Momentum flow correlations from event shapes: Factorized soft gluons and soft-collinear effective theory

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The distributions of two-jet event shapes contain information on hadronization in QCD. Near the two-jet limit, these distributions can be described by convolutions of nonperturbative event shape functions with the same distributions calculated in resummed perturbation theory. The shape functions, in turn, are determined by correlations of momentum flow operators with each other and with lightlike Wilson lines, which describe the coupling of soft, wide-angle radiation to jets. We observe that leading power corrections to the mean values of event shapes are determined by the correlation of a single momentum flow operator with the relevant Wilson lines. This generalizes arguments for the universality of leading power corrections based on the low-scale behavior of the running coupling or resummation. We also show how a study of the angularity event shapes can provide information on correlations involving multiple momentum flow operators, giving a window to the system of QCD dynamics that underlies the variety of event shape functions. In deriving these results, we review, develop and compare factorization techniques in conventional perturbative QCD and soft-collinear effective theory (SCET). We give special emphasis to the elimination of double counting of momentum regions in these two formalisms.

I. INTRODUCTION

In perturbative quantum chromodynamics (QCD), scattering amplitudes computed with massless partons suffer from infrared and collinear singularities, which, however, cancel in suitably-defined infrared safe cross sections. Event shapes, $e(N)$, are numbers associated with final states, N, defined so that the semi-inclusive cross sections with initial state I at center-of-mass energy Q,

$$\frac{d\sigma}{de}(Q) = \frac{1}{2Q^2}\sum_N |M_{I\to N}|^2(2\pi)^4\delta^4(Q-p_N)\delta(e-e(N))$$

(1)

are infrared safe for leptonic initial states (annihilation), and are factorizable into parton distributions and infrared safe short-distance functions for hadronic initial states. Infrared safety requires, in general, that the event shape $e(N)$ be equal for states that differ only by a rearrangement of collinear partons or by the emission or absorption of zero-momentum partons.

Event shapes have been the subject of extensive study, partly as tools to explore the evolution of QCD dynamics from weak to strong coupling (see Ref. [1] for a review). This paper will compare event shapes as treated in full QCD using factorization theorems and in the QCD effective theory, soft-collinear effective theory (SCET). In the process, we will derive a number of new results, relying on our ability to express infrared dynamics in terms of specific matrix elements in the full and effective theories.

Although infrared safe, and therefore calculable self-consistently in perturbation theory, event shape cross sections at all but the very highest energies are not usually well-approximated by perturbative results alone. The relationship between perturbative and nonperturbative contributions to fully inclusive cross sections ($e(N) = 1$) was formulated early in terms of QCD sum rules [2] based on fixed order in perturbation theory and the operator product expansion, and later in terms of its high-order behavior [3]. Eventually, it was realized that a similar analysis can be applied to many semi-inclusive cross sections that are infrared safe.

The leading nonperturbative contributions to the mean values of many event shapes enter at the level of $1/Q$ in leptonic annihilation experiments, with Q the total center-of-mass energy. This dependence was predicted from the high-order behavior of QCD perturbation theory at infrared scales, using analyses based on the running of the coupling [4–7]. These analyses naturally led to the proposal that coefficients of the $1/Q$ power are universal up to calculable factors.

Power corrections in event shapes can be measured by comparing fixed-order perturbative predictions to data from a range of Q, and $1/Q$ corrections are readily seen in such comparisons [1]. For the mean values of many event shapes, these corrections are indeed universal, in a sense that we will review below. This is a striking prediction for nonperturbative effects, based on an analysis of perturbation theory at infrared scales.

Beyond mean values, the distributions of event shapes require, as described below, convolutions of resummed perturbative cross sections with nonperturbative "event shape functions" [8,9], a different one for each event shape. The event shape functions summarize the leading power corrections for all the moments of the event shape in question.

The roles and determinations of event shape functions are in some ways analogous to those of parton distribution functions. Parton distributions can be defined in terms of matrix elements in QCD, and these matrix elements have universality properties based on factorization. As we shall review below, event shape functions may also be defined as matrix elements in QCD or SCET, which describe correlations between energy flow in different directions in the presence of lightlike color sources [9]. The universality properties of event shape functions, however, are certainly less well understood.

These issues can be addressed in the terminology of perturbative QCD, using techniques of factorization and resummation. At the present time, however, a growing body of work in jet physics relies on the language of soft-collinear effective theory to study power corrections [10,11] and perturbative resummations [12,13]. With this in mind, we will use event shapes in the two-jet limit of leptonic annihilation to compare treatments based on factorized cross sections in perturbative QCD [5,8,9] and alternatively in SCET [10,11]. We hope that this dual description will be helpful to some readers. Of course, even for this limited set of observables, a full comparison for both perturbative resummation and nonperturbative power corrections would require a lengthy discussion. In this paper, we will concentrate on power corrections implied by these closely-related formalisms.

Not surprisingly, we will find that the two approaches are equivalent at the level of leading power corrections and shape functions that are associated with soft gluon emission. In both cases, infrared dynamics will be described by matrix elements involving Wilson lines. In particular, we will see that the soft shape functions of the factorization theorem can be defined to correspond directly to the functions that describe ultrasoft gluon radiation in these observables.

Once we have discussed the formalism relating event shapes to matrix elements of momentum flow operators, we will derive a number of results of phenomenological interest. The universality relations between different $1/Q$ corrections proposed in Refs. [4–7] for mean values of event shapes can be derived from the field-theoretic shape functions, without invoking the dominance of single soft gluon emission or related assumptions on the universality of the infrared running coupling [14]. In particular, we will derive the universality of leading power corrections to the average values of a large class of event shapes entirely from factorization and the boost-invariance of products of the relevant lightlike Wilson lines.

To specify the full event shape function for an arbitrary infrared safe observable, it is in principle necessary to know all the energy flow correlators. Relations between event shape functions, however, have been conjectured for a particular class of event shapes, the angularities [15–17], which include the thrust and jet broadening. A scaling relation for the angularity shape functions follows from the assumption of negligible correlations between jet hemispheres for soft radiation. We will use insight gained from comparing factorized QCD and SCET to derive explicit relations between energy correlations and violations of the scaling rule, providing a set of measurements that relate directly to the correlations.

Before going further, it is important to emphasize that the strength of nonperturbative power corrections to any observable depends generically on the definition of perturbation theory chosen for that observable. In particular, in observable-specific perturbative schemes for the coupling based on the method of effective charges [18], as reviewed in Ref. [19], the coefficients of power corrections to average values tend to decrease markedly compared to perturbative expansions in $\overline{\text{MS}}$ definitions. This method incorporates measurements of the observable in question directly into the renormalization scheme. It therefore builds more information into perturbative expansions for these observables than is possible in conventional schemes. Certainly a better understanding of the relationships between process-specific and process-independent approaches to perturbation theory would be helpful. In the discussion of this paper, however, we will assume that the strong coupling, α_s, is defined in a process-independent fashion.

Our discussion begins in the following section with a description of the class of event shapes that we study, and specifically defines the angularity event shapes. In Sec. III, we review and relate factorization formalisms for event shapes in full QCD and SCET, exhibit the matrix elements that determine the soft gluon dynamics and define the event shape functions. We explore this relationship further in Sec. IV, with a comparison of the elimination of double counting through zero-bin subtraction in SCET [20] and eikonal subtractions in QCD. In Sec. V we introduce momentum flow operators and draw the consequences of boost invariance for their correlations in soft functions. We show that power corrections associated with jet functions are subleading for a large class of angularities.

In Section VI we apply the formalism of Sec. V first to the average values of event shapes, demonstrating the universality properties of these mean values. We go on to treat the scaling properties of angularities beyond their mean values, and we discuss the information on momentum flow correlators that is implicit in possible violations of the scaling rule proposed in Refs. [16,17].

II. TWO-JET EVENT SHAPES

In this paper, we study event shapes that can be expressed in the form

$$e = \frac{1}{Q} \sum_{i \in N} |\mathbf{p}_i^\perp| f_e(\eta_i), \tag{2}$$

where the sum is over final state particles, and the trans-

verse momenta and (pseudo-)rapidities $\eta_i = \ln \cot(\theta_i/2)$ are measured with respect to the thrust axis.[1] Two examples of these are the C-parameter and angularities (which include the thrust), which are expressed in terms of the rapidities as [16,21]

$$C = \frac{1}{Q} \sum_{i \in N} \frac{3|\mathbf{p}_i^{\perp}|}{\cosh \eta_i}, \qquad \tau_a = \frac{1}{Q} \sum_{i \in N} |\mathbf{p}_i^{\perp}| e^{-|\eta_i|(1-a)}. \quad (3)$$

In the limit $e \to 0$, the cross sections for all of these event shapes are dominated by final states consisting of two perfectly-collimated jets. For this reason, we refer to them as two-jet event shapes, and the limit $e \to 0$ as the two-jet limit. Power corrections to the distributions for these event shapes enter at the level of $1/(eQ)^n$, starting at $n = 1$, in addition to corrections suppressed by additional powers of Q [4–7,10,11].

Near $e = 0$, all such event shapes generate double logarithms in perturbation theory, which in many cases can be resummed to next-to-leading logarithms [22–24] and beyond [15]. A quantitative description of these distributions, however, requires nonperturbative input, which can be summarized in an *event shape function*, $S_{NP,e}$. The physical cross section is then given as a convolution of this shape function with the resummed perturbative function [8,9]

$$\frac{d\sigma(e, Q)}{de} = \int_0 d\zeta S_{NP,e}(\zeta Q, \Lambda) \frac{d\sigma_{\text{PT}}(e - \zeta, Q, \Lambda)}{de}. \quad (4)$$

The mass Λ, which we may think of as a scale comparable to, but larger than, Λ_{QCD}, represents the boundary between perturbative and nonperturbative contributions. Below, it will appear as the renormalization scale for certain matrix elements that define the nonperturbative function.

The nonperturbative function $S_{NP,e}(\zeta Q, \Lambda)$ for event shape e is independent of the overall momentum scale, Q. Estimating $S_{NP,e}(\zeta Q, \Lambda)$ in Eq. (4) from the plentiful data at the Z pole, for example, allows predictions of cross sections at lower and higher energies. These predictions successfully describe e^+e^- data for the thrust and jet mass distributions over a wide range of center-of-mass energies [8,9,25].

Our discussion below is focused on values of e that describe two-jet events, which dominate final states in leptonic annihilation. Extensive data on these shapes have been recorded, which may in principle be mined for information on the process of hadronization in QCD. Extensions to multijet events are possible, following Refs. [26,27]. Certainly, the reasoning below will require further development for these cases.

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[1]The thrust axis is the choice of axis that minimizes the quantity on the right-hand side of τ_a in Eq. (3) for $a = 0$.

III. FACTORIZATION IN THE TWO-JET LIMIT

In this section, we review basic results on the factorization of cross sections in the two-jet limit. We begin with an outline of the factorization analysis in full QCD, in the notation of Ref. [15], followed by a discussion based on soft-collinear effective theory [10,11]. Our discussion will apply to both perturbative and nonperturbative contributions.

Most of the results of this section have been given previously elsewhere, but we believe that a side-by-side presentation may help to shed light on both formalisms. In particular, a comparison of the formalisms will suggest the importance in both cases of the elimination of double counting. This will be the subject of Sec. IV, where we make use of the SCET discussion of "zero-bin subtractions" in Ref. [20].

A. QCD matrix elements

In the two-jet limit, the differential cross section (distribution) for a two-jet event shape e factorizes into a convolution of functions that characterize the jets with a "soft" function that describes wide-angle gluon emission [15],

$$\frac{d\sigma}{de} = \sigma_0(Q) \int de_n de_{\bar{n}} de_S \delta(e - e_n - e_{\bar{n}} - e_S)$$
$$\times J_n(Q, e_n) J_{\bar{n}}(Q, e_{\bar{n}}) S_{n\bar{n}}(e_S Q) + \mathcal{O}(e^0), \quad (5)$$

where σ_0 carries the overall dimensions, and can be defined as the Born cross section to lowest order in α_s. In perturbation theory this cross section behaves as the order $1/e$ times logarithms as e vanishes, with contributions at order e^0 from wide-angle three-jet events.

In Ref. [15], the perturbative resummation of the angularities was studied, with the functions in Eq. (5) defined in terms of QCD matrix elements. We may start by defining a set of incoming path-ordered exponentials or Wilson lines,

$$\Phi_{\xi_c}^{(\text{f})}(z) = P \exp\left[ig \int_{-\infty}^0 d\lambda \xi_c \cdot \mathcal{A}^{(\text{f})}(\lambda \xi_c + z) \right], \quad (6)$$

where f labels a color representation. The vector ξ_c, which defines the path of the ordering, was taken in Ref. [15] to be off the light-cone, at least to start.

Following Ref. [15], the jet functions of Eq. (5) are defined in terms of matrix elements

$$J_c^{\mu}(Q, e_{J_c}) = \frac{2}{Q^2} \frac{(2\pi)^6}{N_C} \sum_{N_{J_c}} \text{Tr}[\gamma^{\mu} \langle 0| \Phi_{\xi_c}^{(\text{q})\dagger}(0) q(0) |N_{J_c}\rangle$$
$$\times \langle N_{J_c}| \bar{q}(0) \Phi_{\xi_c}^{(\text{q})}(0) |0\rangle] \delta(e_{J_c} - e(N_{J_c}))$$
$$\times \delta(Q - \omega(N_{J_c})) \delta^2(\hat{n}_{J_c} - \hat{n}(N_{J_c})), \quad (7)$$

where $c = n, \bar{n}$ labels the direction of the jet as above. The jet functions are constructed from the squared amplitudes for the quark (or other partonic) field to produce states N_{J_c}

with total energy Q, with a momentum whose direction $\hat{n}$ is in a fixed direction $\hat{n}_{J_c}$, and with a fixed contribution $e_{J_c} \ll 1$ to the event shape in question. The amplitudes are rendered gauge invariant by multiplying the partonic fields by ordered exponentials (6) in the ξ_c directions and in the quark representation.

To define the soft function, we introduce "eikonal" cross sections

$$\bar{\sigma}^{(\mathrm{eik})}(\mu, e) \equiv \frac{1}{N_C} \sum_{N_{\mathrm{eik}}} \langle 0 | \Phi_{\bar{n}}^{(\bar{q})\dagger}(0) \Phi_n^{(q)\dagger}(0) | N_{\mathrm{eik}} \rangle$$
$$\times \langle N_{\mathrm{eik}} | \Phi_n^{(q)}(0) \Phi_{\bar{n}}^{(\bar{q})}(0) | 0 \rangle \delta(e - e(N_{\mathrm{eik}})), \tag{8}$$

in which final states N_{eik} are produced by products of outgoing Wilson lines in directions n^μ and $\bar{n}^\mu$, which we may take to be opposite-moving. These are defined by analogy to Eq. (6) with the "time" variable λ running from zero to infinity. The eikonal cross section, Eq. (8) must be renormalized, with scale μ. It provides a good approximation for soft radiation that is not collinear to these vectors, but it also contains collinear-singular radiation parallel to the directions of the lines. The collinear enhancements are already taken into account in the jet function, and the soft function S in Eq. (5) must be defined in such a way as not to double-count these regions. In fact, it is easier than it might seem to avoid double-counting. This is because we can apply the same factorization to the eikonal as to the full cross section, factoring the *same* soft function S from a set of eikonal jet functions, which can themselves be defined in terms of matrix elements as

$$J_c^{(\mathrm{eik})}(Q, e_c) \equiv \frac{1}{N_C} \sum_{N_c^{(\mathrm{eik})}} \langle 0 | \Phi_{\xi_c}^{(f_c)\dagger}(0) \Phi_{\beta_c}^{(f_c)\dagger}(0) | N_c^{(\mathrm{eik})} \rangle$$
$$\times \langle N_c^{(\mathrm{eik})} | \Phi_{\beta_c}^{(f_c)}(0) \Phi_{\xi_c}^{(f_c)}(0) | 0 \rangle$$
$$\times \delta(e_c - e(N_c^{(\mathrm{eik})})), \tag{9}$$

where the roles of the quark fields are taken by recoilless, lightlike Wilson lines.

We first observe that there is a certain ambiguity in the separation of jet and soft functions. The ambiguity is exhibited clearly by a Laplace transform, where large ν is conjugate to small e. Under the Laplace transform, $d\sigma/de$ in Eq. (5) factorizes into a simple product of jet and soft functions,

$$\tilde{\sigma}(Q, \nu) = \int_0^\infty de\, \mathrm{e}^{-\nu e} \frac{d\sigma(Q)}{de}$$
$$= \sigma_0(Q) \tilde{J}_n(Q, \nu) \tilde{J}_{\bar{n}}(Q, \nu) \tilde{S}(Q/\nu). \tag{10}$$

Notice that dependence on the upper limit of the integral over e is exponentially suppressed at large ν. The product on the right in (10) can be treated in different ways, depending on the task at hand. The potential sources of

double counting are eliminated in this case by defining $\tilde{S}(Q/\nu)$ as the transform-space ratio of the eikonal cross section to the product of eikonal jets,

$$\tilde{S}(Q/\nu) = \frac{\tilde{\sigma}^{(\mathrm{eik})}(Q, \nu)}{\tilde{J}_n^{(\mathrm{eik})}(Q, \nu) \tilde{J}_{\bar{n}}^{(\mathrm{eik})}(Q, \nu)}. \tag{11}$$

In Eq. (11), the eikonal cross section and eikonal jet functions approximate well the wide-angle soft radiation in the full cross section and in the jet functions, respectively. Thus the inverse eikonal jet functions in Eq. (11) cancel the contributions of the soft radiation in the partonic jet functions in Eq. (10). When expanded in the coupling, the denominators can be reinterpreted as sets of nested subtractions.

For the study of power corrections [9], it will be more useful to implement a slightly different organization, found by simply shifting the factors $\tilde{J}_n^{(\mathrm{eik})}(Q, \nu) \tilde{J}_{\bar{n}}^{(\mathrm{eik})}(Q, \nu)$ from the soft function to the jets:

$$\tilde{\sigma}(Q, \nu) = \sigma_0(Q) \tilde{\mathcal{J}}_n(Q, \nu) \tilde{\mathcal{J}}_{\bar{n}}(Q, \nu) \tilde{\sigma}^{\mathrm{eik}}(Q, \nu), \tag{12}$$

where

$$\tilde{\mathcal{J}}_n(\nu) = \frac{\tilde{J}_n(Q, \nu)}{\tilde{J}_n^{(\mathrm{eik})}(Q, \nu)}, \qquad \tilde{\mathcal{J}}_{\bar{n}}(Q, \nu) = \frac{\tilde{J}_{\bar{n}}(Q, \nu)}{\tilde{J}_{\bar{n}}^{(\mathrm{eik})}(Q, \nu)}. \tag{13}$$

The eikonal subtractions serve the same role as above in Eq. (11), but now all double counting is subtracted from the jet functions. We will show in Sec. IV that this method of subtraction is directly related to the "zero-bin subtraction" scheme [20] in SCET. But first let us review the factorization of jet cross sections from the perspective of SCET.

B. Factorization for event shapes in SCET

The SCET analysis begins with the expression for the distribution of event shape e in QCD (the "full theory" from an effective theory point of view),

$$\frac{d\sigma}{de} = \frac{1}{2Q^2} \sum_N |\langle N | J^\mu(0) | 0 \rangle L_\mu|^2 (2\pi)^4 \delta^4(Q - p_N)$$
$$\times \delta(e - e(N)), \tag{14}$$

where $J^\mu = \bar{q} \Gamma^\mu q$ is the production current in QCD. The vector L_μ is the leptonic part of the amplitude for $\mathrm{e}^+\mathrm{e}^- \to \bar{q}q$. SCET organizes QCD in an expansion in powers of a parameter $\lambda \sim \sqrt{\Lambda_{\mathrm{QCD}}/Q}$ [28,29]. The modes in the effective theory we will use are collinear quarks and gluons, $\xi_{n,\bar{n}}$ and $A_{n,\bar{n}}$, and ultrasoft (usoft) gluons, A_{us}. These modes are distinguished by the scaling of the light-cone components $p = (n \cdot p, \bar{n} \cdot p, p_\perp)$ of their momenta, defined with respect to light-cone vectors $n, \bar{n} = (1, \mathbf{0}_\perp, \pm 1)$. Ultrasoft momenta scale as $p_{\mathrm{us}} \sim Q(\lambda^2, \lambda^2, \lambda^2)$. Collinear modes can have momenta with one of two possible scalings: $p_n \sim Q(\lambda^2, 1, \lambda)$ or $p_n \sim Q(\lambda^4, 1, \lambda^2)$, and similarly for $\bar{n}$-collinear momenta $p_{\bar{n}}$. The theory with collinear modes

with the first scaling is called $SCET_I$, and with the second, $SCET_{II}$ [30]. The typical transverse momenta of the collinear particles in the final state determines the correct choice of scalings. (For example, very narrow jets with $p_\perp \sim \Lambda_{QCD}$ must be treated in $SCET_{II}$.) We begin by matching QCD onto $SCET_I$, and consider $SCET_{II}$ when we wish to account for narrower jets.

At leading (zeroth) order in the expansion in λ, the full theory current J^μ matches onto the $SCET_I$ operators [31],

$$j^\mu_{\omega,\omega'} = \bar{\chi}_{n,\omega} \Gamma^\mu \chi_{\bar{n},\omega'}, \qquad (15)$$

where ω, ω' denote label momenta, and the jet fields $\chi_{n,\bar{n}}$ are given by

$$\chi_{n,\omega} = [W_n^\dagger \xi_n]_\omega. \qquad (16)$$

Here W_n is a Wilson line of collinear gluons,

$$W_n(z) = P \exp\left[ig \int_{-\infty}^0 ds\, \bar{n} \cdot A_n(\bar{n}s + z) \right]. \qquad (17)$$

Label momenta are the "large" pieces of collinear momentum. The momentum of a collinear mode splits into this label piece and a residual piece. More precisely, $p_n = \tilde{p}_n + k$, where the label momentum $\tilde{p}_n = \bar{n} \cdot \tilde{p}_n \frac{n}{2} + \tilde{p}_n^\perp$ contains only the $\mathcal{O}(Q)$ piece of $\bar{n} \cdot p_n$ and the $\mathcal{O}(Q\lambda)$ piece of $p_n^\perp$, and the residual momentum k is of order Λ_{QCD} in all components. A collinear field with a label ω creates and destroys only modes with small $\mathcal{O}(\Lambda_{QCD})$ fluctuations about the momentum ω.

The matching between the full and effective theories is performed by matching matrix elements at a scale μ:

$$\langle J^\mu \rangle_{QCD}(\mu) = C(\omega, \omega'; \mu) \langle j^\mu_{\omega,\omega'} \rangle_{SCET}(\mu), \qquad (18)$$

where the labels ω, ω' are summed over. The coefficient $C(\omega, \omega'; \mu)$ is the Wilson coefficient in this matching. The combinations of fields and Wilson lines in these expressions bear a close resemblance to the jet functions of Eq. (7).

We can remove the coupling between ultrasoft gluons and the collinear fields in the $SCET_I$ Lagrangian via the field redefinitions [32],

$$\xi_n = Y_n^\dagger \xi_n', \qquad A_n = Y_n^\dagger A_n' Y_n, \qquad W_n = Y_n^\dagger W_n' Y_n, \qquad (19)$$

and similarly for the $\bar{n}$-collinear fields, using the "outgoing" [33] Wilson line of ultrasoft gluons,

$$Y_n(z) = P \exp\left[ig \int_0^\infty ds\, n \cdot A_{us}(ns + z) \right]. \qquad (20)$$

For instance, the term containing the collinear quark-usoft gluon interaction becomes

$$\bar{\xi}_n in \cdot D_{us} \xi_n \longrightarrow \bar{\xi}_n' in \cdot \partial \xi_n', \qquad (21)$$

where $iD_{us} = i\partial + gA_{us}$. With this separation, we have established an effective theory that captures the dynamics

in the full theory for the two-jet event shape distributions in the two-jet limit. The identification of jet and soft quanta precisely matches the "leading regions" of the full theory, as identified by analysis of momentum-space integrals for arbitrary diagrams [34]. This analysis, of course, is process-dependent, and the validity of this SCET holds up to the same corrections that apply to Eq. (5), and is improvable by adding more jets, for example.

The redefined jet production current becomes

$$j^\mu_{\omega,\omega'} = \bar{\chi}_{n,\omega}' Y_n \Gamma^\mu Y_{\bar{n}}^\dagger \chi_{\bar{n},\omega'}', \qquad (22)$$

where the primes on the redefined fields χ' refer to the use of ξ' and W' in Eq. (16). This leads to a factorization of the differential cross section in Eq. (14), analogous to Eq. (5),

$$\frac{d\sigma}{de} = |C(Q, -Q; \mu)|^2 \int de_J \sigma_J(e_J; \mu) S(e - e_J; \mu), \quad (23)$$

where the function σ_J contains the collinear fields and states:

$$\sigma_J(e_J; \mu) = \frac{1}{2Q^2} \frac{L^2}{3} \sum_{N_{J_n} N_{J_{\bar{n}}}} |\langle N_{J_n} N_{J_{\bar{n}}} | \bar{\chi}_{n,Q}' \Gamma^\mu \chi_{\bar{n},-Q}' |0\rangle|^2 (\mu)$$
$$\times \delta(e_J - e(N_{J_n} N_{J_{\bar{n}}})), \qquad (24)$$

where L^2 is the spin-averaged, squared leptonic amplitude, and where the soft function S contains the usoft fields:

$$S(e; \mu) = \frac{1}{N_C} \text{Tr} \sum_{X_u} |\langle X_u | T[Y_n Y_{\bar{n}}^\dagger] |0\rangle|^2 (\mu) \delta(e - e(X_u)). \qquad (25)$$

We have split up the final state N into the jets $N_{J_{n,\bar{n}}}$ and the usoft sector X_u, imagining them to live in clearly separated regions. The notation in Eq. (23) setting the labels ω, ω' of the collinear fields to $\pm Q$ means that the fields must create a quark and an antiquark jet with total label momenta $n \cdot \tilde{p}_{J_n} = \bar{n} \cdot \tilde{p}_{J_{\bar{n}}} = Q$, and $\tilde{p}_{J_n}^\perp = \tilde{p}_{J_{\bar{n}}}^\perp = 0$.

Before we proceed, let us consider briefly the scale dependence of the jet and soft functions. The matching from QCD to SCET is done at the scale $\mu = Q$, minimizing large logarithms in the Wilson coefficient $C(Q, -Q; \mu)$. The collinear matrix element is naturally evaluated at a scale $\mu_c \sim Q\lambda$, and the soft function at $\mu_s \sim Q\lambda^2$, to minimize logarithms in those functions. The running between these scales and $\mu = Q$ is achieved by renormalization group evolution. The calculations required to do this in SCET are described in Refs. [12,13]. In this paper, we focus not on the details of the perturbative matching and evolution, but rather on the properties of matrix elements at the scale $\mu \sim \Lambda_{QCD}$ that must be treated as nonperturbative quantities.

For observables dominated by fairly wide jets with $p_\perp \sim Q\lambda \sim \sqrt{\Lambda_{QCD}Q}$, the collinear jet function at $\mu_c = Q\lambda$ may be evaluated in perturbation theory, leaving only

the soft function at $\mu_s = \Lambda_{\text{QCD}}$ as a nonperturbative function. For observables in which narrow jets with $p_\perp \sim \Lambda_{\text{QCD}}$ contribute heavily, the collinear matrix elements are naturally evaluated at the scale $\mu_c = \Lambda_{\text{QCD}}$. Between the scales $Q\lambda$ and $Q\lambda^2$, the theory SCET$_I$ at the higher scale must be matched onto SCET$_{II}$ at the lower scale. The form of the collinear operator (22) remains the same; only the momentum scaling of the collinear fields changes. A collinear matrix element in SCET$_{II}$ is a nonperturbative quantity. In Sec. V, we compare the relative sizes of nonperturbative corrections to event shape distributions from the soft and jet functions. For the observables of most interest to us, the power corrections from the soft function will dominate.

Returning now to the jet function in Eq. (24), we can further separate the matrix elements involving the jet quanta into the form:

$$\sigma_J(e_J; \mu) = \frac{1}{2Q^2} \frac{L^2}{3} \text{Tr} \bigg[\Gamma_\mu^\dagger \sum_{N_{J_n}} \langle 0 | \chi'_{n,Q} | N_{J_n} \rangle$$
$$\times \langle N_{J_n} | \bar{\chi}'_{n,Q} | 0 \rangle \Gamma^\mu \sum_{N_{J_{\bar{n}}}} \langle 0 | \bar{\chi}'_{\bar{n}, -Q} | N_{J_{\bar{n}}} \rangle$$
$$\times \langle N_{J_{\bar{n}}} | \chi'_{\bar{n}, -Q} | 0 \rangle \bigg] (\mu) \delta(e_J - e(J_n) - e(J_{\bar{n}})).$$
$$(26)$$

The individual squares of matrix elements are closely related to the jet functions in factorization-based treatments of semi-inclusive cross sections, Eq. (7) [15]. In particular, the operator content, including the presence of the Wilson lines W_n, is essentially equivalent. The choice of vector [ξ_c in Eq. (7)] is somewhat different. Another difference is that here the collinear fields are fully separated from usoft partons, so that the sums over explicit jet final states $|N_{J_{n,\bar{n}}}\rangle$ contain no usoft lines at all, while correspondingly, the virtual states that enter the jet matrix elements have no soft lines. In the soft function as well, jet lines appear neither in the final state $|X_u\rangle$ nor in the matrix elements.

In SCET, this bookkeeping is built in by a "zero-bin subtraction" prescription for collinear fields [20]. For each collinear field in the SCET Lagrangian, there is a sum over all label momenta, which must always be nonzero, to avoid overlap with usoft modes. Thus, regions of collinear loop diagrams or phase space integrals where a collinear particle can become ultrasoft must always be subtracted off.

Each particle in the final state may be placed in a bin determined by its label momentum, leaving it with a residual momentum inside the bin, while those with zero label fall into the zero bin, and are assigned to the usoft sector X_u. The couplings of usoft particles to the states $N_{J_{n,\bar{n}}}$ have already been factored out and accounted for in the soft function S.

A natural question at this point is whether the jet and soft functions so defined are individually infrared safe. The answer is yes, because infrared safety does not require Lorentz invariance, only the hermiticity of the interaction Hamiltonian, and a sufficiently smooth shape function [35]. In our case, the cancellation of soft and collinear singularities in leptonic annihilation processes can be carried out for fixed values of one of the components of light-cone momentum [34]. We thus expect infrared safety for each of the SCET jet and soft functions.

IV. ELIMINATING DOUBLE COUNTING

In this section, we continue our comparison of the SCET treatment of event shapes to analyses based directly on factorization [9,15], concentrating on the elimination of double counting. The observations below are relatively simple, but to our knowledge they have not yet been made in the literature in this context.

Proofs of factorization in perturbative QCD typically make use of Ward identities and subtractions to avoid double counting of leading configurations while maintaining infrared safety [15,36]. Unlike their SCET analogs, the matrix elements in Sec. III above have no restrictions on their momentum integrals. The jet functions thus include a considerable contribution from what in SCET are classified as usoft gluons.[2] The soft function S was correspondingly defined to avoid double-counting of collinear gluons by the soft function, and wide-angle soft gluons by the jet functions. As noted above, the soft function is constructed from Wilson lines in a manner precisely analogous to the SCET soft function, Eq. (25). From Eqs. (12) and (13), with double-counted soft modes subtracted from the collinear jets, we see that the eikonal cross section of Eq. (8) is exactly the same as the SCET soft function (25), except that the former does not include an explicit limitation to usoft final states, X_u.

The subtraction of double-counted soft modes from the jet functions is precisely what is achieved in SCET by the zero-bin subtraction for collinear fields (cf. Eqs. (74) and (75) in Ref. [20]). Although there is a restriction to ultrasoft states in (25), the phase space integrals implicit in the sum still cover the entire region of momentum space, so that the eikonal soft function in QCD and the SCET shape function (25) are practically equivalent. It is not inconsistent to integrate up to infinite momenta in a usoft integral. One is first taking the limit $\lambda \to 0$, i.e. $Q \to \infty$, then integrating up to infinity the usoft momenta, which remain always formally much smaller than Q.[3] It is important only

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[2]They also include quanta collinear to the opposite-moving jet. In SCET, these quanta are simply not present in the jet lines. In the factorized jet functions of Eq. (7), although present they do not produce collinear singularities because the Wilson lines Φ_{ξ_c} are defined with respect to vectors ξ_c that are not lightlike.

[3]We thank A. Manohar for discussions on this point. See related remarks in Ref. [11].

that the soft functions for the observables we calculate are dominated by those particles which have "truly" usoft momenta. Below, we will examine the validity of this assumption for the event shapes we consider. First, let us elaborate on the connection between double-counting subtraction procedures in QCD and SCET.

A. Zero-bin subtractions in factorization

The leading-order (in λ) Lagrangian of SCET contains several parts (see, e.g., Ref. [32]). First, the purely ultrasoft Lagrangian for quarks and gluons is identical to that in QCD:

$$\mathcal{L}_{\text{us}} = \bar{q}_{\text{us}} i \slashed{D}_{\text{us}} q_{\text{us}} - \frac{1}{2} G^{\mu\nu} G_{\mu\nu} + \mathcal{L}_{\text{us}}^{\text{g.f.}}, \quad (27)$$

where the ultrasoft field strength, $G^{\mu\nu} = \frac{i}{g}[D_{\text{us}}^{\mu}, D_{\text{us}}^{\nu}]$, is given in terms of the ultrasoft covariant derivative, $iD_{\text{us}}^{\mu} = i\partial^{\mu} + gA_{\text{us}}^{\mu}$. The usoft gauge fixing and ghost terms are represented by $\mathcal{L}_{\text{us}}^{\text{g.f.}}$. Meanwhile, the Lagrangian for collinear quarks is

$$\mathcal{L}_c^{(q)} = \bar{\xi}_{n,p'}[in \cdot D_{\text{us}} + gn \cdot A_{n,q} + (\mathcal{P}_\perp + g\slashed{A}_{n,q}^\perp) \\ \times W \frac{1}{\bar{\mathcal{P}}} W^\dagger (\mathcal{P}_\perp + g\slashed{A}_{n,q'}^\perp)] \frac{\bar{\slashed{n}}}{2} \xi_{n,p}, \quad (28)$$

where $\bar{\mathcal{P}}$ and $\mathcal{P}_\perp$ are label momentum operators which pick out the $\mathcal{O}(Q)$ piece of the $\bar{n} \cdot p$ component and the $\mathcal{O}(Q\lambda)$ piece of the $p_\perp$ component of the collinear momenta. Here W is the Wilson line defined in Eq. (17), organizing collinear gluons $\bar{n} \cdot A$, each of which are leading power in λ [37]. There is also an implicit sum over all labels and conservation of label momentum in each term [37]. Finally, the collinear gluon Lagrangian is

$$\mathcal{L}_c^{(g)} = \frac{1}{2g^2} \text{Tr}\{[i\mathcal{D}^\mu + gA_{n,q}^\mu, i\mathcal{D}^\nu + gA_{n,q'}^\nu]^2\} + \mathcal{L}_c^{\text{g.f.}}, \quad (29)$$

where $i\mathcal{D}^\mu = \frac{n^\mu}{2}\bar{\mathcal{P}} + \mathcal{P}_\perp^\mu + \frac{\bar{n}^\mu}{2}in \cdot D_{\text{us}}$, and $\mathcal{L}_c^{\text{g.f.}}$ contains the collinear gauge fixing and ghost terms.

The component $n \cdot A_{\text{us}}$ of the usoft gluon field appears in the collinear quark and gluon Lagrangians above. The field redefinition (19) removes this interaction, through the identities $in \cdot D_{\text{us}} Y_n^\dagger = 0$ and $i\mathcal{D}^\mu = Y_n^\dagger i\mathcal{D}_{(0)}^\mu Y_n$, where $i\mathcal{D}_{(0)}^\mu = \frac{n^\mu}{2}\bar{\mathcal{P}} + \mathcal{P}_\perp^\mu + \frac{\bar{n}^\mu}{2}in \cdot \partial$. This gives the redefined collinear quark Lagrangian

$$\mathcal{L}_c^{\prime(q)} = \bar{\xi}_{n,p'}'[in \cdot \partial + gn \cdot A_{n,q}' \\ + (\mathcal{P}_\perp + g\slashed{A}_{n,q}'^\perp) W \frac{1}{\bar{\mathcal{P}}} W^\dagger (\mathcal{P}_\perp + g\slashed{A}_{n,q'}'^\perp)] \frac{\bar{\slashed{n}}}{2} \xi_{n,p}', \quad (30)$$

and collinear gluon Lagrangian

$$\mathcal{L}_c^{\prime(g)} = \frac{1}{2g^2} \text{Tr}\{[i\mathcal{D}_{(0)}^\mu + gA_{n,q}'^\mu, i\mathcal{D}_{(0)}^\nu + gA_{n,q'}'^\nu]^2\} \\ + \mathcal{L}_c^{\prime\text{g.f.}}. \quad (31)$$

It is in the theory described by the redefined Lagrangian $\mathcal{L}_{(c)}'$ that we derived factorization for the two-jet event shape distributions. As given in Eq. (23), the full distributions are convolutions of soft and separate jet functions for the two jets, which are easily abstracted from Eq. (26). For example, in terms of the collinear SCET quark field, ξ_n' in (30), we can define a jet function $J_n^c(e)$, as

$$J_n^c(e)\left(\frac{\slashed{n}}{2}\right) = \sum_{N_{J_n}} |\langle N_{J_n} | [\bar{\xi}_n' W_n']_{p_n} |0\rangle|_{\mathcal{L}_c'}^2 \delta(e - e(N_{J_n})), \quad (32)$$

where the total label momentum of the collinear fields must equal the label momentum $\tilde{p}_n$ of the states N_{J_n}, which are free of quanta in the usoft sector. The right-hand side is proportional to the matrix $\slashed{n}/2$, as shown, and $J_n^c(e)$ is a remaining scalar function. Here we include a subscript on the matrix elements to emphasize the Lagrangian in question. This jet function contains contributions only from truly collinear particles with nonzero label momenta, and all usoft particle contributions are accounted for in the soft function S. Restricting collinear label momenta to nonzero values avoids double-counting of usoft contributions in these two functions. It may be convenient, however, to allow integrals over collinear momenta to include the "zero-bin". One must then define a prescription to subtract it out again. This can be achieved by the zero-bin subtraction procedures illustrated by Manohar and Stewart in Ref. [20], or by the method of factoring out eikonal jet functions described by Ref. [15].

In order to establish the connection between the zero-bin and eikonal jet methods for subtraction, let us take another look at the collinear quark Lagrangian in SCET, now with sums over labels explicit:

$$\mathcal{L}_c^{\prime(q)}(x) = \sum_{\tilde{p}, \tilde{p}' \neq 0} e^{i(\tilde{p}' - \tilde{p}) \cdot x} \bar{\xi}_{n,p'}'(x) \left[in \cdot \partial \\ + g \sum_{\tilde{q} \neq 0} e^{-i\tilde{q} \cdot x} n \cdot A_{n,q}'(x) \right] \frac{\bar{\slashed{n}}}{2} \xi_{n,p}'(x). \quad (33)$$

We have dropped for now the terms with $\slashed{A}^\perp$ gluons, since they are irrelevant for the remainder of this discussion.[4] The tildes denote label momenta containing only the large $\mathcal{O}(Q)$ and $\mathcal{O}(Q\lambda)$ pieces, $\tilde{p}^\mu = \bar{n} \cdot \tilde{p}\frac{n^\mu}{2} + \tilde{p}_\perp^\mu$, and the remaining x dependence of collinear fields fluctuates on a scale described by residual momenta k of order $Q\lambda^2$. The sums over collinear labels are restricted to nonzero values to avoid double counting the usoft modes, and we have shown explicitly the phases that enforce label momentum

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[4]The perp gluons in the zero-bin overlap with usoft gluons, but these interact with collinear fields only at subleading order in λ.

conservation. As in the zero-bin prescription of Ref. [20], the sums over labels can be allowed to include the zero bin if we add a term that subtracts it out again. Let us worry about this only for the gluon field, since the interaction of usoft quarks with collinear fields is subleading in λ. These steps can be achieved beginning with a Lagrangian $\mathcal{L}''$ that includes zero-bin collinear gluons:

$$\mathcal{L}_c''^{(q)}(x) = \sum_{\tilde{p}, \tilde{p}' \neq 0} e^{i(\tilde{p}' - \tilde{p}) \cdot x} \bar{\xi}_{n,p'}''(x) \left[in \cdot \partial \right.$$
$$\left. + g \sum_{\tilde{q}} e^{-i\tilde{q} \cdot x} n \cdot A_{n,q}''(x) \right] \frac{\bar{\slashed{n}}}{2} \xi_{n,p}''(x). \quad (34)$$

We then perform a field redefinition on (nonzero-bin) collinear fields:

$$\xi_{n,p}'' = U_n^\dagger \xi_{n,p}', \qquad A_{n,p}'' = U_n^\dagger A_{n,p}' U_n, \qquad (35)$$
$$W_n'' = U_n^\dagger \tilde{W}_n' U_n,$$

where U_n is a "zero-bin" Wilson line:

$$U_n(x) = P \exp\left[ig \int_0^\infty ds\, n \cdot A_{n,0}''(ns + x) \right], \quad (36)$$

mimicking the field redefinition (19) with the usoft Wilson line. The Lagrangian obtained from $\mathcal{L}_c''^{(q)}$ by the redefinition (35) is precisely $\mathcal{L}_c'^{(q)}$, with the $\tilde{q} = 0$ term subtracted off. The redefinition of the gluon field removes the zero-bin gluons from the corresponding collinear gluon Lagrangian. Note, however, that it does not yet remove zero-bin $\bar{n} \cdot A_n$ fields in W''. Thus, the Wilson line $\tilde{W}'$ is not yet W', when the latter is defined to be free of zero-bin gluons.

The jet function $J_n^c(e)$ of Eq. (32) is, as indicated, calculated in the theory described by $\mathcal{L}_c'$ in order to avoid double-counting the usoft modes. However, if we for convenience decide to include the zero-bin in all the collinear momentum integrals, we are actually calculating a slightly different jet function, J_n, in the theory described by $\mathcal{L}_c''$. Via the field redefinition (35) we can move back to the Lagrangian $\mathcal{L}_c'$, but then we must include the U_n Wilson line in the current:

$$J_n(e)\left(\frac{\slashed{n}}{2}\right) = \sum_{N_{J_n}} |\langle N_{J_n} | [\bar{\xi}_n'' W_n'']_{p_n} |0\rangle|^2_{\mathcal{L}_c''} \delta(e - e(N_{J_n}))$$
$$= \sum_{N_{J_n}} |\langle N_{J_n} | [\bar{\xi}_n' \tilde{W}_n']_{p_n} U_n |0\rangle|^2_{\mathcal{L}_c'} \delta(e - e(N_{J_n})). \quad (37)$$

In the Lagrangian $\mathcal{L}_c'$, however, the zero-bin gluons do not interact with the collinear fields at all. Once the $\bar{n} \cdot A_{n,0}$ zero-bin gluons have been removed from $\tilde{W}'$, this jet function factorizes into "purely" collinear and "zero-bin" parts. To accomplish this, let us also split the collinear Wilson line $\tilde{W}_n'$ into its zero-bin and purely collinear parts,

$$\tilde{W}_n'(z) = W_n'(z)\Omega_n(z), \quad (38)$$

where

$$\Omega_n(z) = P \exp\left[ig \int_{-\infty}^0 ds\, \bar{n} \cdot A_{n,0}(\bar{n}s + z) \right], \quad (39)$$

and W_n' is the collinear Wilson line with restrictions to nonzero label momenta on all gluons. Then the alternative ($\mathcal{L}''$) jet function factorizes into

$$J_n(e)\left(\frac{\slashed{n}}{2}\right) = \sum_{N_{J_n^c}, N^{(\text{eik})}} |\langle N_{J_n^c} | [\bar{\xi}_n' W_n']_{p_n} |0\rangle\langle N^{(\text{eik})} | \Omega_n U_n |0\rangle|^2_{\mathcal{L}_c'}$$
$$\times \delta(e - e(N_{J_n^c}) - e(N^{(\text{eik})}))$$
$$= \int de_c J_n^c(e_c) J_n^{(\text{eik})}(e - e_c)\left(\frac{\slashed{n}}{2}\right), \quad (40)$$

where J_n^c is the jet function calculated in the purely-collinear theory described by $\mathcal{L}_c'$, and $J_n^{(\text{eik})}$ is the eikonal jet function. The collinear fields in J_n^c can only produce gluons with nonzero label momenta, while the Wilson lines Ω_n, U_n in $J_n^{(\text{eik})}$ produce gluons in the zero bin. It is closely related to the eikonal jet function defined above, in Eq. (9).

By taking the Laplace transform of the jet functions in this convolution, we may solve for the purely-collinear jet function

$$\tilde{J}_n^c(\nu) = \frac{\tilde{J}_n(\nu)}{\tilde{J}_n^{(\text{eik})}(\nu)}, \quad (41)$$

which is precisely Eq. (13). This relation tells us that we may calculate the jet function on the left, J_n^c, by calculating instead the jet function on the right, J_n, which include the zero-bins in collinear integrals, if we then subtract out the double-counted usoft contributions with the eikonal jet function, which is precisely the prescription of Ref. [15]. We emphasize that the alternative SCET jet function, J_n, is directly analogous to the pQCD jet function defined in Eq. (7). There remain technical differences associated with the states $|N_{J_c}\rangle$ in (7), which can also contain energetic lines collinear to the opposite-moving jet, for example. As noted above, the effect of such lines is perturbatively calculable, so that these remaining differences are perturbatively calculable. We have thus shown its equivalence to the zero-bin subtraction procedure of Ref. [20] in SCET.

B. The ultrasoft region in SCET

For the purposes of our discussion below, we need to consider in some detail the precise definition of our usoft region. In each of the two-jet event shapes of Sec. II, the contributions of particles in the far forward and backward regions are suppressed exponentially in the absolute value of their rapidity, and hence as a power of their larger light-cone momentum (order Q). The contributions of jetlike particles at low (order Λ) transverse momenta are suppressed by a power of Q compared to those emitted with

similar low transverse momenta at wide angles. For this reason, to leading powers in $1/(eQ)$, we need not include separate nonperturbative functions for the jets [5], which may be expected to enter beginning at the level of $1/(eQ^{1+b})$, where b depends on the event shape at hand. For the angularities, $b = 1 - a$. We will give a more formal argument for this result in the next section. The soft shape function organizes all powers like $1/(eQ)^n$, while neglecting all further suppression by powers of Q.

Our analysis of power corrections from the soft function will depend crucially on the boost invariance of the usoft Wilson lines appearing in S, and, in the sum over usoft states in (25), we integrate over all momenta. With this in mind, we will want our ultrasoft region formally to include all gluons with small transverse momenta but boosted to arbitrarily large rapidities. The light-cone components $k^{\pm}$ of their momenta are then, strictly speaking, larger than the typical usoft scaling $Q\lambda^2$. Formally, the usoft modes in SCET do cover all momenta up to the scale $\mu = Q$, although the effective theory Lagrangian is a good approximation to the dynamics only of the truly usoft particles. As long as we pick observables to which only these truly usoft particles contribute significantly, it is safe to include the highly-boosted particles as well in the usoft sector of the theory. Indeed, for event shapes such as the angularities, the contribution of particles in the far-forward and far-backward regions are power-suppressed.

Consider, for example, the shape function (25) for the angularities as defined in Eq. (3). Do the usoft modes in SCET correctly describe the dynamics of all the small transverse-momentum particles that make a non-negligible contribution to the observable τ_a? Let us say that if the exponential factor, $\exp[-|\eta_i|(1 - a)]$, for particle i is of order λ or smaller, then its contribution is negligible. Then the largest-rapidity particle making a non-negligible contribution to the event shape has

$$|\eta_i| \sim -\frac{1}{1-a} \ln\lambda, \tag{42}$$

or, defining the rapidity as $\eta = \frac{1}{2}\ln(\bar{n} \cdot k / n \cdot k)$,

$$\max\left(\frac{\bar{n} \cdot k}{n \cdot k}, \frac{n \cdot k}{\bar{n} \cdot k}\right) \sim \lambda^{-2/(1-a)}. \tag{43}$$

Now, because $n \cdot k \, \bar{n} \cdot k \sim k_\perp^2$, and for usoft particles, $k_\perp \sim Q\lambda^2$, Eq. (43) implies that usoft particles with the larger of their light-cone momenta up to the order $Q\lambda^{2-1/(1-a)}$ contribute non-negligibly to the event shape τ_a. For $a < 1/2$, this light-cone momentum is still smaller than the corresponding component of a collinear momentum, which is order Q (e.g. for the thrust, $a = 0$. With $Q \sim 100$ GeV and $\Lambda_{\mathrm{QCD}} \sim 1$ GeV, we have $\lambda = 0.1$, so the largest usoft light-cone momentum that contributes to τ_0 is 10 GeV, still well below Q.) As long as this hierarchy of scales holds for the large light-cone components of usoft and collinear momenta, the only component of usoft gluons that interacts

with collinear modes in the n direction in the leading-order SCET Lagrangian is the $n \cdot A_{\mathrm{us}}$ component ($\bar{n} \cdot A_{\mathrm{us}}$ in the $\bar{n}$ direction), so that these interactions can be removed by the field redefinitions with the Wilson lines $Y_{n,\bar{n}}$. This guarantees the form of our usoft shape function (25), and ensures that only the truly usoft particles contribute to the sum. We can extend the range of allowed values of a beyond $a < 1/2$ (but only up to $a < 1$) by relaxing our criterion for the size of ''non-negligible'' terms (i.e. allowing $\exp[-|\eta_i|(1 - a)]$ to be larger than λ but smaller than 1). For the C-parameter, we need $1/\cosh\eta_i < \lambda$, which for the values chosen above translates to $n \cdot k$, $\bar{n} \cdot k \lesssim 20$ GeV.

Thus, in the following we may safely incorporate the power-suppressed contributions of the very far-forward and far-backward radiation in the SCET shape function, Eq. (25), and identify it with the eikonal cross section, Eq. (8), evaluated at the corresponding scale. This means that in both SCET and full QCD we may treat the sum over states in the shape function as boost-invariant. It is this result that will lead us to demonstrate universality properties below.

V. IDENTIFYING POWER CORRECTIONS

We are now ready to identify the power corrections that arise naturally when the soft and jet functions are evaluated at scales of order Λ_{QCD}. In doing so, we set aside issues of perturbative resummation and of matching, treated in full QCD for the angularities in Ref. [15], and very recently in SCET for the closely-related jet cross sections by Refs. [12,13]. In the discussion of this section, we will find useful a variant of the energy flow operators introduced in Refs. [38] and applied in this context by Ref. [9]. This operator is clearly also closely related to the energy-energy correlations of Refs. [39].

As mentioned earlier, the typical transverse momenta of the collinear particles in the jets which contribute to a given observable determine whether they should be treated in the theory $\mathrm{SCET_I}$, in which collinear momenta scale as $p_c = (n \cdot p_c, \bar{n} \cdot p_c, p_c^\perp) \sim Q(\lambda^2, 1, \lambda)$ or $Q(1, \lambda^2, \lambda)$, recalling that $\lambda \sim \sqrt{\Lambda_{\mathrm{QCD}}/Q}$. The typical virtuality of such particles being $p_c^2 = Q\Lambda_{\mathrm{QCD}}$, jet functions in this theory can be calculated perturbatively. However, some event shapes may weight much narrower jets more heavily, in which jet constituents with transverse momenta of order $Q\lambda^2 \sim \Lambda_{\mathrm{QCD}}$ become important. These degrees of freedom must be treated as collinear particles in $\mathrm{SCET_{II}}$, in which collinear momentum scale as $p_c \sim Q(\lambda^4, 1, \lambda^2)$ or $Q(1, \lambda^4, \lambda^2)$. These particles have virtualities of order $p_c^2 \sim \Lambda_{\mathrm{QCD}}^2$, and so give rise to nonperturbative effects in addition to those from soft particles. For such event shapes, nonperturbative power corrections to the jet functions may compete with (or even dominate) those in the soft functions.

We will now compare the size of power corrections from the soft and collinear functions for the event shapes in question. We will give our arguments in terms of SCET matrix elements, keeping in mind that they can be presented in terms of matrix elements in full QCD in a similar manner. For the C-parameter and angularities τ_a with $a < 1$, the dominant power corrections (of the order Λ_{QCD}/Q) will come only from the effect of usoft particles whose momenta are of $\mathcal{O}(Q\lambda^2)$. Power corrections from collinear particles will be found to scale as $(\Lambda_{\text{QCD}}/Q)^{2-a}$, which then dominate for $a \geq 1$. However, for $a \geq 1$, there are also other power corrections, for example, due to the shift in the thrust axis itself caused by the soft radiation [15,23,40]. The inclusion of these effects, while necessary for a complete treatment of power corrections to τ_a with $a \geq 1$, is outside the scope of this paper.

Consider the distribution of an event shape of the form in Eq. (2), given in SCET by Eqs. (23)–(25). The collinear cross section (24) is, writing out the general event shape of Eq. (2) in the delta function explicitly

$$\sigma_J(e_J; \mu_c) = \frac{1}{2Q^2} \sum_{N_{J_n} N_{J_{\bar{n}}}} |\langle N_{J_n} N_{J_{\bar{n}}}| \bar{\chi}_{n,Q} \Gamma^\mu \chi_{\bar{n},-Q}|0\rangle|^2(\mu_c)$$
$$\times \delta\left(e_J - \frac{1}{Q} \sum_{i \in N_{J_n} N_{J_{\bar{n}}}} |\mathbf{p}_i^\perp| f_e(\eta_i)\right), \quad (44)$$

while the soft function is

$$S(e; \mu_s) = \frac{1}{N_c} \text{Tr} \sum_{X_u} |\langle X_u| Y_n \bar{Y}_{\bar{n}}|0\rangle|^2(\mu_s)$$
$$\times \delta\left(e - \frac{1}{Q} \sum_{i \in X_u} |\mathbf{k}_i^\perp| f_e(\eta_i)\right), \quad (45)$$

where we have now chosen to denote explicitly the dependence of the jet and soft functions on the scales μ_c, μ_s. Also, we have suppressed the factor associated with the leptonic part, and we have removed the time-ordering operator that was in the soft function in Eq. (25) by using the Wilson line $\bar{Y}_{\bar{n}}$, where the bar denotes the antifundamental representation of $SU(N_c)$ [11]. For event shapes such as τ_a for $a < 1$, the collinear scale μ_c can be chosen at a perturbative scale $\mu_c \sim Q\lambda$, and we are in SCET$_\text{I}$. For $a > 1$, the event shapes pick out narrower jets so that the collinear scale is determined to be of order $\mu_c \sim \Lambda_{\text{QCD}}$, putting us in SCET$_\text{II}$, where the jet function is nonperturbative.

We may express the delta functions in Eqs. (44) and (45) in operator form by making use of a transverse energy flow operator, defined by its action on states N:

$$\mathcal{E}_T(\eta)|N(k_i)\rangle = \sum_{i \in N} |\mathbf{k}_i^\perp| \delta(\eta - \eta_i)|N(k_i)\rangle, \quad (46)$$

where the sum is over the particles i in state N. This is equivalent to the energy flow operators discussed in Refs. [9,38,41]. In terms of this operator, the collinear

and soft functions (44) and (45) can be written as

$$\sigma_J(e_J; \mu_c) = \frac{1}{2Q^2} \sum_{N_{J_n} N_{J_{\bar{n}}}} \left\langle 0 \middle| \bar{\chi}_{\bar{n},-Q} \bar{\Gamma}^\mu \chi_{n,Q} \right.$$
$$\times \delta\left(e_J - \frac{1}{Q} \int_{-\infty}^\infty d\eta f_e(\eta) \mathcal{E}_T(\eta)\right) \middle| N_{J_n} N_{J_{\bar{n}}} \right\rangle$$
$$\times \langle N_{J_n} N_{J_{\bar{n}}}| \bar{\chi}_{n,Q} \Gamma^\mu \chi_{\bar{n},-Q}|0\rangle, \quad (47)$$

and

$$S_e(e; \mu_s) = \frac{1}{N_C} \text{Tr} \sum_{X_u} \left\langle 0 \middle| \bar{Y}_{\bar{n}}^\dagger Y_n^\dagger \right.$$
$$\times \delta\left(e - \frac{1}{Q} \int_{-\infty}^\infty d\eta f_e(\eta) \mathcal{E}_T(\eta)\right) \middle| X_u \right\rangle$$
$$\times \langle X_u| Y_n \bar{Y}_{\bar{n}}|0\rangle. \quad (48)$$

We can expand the delta functions in power series to identify the power corrections. If we first factor out the overall, canonical factor of $1/e$, shared with perturbation theory, and assume that the matrix elements are of the order of the momentum components of the usoft gluons, $Q\lambda^2 \sim \Lambda_{\text{QCD}}$, we derive a power series in $\Lambda_{\text{QCD}}/(eQ)$. Indeed, the purpose of event shape functions is to organize all terms in this series when $\Lambda_{\text{QCD}}/(eQ) \sim 1$ and all such power corrections are comparable. These power corrections are particularly clearly exhibited by Laplace transforms, Eq. (10), of the soft function at low scales [9],

$$\tilde{S}_e(\nu; \mu_s) = \int_0^\infty de \exp[-\nu e] S_e(e; \mu_s)$$
$$= \frac{1}{N_C} \text{Tr} \langle 0| \bar{Y}_{\bar{n}}^\dagger Y_n^\dagger \exp\left[-\frac{\nu}{Q}\right.$$
$$\times \int_{-\infty}^\infty d\eta f_e(\eta) \mathcal{E}_T(\eta)\right] Y_n \bar{Y}_{\bar{n}}|0\rangle, \quad (49)$$

where we have summed over the complete set of intermediate states in Eq. (48), as argued in Sec. IV B. Expanding the exponential, we find a series in powers of the Laplace variable ν divided by Q.

For the soft function, according the discussion in the previous section, the sum over usoft states is unrestricted, as is the integral over rapidities inside the delta function. In the collinear function, choosing e_J to be close to the two-jet limit $e_J = 0$, or specifying a jet definition to pick out two-jet events, restricts the phase space integrals in the collinear cross section to those with large rapidities, effectively limiting the range of the rapidity integral as well. The rapidities η can be written in terms of the light-cone momenta of final state partons, $\eta = \frac{1}{2} \ln(\bar{n} \cdot p / n \cdot p)$. For a usoft parton, the ratio $n \cdot p_{\text{us}}/\bar{n} \cdot p_{\text{us}} \sim 1$, as all momentum components are $\mathcal{O}(Q\lambda^2)$, so $|\eta| \sim 0$, while for collinear partons in SCET$_\text{II}$, one light-cone component is $\mathcal{O}(Q)$ while the other is $\mathcal{O}(Q\lambda^4)$. Thus, $n \cdot p_c/\bar{n} \cdot p_c \sim \lambda^4$ or λ^{-4}, so $\eta \sim \pm \ln\lambda^2$. Consider what this implies for the collinear and soft functions in the case of the angular-

ities. The function $f_e(e)$ for $e = \tau_a$ is $f_{\tau_a}(\tau_a) = \mathrm{e}^{-|\eta|(1-a)}$. In the usoft function, this factor is of $\mathcal{O}(1)$. In the collinear integral, the phase space restrictions limit the rapidity integral to $|\eta| \gtrsim \ln(1/\lambda^2)$, so that the collinear function is effectively

$$\sigma_J(\tau_a) = \frac{1}{2Q^2} \sum_{N_{J_n} N_{J_{\bar{n}}}} \left\langle 0 \left| \bar{\chi}_{\bar{n},-Q} \bar{\Gamma}^\mu \chi_{n,Q} \right. \right.$$
$$\times \delta\left(\tau_a - \frac{2}{Q} \int_{\ln(1/\lambda^2)}^\infty d\eta\, \mathrm{e}^{-(1-a)\eta} \mathcal{E}_T(\eta) \right) \left| N_{J_n} N_{J_{\bar{n}}} \right\rangle$$
$$\times \langle N_{J_n} N_{J_{\bar{n}}} | \bar{\chi}_{n,Q} \Gamma^\mu \chi_{\bar{n},-Q} | 0 \rangle. \tag{50}$$

Although we cannot compute these nonperturbative matrix elements at the scale $\mu_c \sim Q\lambda^4$, we can estimate their dependence on λ from dimensional analysis. Matrix elements of powers of the operator $\mathcal{E}_T(\eta)$ in collinear states in SCET$_{\mathrm{II}}$ should vary as corresponding powers of $Q\lambda^2$. Similarly, each rapidity integral should behave as $\lambda^{2(1-a)}$. Combined with the factor $1/Q$ in front of the rapidity integral, power corrections to the collinear jet function occur as powers of $\lambda^{4-2a}/\tau_a = (1/\tau_a)(\Lambda_{\mathrm{QCD}}/Q)^{2-a}$. Correspondingly, in Laplace moment space, this becomes a power series in $\nu(\Lambda_{\mathrm{QCD}}/Q)^{2-a}$. The latter is also the only argument for the jet function that serves as a boundary condition in the perturbative QCD resummation of Ref. [15].[5] As long as $a < 1$, we may consider these to be subleading compared to the power corrections of the soft function, which are powers of Λ_{QCD}/Q. For $a \gtrsim 1$, we must take them into account, along with the recoil corrections mentioned above [16,17,42].

From now on, we consider only observables that pick out jets with typical transverse momenta well above the nonperturbative scale. In the language of SCET, this allows us to work in the theory SCET$_{\mathrm{I}}$ and consider power corrections only from the soft function.

VI. MOMENTUM FLOW OPERATORS, UNIVERSALITY AND SCALING

A. Nonperturbative universality from perturbative QCD

A striking prediction from the analysis of event shapes in perturbation theory, including those given in Eq. (3), is the universality of power corrections to their mean values [5–7,9,15–17,21,41,43,44],

$$\langle e \rangle = \langle e \rangle_{\mathrm{PT}} + c_e \frac{\mathcal{A}}{Q}. \tag{51}$$

In this expression, $\mathcal{A}$ is a universal parameter and c_e is a calculable coefficient that depends on the observable, as we shall see below. The same reasoning that leads to (51), when applied to the event shape distributions, produces a

shift in the resummed perturbative cross section

$$\frac{d\sigma}{de}(e) \bigg|_{\mathrm{PT}} \overrightarrow{\mathrm{NP}} \frac{d\sigma}{de}\left(e - c_e \frac{\mathcal{A}}{Q} \right) \bigg|_{\mathrm{PT}}. \tag{52}$$

These relations were derived in Refs. [6,7,43] from the assumption of a "dispersive" representation for $\alpha_s(\mu^2)$ considered as an analytic function of the scale μ, and in Refs. [5] they were abstracted directly from the form of resummed perturbation theory.

A more general approach [9,16,17] replaces the shift of Eq. (52) by a convolution with a shape function defined as above, which reduces to a product in Laplace moment space, Eq. (10). As we have noted, these shape functions are all different, but for the angularities a generalization of the universality of Eq. (51) has been suggested, in the form of a scaling relation. The Laplace-transformed shape function for angularity distributions arising from resummed perturbation theory at next-to-leading logarithm (NLL) [16,17] displays a simple scaling with the parameter a:

$$\ln S_a(\nu) = \frac{1}{1-a} \sum_{n=1}^\infty \lambda_n \left(-\frac{\nu}{Q} \right)^n, \tag{53}$$

where λ_n is independent of a. If we keep only the linear, ν/Q, term in the shape function, its inverse Laplace transform gives a delta function, which in the convolution of Eq. (5) leads immediately to the shift of Eq. (52). As noted above, we limit our attention to angularities for $a < 1$. The values of the coefficients λ_n of Eq. (53), of course, must be abstracted from a combination of experiment and resummed perturbation theory.

Event shape functions derived from resummation organize all corrections in ν/Q that are implied by perturbation theory. Formally, the coefficients λ_n are given in the NLL resummed cross section by

$$\lambda_n = \frac{2}{nn!}\left(-\frac{\nu}{Q} \right)^n \int_0^{\kappa^2} \frac{dp_T^2}{p_T^2} p_T^n A(\alpha_s(p_T)), \tag{54}$$

where $A(\alpha_s(p_T)) = C_i(\alpha_s/\pi) + \ldots$ is the residue of the $1/(1-x)$ pole in the splitting function for the parton, $i =$ quark or gluon, that initiates the jet, and κ is an infrared factorization scale. In this picture of power corrections, the coefficients λ_n are independent of a. The coefficient of the lowest power, $n = 1$, is equivalent to an integral over the running coupling, defined in a scheme where the coupling incorporates all higher powers of $A(\alpha_s)$ in $\overline{\mathrm{MS}}$ [45]. This approach generalizes the dispersive treatment of Refs. [7,43,44] to higher power corrections, but shares with it a reliance on (exponentiated) low-order gluon emission.

Analyses based on a dispersive coupling or on resummation rely to a greater or lesser extent on the kinematics of single soft gluon emission in the final state, and the universality relations follow from the boost invariance of these emission cross sections. The "Milan factor" [44] of

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[5]See, for example, Eqs. (67) and (74) of Ref. [15].

the dispersive approach accounts for effects at next order in α_s, where boost invariance and the resultant universalities can be maintained. We are about to show that the boost invariance of Wilson lines in the soft shape function Eq. (25) is by itself enough to prove the universality relation for the mean values without further assumptions. In Refs. [9,41] the role of energy flow was explored in a manner closely related to our discussion below.

B. Universality in average event shapes from the soft function

We continue to limit our attention to event shapes for which the dominant power corrections come from the soft function. For the shape function in the form given in Eq. (48), the operators in the matrix element no longer contain any reference to the final state X_u, so, as in Eq. (49), we may perform the sum over intermediate ultra-soft states, leaving

$$S_e(e) = \frac{1}{N_C} \text{Tr}\langle 0|\bar{Y}_{\bar{n}}^\dagger Y_n^\dagger \delta\left(e - \frac{1}{Q}\right.$$
$$\times \left. \int d\eta \, \mathcal{E}_T(\eta) f_e(\eta)\right) Y_n \bar{Y}_{\bar{n}}|0\rangle. \quad (55)$$

From now on we drop the explicit dependence of the soft function on the scale μ_s. In (55), we insert factors of $U(\Lambda(\eta'))^\dagger U(\Lambda(\eta')) = 1$, implementing a Lorentz boost of each operator in the z-direction with a rapidity η'. The vacuum $|0\rangle$ is invariant under Lorentz boosts, and the Wilson lines are also invariant:

$$U(\Lambda(\eta'))Y_n(0)U(\Lambda(\eta'))^\dagger$$

$$= U(\Lambda(\eta'))P\exp\left[ig\int_0^\infty ds\, n \cdot A_{\text{us}}(ns)\right]U(\Lambda(\eta'))^\dagger$$

$$= P\exp\left[ig\int_0^\infty ds\, \alpha n \cdot A_{us}(\alpha ns)\right] = Y_n(0), \quad (56)$$

where $\alpha = e^{-\eta'}$, as $n \to \alpha n$ and $\bar{n} \to \alpha^{-1}\bar{n}$. (This is also known in SCET as type-III reparametrization invariance [46].) The only change is in the operator $\mathcal{E}_T(\eta)$:

$$U(\Lambda(\eta'))\mathcal{E}_T(\eta)U(\Lambda(\eta'))^\dagger = \mathcal{E}_T(\eta + \eta'), \quad (57)$$

which follows from the defining relation for the $\mathcal{E}_T$ operators, Eq. (46). Thus, the argument of the operator $\mathcal{E}_T(\eta)$ in the shape function in Eq. (55) may be shifted to any value of rapidity, $\mathcal{E}_T(\eta) \to \mathcal{E}_T(\eta + \eta')$. At this stage, this does not yet allow us to perform the rapidity integral of $f_e(\eta)$ inside the delta function. Thus we do not find that the leading power correction simply shifts the argument of the perturbative event shape distributions, as the delta function is a highly nonlinear function of the energy flow operator and sits sandwiched between Wilson lines in the matrix element. If we do neglect correlations between these operators, we derive a delta function for the shape function, and reproduce the shift in the distribution, Eq. (52) [9,41].

The boost property (57) of a single operator, however, gives a strong result when applied to the first moment of an event shape distribution [14]. Taylor expanding the delta function in Eq. (55) (which is valid if we integrate the distribution over a sufficiently large region near the end-point), we find

$$S_e(e) = \delta(e) - \delta'(e)\frac{1}{Q}\int d\eta f_e(\eta)\frac{1}{N_C}$$
$$\times \text{Tr}\langle 0|\bar{Y}_{\bar{n}}^\dagger Y_n^\dagger \mathcal{E}_T(\eta + \eta')Y_n\bar{Y}_{\bar{n}}|0\rangle + \cdots. \quad (58)$$

Recalling the boost properties of the Wilson lines and the energy flow operators $\mathcal{E}_T(\eta)$, we are free to choose any value for η' in this expression. Then, choosing $\eta' = -\eta$, we find that, remarkably, we may take the matrix element of the $\mathcal{E}_T$ operator out of the integral over η, leaving the result

$$S_e(e) = \delta(e) - \delta'(e)c_e\frac{\mathcal{A}}{Q} + \cdots, \quad (59)$$

where the coefficient c_e is given by the integral

$$c_e = \int_{-\infty}^\infty d\eta f_e(\eta), \quad (60)$$

and the universal quantity $\mathcal{A}$ is

$$\mathcal{A} = \frac{1}{N_C}\text{Tr}\langle 0|\bar{Y}_{\bar{n}}^\dagger Y_n^\dagger \mathcal{E}_T(0)Y_n\bar{Y}_{\bar{n}}|0\rangle. \quad (61)$$

For the C-parameter and angularities τ_a, the integrals of the corresponding weight functions

$$f_C(\eta) = \frac{3}{\cosh\eta}, \qquad f_{\tau_a} = e^{-|\eta|(1-a)}, \quad (62)$$

over all rapidities give the coefficients

$$c_C = 3\pi, \qquad c_{\tau_a} = \frac{2}{1-a}. \quad (63)$$

When convoluted with the perturbative distribution, $S_e(e)$ reproduces the universality relations of Eq. (51) for the first moments of the distributions. We have thus established these results without appealing to a one-gluon or related approximation. All higher-order corrections due to multiple-gluon emission separate from the observable-dependent factor c_e, which can be computed in a "naive" fashion [1] as in Eq. (60) above.

The result for the C-parameter may be extended to a larger class of related event shapes by defining functions, $f_{C_a}(\eta) = 3/\cosh^a\eta$, by analogy to the angularities. The integral over rapidities of this function gives the coefficient $c_{C_a} = 3B(a/2, 1/2)$, where $B(x, y)$ is the beta function. In like manner, various new event shapes may be defined by appropriate choices for the function $f_e(\eta)$.

C. Angularity distributions and momentum flow

The expression (55) for the shape function in terms of energy flow operators enables us to put the power expansion of Eq. (53) into a more general field-theoretic context, and to discuss the possible significance of scaling violation.

Let us compare Eq. (53), derived from resummed perturbation theory, with the Laplace transform of the corresponding shape function in Eq. (55) [41]. This is given by

$$\tilde{S}_a(\nu) = \frac{1}{N_C} \text{Tr}\langle 0| \bar{Y}_{\bar{n}}^\dagger Y_n^\dagger \exp\left[-\frac{\nu}{Q} \right. $$
$$\left. \times \int d\eta e^{-|\eta|(1-a)} \mathcal{E}_T(\eta) \right] Y_n \bar{Y}_{\bar{n}} |0\rangle, \qquad (64)$$

which can be reexpressed as an expansion in cumulants,

$$\ln\left[\frac{\tilde{S}_a(\nu)}{\tilde{S}_a(0)}\right] = \sum_{n=1}^\infty \frac{1}{n!} \left(-\frac{\nu}{Q}\right)^n \left\langle\!\!\left\langle \left[\int d\eta e^{-|\eta|(1-a)} \mathcal{E}_T(\eta)\right]^n \right\rangle\!\!\right\rangle$$
$$\equiv \sum_{n=1}^\infty \frac{1}{n!} \left(-\frac{\nu}{Q}\right)^n \mathcal{A}_n(a). \qquad (65)$$

Here, and below, in the cumulants the Wilson lines Y_n and $Y_{\bar{n}}$ are understood. With this normalization, the coefficient $\mathcal{A}_1(a)$ for the angularities is related to the universal coefficient $\mathcal{A}$ in Eq. (61) by $\mathcal{A}_1(a) = 2\mathcal{A}/(1-a)$. The factor of $\tilde{S}_a(0)$ on the left-hand side of Eq. (65) correctly accounts for the normalization of the soft function. [Of course, from Eq. (64), we see that $\tilde{S}_a(0) = 1$, but the normalization would not be trivial in the analogous equation for the jet function, for instance.] In terms of the matrix elements above, we find a general form for the coefficients λ_n, which is not limited to NLL resummation,

$$\lambda_n(a) = \frac{1-a}{n!} \mathcal{A}_n(a), \qquad (66)$$

which, in the general case for $n > 1$, may still depend upon a, as indicated.

To explore the information contained in the cumulants, $\mathcal{A}_n$, let us study the a dependence of the parameters λ_n in Eq. (66) for low n. The $n = 1$ term, $\lambda_1(a)$, is independent of a, as we showed in the previous section, in agreement with the resummed perturbation theory result, Eq. (54). The a dependence of the second and higher terms, however, differs in general. Nevertheless, boost invariance always allows us to perform one rapidity integral in the cumulant matrix elements. For the case $n = 2$, we have

$$\lambda_2(a) = \frac{1}{2} \int_{-\infty}^\infty d\eta [1 + (1-a)|\eta|]$$
$$\times e^{-|\eta|(1-a)} \langle\langle \mathcal{E}_T(0)\mathcal{E}_T(\eta)\rangle\rangle. \qquad (67)$$

Under certain conditions, the a dependence of this expression also disappears. In Ref. [16], it was observed that the scaling rule for the nth cumulant term in Eq. (65) is good when the energy flow correlations are negligible for rapid-

ity intervals larger than a range $\Delta\eta \sim 1/[n(1-a)]$. Assume, then, that the correlator $\langle\langle \mathcal{E}_T(0)\mathcal{E}_T(\eta)\rangle\rangle$ is nonzero only for $\eta \ll \frac{1}{2(1-a)}$. Then we may Taylor expand the remainder of the integrand in Eq. (67) about $\eta = 0$:

$$\int_0^\infty d\eta \left\{ 1 - \frac{1}{2}[(1-a)\eta]^2 + \frac{1}{3}[(1-a)\eta]^3 + \cdots \right\}$$
$$\times \langle\langle \mathcal{E}_T(0)\mathcal{E}_T(\eta)\rangle\rangle. \qquad (68)$$

Insofar as the correlator $\langle\langle \mathcal{E}_T(0)\mathcal{E}_T(\eta)\rangle\rangle$ has support only over a region $\eta \ll \frac{1}{1-a}$, the leading term of the expansion dominates, and we recover the a-independence of λ_2. Interestingly, there is no $\mathcal{O}(\eta)$ term in the expansion multiplying the correlator, so that violations of the scaling rule should be even smaller than one might initially expect, at least for moderate values of $1 - a$.

We must wait on the analysis of data to interpret the significance of scaling violation for the angularities. Supposing, however, that substantial scaling violation were found in the power $(\nu/Q)^2$ in the shape functions for angularities, we can learn about nonperturbative correlations in energy flow through Laplace moments of the cumulants. For example, using Eq. (67), we observe that

$$\mathcal{C}_2(a) - \frac{\partial}{\partial \ln(1-a)} \mathcal{C}_2(a) = \lambda_2(a), \qquad (69)$$

where $\mathcal{C}_2(a)$ is a direct Laplace moment of the correlator of energy flow operators in terms of their rapidity separation, with one fixed at rapidity zero,

$$\mathcal{C}_2(a) \equiv \frac{1}{2} \int_{-\infty}^\infty d\eta e^{-(1-a)|\eta|} \langle\langle \mathcal{E}_T(0)\mathcal{E}_T(\eta)\rangle\rangle. \qquad (70)$$

Assuming that the correlations vanish for $a \to -\infty$, the solution to Eq. (69) gives these Laplace moments directly in terms of the cumulants $\mathcal{A}_2$, and hence in terms of the coefficient $\lambda_2(a)$, which is, in principle, an observable

$$\mathcal{C}_2(a) = (1-a) \int_{-\infty}^a \frac{da'}{(1-a')^2} \lambda_2(a'). \qquad (71)$$

Furthermore, derivatives of $\mathcal{C}_2(a)$ with respect to a provide information on Laplace moments of the cumulants supplemented by powers.

For higher n, the situation becomes somewhat more complex, but continues to encode potentially interesting physical information. For the coefficient $\mathcal{A}_3$ in Eq. (65), we can similarly perform one of the three rapidity integrals and obtain

$$\mathcal{A}_3(a) = \frac{4}{1-a} \int_0^\infty d\eta_2 \int_0^\infty d\eta_3 e^{-(\eta_2+\eta_3)(1-a)}$$
$$\times [3\langle\langle \mathcal{E}_T(0)\mathcal{E}_T(-\eta_2)\mathcal{E}_T(\eta_3)\rangle\rangle$$
$$- \langle\langle \mathcal{E}_T(0)\mathcal{E}_T(\eta_2)\mathcal{E}_T(\eta_3)\rangle\rangle], \qquad (72)$$

which again respects the $\frac{1}{1-a}$ scaling under the assumption that the correlators are nonzero only for $\eta_{2,3} \ll \frac{1}{1-a}$.

VII. CONCLUSION

We have explored power corrections for event shapes using factorization theorems in both full QCD and SCET. In this context, we have shown that the formalisms lead to equivalent event shape functions that summarize nonperturbative effects of soft gluon emission on event shape distributions for two-jet events. We have shown how the boost invariance of lightlike Wilson lines implies the universality of the leading $1/Q$ corrections to the mean values of the event shapes, without relying on low-order or even resummed perturbative calculations.

In addition, we have used the field-theoretic formalism to interpret potential violations of the scaling rule for angularity shape functions in terms of correlations between energy flow operators for soft gluon radiation. Using $1/Q^2$ corrections in shape functions as an example, we have demonstrated how, in principle, a violation of scaling for the angularities can provide information on specific matrix elements in the effective theory. The analysis of existing extensive and high-quality data from leptonic annihilation experiments could, in this way, provide a new experimental window into the process of hadronization in quantum chromodynamics.

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