

Monopole wall

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We construct, numerically, a solution of the SU(2) Bogomolny equations corresponding to a sheet of BPS monopoles. It represents a domain wall between a vacuum region and a region of constant energy density, and it is the smoothed-out version of the planar sheet of Dirac monopoles obtained by linear superposition.

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I. INTRODUCTION

Bogomolny-Prasad-Sommerfield (BPS) monopoles have long been of considerable interest (for reviews, see Refs. [1,2]), and recently have found a new interpretation in the context of D -branes. This D -brane connection is partly responsible for a particular interest in periodic assemblages of monopoles. For example, there have been studies of monopole chains [3], where the underlying theory (such as the Nahm transform) is well-developed. However, for monopole sheets or walls, where the fields are doubly-periodic, much less is known.

It has been suggested [4,5] that the (essentially Abelian) homogeneous solution on \mathbb{R}^3 may be viewed as a monopole sheet, and there is a sense in which such an interpretation is meaningful. If, however, one constructs a sheet of Dirac monopoles, as a double series, then the resulting field looks rather different. The purpose of this note is to present numerical evidence for the existence of an SU(2) monopole sheet which is a smoothed-out version of the Dirac sheet. As we shall see, it resembles a domain wall between a vacuum region and a region of homogeneous phase with constant energy density.

II. THE U(1) CASE: SHEET OF DIRAC MONOPOLES

Let $\mathbf{r} = (x, y, z)$ denote the usual position vector in \mathbb{R}^3 , and let P be the square lattice in the xy -plane consisting of points of the form $\mathbf{r} = (j, k, 0)$, with $j, k \in \mathbb{Z}$. Suppose we place a unit-charge Dirac monopole at each site of this lattice. The resulting magnetic field is, at least formally,

$$\mathbf{B}(x, y, z) = \frac{1}{2} \sum_{j,k \in \mathbb{Z}} (x-j, y-k, z)/r_{jk}^3, \quad (1)$$

where $r_{jk}^2 = (x-j)^2 + (y-k)^2 + z^2$. The component B_z is a positive series which converges absolutely, and the other two components B_x and B_y are conditionally convergent (except, of course, at the points of P , where \mathbf{B} is singular). For fixed (x, y) , we have $\mathbf{B} \rightarrow (0, 0, \pm\pi)$ as $z \rightarrow \pm\infty$; this was checked by numerical summation of the

series, and can also be established by a rough analytic approximation.

Similarly, we may obtain a gauge potential \mathbf{A} by taking a gauge potential for each monopole and summing these; for example,

$$\mathbf{A}(x, y, z) = \sum_{j,k \in \mathbb{Z}} \frac{(k-y, x-j, 0)}{2r_{jk}(z+r_{jk})}, \quad (2)$$

which is valid in the region $z > 0$. The asymptotic behavior of (2) is $\mathbf{A} \rightarrow \frac{\pi}{2}(-y, x, 0)$ as $z \rightarrow \infty$.

In general, if we have a doubly-periodic vector field \mathbf{B} which is smooth for $|z| \geq c$, then we can define its charge (N_-, N_+) by

$$N_{\pm} = \frac{1}{2\pi} \int_0^1 dx \int_0^1 dy B_z(x, y, \pm c); \quad (3)$$

by continuity, N_{\pm} do not change if c is increased. If \mathbf{B} is genuinely a doubly-periodic magnetic field, in other words if its gauge potential is doubly-periodic up to a gauge transformation, then N_- and N_+ will both be integers: this is the Dirac quantization condition for a magnetic field on the torus T^2 . The field (1), in view of its asymptotic behavior, clearly has charge $(-\frac{1}{2}, \frac{1}{2})$, and so it does not quite satisfy this quantization condition. (Of course, taking an array of monopoles of charge 2 would have the effect of multiplying all the expressions by 2, and so this would give a genuinely doubly-periodic magnetic field.)

Another example of a ‘‘half-doubly-periodic’’ field is the homogeneous solution

$$\mathbf{B}(x, y, z) = (0, 0, \pi), \quad \mathbf{A}(x, y, z) = \frac{\pi}{2}(-y, x, 0), \quad (4)$$

which has charge is $(\frac{1}{2}, \frac{1}{2})$. If we add Eq. (4) to Eq. (1), then we obtain a doubly-periodic magnetic field of charge $(0, 1)$, with

$$\mathbf{B} \rightarrow \begin{cases} 2\pi & \text{as } z \rightarrow \infty, \\ 0 & \text{as } z \rightarrow -\infty. \end{cases} \quad (5)$$

This is the prototype of our monopole sheet: it separates a vacuum region with energy density approximately zero (for $z \ll -1$), from a region of homogeneous phase with energy density approximately constant (for $z \gg 1$). In

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what follows, we will show that it has a non-Abelian version in which the singularities are smoothed out.

III. THE SU(2) CASE: SETUP

Let $\{A_j(x, y, z), \Phi(x, y, z)\}$ denote an SU(2) Yang-Mills-Higgs field, and define $D_j\Phi := \partial_j\Phi + [A_j, \Phi]$, $B_j := \frac{1}{2}\varepsilon_{jkl}(\partial_k A_l - \partial_l A_k + [A_k, A_l])$, as usual. We impose the global conditions:

- (a) A_j and Φ are 2×2 anti-Hermitian trace-free matrices, and are smooth on \mathbb{R}^3 ;
- (b) A_j and Φ are periodic in both x and y (actually periodic, not merely up to a gauge transformation), with unit periods.

In addition, we need boundary conditions as $z \rightarrow \pm\infty$, and these can be formulated as follows. If the restriction $\Phi_c := \Phi|_{z=c}$ is nowhere-zero (as a function of x and y), then the normalized Higgs field $\hat{\Phi}_c := \Phi_c/|\Phi_c|$ is well-defined, and is a map from the torus T^2 to the 2-sphere S^2 . Here $|\Phi|^2 := -\frac{1}{2}\text{tr}(\Phi^2)$. So $\hat{\Phi}_c$ has a degree (winding number) $N_c \in \mathbb{Z}$. If Φ_z is nowhere-zero for $z \geq c$, then by continuity the degree N_z is independent of z , and is denoted N_+ ; similarly for N_- . The boundary condition, motivated by the U(1) case, is:

- (c) $D_x\Phi \rightarrow 0$ and $D_y\Phi \rightarrow 0$ as $z \rightarrow \infty$; if $N_+ \neq 0$, then $|\Phi|/z \rightarrow 2\pi|N_+|$, and $|D_z\Phi|$ is bounded, as $z \rightarrow \infty$; if $N_+ = 0$, then $|\Phi| \rightarrow \text{const}$ and $|D_z\Phi| \rightarrow 0$ as $z \rightarrow \infty$; and similarly as $z \rightarrow -\infty$.

We say that such a field has charge $(-N_-, N_+)$.

For fields satisfying conditions (a)–(c), there is a topological lower bound on the energy. The derivation of this is analogous to the one used for monopoles localized in \mathbb{R}^3 . In order to get finite energy, we need to restrict to a finite cylinder $-L \leq z \leq L$; the condition (c) is adapted in the obvious way to become a condition at $z = \pm L$. The energy density is

$$\mathcal{E} := -\frac{1}{2}\text{tr}[(D_j\Phi)^2 + (B_j)^2]. \quad (6)$$

The first observation is that

$$N_c = \frac{1}{4\pi} \int_{z=c} \text{tr}[\hat{\Phi} B_z] dx dy, \quad (7)$$

which is a standard calculation [6]. The energy is

$$\begin{aligned} E_L &:= \int_{-L}^L dz \int dx dy \mathcal{E} \\ &= -\frac{1}{2} \int \text{tr}(D_j\Phi + B_j)^2 d^3x + \int \text{tr}(D_j\Phi \cdot B_j) d^3x. \end{aligned}$$

Assuming for simplicity that $N_+ \geq 0 \geq N_-$, and using Stokes's theorem, this leads to the inequality

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$$E_L \geq 8\pi^2 L(N_+^2 + N_-^2), \quad (8)$$

with equality if and only if the Bogomolny equations

$$D_j\Phi = -B_j \quad (9)$$

are satisfied.

There is an exact solution of (9) representing a field of charge (1, 1)—in other words, with $N_+ = 1 = -N_-$. This is obtained [4,5] by embedding the homogeneous Abelian field (4), multiplied by a factor of -2 , into SU(2):

$$\Phi = 2i\pi z \sigma^3, \quad A_j = i\pi(y, -x, 0)\sigma^3; \quad (10)$$

and then applying an SU(2) gauge transformation (which it is not hard to write down explicitly), in order to make the fields periodic. The energy density of Eq. (10) is easily read off, and is $\mathcal{E} = 8\pi^2$; so the Bogomolny bound (8) is saturated, as expected.

IV. THE SU(2) CASE: NUMERICAL SOLUTION

The problem is to investigate whether there is a SU(2) solution of the Bogomolny Eqs. (9) with charge (0, 1)—and to see what it looks like. No existence theorem is currently available, and the best that can be done here is to use a numerical approach. The idea is to discretize the system as a lattice gauge theory, and to minimize the energy E_L numerically. A minimum which saturates the lower bound (8), in other words with $E_L = 8\pi^2 L$, should then be a solution.

So we replace \mathbb{R}^3 , or rather the region $(0, 1] \times (0, 1] \times [-L, L]$, with a cubic $M \times M \times (2LM + 1)$ lattice Γ , so the lattice spacing is $h = 1/M$. The gauge potential is represented, in the standard way, by assigning an element \mathcal{A} of SU(2) to each link of Γ ; the curvature is then an element \mathcal{B} of SU(2) associated with each face. We represent the Higgs field by assigning an element ϕ of SU(2) to each site of Γ ; the covariant derivatives $D\phi$ are then elements of SU(2) associated with each link. The energy is given by the “Wilson action”

$$E_L = \frac{2}{h} \sum_{\text{faces}} (1 - \text{tr}\mathcal{B}) + \frac{2}{h} \sum_{\text{links}} (1 - \text{tr}D\phi), \quad (11)$$

which is gauge-invariant on the lattice.

The boundary conditions, which are also gauge-invariant, are as follows:

- (i) at $z = L$, we impose $\frac{1}{2}\text{tr}\phi = \cos(2\pi hL)$;
- (ii) at $z = -L$, we impose $\mathcal{B} = 1$.

These correspond, respectively, to the conditions $|\Phi| = 2\pi L$ at $z = L$, and $\mathbf{B} = \mathbf{0}$ at $z = -L$, in the continuum version.

The initial configuration was constructed by starting with the lattice version of the homogeneous solution, namely

$$\begin{aligned}\phi &= \exp(2\pi i h z \sigma^3), & \mathcal{A}_x &= \exp(\pi i h y \sigma^3), \\ \mathcal{A}_y &= \exp(-\pi i h x \sigma^3), & \mathcal{A}_z &= 1;\end{aligned}$$

then gauge-transforming so as to make these periodic; and finally adjusting by setting $\phi = 1$ for $z < 0$, and interpolating so that \mathcal{A}_x and \mathcal{A}_y go to 1 as z goes from 0 to $-L$. The resulting lattice field has charge $(0, 1)$.

Starting with this initial configuration, the energy (11) was minimized using a conjugate-gradient method, while maintaining the boundary conditions (and also, of course, the condition that \mathcal{A} and ϕ should be $SU(2)$ -valued). This was done for various values of M (or equivalently h) and L . For $M \geq 12$, the results are already of high accuracy, in the sense that the value of E_L at its minimum is within 0.5% of the Bogomolny bound, and it changes by less than that amount if M is increased. The graphs in Fig. 1 depict the minimum-energy configuration with $M = 14$ (so the lattice spacing is $h = 0.071$), and $L = 2$. Its lattice energy is $E_L = 0.9997 \times 8\pi^2 L$. The upper-left-hand graph shows the topological charge (winding number) N of $\Phi|_z$, as a function of z ; in particular, we see that the solution does indeed have charge $(0, 1)$. The norm of the Higgs field Φ (at $x = y = 1$, and as a function of z) is shown in the upper-right-hand graph; as expected, it approaches a constant value as $z \rightarrow -\infty$, and depends linearly on z as $z \rightarrow \infty$. The lower-left-hand graph shows the energy density \mathcal{E} (or rather its lattice version) summed over x and y , as a function of z . We see that the energy density tends to zero as $z \rightarrow -\infty$, approaches a constant value as $z \rightarrow \infty$, and is peaked at $z \approx 0.4$. Finally, the lower-right-hand subfigure plots the energy density at $z = 0.4$, as a function of x and y . For $|z| \geq 1$, the corresponding energy plots are essentially constant in x and y , but at the location of the monopole sheet (which here is at $z \approx 0.4$) we see the

individual monopoles (one in each fundamental cell of the lattice, of which four are shown).

V. CONCLUDING REMARKS AND OPEN QUESTIONS

It is straightforward to change the x, y -periods by scaling: in fact, if $\{\Phi, A_j\}$ is a solution with unit periods in x and y , then $\{\tilde{\Phi}(\mathbf{r}) := \lambda\Phi(\mathbf{r}/\lambda), \tilde{A}_j(\mathbf{r}) := \lambda A_j(\mathbf{r}/\lambda)\}$ is a solution with periods λ . More generally, it should be possible (although this has not been investigated) to construct analogous solutions in which the monopole sheet has a different lattice structure, for example, hexagonal. (Compare the case of the Skyrme system, where a hexagonal lattice is slightly more energetically favorable than a square lattice [1,7].)

If one takes the gauge group to be $SO(3)$ rather than $SU(2)$, then the topological classification of doubly-periodic monopoles has an additional feature, since $SO(3)$ bundles over the torus T^2 are not necessarily trivial—they are classified by \mathbb{Z}_2 . Essentially, the consequence is that the charges (N_-, N_+) of $SO(3)$ monopole sheets need not be integers, but may be half-integers. The simplest example [5] is the homogeneous solution of charge $(\frac{1}{2}, \frac{1}{2})$, analogous to Eq. (11), namely

$$\Phi^{(0)} = i\pi z \sigma^3, \quad A_j^{(0)} = \frac{1}{2}i\pi(y, -x, 0)\sigma^3; \quad (12)$$

which, in effect, lives on a nontrivial $SO(3)$ bundle over $T^2 \times \mathbb{R}$. There should exist an $SO(3)$ domain wall solution of charge $(0, \frac{1}{2})$ which is analogous to the charge $(0, 1)$ solution. More generally, there should exist sheets of charge $(p/2, q/2)$, where $p, q \in \mathbb{Z}$. If p and q are distinct, then such a solution represents a wall between two distinct homogeneous phases; one of these could be the vacuum, if the corresponding charge is zero.

The only “visible” parameters in the numerical solution presented above are those corresponding to translations in x, y , and z . (The position of the sheet, which turned out to be at $z \approx 0.4$, is in effect set by the boundary condition at $z = L$: in fact, by the fixed value of $|\Phi|$ at $z = L$.) Whether there are additional moduli is not known. For the homogeneous solutions, the moduli can be computed explicitly [4,5]: the solutions of charge $(p/2, p/2)$, with $p \in \mathbb{Z}$, depend on $4p$ parameters; and the perturbations (normalizable zero-modes) can be written down explicitly in terms of theta-functions. (Only the $p = 1$ case was presented in [5], but its generalization to $p \geq 2$ is straightforward.) This suggests that there might also be additional moduli in the general case of charge $(p/2, q/2)$.

The main tool in the analysis of BPS monopoles has been the Nahm transform [8,9]. The general pattern [10] suggests that the Nahm transform of a monopole sheet (a solution on $T^2 \times \mathbb{R}$), will be another solution on $T^2 \times \mathbb{R}$ —in other words, that monopole sheets are “self-reciprocal” under the Nahm transform. The only current evidence in

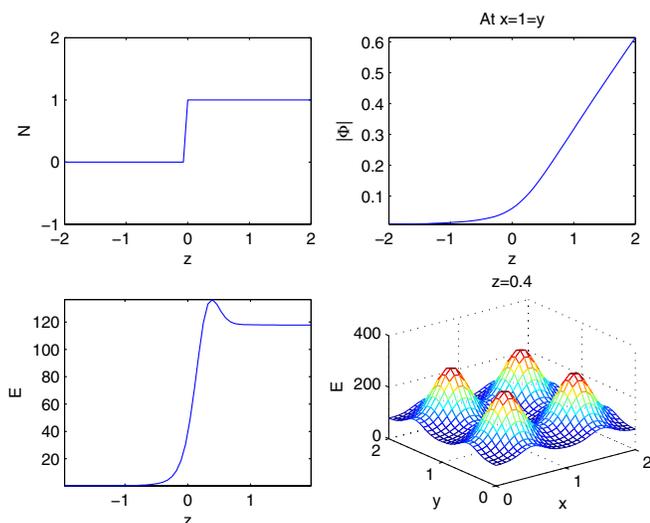


FIG. 1 (color online). Charge, Higgs field, and energy density of $SU(2)$ monopole sheet.

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favor of this comes from the class of homogeneous solutions: the U(1) case was described in [5], and the SU(2) case is similar.

Solutions (with prescribed singularities) of the Bogomolny equations on $C \times I$, where C is a Riemann surface and I is an interval, crop up in the area of supersymmetric gauge theory and branes (see, for example, Ref. [11]). The context of the present paper is rather different, but the solutions described here could be reinterpreted in D -brane language.

There are many similarities between BPS monopoles and Skyrmions (see, for example, Ref. [12]). In the Skyrme system, there is a wall-like solution analogous to a graphene sheet, and one may view fullerenelike Skyrmions as being shells constructed from such sheets [7]; in fact, the sheet separates a region of ‘‘Skyrmion core’’ from the region outside the Skyrmion. The monopole sheet described above appears at first sight to be rather different in nature, since neither side of the wall corresponds to the

field outside a BPS monopole (which, of course, is that of a Dirac monopole). It may, however, turn out to be relevant as the wall of a magnetic bag [13].

To summarize: we have constructed, numerically, a solution of the Bogomolny equations representing a sheet of BPS monopoles: in general, it is a domain wall between two regions of constant energy density, either of which can be a vacuum region. A purely-magnetic wall separating regions of different energy density would be unstable; but in the BPS system studied here, the pressure on the wall from the Higgs field is exactly opposite to that from the magnetic field, and so there is no obvious instability.

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- [1] N.S. Manton and P.M. Sutcliffe, *Topological Solitons* (Cambridge University Press, Cambridge, 2004).
 - [2] E. Weinberg and P. Yi, hep-th/0609055.
 - [3] S. Cherkis and A. Kapustin, *Commun. Math. Phys.* **218**, 333 (2001).
 - [4] K. Lee, *Phys. Lett. B* **445**, 387 (1999).
 - [5] R.S. Ward, *Phys. Lett. B* **619**, 177 (2005).
 - [6] A. Jaffe and C. Taubes, *Vortices and Monopoles* (Birkhäuser, Boston, 1980).
 - [7] R.A. Battye and P.M. Sutcliffe, *Phys. Lett. B* **416**, 385 (1998).
 - [8] W. Nahm, in *Monopoles in Quantum Field Theory*, edited by N.S. Craigie, P. Goddard, and W. Nahm (World Scientific, Singapore, 1982); in *Group Theoretical Methods in Physics*, Lecture Notes in Physics Vol. 201 (Springer, New York, 1984).
 - [9] E.F. Corrigan and P. Goddard, *Ann. Phys. (N.Y.)* **154**, 253 (1984).
 - [10] M. Jardim, *J. Geom. Phys.* **52**, 313 (2004).
 - [11] A. Kapustin and E. Witten, hep-th/0604151.
 - [12] C.J. Houghton, N.S. Manton, and P.J. Sutcliffe, *Nucl. Phys.* **B510**, 507 (1998).
 - [13] S. Bolognesi, *Nucl. Phys.* **B752**, 93 (2006).