalent expression for r:

$$r = a(1 - e\cos E). \tag{4.8}$$

The eccentric anomaly E is obtained from the unperturbed mean anomaly by

$$E - e\sin E = M = n(t - t_p), \tag{4.9}$$

where t_p is the time of perihelion passage and $n = (GM_\odot/a^3)^{1/2}$ is the planetary mean motion.

From Eq. (4.6) and (4.7), expressions for the ecliptic coordinates of the Earth and of Mercury may be obtained, leading to the following expression for the range ρ:

$$\rho^2 = (\mathbf{r}_1 - \mathbf{r}_2)^2 = r_1^2 + r_2^2 - 2r_1r_2\cos\theta_{12}, \tag{4.10}$$

where

$$\cos\theta_{12} = \cos[f_1 + (\tilde{\omega}_1 - L_{10}) - (\Omega_2 - L_{20}) - (L_{20} - L_{10})]\cos[f_2 + (\tilde{\omega}_2 - L_{20}) - (\Omega_2 - L_{20})] + \cos I_2\sin[f_1 + (\tilde{\omega}_1 - L_{10}) - (\Omega_2 - L_{20}) - (L_{20} - L_{10})]\sin[f_2 + (\tilde{\omega}_2 - L_{20}) - (\Omega_2 - L_{20})] \tag{4.11}$$

is the angle between the radius vectors from the Sun to the two planets. This result displays explicitly the dependence of ρ on the chosen set of nine Keplerian elements. Since $\partial\rho/\partial f$ does not appear in Eq. (4.5), the true anomaly f is to be regarded as a function of e and M, consistent with Eqs. (4.7)–(4.9).

The partial derivatives needed in Eq. (4.5) may now be evaluated from Eqs. (4.10) and (4.11). In calculating these derivatives, we are at this stage regarding M as independent of $\tilde{\omega}$ and a, although these unperturbed elements are related by

$$M = nt + L_0 - \tilde{\omega} = (GM_\odot/a^3)^{1/2}t + L_0 - \tilde{\omega}. \quad (4.12)$$

We shall later take this into account when determining partial derivatives with respect to the unperturbed elements. For derivatives with respect to a_1 and a_2, we find

$$\frac{\partial\rho}{\partial a_1} = r_1(r_1 - r_2\cos\theta_{12})/(a_1\rho), \quad (4.13)$$

$$\frac{\partial\rho}{\partial a_2} = r_2(r_2 - r_1\cos\theta_{12})/(a_2\rho). \quad (4.14)$$

Derivatives with respect to the eccentricities are

$$\frac{\partial\rho}{\partial e_1} = [-a_1\cos f_1(r_1 - r_2\cos\theta_{12}) + (x_1y_2 - y_1x_2) \times \sin f_1(2 + e_1\cos f_1)/(1 - e_1^2)]/\rho, \quad (4.15)$$

$$\frac{\partial\rho}{\partial e_2} = [-a_2\cos f_2(r_2 - r_1\cos\theta_{12}) + (x_1v_{2x} + y_1v_{2y}) \times \sin f_2(2 + e_2\cos f_2)/(1 - e_2^2)]/\rho, \quad (4.16)$$

where

$$v_{2x} = r_2[\cos\Omega_2\sin(f_2 + \omega_2) + \cos I_2\sin\Omega_2\cos(f_2 + \omega_2)],$$
$$v_{2x} = r_2[\sin\Omega_2\sin(f_2 + \omega_2) - \cos I_2\cos\Omega_2\cos(f_2 + \omega_2)], \quad (4.17)$$

and $(x_1, y_1, 0)$ and (x_2, y_2, z_2) are ecliptic coordinates of the Earth and Mercury, respectively. For the derivatives with respect to longitude of perihelion, we have

$$\frac{\partial\rho}{\partial\tilde{\omega}_1} = (x_2y_1 - y_2x_1)/\rho, \quad (4.18)$$

$$\frac{\partial\rho}{\partial\tilde{\omega}_2} = (v_{2y}y_1 + v_{2x}x_1)/\rho. \quad (4.19)$$

The vanishing of I_1 requires that care be taken in order to avoid ambiguity in the calculation of perturbations arising through $\Delta\Omega_1$. Thus,

$$\lim_{I_1\to 0}\frac{1}{\sin I_1}\frac{\partial\rho}{\partial\Omega_1} = r_1\cos(f_1 + \omega_1)z_2/\rho, \quad (4.20)$$

$$\frac{\partial\rho}{\partial\Omega_2} = -[x_1(v_{2x} - y_2) + y_1(v_{2y} + x_2)]/\rho, \quad (4.21)$$

$$\frac{\partial\rho}{\partial I_1} = -r_1\sin(f_1 + \omega_1)z_2/\rho, \quad (4.22)$$

$$\frac{\partial\rho}{\partial I_2} = -r_2\sin(f_2 + \omega_2)\sin I_2(x_1\sin\Omega_2 - y_1\cos\Omega_2)/\rho, \quad (4.23)$$

$$\frac{\partial\rho}{\partial M_1} = (r_1 - r_2\cos\theta_{12})a_1e_1\sin f_1/[\rho(1 - e_1^2)^{1/2}] - (x_1y_2 - y_1x_2)(1 - e_1^2)^{1/2}a_1^2/(\rho r_1^2), \quad (4.24)$$

$$\frac{\partial\rho}{\partial M_2} = (r_2 - r_1\cos\theta_{12})a_2e_2\sin f_2/[\rho(1 - e_2^2)^{1/2}] + (x_1v_{2x} + y_1v_{2y})(1 - e_2^2)^{1/2}a_2^2/(\rho r_2^2). \quad (4.25)$$

The results, Eqs. (4.12)–(4.23), will be used in equations analogous to (4.5) to find $\partial\rho/\partial d_i$ for the parameters $d_i = \gamma$, β, J_2, α_i, ξ, λ_G, and η. The results for Δa_i, etc. are derived in the appendix. These partial derivatives will then be used in Eq. (3.4) to find the matrix C_{kj}.

The derivatives given by Eqs. (4.20) and (4.22) appear to depend on an undefined quantity, ω_1. However it can be easily verified that when the contributions to $\partial\rho/\partial d_i$ from both $\Delta\Omega_1$ and ΔI_1 are combined using any of the integrated results of the Appendix, the combination is independent of ω_1. This is so for the perturbations due to α_1, α_2, α_3, and ξ. For the perturbations due to γ, β, λ_G, and η, $\Delta\Omega = \Delta I = 0$ anyway. For the perturbations due to J_2, all the calculations are done in the solar equatorial coordinate system (where no ambiguities arise since in this system $I_1 \neq 0$).

So far, we have considered the derivatives of the range with respect to the parameters γ, β, α_i, J_2, ξ, λ_G, and η. As discussed in Sec. III B, a set of unperturbed orbital parameters will also be included in the analysis. This requires that we find $\partial\rho/\partial d_i$ for the additional 10 parameters: $d_j = GM_\odot$, a_i, e_i, $\tilde{\omega}_i - L_{i0}$, $\Omega_2 - L_{20}$, I_2, $L_{20} - L_{10}$ $(i = 1, 2)$. These partial derivatives will be simple linear combinations of the derivatives (4.12)–(4.23). We shall denote partial derivatives with respect to any of the selected set of unperturbed Keplerian elements $(d_1, d_2, \ldots, d_9)$ by means of a subscript K.

In evaluating partial derivatives with respect to any of the chosen Keplerian parameters, particular care must be exercised to account for the dependence of the unperturbed mean anomaly on semimajor axis, perihelion position, and initial longitude, Eq. (4.12). For this portion of the calculation M is no longer regarded as an independent variable but as an intermediate variable, which can be affected by changes in a, $\tilde{\omega}_i - L_0$, and $GM_\odot$. Thus, for example, changes in values of the semimajor axes will affect the mean motion, n_i, in Eq. (4.24), which must be accounted for in the total partial derivative with respect to a_i. In this case, the total derivatives are given by

$$\left(\frac{\partial\rho}{\partial a_1}\right)_K = \frac{\partial\rho}{\partial a_1} - \frac{3}{2}\frac{n_1t}{a_1}\frac{\partial\rho}{\partial M_1}, \quad (4.26)$$

$$\left(\frac{\partial\rho}{\partial a_2}\right)_K = \frac{\partial\rho}{\partial a_2} - \frac{3}{2}\frac{n_2t}{a_2}\frac{\partial\rho}{\partial M_2}, \quad (4.27)$$

where $\partial\rho/\partial a_i$ and $\partial\rho/\partial M_i$ are given by Eqs. (4.13), (4.14), (4.24), and (4.25).

Similarly, for the determination of the perihelion position variables $\tilde{\omega}_i - L_{i0}$ we have

$$\left(\frac{\partial\rho}{\partial(\tilde{\omega}_1 - L_{10})}\right)_K = \frac{\partial\rho}{\partial\tilde{\omega}_1} - \frac{\partial\rho}{\partial M_1}, \qquad (4.28)$$

$$\left(\frac{\partial\rho}{\partial(\tilde{\omega}_2 - L_{20})}\right)_K = \frac{\partial\rho}{\partial\tilde{\omega}_2} - \frac{\partial\rho}{\partial M_2}, \qquad (4.29)$$

the minus signs arising from the negative coefficient of $\tilde{\omega}$ in Eq. (4.17).

No such corrections are necessary to obtain partial derivatives with respect to the eccentricities:

$$\left(\frac{\partial\rho}{\partial e_1}\right)_K = \frac{\partial\rho}{\partial e_1}, \qquad \left(\frac{\partial\rho}{\partial e_2}\right)_K = \frac{\partial\rho}{\partial e_2}. \qquad (4.30)$$

For the other parameters we have

$$\left(\frac{\partial\rho}{\partial(\Omega_2 - L_{20})}\right)_K = \frac{\partial\rho}{\partial\Omega_2}, \qquad \left(\frac{\partial\rho}{\partial I_2}\right)_K = \frac{\partial\rho}{\partial I_2}, \qquad (4.31)$$

$$\left(\frac{\partial\rho}{\partial(L_{20} - L_{10})}\right)_K = -\frac{\partial\rho}{\partial\tilde{\omega}_1}. \qquad (4.32)$$

In Eqs. (4.26)–(4.32), the derivatives occurring on the right-hand sides (without subscripts K) are those calculated in Eqs. (4.13)–(4.25).

V. APPLICATION TO A MERCURY ORBITER MISSION

A. Calculations

Software has been written that calculates all partial derivatives, computes the matrices B_{mn} and C_{mn}, and solves for the uncertainties δd_l, Eq. (3.13). This program has been used to evaluate specific Earth-Mercury ranging missions, all with arbitrarily chosen initial epoch $t_0 = 2012$ January 0.5 E.T. (Julian Date 2455928.0). (Other choices for the initial epoch lead to variations in the estimated uncertainties of no more than 10% to 15% from the results quoted here.) We have assumed a range data point once per day, except that any point arising when the Earth-Mercury vector is aligned within five degrees of the Sun has been deleted. This omission originally was intended to allow for the noise at S-band, which swamps the signal when ranging too near the Sun. It has been left in, even though current proposals are to use Ka-band and X-band ranging signals. The principal important effect of this assumption on our results is to limit the observable magnitude of the time delay effect and hence the accuracy of the determination of γ. However, this bound on γ is not critically sensitive to the particular choice of five degrees since the time delay depends only on the logarithm of the distance of closest approach of the signal to the Sun.

TABLE I. Values of the unperturbed Keplerian elements (all angles are in radians).

Parameter	Mercury	Earth
a (meters)	5.79×10^{10}	1.496×10^{11}
e	0.20563	0.0167
n (rad/day)	0.0714	0.0172
L_0 (epoch JD 2455928.0)	1.7521	3.2982
$\tilde{\omega}$	1.3452	1.793
Ω	0.8433	0.00
I	0.1222	0.00
$GM_\odot$	9.983×10^{29} m^3/day^2	
c (speed of light)	2.592×10^{13} m/day	

In calculations as complicated as the ones reported here, it is realistic to anticipate the possibility of algebraic or computational errors. The expressions given in the Appendix for the orbital perturbations due to γ, β, α_i, ξ, and λ_G, as well as expressions given by Fitzpatrick [30] for the solar quadrupole moment perturbations, have been encoded in FORTRAN and used in calculations of the matrix B_{mn}. Independently, and working on a second computer in a different programming language, the Lagrangian perturbation equations were integrated numerically using a very accurate scheme based on Bode's Rule [41]. These independent calculations of the orbital perturbations were found to agree to 1 part in 10^{10}. Similarly, independent calculations of the partial derivatives of the range with respect to the orbital elements and to the Nordtvedt parameter were found to be in agreement. Thus we feel confident that no algebraic or computational errors notably affect our results.

Values of the orbital elements of the Earth and of Mercury at the initial epoch are listed in Table I together with values of some constants used in the calculations. The

TABLE II. Assumed present uncertainties in the nonorbital parameters for our Mercury Orbiter–Earth ranging study. For information on how most of these uncertainties were determined, see Ref. [42].

Parameter	Limit
$\gamma - 1$	2.3×10^{-5}
$\beta - 1$	3×10^{-3}
ξ	4×10^{-7}
α_1	10^{-4}
α_2	4×10^{-7}
α_3	2×10^{-20}
η	4.5×10^{-4}
ζ_1	2×10^{-2}
ζ_2	4×10^{-5}
ζ_3	10^{-8}
ζ_4	...
$\dot{G}/G$ (yr^{-1})	0.9×10^{-12}

TABLE III. Results for the modified worst-case systematic error limits on the parameters for various mission lengths, assuming general relativity is correct, but $\dot{G}/G$ might not be zero. The rms error is assumed to be 4.5 cm for ranging between the Earth and Mercury. The parameter uncertainties given correspond to one-third of the worst-case results. Angles are measured in radians.

Parameter	1 yr	2 yr	8 yr
		12-parameter case	
a_1 (meters)	0.043	0.020	0.016
a_2 (meters)	0.044	0.040	0.028
e_1	4.3×10^{-13}	3.5×10^{-13}	3.4×10^{-13}
e_2	7.7×10^{-13}	6.5×10^{-13}	5.9×10^{-13}
$\tilde{\omega}_1 - L_{10}$	2.6×10^{-11}	2.2×10^{-11}	2.6×10^{-11}
$\tilde{\omega}_2 - L_{20}$	6.7×10^{-12}	4.2×10^{-12}	4.4×10^{-12}
$\Omega_2 - L_{20}$	4.8×10^{-11}	4.6×10^{-11}	4.7×10^{-11}
I_2	5.6×10^{-12}	5.5×10^{-12}	5.1×10^{-12}
$L_{20} - L_{10}$	1.3×10^{-12}	1.2×10^{-12}	1.1×10^{-12}
$GM_{\odot}$ (fractional)	1.9×10^{-12}	1.8×10^{-12}	1.3×10^{-12}
J_2	1.4×10^{-9}	9.6×10^{-10}	6.4×10^{-10}
$\dot{G}/G$ (yr^{-1})	3.0×10^{-13}	6.3×10^{-14}	3.7×10^{-15}
		4-parameter case	
a_1 (meters)	0.017	0.015	0.014
a_2 (meters)	0.024	0.024	0.018
$GM_{\odot}$ (fractional)	1.1×10^{-12}	1.1×10^{-12}	8.5×10^{-13}
J_2	6.8×10^{-10}	6.0×10^{-10}	4.0×10^{-10}

results are shown in Tables II and III, and are discussed in the next section.

B. Present uncertainties in gravitational physics parameters

An updated general discussion of the uncertainties in all of the PPN parameters has been given fairly recently by Will [42]. He includes the post-Newtonian "conservation law" parameters ζ_1, ζ_2, ζ_3, and ζ_4, in addition to the ones we have discussed earlier. We will use the uncertainties listed in his Table IV, except as discussed below.

For γ, Doppler measurements of the time delay during a passage of the Cassini spacecraft behind the Sun have been used recently [43] to determine that

$$\gamma - 1 = (2.1 \pm 2.3) \times 10^{-5}. \tag{5.1}$$

For ξ, we have listed the uncertainty as 4×10^{-7}, since Nordtvedt [44] notes that the uncertainty for ξ should be comparable with that for α_2, based on the spin-axis orientation for the Sun. Also, improvements in the analysis of lunar laser ranging data now give [27,28,45] for the Nordtvedt parameter η and for $(dG/dt)/G$:

$$\eta = (4.4 \pm 4.5) \times 10^{-4}, \tag{5.2}$$

$$|\dot{G}|/G < 0.9 \times 10^{-12}/\mathrm{yr}. \tag{5.3}$$

Since the complete PPN expression for η is

$$\eta = 4\beta - \gamma - 3 - \frac{10\xi}{3} - \alpha_1 - \frac{2\alpha_2}{3} - \frac{2\zeta_1}{3} - \frac{\zeta_2}{3}, \tag{5.4}$$

and Will lists the uncertainty in ζ_1 as 2×10^{-2}, we will assume that η can be used in determining the other parameters only if the conservation law parameters are being assumed to be zero. The corresponding cases will be referred to as "metric theory" cases, as was done by Milani *et al.* [7]. The only conservation law parameter entering into the perihelion advance expression is ζ_2, and its listed uncertainty of 4×10^{-5} is small enough so that it will not affect any of our results.

With the above modifications, our assumed present uncertainties in the parameters are given in Table II.

C. Results

We have considered four main cases. Results for all of these cases will be given for mission lengths of one, two, and eight years. They are listed in Tables III and IV. The one-year results correspond to the approximate length of the nominal BepiColombo mission. The two-year results demonstrate the improvement possible with an extended BepiColombo mission, while the eight-year results indicate what might be achieved with a much longer mission or with the combination of data from two missions.

The results given in Tables III and IV are the modified worst-case estimates, which are the worst-case uncertainties reduced by a factor of 3, as discussed in Sec. III A. These results assume an rms error of $\sigma = 4.5$ cm [see Eq. (3.10)]. Since the results are directly proportional to σ, from the quoted values one can derive results corresponding to other assumed values of σ by multiplication by the appropriate factor. The actual worst-case estimates can be obtained by multiplying by the factor of 3, and the uncertainties that would arise under the assumption of random and uncorrelated errors can be derived by multiplying by the factor $3/\sqrt{N}$, where N is the number of data points. For 1, 2, and 8 yr, respectively, the factor $3/\sqrt{N} = 0.164$, 0.116, and 0.0584, respectively.

The first case corresponds to assuming general relativity is valid, but that the gravitational constant may change with time. There are 12 parameters involved, including the 9 orbit parameters, $GM_{\odot}$, J_2, and $\dot{G}/G$. The results are given in Table III. A 4-parameter second case which is useful in understanding the limits on determining J_2 and other parameters also is included in Table III.

The third and fourth cases assume that general relativity may be wrong, and that the correct theory may be nonmetric. Thus Eq. (5.4) with ζ_1 and ζ_2 set equal to zero cannot be used to give a relationship between η and the metric theory relativity parameters. For the third case, spatial isotropy is assumed, so that $\alpha_1 = \alpha_2 = \alpha_3 = \zeta = 0$. For this case there are 15 parameters, including $GM_{\odot}$, γ, β, J_2, $\dot{G}/G$, and η. The results for 1 yr, 2 yr and 8 yr of simulated observations are given in columns 1, 3, and 5 of Table IV.

TABLE IV. Nonmetric theory results, with the Nordtvedt parameter η treated as an independent parameter. The assumed rms error is 4.5 cm.

Parameter	1-year spatial isotropy	1-year general case	2-year spatial isotropy	2-year general case	8-year spatial isotropy	8-year general case
a_1 (meters)	1.4	2.0	0.48	0.67	0.25	0.27
a_2 (meters)	0.68	0.95	0.27	0.34	0.18	0.19
e_1	9.7×10^{-12}	1.4×10^{-11}	3.4×10^{-12}	4.4×10^{-12}	1.4×10^{-12}	1.7×10^{-12}
e_2	7.4×10^{-12}	9.3×10^{-12}	2.4×10^{-12}	3.6×10^{-12}	1.3×10^{-12}	1.7×10^{-12}
$\tilde{\omega}_1 - L_{10}$	2.5×10^{-10}	3.3×10^{-10}	5.1×10^{-11}	1.1×10^{-10}	2.8×10^{-11}	3.7×10^{-11}
$\tilde{\omega}_2 - L_{20}$	5.5×10^{-11}	7.9×10^{-11}	1.8×10^{-11}	2.6×10^{-11}	1.0×10^{-11}	1.2×10^{-11}
$\Omega_2 - L_{20}$	7.0×10^{-11}	9.0×10^{-11}	4.7×10^{-11}	7.3×10^{-11}	4.8×10^{-11}	5.6×10^{-11}
I_2	1.4×10^{-11}	1.6×10^{-11}	7.8×10^{-12}	9.3×10^{-12}	7.2×10^{-12}	8.2×10^{-12}
$L_{20} - L_{10}$	1.5×10^{-12}	4.2×10^{-12}	1.3×10^{-12}	1.9×10^{-12}	1.1×10^{-12}	1.4×10^{-12}
$GM_\odot$ (fractional)	1.6×10^{-11}	2.3×10^{-11}	5.0×10^{-12}	7.0×10^{-12}	1.5×10^{-12}	2.5×10^{-12}
γ	2.7×10^{-5}	2.9×10^{-5}	2.2×10^{-5}	2.3×10^{-5}	2.2×10^{-5}	2.2×10^{-5}
β	5.7×10^{-4}	7.2×10^{-4}	1.8×10^{-4}	2.2×10^{-4}	5.7×10^{-5}	6.0×10^{-5}
J_2	6.8×10^{-8}	7.8×10^{-8}	2.1×10^{-8}	2.3×10^{-8}	4.6×10^{-9}	4.9×10^{-9}
$\dot{G}/G$ (yr^{-1})	3.4×10^{-13}	4.2×10^{-13}	6.8×10^{-14}	7.0×10^{-14}	3.7×10^{-15}	3.8×10^{-15}
η	2.1×10^{-3}	2.6×10^{-3}	4.4×10^{-4}	8.5×10^{-4}	8.6×10^{-5}	9.4×10^{-5}
α_1		2.1×10^{-5}		6.2×10^{-6}		8.6×10^{-7}
α_2		2.9×10^{-6}		1.8×10^{-6}		1.2×10^{-6}
ξ		3.9×10^{-6}		3.9×10^{-6}		1.0×10^{-6}

The fourth case does not assume spatial isotropy. Initially α_1, α_2, α_3, and ζ were solved for, as well as the parameters for the third case. However, it was found that the results for α_3 were about 10 orders of magnitude worse than the *a priori* uncertainty. Thus α_3 was dropped from the list of parameters, and 18 parameters were solved for. The results are given in columns 2, 4, and 6 of Table IV.

D. Discussion of results

1. Case with general relativity correct

Assuming that general relativity is correct, the most important result concerns the accuracy with which J_2 can be determined. From Table III, the expected accuracy reaches 1.4×10^{-9} after 1 yr and 9.6×10^{-10} after 2 yr. Duval *et al.* [31] quote an uncertainty of 4×10^{-8} for J_2 from helioseismology. However, it may be difficult for helioseismic studies to improve considerably on this result. Brown *et al.* [32] find that the solar equatorial angular velocity rises gradually between $r \approx 0.5R_\odot$ and the surface, with the variation being less than 10%. However, the solar rotation velocity has not yet been determined for $r \leq 0.2R_\odot$ at low latitudes and for $r \leq 0.5R_\odot$ elsewhere. For this reason, they state that it is difficult to make a realistic estimate of the uncertainty in J_2. They also discuss possible differences in the rotation rate with latitude below the base of the convection zone, as suggested by solar dynamo calculations. A J_2 measurement with 2×10^{-9} or better accuracy would provide an important input to solar rotation theories.

2. Cases with general relativity not assumed

The logarithmic dependence of the time delay, Eq. (2.27), is unlike any other time signature in the ranging signal. As a result, the determination of γ is due mainly to the time delay, and the modified worst-case uncertainty in γ even for the general case approaches a lower limit of 2.2×10^{-5} after only about six of Mercury's orbital periods. After this time, additional ranging does not result in further appreciable improvement in estimates of γ, as shown in Table IV. If ranging when the sight path comes within 2° of the Sun is allowed, the uncertainty in γ drops to 1.5×10^{-5} after a 1-year mission for the isotropic case. The present uncertainty for γ, as determined from the Cassini relativistic time delay measurements [43] is 2.3×10^{-5}. Comparable accuracy is expected from determining the geodetic precession during the Gravity Probe B mission.

For $\dot{G}/G$, Table IV gives the accuracy for the general case as 4.2×10^{-13}/yr for a 1 yr mission, 7.0×10^{-14}/yr for a 2 yr mission, and 3.8×10^{-15}/yr for an 8 yr mission. These results for 2 yr or longer are of course much better than the present uncertainty in $\dot{G}/G$ of 1.1×10^{-12}/yr from lunar laser ranging [27,28]. The results are not much worse than those in Table III, where general relativity was assumed to be correct. Because a secular change in G results in a perturbation of the mean anomaly which is quadratic in time, a long-term ranging mission is one of the best ways to obtain information about this effect. However, caution is in order, since other effects also can give quadratic variations with time, as discussed later.

When both J_2 and β are included, as well as γ and $\dot{G}/G$, the results from Table IV show that the uncertainty for J_2 is increased to 7.8×10^{-8} after 1 yr, 2.3×10^{-8} after 2 yr, and 4.9×10^{-9} after 8 yr, for the general case. This last uncertainty is about 3% of the rigid rotation value for J_2. However, these uncertainties are much worse than those from Table III, which were obtained assuming that general relativity is correct. The reason for this will be discussed later.

The corresponding uncertainties for β are 7.2×10^{-4}, 2.2×10^{-4}, and 6.0×10^{-5}, respectively. The present uncertainty in β, not considering the Nordtvedt effect, is 3×10^{-3}. Thus the improvement in β, assuming nonmetric theories, would be substantial. The differences between the results for the isotropic and general cases for $\dot{G}/G$, J_2, and β are fairly small.

For the preferred frame parameters an α_1 and α_2, and the Whitehead parameter ξ, the only potential improvement over the present uncertainties would be for α_1. Here the expected uncertainty after 1 yr is 2.1×10^{-5}, considerably better than the present uncertainty of roughly 1×10^{-4}. In addition, the uncertainty would be reduced to 6.2×10^{-6} after 2 yr.

The results for the Nordtvedt parameter η, solved for from the perturbations due to Jupiter, are given in Table IV. η is just solved for along with the 14 other parameters for the spatial isotropy case or the 17 other parameters from the nonisotropic case, and no relationship of η to other parameters is included. The uncertainties for η for the isotropic case are 2.1×10^{-3} for 1 yr, 4.4×10^{-4} for 2 yr, and 8.6×10^{-5} for 8 yr. Thus it would take 2 yr for the uncertainty in η to get down to about the same level as the present value of 4.5×10^{-4}. However, it seems likely that the uncertainty in the results from lunar laser ranging will be reduced considerably in the next few years if substantial improvements in the measurement accuracy are achieved [28].

Our results from Table IV can be compared with those of Milani *et al.* [7] for the 1 yr nominal lifetime of the BepiColombo mission. For their experiment A (nonmetric, isotropic) both formal rms uncertainties and estimates of the realistic uncertainties are given in their Table II. Experiment A involves the same parameters as those included in our Table IV, except that γ is not considered. Based on the realistic estimates, their uncertainty for $(dG/dt)/G$ is about the same as ours. For β, J_2, and $GM_\odot$, their uncertainties are about a factor 2.5 smaller than ours for 1 yr of data, but quite similar to what we find for 2 yr of data. Since their Table III shows high correlations between these three parameters, as expected, it is not surprising that the ratios of uncertainties are nearly the same for all three. The results of Milani *et al.* [7] for their experiment C (nonmetric, non isotropic), which has α_1 and α_2 added to the parameters for experiment A, are fairly close to our results, except for η. Unfortunately, the ratio of uncertainties is large for η. Here, the uncertainty found by Milani *et al.* [7] in experiment A is a factor 6 smaller than we found, even for 8 yr of data, and is much smaller than our result for 1 yr of data.

The difference in the results for all but η is believed to be due to the difference in the way systematic errors are handled in the two studies. The question is why does our modified worst-case error model predict more accuracy loss for a number of the interesting parameters than does the larger long-term time delay drift in the ranging transponders and the accelerometer systematic errors included by Milani *et al.* [7]? This is presumably due to the presence of critical time signatures in the perturbations arising from some of the interesting parameters, and the relatively small amplitude of these signatures in the error model of Milani *et al.* However, it is quite possible that systematic errors at some of the characteristic frequencies of concern in our approach can be shown to be less than we have allowed for. Thus further studies of the achievable ranging system accuracy for critical time signatures appear to be needed.

For η, we have looked for a problem with our results, but have not been able to find one. From Eqs. (2.23), (2.24), and (2.25), we have estimated the rms value of the signature for η as being roughly 400 m. Thus the smallest worst-case uncertainly we could expect for η with 4.5 cm rms range uncertainty, even for long observation times, is about 1×10^{-4}. This would give 3×10^{-5} for the modified worst-case uncertainty, which is only a factor 3 better than what we find for 8 yr of observations. Quite a lot of the signature has the frequency of the synodic period of the Earth or Mercury with respect to Jupiter, and it would take observations lasting roughly Jupiter's orbital period to separate these frequencies from the orbital frequencies of the Earth and Mercury. Thus our uncertainty of 2.1×10^{-3} for 1 yr does not seem surprisingly large. If we multiply our modified worst-case uncertainty by the factor 0.164 given in Sec. V C to convert it to an rms error, not considering systematic errors at all, the resulting uncertainly is 3.4×10^{-4}.

E. Interpretation of the results for J_2 and β

It is useful at this point to examine the question of how J_2 and a few of the other parameters that are likely to be correlated with it are determined by the data. Neglecting γ, dG/dt, η, and the preferred frame parameters, the five parameters that have secular terms in their partial derivatives are J_2, β, $GM_\odot$, a_1, and a_2. Two very well-determined quantities from the Earth-Mercury distance data are the mean motion of Mercury with respect to its perihelion, dM_2/dt, and the rate of change of the difference in mean longitudes, $d(L_2 - L_1)/dt$. The uncertainties for these quantities decrease inversely as the observation time. Thus two linear combinations of the above five parameters are accurately determined. However, periodic terms or much smaller secular terms have to be used to separate individual parameters if more than two are included in a solution.

Both a_1 and a_2 have strong periodic terms in their signatures. However, the uncertainties in the five parameters are much larger than would be found if only two parameters were included in the parameter set. This is because of the extremely high correlations in the secular parts of the time signatures arising from these parameters. If β and $GM_{\odot}$ were the only parameters solved for, then after 8 yr the modified worst-case uncertainty in β would be 2.2×10^{-7}, 10^4 times less than the current uncertainty in β.

We consider now the 4-parameter and 12-parameter cases in Table III, where β is not included. For the 4-parameter case, only J_2, $GM_{\odot}$, a_1, and a_2 are adjusted. One feature of the results for J_2 for this case, which is somewhat surprising at first, is that the uncertainty does not decrease by anywhere near a factor 4 in going from a 2-yr observing period to 8 yr. For the observation times considered, a_1 and a_2 are determined mainly from the periodic terms in their signatures, and their uncertainties limit the accuracy for determining J_2 and $GM_{\odot}$. Thus the uncertainties will not continue decreasing strongly with time in a modified worst-case analysis. The results for J_2, for the 12-parameter case, agree within a factor 2 with those for the 4-parameter case, and show similar behavior in going from a 2-yr observing period to 8 yr.

When β is included instead of J_2 in 4-parameter and 12-parameter solutions like those in Table III, the uncertainties in the other parameters do not change by more than about 30%. Also a 5-parameter solution with J_2, β, $GM_{\odot}$, a_1, and a_2 has only at most a factor of 2.3 larger errors for the common parameters than the 4-parameter solution in Table III. We have not found a third secular term large enough to explain the uncertainties obtained in the 5-parameter case. Thus a third fairly large periodic signature must be responsible for the results.

Unfortunately, when the other seven orbit elements and $\dot{G}/G$ are added to the 5-parameter case to give a 13-parameter case including both J_2 and β, the results for 8 yr are about a factor of 7 worse for J_2 than in the 12-parameter case from Table III. For this 13-parameter case, the fifth important signature to break the correlations between J_2, β, $GM_{\odot}$, a_1, and a_2 comes mainly from the motion of the node of Mercury's orbit on the solar equator caused by J_2. The motion of the node on the solar equator gives a change in the inclination of Mercury's orbit with respect to the ecliptic, which produces a substantial periodic term in the range, with secularly varying amplitude. However, this term is not large enough to prevent the factor 7 loss in accuracy when β and J_2 have to be separated and many other parameters are present.

The results for the 13-parameter case are similar to those of the 15-parameter case in Table IV for J_2 and $GM_{\odot}$. The 40% difference for β probably is due to the fact that a linear combination of β and γ is determined from the analysis if the time delay signature is excluded, and that the uncertainty in γ from the time delay is starting to limit the accuracy with which β itself can be determined from the linear combination. The results for a_1 and a_2 for the 8 yr case are worse for the 15-parameter case by factors of 2.5 and 2.7, respectively, and a few of the Newtonian parameters have larger uncertainties also. Because the secular signature due to motion of the node is important, the accuracy increase for both J_2 and β is a factor of 2 or 3 in going from 2 yr to 8 yr.

F. Limitations due to asteroidal perturbations

The main limitation on the Viking lander determination of $\dot{G}/G$ at about 1×10^{-11}/yr was due to uncertainty in the perturbations of the mean anomaly of Mars by asteroids with nearly commensurable periods [4,46,47]. By commensurable periods, we mean that the orbital period for the asteroid is a factor (n/m) times the planetary orbital period, where n and m are integers. The degree k of the commensurability is given by $k = n - m$. Although the Viking data and other ranging data gave improved masses for a few of the largest asteroids, and improved density estimates for additional groups of asteroids, there were too many smaller asteroids giving significant perturbations to permit solving for many of their masses. Asteroids with near commensurabilities of degree 1, 2, and 3 appeared to be important.

A great deal of progress has been made on determining solar system ephemerides since the Viking era. A recent paper by Konopliv *et al.* [48] describes results for Mars and the Earth from tracking a number of Mars orbiters and the Mars Pathfinder landers. From the data, it was possible to solve for the masses of 63 individual asteroids, as well as the mean densities for three classes of other asteroids. This might lead to an order-of-magnitude improvement in the determination of $\dot{G}/G$. However, $\dot{G}/G$ apparently has not been solved for recently from Mars data, probably because it is quite well determined from lunar laser ranging data.

For Earth-Mercury distance measurements, the limitation on $\dot{G}/G$ is much weaker, particularly because of the shorter orbital periods compared with Mars, and therefore the higher degree commensurabilities involved. We have made a very preliminary estimate of the relative size of perturbations of the longitudes of the Earth and Mars with periods of 10 yr or longer due to Ceres, Pallas, and Vesta. A partial listing of perturbations of the Earth's longitude provided by J. G. Williams was compared with results for Mars from [46]. This comparison indicated that the accuracy achievable for $\dot{G}/G$ from Earth-Mercury range data might be as much as 2 orders of magnitude better than from Earth-Mars range data, or perhaps roughly 3×10^{-14}/yr. This is a factor 8 worse than the accuracy of about 4×10^{-15}/yr shown in Tables III and IV for an 8-yr Mercury mission.

The estimate given above is included only because better information is not available. Our estimate could be off in either direction. The perturbations of the Earth's orbit due to a few of the large asteroids probably can be determined better from the Earth-Mercury range data. However, the real limitation is likely to come from smaller asteroids, which happen to give substantial long-period perturbations. A careful study of the asteroidal perturbations is needed before anything definite can be said about the accuracy for determining $\dot{G}/G$ from Earth-Mercury range data.

The main reason for discussing the interpretation of the results for J_2 and β in some detail earlier was to provide a basis for considering the effects of asteroidal perturbations. It seems unlikely that uncertainties in the asteroidal perturbations would mimic periodic terms such as those in the signatures for a_1 and a_2. Thus the main question is how large the effect on secular terms in the various signatures may be.

For the 15-parameter case shown in Table IV, where general relativity is not assumed to be correct, the main sensitivity to uncertainties in the planetary and asteroidal perturbations appears to come from a secular term. As discussed above, the accuracy for separating J_2 and β depends strongly on the motion of the node of Mercury's orbit on the solar equator due to J_2, which causes a secular change in Mercury's inclination with respect to the Earth's orbit. However, the asteroids will cause secular changes in the orbit planes for both planets, and the magnitude of the resulting secular change in relative inclination will be uncertain because of the residual uncertainties in the masses of the asteroids.

We have made a crude estimate of the size of the problem by using the rate of change of the plane of the Earth's orbit due to Ceres, Vesta, and Pallas. The largest effect is about an order of magnitude larger than the signature used to separate corrections to J_2 and β in the 15-parameter case. However, as mentioned earlier, the periodic perturbations from some of the most massive asteroids should be large enough and distinct enough so that their masses can be determined even better than they are currently known. We estimate, very roughly, that the effects of the remaining asteroidal mass uncertainties will not increase the uncertainties in J_2 and β by more than a factor of 2 or 3 above those shown in Table IV. Whether the results for the 18-parameter anisotropic case will be affected more is not known. However, from the earlier results, only the extra parameter α_1 probably needs to be included in future solutions.

Finally, we consider the 12-parameter case shown in Table III, where general relativity is assumed to be correct. For this case, the secular effects due to Ceres, Vesta, and Pallas can be compared with the equivalent effects from J_2 for dM_2/dt and $d(L_2 - L_1)/dt$. Assuming still that the current accuracy for some asteroidal masses can be improved, we again estimate that the results will not be degraded by more than a factor of 2 or 3. Thus the expected accuracy for J_2, assuming general relativity is correct, would be better than 4×10^{-9} for a 1 yr mission, or 3×10^{-9} for a 2 yr mission.

It is clear from the above discussion of J_2, and from the earlier discussion of $\dot{G}/G$, that a much more complete study of the achievable accuracy will be needed in the future. The present theoretical framework can be used in that study, but partial derivatives with respect to the masses or densities of many selected asteroids will have to be included. Such an extension of the calculations was beyond the scope of the present investigation.

VI. DISCUSSION

The studies described in the present paper were started a number of years ago to give an initial assessment of the accuracy with which general relativity can be tested and the solar quadrupole moment can be determined with a high-accuracy dual-frequency ranging transponder on a Mercury orbiter. A circular and nearly polar orbit at an altitude equal to the planetary radius was chosen in order to minimize the problem of determining the spacecraft orbit. But the results of Milani *et al.* [6] indicate that the orbit determination problem can be handled adequately even for the considerably lower periapsis altitude and eccentric orbit of the BepiColombo Planetary Orbiter.

The results in the main part of the paper are limited in their applicability to the real planetary system because of the neglect of the asteroid-planet perturbations. However, attempts are made to provide crude estimates of limits on the size of the uncertainties due to such effects.

Despite the limitations introduced by uncertainties in the asteroidal perturbations and the possibly somewhat pessimistic nature of our modified worst-case analysis, the present investigation still provides encouraging information concerning the scientific results achievable from high-accuracy tracking of the BepiColombo Planetary Orbiter. In particular, the estimated accuracy for determining J_2 for the Sun, assuming general relativity is correct, would be roughly 1.4×10^{-9} to 4×10^{-9} for the nominal 1 yr mission and 1×10^{-9} to 3×10^{-9} for a 2 yr extended mission. Such results would be an important contribution to understanding the internal rotation of the Sun. For β, the uncertainty for isotropic theories could be reduced to between 2×10^{-4} and 5×10^{-4} if a 2 yr or longer mission is achieved. For $(dG/dt)/G$, the asteroidal mass uncertainties are crucial, but the accuracy achievable for a 2 yr mission still seems likely to be better than 1×10^{-13} yr^{-1}.

APPENDIX: INTEGRATION OF LAGRANGIAN PERTURBATION EQUATIONS

In this appendix we develop the algebraic expressions for the perturbations of the Keplerian elements a, e, Ω, I,

$\tilde{\omega}$, and M arising from the parameters γ, β, α_1, α_2, α_3, ξ, and λ_G. The Lagrangian perturbation equations [30] are integrated using as independent variables either the time t, the eccentric anomaly E, or the true anomaly f. These variables are related by the equations

$$E - e\sin E = nt + L_0 - \tilde{\omega}, \tag{A1}$$

$$\tan\frac{f}{2} = \left(\frac{1+e}{1-e}\right)^{1/2}\tan\frac{E}{2}, \tag{A2}$$

which may be differentiated to yield the following forms useful for changing variables:

$$(1 - e\cos E)dE = ndt = (1-e^2)^{3/2}df/(1+e\cos f)^2. \tag{A3}$$

We shall first tabulate a number of useful results in terms of which perturbations of the Keplerian elements can be compactly expressed. Defining the difference $\Delta(F) = F(t) - F(t=0)$ as the total change in some quantity F between the initial instant at which the perturbation is applied (taken to be $t = 0$), and the time of observation t, we write for example

$$\Delta(f) = f - f_0, \qquad \Delta(\sin E) = \sin E - \sin E_0,$$

and so forth.

The perturbations due to β, γ, α_1, α_2, α_3, ξ, and λ_G can be expressed exactly in terms of the following integrals:

$$S_{IJ}(f) = \int_{f_0}^{f}(\sin f)^I(\cos f)^J df; \tag{A4}$$

$$F_{IJK} = \int_{f_0}^{f}(\sin f)^I(\cos f)^J(1+e\cos f)^K df; \tag{A5}$$

$$G_{IJ} = \int_{f_0}^{f}\frac{(\sin f)^I(\cos f)^J}{1+e\cos f}df; \tag{A6}$$

$$H_{IJ} = \int_{f_0}^{f}\frac{(\sin f)^I(\cos f)^J}{(1+e\cos f)^2}df; \tag{A7}$$

$$I_{IJ} = \int_{f_0}^{f}\frac{(\sin f)^I(\cos f)^J}{(1+e\cos f)^3}df. \tag{A8}$$

In writing out the specific integrals required for these calculations, it is convenient to introduce the abbreviation

$$\varepsilon = (1-e^2)^{1/2}. \tag{A9}$$

The required integrals can be reduced to well-known integrals by using trigonometric identities.

1. Perturbations due to γ

Integration of the Lagrangian perturbation equations for the contribution to $\delta\mathbf{a}_{\rm PPN}$, which is directly proportional to γ, gives the following perturbations in the orbital elements:

$$\begin{aligned}
\Delta a &= 2me[(3+e^2)S_{10} + 4eS_{11}]/\varepsilon^4,\\
\Delta e &= m[(1+3e^2)S_{10} + 4eS_{11}]/(a\varepsilon^2),\\
\Delta\Omega &= \Delta I = 0,\\
\Delta\tilde{\omega} &= m[4S_{20} - \varepsilon^2 S_{01}/e]/(a\varepsilon^2),\\
\Delta M &= -3m[1 + e^2 + e(3+e^2)\\
&\quad \times\cos f_0 + 2e^2\cos^2 f_0]n\Delta(t)/(a\varepsilon^4) - m\Delta E/a\\
&\quad + m[(1+3e^2)S_{01}/e + 2\Delta(\sin f\cos f)]/(a\varepsilon).
\end{aligned}$$

Here, expressions such as Δe represent partial derivatives with respect to the parameter; e.g., Δe is shorthand for $\partial e/\partial\gamma$.

2. Perturbations due to β

For the contribution to $\delta\mathbf{a}_{\rm PPN}$, that is proportional to β, the perturbations in the orbital elements are

$$\begin{aligned}
\Delta a &= 2me[2S_{10} + 2eS_{11}]/\varepsilon^4,\\
\Delta e &= \varepsilon^2\Delta a/2ea,\\
\Delta\Omega &= \Delta I = 0,\\
\Delta\tilde{\omega} &= m[-S_{00} - 2S_{01}/e - \Delta(\sin f\cos f)]/(a\varepsilon^2),\\
\Delta M &= -3m(1+e\cos f_0)^2 n\Delta(t)/(a\varepsilon^4)\\
&\quad + m[2S_{01}/e + \Delta(\sin f\cos f)]/(a\varepsilon).
\end{aligned}$$

3. Perturbations due to α_1

These perturbations depend on the velocity, $\mathbf{w}$, of the solar system with respect to the preferred frame. The results take their most compact form when this velocity is expressed in a coordinate system with the Z axis normal to the plane of the unperturbed planetary orbit, and the X axis in the plane of the orbit oriented to pass through the position of perihelion. If W_1, W_2, W_3 denote the ratio of the Cartesian coordinates of $\mathbf{w}$ in this coordinate system to the speed of light, then we have

$$\begin{aligned}
W_1 &= \{w_x(\cos\omega\cos\Omega - \sin\omega\sin\Omega\cos I)\\
&\quad + w_y(\cos\omega\sin\Omega + \sin\omega\cos\Omega\cos I)\\
&\quad + w_z(\sin\omega\sin I)\}/c,\\
W_2 &= \{w_x(-\sin\omega\cos\Omega - \cos\omega\sin\Omega\cos I)\\
&\quad + w_y(-\sin\omega\sin\Omega + \cos\omega\cos\Omega\cos I)\\
&\quad + w_z(\cos\omega\sin I)\}/c,\\
W_3 &= \{w_x\sin\Omega\sin I - w_y\cos\Omega\sin I + w_z\cos I\}/c.
\end{aligned} \tag{A10}$$

The perturbations due to α_1 can be expressed as follows:

$$\Delta a = 0,$$
$$\Delta e = (m/a)^{1/2}\varepsilon(-W_1 G_{20} + W_2 G_{11})/2,$$
$$\sin I \Delta\Omega = -(m/a)^{1/2} e W_3(\cos\omega G_{20} + \sin\omega G_{11})/2\varepsilon,$$
$$\Delta I = (m/a)^{1/2} e W_3(\sin\omega G_{20} + \cos\omega G_{11})/2\varepsilon,$$
$$\Delta\tilde{\omega} = (1-\cos I)\Delta\Omega + (m/a)^{1/2}\{W_1[(1+e^2)G_{11} + 2eG_{10}] - W_2[(1+e^2)G_{02} + 2eG_{01}]\}/(2e\varepsilon),$$
$$\Delta M = (m/a)^{1/2}\varepsilon^2[-W_1 G_{11} + W_2 G_{02}]/2e.$$

4. Perturbations due to α_2

$$\Delta a = a\{2e[W_1^2 S_{12} + 2W_1W_2 S_{21} + W_2^2 S_{30}] + (W_1^2 - W_2^2)F_{111} + W_1W_2(F_{201} - F_{021})\}/\varepsilon^2,$$
$$\Delta e = 3W_1^2 S_{12}/2 + W_1W_2(5S_{21} - S_{03})/2 + W_2^2\left(S_{30} + \frac{1}{2}S_{12}\right) + \frac{1}{2}W_1W_2[G_{21} - G_{03} + e(G_{20} - G_{02})] - \frac{1}{2}(W_2^2 - W_1^2)(G_{12} + eG_{11}),$$
$$\sin I\Delta\Omega = -\frac{1}{2}W_3[W_2\cos\omega G_{20} + W_1\sin\omega G_{02} + (W_1\cos\omega + W_2\sin\omega)G_{11}],$$
$$\Delta I = \frac{1}{2}W_3[-W_1\cos\omega G_{02} + W_2\sin\omega G_{20} + (W_1\sin\omega - W_2\cos\omega)G_{11}],$$
$$\Delta\tilde{\omega} = (1-\cos I)\Delta\Omega - \frac{1}{e}\left\{W_1^2 S_{03} + 2W_1W_2 S_{12} + W_2^2 S_{21} + \frac{1}{2}(W_2^2 - W_1^2)(G_{21} + S_{21}) - \frac{1}{2}W_1W_2(G_{30} + S_{30} - G_{12} - S_{12})\right\},$$
$$\Delta M = \varepsilon\left\{\frac{1}{2}e(W_1^2 - W_2^2)(H_{03} - \cos^3 f_0 H_{00}) - 3eW_1W_2(H_{30} - \sin^3 f_0 H_{00}) - 3(W_2^2 - W_1^2)(H_{02} - \cos^2 f_0 H_{00})/4 + 3eW_2^2(H_{01} - \cos f_0 H_{00}) + 3W_1W_2(H_{11} - \sin f_0\cos f_0 H_{00} + eH_{10} - e\sin f_0 H_{00})/2 - 2W_1^2(G_{02} - S_{03}/2e) - W_2^2(2G_{20} - S_{21}/e) - W_1W_2(4G_{11} - 2S_{12}/e) - W_1W_2(G_{30} + S_{30} - G_{12} - S_{12})/2e + (W_2^2 - W_1^2)(G_{21} + S_{21})/2e\right\}.$$

5. Perturbations due to α_3

We denote the components of the acceleration $\mathbf{A}$ in the XYZ coordinate system by A_1, A_2, A_3. Then we find

$$\Delta a = \frac{2a^3\varepsilon^2}{GM_\odot}\{A_1(eH_{11} - G_{10}) + A_2(eH_{20} + G_{01})\}, \qquad \Delta e = \frac{a^2\varepsilon^4}{GM_\odot}\{-A_1(eI_{10} + I_{11}) + A_2(H_{00} + I_{02} + eI_{01})\},$$
$$\sin I\Delta\Omega = \frac{A_3 a^2\varepsilon^4}{GM_\odot}\{\cos\omega I_{10} + \sin\omega I_{01}\}, \qquad \Delta I = \frac{A_3 a^2\varepsilon^4}{GM_\odot}\{-\sin\omega I_{10} + \cos\omega I_{01}\},$$
$$\Delta\tilde{\omega} = (1-\cos I)\Delta\Omega + \frac{a^2\varepsilon^4}{GM_\odot e}\{-A_1(H_{00} + I_{20}) + A_2 I_{11}\},$$
$$\Delta M = \frac{3a^2}{GM_\odot}\left\{A_1[I_{00}\varepsilon^5 - n\Delta t\varepsilon^2/(1 + e\cos f_0)]/e + (A_2\varepsilon/e)\left[\frac{1}{2}\Delta(E - e\sin E)^2 + e\Delta(\cos E) - (E_0 - e\sin E_0)\Delta(E - e\sin E) - \Delta(E)(E - e\sin E) + \frac{1}{2}\Delta(E^2)\right]\right\} - \frac{a^2\varepsilon^5}{GM_\odot}[A_1(2I_{01} - I_{20}/e - H_{00}/e) + A_2(2I_{10} + I_{11}/e)].$$

6. Perturbations due to ξ

We define the dimensionless quantity $Z = GM_G/c^2R_G$ and let N_1, N_2, N_3 denote the (XYZ) components of the vector $\boldsymbol{R}_G/R_G$. Then we have the following results:

$$\Delta a = \frac{2Za}{\varepsilon^2}\{2N_1N_2(F_{021} - F_{201}) + 2(N_2^2 - N_1^2)F_{111} - e(N_1^2S_{12} + 2N_1N_2S_{21} + N_2^2S_{30})\},$$

$$\Delta e = Z\{-N_1^2S_{12} - N_2^2S_{30} + 2N_1N_2[S_{03} + G_{03} - G_{21} + e(G_{02} - G_{20}) - 2S_{21}] + 2(N_2^2 - N_1^2)(G_{12} + eG_{11} - S_{12})\},$$

$$\sin I\Delta\Omega = 2ZN_3\{N_1(\cos\omega G_{11} + \sin\omega G_{02}) + N_2(\sin\omega G_{11} + \cos\omega G_{20})\},$$

$$\Delta I = 2ZN_3\{N_1(-\sin\omega G_{11} + \cos\omega G_{02}) + N_2(\cos\omega G_{11} - \sin\omega G_{20})\},$$

$$\Delta\tilde{\omega} = (1 - \cos I)\Delta\Omega + (Z/e)\{N_1^2(S_{03} - 2S_{21} - 2G_{21}) + N_2^2(3S_{21} + 2G_{21}) + 2N_1N_2(2S_{12} - S_{30} + G_{12} - G_{30})\},$$

$$\begin{aligned}\Delta M = Z\varepsilon\{&N_1^2(2G_{02} - S_{03}/e - eH_{03}) + 2N_1N_2(2G_{11} - S_{12}/e + 3eH_{30} - 3H_{11} - 3eH_{10})\\ &+ N_2^2(2G_{20} - S_{21}/e + eH_{03} - 3eH_{01}) - (2/e)(N_1N_2)(G_{12} - G_{30} + S_{12} - S_{30})\\ &+ (N_2^2 - N_1^2)(-2G_{21}/e - 2S_{21}/e + 3H_{02} + 2eH_{03})\} + \frac{Zn\Delta t}{\varepsilon^2}\{N_1^2(e\cos^3 f_0) + N_2^2(3e\cos f_0 - e\cos^3 f_0)\\ &+ 6N_1N_2(\sin f_0\cos f_0 + e\sin f_0 - e\sin^3 f_0) - (N_2^2 - N_1^2)(3\cos^2 f_0 + 2e\cos^3 f_0)\}.\end{aligned}$$

7. Perturbations due to $\lambda_G = -\dot{G}/G$

$$\Delta a = -2a[\Delta t(1 + e\cos f)/\varepsilon^2 - (\Delta E/n)], \qquad \Delta e = (\varepsilon^2)\Delta a/2ae, \qquad \Delta\Omega = \Delta I = 0,$$

$$\Delta\tilde{\omega} = \frac{1}{e}[\Delta(t\sin f) + (\varepsilon/m)\Delta(\cos E)],$$

$$\Delta M = \Delta(tE) + 3E_0\Delta(t) - 2\Delta(E^2)/n + 4e\Delta(E\sin E + \cos E)/n + \varepsilon\Delta(t\sin f)/e + (\varepsilon^2/ne)\Delta(\cos E).$$

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