

Precise mass-dependent QED contributions to leptonic $g - 2$ at order α^2 and α^3

M. Passera

Dipartimento di Fisica “G. Galilei”, Università di Padova and INFN, Sezione di Padova, I-35131, Padova, Italy
(Received 3 August 2006; published 4 January 2007)

Improved values for the two- and three-loop mass-dependent QED contributions to the anomalous magnetic moments of the electron, muon, and τ lepton are presented. The standard model prediction for the electron ($g - 2$) is compared with its most precise recent measurement, providing a value of the fine-structure constant in agreement with a recently published determination. For the τ lepton, differences with previously published results are found and discussed.

DOI: [10.1103/PhysRevD.75.013002](https://doi.org/10.1103/PhysRevD.75.013002)

PACS numbers: 12.20.Ds, 06.20.Jr, 13.40.Em, 14.60.-z

I. INTRODUCTION

The QED part of the anomalous magnetic moment $a_l = (g_l - 2)/2$ of a charged lepton $l = e, \mu$ or τ arises from the subset of standard model (SM) diagrams containing only leptons and photons. For each of the three leptons l , of mass m_l , this dimensionless quantity can be cast in the general form [1]

$$a_l^{\text{QED}} = A_1 + A_2 \left(\frac{m_l}{m_j} \right) + A_2 \left(\frac{m_l}{m_k} \right) + A_3 \left(\frac{m_l}{m_j}, \frac{m_l}{m_k} \right), \quad (1)$$

where m_j and m_k are the masses of the other two leptons. The term A_1 , arising from diagrams containing photons and leptons of only one flavor, is mass and flavor independent. In contrast, the terms A_2 and A_3 are functions of the indicated mass ratios, and are generated by graphs containing also leptons of flavors different from l . The contribution of a lepton j to a_l^{QED} is suppressed by (m_l^2/m_j^2) if $m_j \gg m_l$, while it contains a logarithmic enhancement factor $\ln(m_l/m_j)$ if $m_j \ll m_l$. The muon contribution to a_e^{QED} is thus power suppressed by a factor $(m_e^2/m_\mu^2) \sim 10^{-5}$; nonetheless, as we will discuss in Sec. II, this effect is much larger than the tiny uncertainty very recently achieved in the measurement of a_e [2]. On the contrary, the QED parts of $a_{\mu,\tau}$ beyond one-loop are dominated by the mass-dependent terms.

The functions A_i ($i = 1, 2, 3$) can be expanded as power series in α/π and computed order-by-order

$$A_i = A_i^{(2)} \left(\frac{\alpha}{\pi} \right) + A_i^{(4)} \left(\frac{\alpha}{\pi} \right)^2 + A_i^{(6)} \left(\frac{\alpha}{\pi} \right)^3 + A_i^{(8)} \left(\frac{\alpha}{\pi} \right)^4 + \dots \quad (2)$$

Only one diagram is involved in the evaluation of the lowest-order (first-order in α , second-order in the electric charge) contribution—it provides the famous result by Schwinger $A_1^{(2)} = 1/2$ [3]. The mass-dependent coefficients A_2 and A_3 are of higher order; the goal of this letter is to provide precise numerical values for their $O(\alpha^2)$ and $O(\alpha^3)$ terms. The relevance of the results and the improvements with respect to earlier ones will be discussed separately for each lepton. All results were derived using the

latest CODATA [4] recommended mass ratios: $m_e/m_\mu = 4.83633167(13) \times 10^{-3}$, $m_e/m_\tau = 2.87564(47) \times 10^{-4}$, $m_\mu/m_e = 206.7682838(54)$, $m_\mu/m_\tau = 5.94592(97) \times 10^{-2}$, $m_\tau/m_e = 3477.48(57)$, $m_\tau/m_\mu = 16.8183(27)$. (The value for m_τ adopted by CODATA in Ref. [4] ($m_\tau = 1776.99(29)$ MeV) is based on the 2002 PDG result [5]. This PDG result remains unchanged to date [6,7]).

II. ELECTRON

A. Two-loop contributions

Seven diagrams contribute to the fourth-order coefficient $A_1^{(4)}$, one to $A_2^{(4)}(m_e/m_\mu)$ and one to $A_2^{(4)}(m_e/m_\tau)$. As there are no two-loop diagrams contributing to a_e^{QED} that contain both virtual muons and taus, $A_3^{(4)}(m_e/m_\mu, m_e/m_\tau) = 0$. The mass-independent coefficient has been known for almost 50 years [8]:

$$\begin{aligned} A_1^{(4)} &= \frac{197}{144} + \frac{\pi^2}{12} + \frac{3}{4} \zeta(3) - \frac{\pi^2}{2} \ln 2 \\ &= -0.32847896557919378\dots, \end{aligned} \quad (3)$$

where $\zeta(s)$ is the Riemann zeta function of argument s . The coefficient of the two-loop mass-dependent contribution to a_l^{QED} , $A_2^{(4)}(1/x)$, with $x = m_j/m_l$, is generated by the diagram in Fig. 1, where j is the virtual lepton in the vacuum polarization subgraph. This coefficient was first computed in the late 1950s for the muon $g - 2$ with $x = m_e/m_\mu \ll 1$, neglecting terms of $O(x)$ [9]. The exact

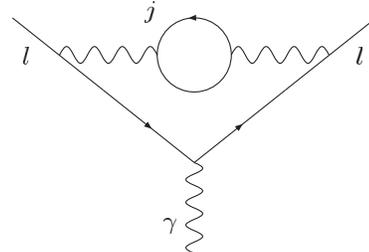


FIG. 1. The QED diagram generating the mass-dependent part of a_l^{QED} in order α^2 .

expression for $0 < x < 1$ was reported by Elend in 1966 [10]. However, its numerical evaluation was considered tricky because of large cancellations and difficulties in the estimate of the accuracy of the results, so that common practice was to use series expansions instead [11–13]. Taking advantage of the properties of the dilogarithm $\text{Li}_2(z) = -\int_0^z (dt/t) \ln(1-t)$ [14], the exact result was cast in [15] in a very simple and compact analytic form, valid, contrary to the one in [10], also for $x \geq 1$ (the case relevant to a_e^{QED} and part of a_μ^{QED}):

$$A_2^{(4)}(1/x) = -\frac{25}{36} - \frac{\ln x}{3} + x^2(4 + 3 \ln x) + \frac{x}{2}(1 - 5x^2) \\ \times \left[\frac{\pi^2}{2} - \ln x \ln\left(\frac{1-x}{1+x}\right) - \text{Li}_2(x) + \text{Li}_2(-x) \right] \\ + x^4 \left[\frac{\pi^2}{3} - 2 \ln x \ln\left(\frac{1}{x} - x\right) - \text{Li}_2(x^2) \right]. \quad (4)$$

For $x = 1$, Eq. (4) gives $A_2^{(4)}(1) = 119/36 - \pi^2/3$; of course, this contribution is already part of $A_1^{(4)}$ in Eq. (3). Numerical evaluation of Eq. (4) with the mass ratios given in Sec. I yields

$$A_2^{(4)}(m_e/m_\mu) = 5.19738670(28) \times 10^{-7} \quad (5)$$

$$A_2^{(4)}(m_e/m_\tau) = 1.83762(60) \times 10^{-9}, \quad (6)$$

where the standard errors are only due to the uncertainties of the mass ratios. The results of Eqs. (5) and (6) are equal to those obtained with a series expansion in powers of y and $\ln y$, with $y \ll 1$ [4].

Adding up Eqs. (3), (5), and (6) we get the two-loop QED coefficient

$$C_e^{(4)} = A_1^{(4)} + A_2^{(4)}(m_e/m_\mu) + A_2^{(4)}(m_e/m_\tau) \\ = -0.32847844400290(60). \quad (7)$$

The mass-dependent part of $C_e^{(4)}$ is small but not negligible, giving a relative contribution to the theoretical prediction of the electron $g - 2$ of 2.4 ppb (ppb). This value is much larger than the fabulous 0.7 ppb relative uncertainty very recently achieved in the measurement of a_e [2]. The uncertainties in $A_2^{(4)}(m_e/m_\mu)$ and $A_2^{(4)}(m_e/m_\tau)$ are dominated by the latter and were added in quadrature. The resulting error $\delta C_e^{(4)} = 6 \times 10^{-13}$ leads to a totally negligible $O(10^{-18})$ uncertainty in the a_e^{QED} prediction.

B. Three-loop contributions

More than 100 diagrams are involved in the evaluation of the three-loop (sixth-order) QED contribution. Their analytic computation required approximately three decades, ending in the late 1990s. The coefficient $A_1^{(6)}$ arises from 72 diagrams. Its exact expression, mainly due to Remiddi and his collaborators, reads [16,17]:

$$A_1^{(6)} = \frac{83}{72} \pi^2 \zeta(3) - \frac{215}{24} \zeta(5) - \frac{239}{2160} \pi^4 + \frac{28259}{5184} \\ + \frac{139}{18} \zeta(3) - \frac{298}{9} \pi^2 \ln 2 + \frac{17101}{810} \pi^2 \\ + \frac{100}{3} \left[\text{Li}_4(1/2) + \frac{1}{24} (\ln^2 2 - \pi^2) \ln^2 2 \right] \\ = 1.181241456587 \dots \quad (8)$$

This value is in very good agreement with previous results obtained with numerical methods [18].

The calculation of the exact expression for the coefficient $A_2^{(6)}(m_l/m_j)$ for arbitrary values of the mass ratio m_l/m_j was completed in 1993 by Laporta and Remiddi [19,20] (earlier works include Refs. [21]). Let us focus on a_e^{QED} ($l = e, j = \mu, \tau$). This coefficient can be further split into two parts: the first one, $A_2^{(6)}(m_l/m_j, \text{vac})$, receives contributions from 36 diagrams containing either muon or τ vacuum polarization loops [19], whereas the second one, $A_2^{(6)}(m_l/m_j, \text{lbl})$, is due to 12 light-by-light scattering diagrams with either muon or τ loops [20]. The exact expressions for these coefficients are rather complicated, containing hundreds of polylogarithmic functions up to fifth degree (for the light-by-light diagrams) and complex arguments (for the vacuum polarization ones). Indeed, they were too long to be listed in [19,20] (but were kindly provided by their authors), although series expansions were given for the cases of physical relevance. The exact expressions for the light-by-light contributions also contain a few pentalogarithms in integral form. We expressed these integrals in terms of harmonic polylogarithms (introduced by Remiddi and Vermaseren in [22]), thus avoiding their numerical integration.

The numerical evaluations of the exact expressions for $A_2^{(6)}(m_l/m_j, \text{vac})$ and $A_2^{(6)}(m_l/m_j, \text{lbl})$ require some care, as the presence of large cancellations makes them prone to potentially large roundoff errors. For this reason, numerical evaluations were carried out with *Mathematica* codes employing exclusively arbitrary-precision numbers, keeping track of precision at all points [23]. Harmonic polylogarithms were implemented via the *Mathematica* package HPL [24]. Using the recommended mass ratios given in Sec. I, we obtain the following values:

$$A_2^{(6)}(m_e/m_\mu, \text{vac}) = -2.17684015(11) \times 10^{-5} \quad (9)$$

$$A_2^{(6)}(m_e/m_\mu, \text{lbl}) = 1.439445989(77) \times 10^{-5} \quad (10)$$

$$A_2^{(6)}(m_e/m_\tau, \text{vac}) = -1.16723(36) \times 10^{-7} \quad (11)$$

$$A_2^{(6)}(m_e/m_\tau, \text{lbl}) = 5.0905(17) \times 10^{-8}. \quad (12)$$

The sums of Eqs. (9) and (10) and Eqs. (11) and (12) are

$$A_2^{(6)}(m_e/m_\mu) = -7.37394164(29) \times 10^{-6} \quad (13)$$

$$A_2^{(6)}(m_e/m_\tau) = -6.5819(19) \times 10^{-8}. \quad (14)$$

Eqs. (9)–(14) provide the first evaluation of the full analytic expressions for these coefficients with the CODATA mass ratios of [4]; they are almost identical to the results $A_2^{(6)}(m_e/m_\mu) = -7.37394158(28) \times 10^{-6}$ and $A_2^{(6)}(m_e/m_\tau) = -6.5819(19) \times 10^{-8}$ obtained in [4] via the approximate series expansions in the mass ratios. The small difference between $A_2^{(6)}(m_e/m_\mu)$ of [4] and Eq. (13) mainly originates from the $O((m_e/m_\mu)^6)$ term in the series expansion of $A_2^{(6)}(m_e/m_\mu, \text{lbl})$; indeed, due to its smallness, this term was neglected in the expansions [20] used in [4]. Expanding the exact Laporta-Remiddi expression for the sum of light-by-light and vacuum polarization contributions, for $r = m_l/m_j \ll 1$, we get

$$A_2^{(6)}(r) = \sum_{i=1}^4 r^{2i} f_{2i}(r) + O(r^{10} \ln^2 r), \quad (15)$$

$$f_2(r) = \frac{23 \ln r}{135} + \frac{3\zeta(3)}{2} - \frac{2\pi^2}{45} - \frac{74957}{97200}, \quad (16)$$

$$f_4(r) = -\frac{4337 \ln^2 r}{22680} + \frac{209891 \ln r}{476280} + \frac{1811\zeta(3)}{2304} - \frac{1919\pi^2}{68040} - \frac{451205689}{533433600}, \quad (17)$$

$$f_6(r) = -\frac{2807 \ln^2 r}{21600} + \frac{665641 \ln r}{2976750} + \frac{3077\zeta(3)}{5760} - \frac{16967\pi^2}{907200} - \frac{246800849221}{480090240000}, \quad (18)$$

$$f_8(r) = -\frac{55163 \ln^2 r}{594000} + \frac{24063509989 \ln r}{172889640000} + \frac{9289\zeta(3)}{23040} - \frac{340019\pi^2}{24948000} - \frac{896194260575549}{2396250410400000}. \quad (19)$$

The functions $f_2(r)$ and $f_4(r)$ coincide with the expansions provided in [20], and $f_6(r)$ agrees with the combination of parts from [19] (for the vacuum polarization contribution) and [25] (heavy-mass expansions for the light-by-light diagrams); $f_8(r)$ is new. The value of $A_2^{(6)}(m_e/m_\mu)$ obtained with Eq. (15) perfectly agrees with that in Eq. (13) determined with the exact formulae. Indeed, their difference is of $O(10^{-23})$, to be compared with the $O(10^{-13})$ error $\delta A_2^{(6)}(m_e/m_\mu)$ due to the present uncertainty of the ratio m_e/m_μ . Therefore, it will be possible to compute $A_2^{(6)}(m_e/m_\mu)$ with the simple expansion in Eq. (15)—thus

avoiding the complexities of the exact expressions—even if the precision of the ratio m_e/m_μ will improve in the future by orders of magnitude.

The contribution of the three-loop diagrams with both μ and τ loop insertions in the photon propagator can be calculated numerically from the integral expressions of Ref. [11]. We get

$$A_3^{(6)}(m_e/m_\mu, m_e/m_\tau) = 1.90945(62) \times 10^{-13}, \quad (20)$$

a totally negligible $O(10^{-21})$ contribution to a_e^{QED} . Adding up Eqs. (8), (13), (14), and (20) we obtain the three-loop QED coefficient

$$C_e^{(6)} = 1.181234016827(19). \quad (21)$$

The relative contribution to a_e^{QED} of the mass-dependent part of this coefficient is ~ 0.1 ppb. This is smaller than the present ~ 0.7 ppb experimental uncertainty [2]. The error 1.9×10^{-11} in Eq. (21) leads to a totally negligible $O(10^{-19})$ uncertainty in a_e^{QED} .

C. Determination of α from the electron $g - 2$

Very recently, a new measurement of the electron anomalous magnetic moment by Gabrielse and his collaborators achieved the fabulous relative uncertainty of 0.66 ppb [2],

$$a_e^{\text{EXP}} = 115\,965\,218.085(76) \times 10^{-12}. \quad (22)$$

This uncertainty is nearly 6 times smaller than that of the last measurement of a_e reported back in 1987, $a_e^{\text{EXP}} = 1159652188.3(4.2) \times 10^{-12}$ [4,26]. These two measurements differ by 1.7 standard deviations.

The fine-structure constant α can be determined equating the theoretical SM prediction of the electron $g - 2$ with its measured value

$$a_e^{\text{SM}}(\alpha) = a_e^{\text{EXP}}. \quad (23)$$

The SM prediction contains the QED contribution $a_e^{\text{QED}}(\alpha) = \sum_{i=1}^5 C_e^{(2i)}(\alpha/\pi)^i$ (higher-order coefficients are assumed to be negligible), plus small weak and hadronic loop effects: $a_e^{\text{SM}}(\alpha) = a_e^{\text{QED}}(\alpha) + a_e^{\text{EW}} + a_e^{\text{HAD}}$ (the dependence on α of any contribution other than a_e^{QED} is negligible). The electroweak contribution is [4]:

$$a_e^{\text{EW}} = 0.0297(5) \times 10^{-12}; \quad (24)$$

this precise value includes the two-loop contributions calculated in Ref. [27]. The hadronic term is [4,28]:

$$a_e^{\text{HAD}} = 1.671(19) \times 10^{-12}. \quad (25)$$

The latest value for the four-loop QED coefficient is $C_e^{(8)} = -1.7283(35)$ [29]. Following the argument of [4], we adopt the educated guess $C_e^{(10)} = 0.0(3.8)$ for the five-loop coefficient. The errors $\delta C_e^{(8)} = 0.0035$ and $\delta C_e^{(10)} = 3.8$ lead to an uncertainty of 0.1×10^{-12} and 0.3×10^{-12}

in a_e^{QED} , respectively. Solving Eq. (23) with the new measured value of Eq. (22), we obtain

$$\begin{aligned}\alpha^{-1} &= 137.035999709(12)(30)(2)(90) \\ &= 137.035999709(96)[0.70 \text{ ppb}].\end{aligned}\quad (26)$$

The first and second errors are due to the uncertainties of the four- and five-loop QED coefficient $\delta C_e^{(8)}$ and $\delta C_e^{(10)}$, respectively; the third one is caused by the tiny δa_e^{HAD} , and the last one (90×10^{-9}) is from the experimental δa_e^{EXP} in Eq. (22). The uncertainty of the electroweak and two/three-loop QED contributions are totally negligible at present. The determination in Eq. (26) is in perfect agreement with the new result of Ref. [30],

$$\alpha^{-1} = 137.035999710(96) \quad (27)$$

(also based on the new measurement of Ref. [2]), whose great precision represents the first significant improvement of this fundamental constant in a decade. The totally negligible difference between Eqs. (26) and (27) is due to the rounded value $a_e^{\text{EW}} = 0.030(1) \times 10^{-12}$ [4] employed by the authors of Ref. [30] instead of Eq. (24).

At present, the best determinations of α independent of the electron $g - 2$ are

$$\alpha^{-1}(\text{Rb}) = 137.03599878(91)[6.7 \text{ ppb}], \quad (28)$$

$$\alpha^{-1}(\text{Cs}) = 137.0360000(11)[8.0 \text{ ppb}] \quad (29)$$

they are less precise by roughly a factor of 10. The value $\alpha^{-1}(\text{Rb})$ was deduced from the measurement of the ratio h/M_{Rb} based on Bloch oscillations of Rb atoms in an optical lattice (h is the Planck constant and M_{Rb} is the mass of the Rb atom) [31], while $\alpha^{-1}(\text{Cs})$ was determined from the measurement of the ratio h/M_{Cs} (M_{Cs} is the mass of the Cs atom) via cesium recoil measurement techniques [32,33]. These two determinations of α also rely on the precisely known Rydberg constant and relative atomic masses of the electron, Rb and Cs atoms [4,34]. The values of α in Eqs. (28) and (29) are in good agreement with the result of Eq. (26), differing from the latter by -1.0 and $+0.3$ standard deviations, respectively. This comparison provides a beautiful test of the validity of QED. It also probes for possible electron substructure [30].

III. TAU

The two-loop mass-dependent QED contributions to the anomalous magnetic moment of the τ , obtained by direct evaluation of the exact formula in Eq. (4), are

$$A_2^{(4)}(m_\tau/m_e) = 2.024284(55), \quad (30)$$

$$A_2^{(4)}(m_\tau/m_\mu) = 0.361652(38). \quad (31)$$

These two values are very similar to those computed via a dispersive integral in Ref. [35] (which, however, contain no

estimates of the uncertainties). Eqs. (30) and (31) are also in agreement (but more accurate) with those of Ref. [12]. Adding up Eqs. (3), (30), and (31) we get

$$C_\tau^{(4)} = 2.057457(93) \quad (32)$$

(note that the uncertainties in m_τ/m_e and m_τ/m_μ are correlated). The resulting error 9.3×10^{-5} leads to a 5×10^{-10} uncertainty in a_τ^{QED} .

We computed the three-loop mass-dependent contributions by direct numerical evaluation of the exact analytic expressions (see Sec. II B). The results are:

$$A_2^{(6)}(m_\tau/m_e, \text{vac}) = 7.25699(41) \quad (33)$$

$$A_2^{(6)}(m_\tau/m_e, \text{lbl}) = 39.1351(11) \quad (34)$$

$$A_2^{(6)}(m_\tau/m_\mu, \text{vac}) = -0.023554(51) \quad (35)$$

$$A_2^{(6)}(m_\tau/m_\mu, \text{lbl}) = 7.03376(71). \quad (36)$$

Employing the approximate series expansions (see Sec. II B) we obtain almost identical values: 7.25699(41), 39.1351(11), $-0.023564(51)$, 7.03375(71). The estimates of Ref. [35] were: 10.0002, 39.5217, 2.9340, and 4.4412 (no error estimates were provided), respectively; they are at variance with our results, Eqs. (33)–(36), derived from the exact analytic expressions. The estimates of Ref. [36] compare slightly better: 7.2670, 39.6, -0.1222 , 4.47 (no errors provided). In the specific case of $A_2^{(6)}(m_\tau/m_\mu, \text{lbl})$ it is easy to check that the values of Refs. [35,36] differ from Eq. (36) because their derivations did not include terms of $O(m_\mu/m_\tau)$, which turn out to be unexpectedly large. The sums of Eqs. (33) and (34) and Eqs. (35) and (36) are

$$A_2^{(6)}(m_\tau/m_e) = 46.3921(15), \quad (37)$$

$$A_2^{(6)}(m_\tau/m_\mu) = 7.01021(76). \quad (38)$$

The contribution of the three-loop diagrams with both electron- and muon-loop insertions in the photon propagator can be calculated numerically from the integral expressions of [11]. We get

$$A_3^{(6)}(m_\tau/m_e, m_\tau/m_\mu) = 3.34797(41). \quad (39)$$

This value disagrees with the results of Refs. [36] (1.679) and [35] (2.75316). Combining the three-loop results of Eqs. (8) and (37)–(39) we find the sixth-order QED coefficient

$$C_\tau^{(6)} = 57.9315(27). \quad (40)$$

The error 2.7×10^{-3} induces a 3×10^{-11} uncertainty in a_τ^{QED} . The order of magnitude of the three-loop contribution to a_τ^{QED} , dominated by the mass-dependent terms, is comparable to that of electroweak and hadronic effects.

Adding up all the above contributions and using the new value $\alpha^{-1} = 137.035999710(96)$ [30] (or the value derived in Eq. (26), $\alpha^{-1} = 137.035999709(96)$ —the difference is totally negligible) we obtain the total QED contribution to the $g - 2$ of the τ :

$$a_\tau^{\text{QED}} = 117324(2) \times 10^{-8}. \quad (41)$$

The error $\delta a_\tau^{\text{QED}}$ is the uncertainty $\delta C_\tau^{(8)}(\alpha/\pi)^4 \sim \pi^2 \ln^2(m_\tau/m_e)(\alpha/\pi)^4 \sim 2 \times 10^{-8}$ which we assigned to a_τ^{QED} for uncalculated four-loop contributions. As we mentioned earlier, the errors due to the uncertainties of the $O(\alpha^2)$ and $O(\alpha^3)$ terms are negligible. The error induced by the uncertainty of α is only 8×10^{-13} (and thus totally negligible).

The $g - 2$ of the τ is a very interesting observable, even if the short lifetime of this lepton makes its measurement very difficult at present. The possibility to improve the recent experimental bounds [7] is certainly not excluded.

IV. MUON

This final section reports the results relevant to a_μ^{QED} (see [37] for recent reviews of the entire SM prediction). Some of them were already presented in [15]. The two-loop contributions are

$$A_2^{(4)}(m_\mu/m_e) = 1.0942583111(84), \quad (42)$$

$$A_2^{(4)}(m_\mu/m_\tau) = 0.000078064(25). \quad (43)$$

The sum of Eqs. (3), (42), and (43) provides the coefficient $C_\mu^{(4)} = 0.765857410(27)$. The value $\delta C_\mu^{(4)} = 2.7 \times 10^{-8}$ was obtained adding in quadrature the errors in Eqs. (42) and (43). It produces a tiny 1.4×10^{-13} uncertainty in a_μ^{QED} . The three-loop contributions are

$$A_2^{(6)}(m_\mu/m_e, \text{vac}) = 1.920455130(33) \quad (44)$$

$$A_2^{(6)}(m_\mu/m_e, \text{lbl}) = 20.94792489(16) \quad (45)$$

$$A_2^{(6)}(m_\mu/m_\tau, \text{vac}) = -0.00178233(48) \quad (46)$$

$$A_2^{(6)}(m_\mu/m_\tau, \text{lbl}) = 0.00214283(69) \quad (47)$$

$$A_2^{(6)}(m_\mu/m_e) = 22.86838002(20) \quad (48)$$

$$A_2^{(6)}(m_\mu/m_\tau) = 0.00036051(21). \quad (49)$$

The analytic calculation of the three-loop diagrams with both electron and τ loop insertions in the photon propagator became available in 1999 [13] and was confirmed more recently [38]. This analytic result yields the numerical value [15]

$$A_3^{(6)}(m_\mu/m_e, m_\mu/m_\tau) = 0.00052766(17), \quad (50)$$

providing a small 0.7×10^{-11} contribution to a_μ^{QED} . The error 1.7×10^{-7} is caused by the uncertainty of the ratio m_μ/m_τ . Combining the three-loop results of Eqs. (8) and (48)–(50) we get the three-loop coefficient $C_\mu^{(6)} = 24.05050964(43)$. The error 4.3×10^{-7} induces a negligible $O(10^{-14})$ uncertainty in a_μ^{QED} .

Adding the four-loop and leading five-loop contributions computed by Kinoshita and Nio, $C_\mu^{(8)} = 130.9916(80)$ [29,39] and $C_\mu^{(10)} = 663(20)$ [40] (estimates obtained with the renormalization-group method agree with this five-loop result [41]), and using the new value $\alpha^{-1} = 137.035999710(96)$ [30] (or the value derived in Eq. (26), $\alpha^{-1} = 137.035999709(96)$ —the difference is negligible) we get the new total QED contribution to the muon $g - 2$,

$$a_\mu^{\text{QED}} = 116584718.09(14)(08) \times 10^{-11}. \quad (51)$$

The first error is determined by the uncertainties of the QED coefficients (dominated by the five-loop one, $\delta C_\mu^{(10)} = 20$), while the second is caused by the tiny uncertainty $\delta\alpha$. Equation (51) is in good agreement with the recent value $a_\mu^{\text{QED}} = 116584717.62(14)(85) \times 10^{-11}$ [40], and the uncertainty due to $\delta\alpha$ is strongly reduced.

ACKNOWLEDGMENTS

I wish to thank A. Ferroglia, M. Giacomini, T. Kinoshita, D. Maître, P. Minkowski, and P. J. Mohr for useful discussions and correspondence.

-
- [1] T. Kinoshita and W. J. Marciano, in *Quantum Electrodynamics*, edited by T. Kinoshita (World Scientific, Singapore, 1990), pp. 419–478.
[2] B. Odom, D. Hanneke, B. D’Urso, and G. Gabrielse, *Phys. Rev. Lett.* **97**, 030801 (2006).
[3] J. S. Schwinger, *Phys. Rev.* **73**, 416 (1948).
[4] P. J. Mohr and B. N. Taylor, *Rev. Mod. Phys.* **77**, 1 (2005).

- [5] K. Hagiwara *et al.* (Particle Data Group), *Phys. Rev. D* **66**, 010001 (2002).
[6] S. Eidelman *et al.* (Particle Data Group), *Phys. Lett. B* **592**, 1 (2004).
[7] W.-M. Yao *et al.* (Particle Data Group), *J. Phys. G* **33**, 1 (2006).
[8] C. M. Sommerfield, *Phys. Rev.* **107**, 328 (1957); *Ann.*

- Phys. (N.Y.) **5**, 26 (1958); A. Petermann, *Helv. Phys. Acta* **30**, 407 (1957); *Nucl. Phys.* **5**, 677 (1958).
- [9] H. Suura and E. Wichmann, *Phys. Rev.* **105**, 1930 (1957); A. Petermann, *Phys. Rev.* **105**, 1931 (1957).
- [10] H.H. Elend, *Phys. Lett.* **20**, 682 (1966); **21**, 720(E) (1966).
- [11] M. A. Samuel and G. Li, *Phys. Rev. D* **44**, 3935 (1991); **46**, 4782(E) (1992); **48**, 1879(E) (1993).
- [12] G. Li, R. Mendel, and M. A. Samuel, *Phys. Rev. D* **47**, 1723 (1993).
- [13] A. Czarnecki and M. Skrzypek, *Phys. Lett. B* **449**, 354 (1999).
- [14] L. Lewin, *Polylogarithms and Associated Functions* (Elsevier, North Holland, 1981).
- [15] M. Passera, *J. Phys. G* **31**, R75 (2005).
- [16] J. A. Mignaco and E. Remiddi, *Nuovo Cimento A* **60**, 519 (1969); R. Barbieri and E. Remiddi, *Phys. Lett. B* **49**, 468 (1974); R. Barbieri, M. Caffo, and E. Remiddi, *ibid.* **57**, 460 (1975); M.J. Levine, E. Remiddi, and R. Roskies, *Phys. Rev. D* **20**, 2068 (1979); S. Laporta and E. Remiddi, *Phys. Lett. B* **265**, 182 (1991); S. Laporta, *Phys. Rev. D* **47**, 4793 (1993); *Phys. Lett. B* **343**, 421 (1995); S. Laporta and E. Remiddi, *Phys. Lett. B* **356**, 390 (1995).
- [17] S. Laporta and E. Remiddi, *Phys. Lett. B* **379**, 283 (1996).
- [18] T. Kinoshita, *Phys. Rev. Lett.* **75**, 4728 (1995); *Quantum Electrodynamics*, edited by T. Kinoshita (World Scientific, Singapore, 1990), pp. 218–321.
- [19] S. Laporta, *Nuovo Cimento A* **106**, 675 (1993).
- [20] S. Laporta and E. Remiddi, *Phys. Lett. B* **301**, 440 (1993).
- [21] T. Kinoshita, *Nuovo Cimento B* **51**, 140 (1967); B.E. Lautrup and E. de Rafael, *Phys. Rev.* **174**, 1835 (1968); R. Barbieri and E. Remiddi, *Nucl. Phys.* **B90**, 233 (1975); B.E. Lautrup and M. A. Samuel, *Phys. Lett. B* **72**, 114 (1977).
- [22] E. Remiddi and J. A. M. Vermaseren, *Int. J. Mod. Phys. A* **15**, 725 (2000); T. Gehrmann and E. Remiddi, *Comput. Phys. Commun.* **141**, 296 (2001); **144**, 200 (2002).
- [23] S. Wolfram, *The Mathematica Book* (Wolfram Media/Cambridge University Press, Cambridge, England, 2003), p. 731 5th ed.
- [24] D. Maître, *Comput. Phys. Commun.* **174**, 222 (2006).
- [25] J.H. Kühn *et al.*, *Phys. Rev. D* **68**, 033018 (2003).
- [26] R. S. Van Dyck, P.B. Schwinberg, and H.G. Dehmelt, *Phys. Rev. Lett.* **59**, 26 (1987).
- [27] A. Czarnecki, B. Krause, and W.J. Marciano, *Phys. Rev. Lett.* **76**, 3267 (1996).
- [28] M. Davier and A. Höcker, *Phys. Lett. B* **435**, 427 (1998); B. Krause, *Phys. Lett. B* **390**, 392 (1997).
- [29] T. Kinoshita and M. Nio, *Phys. Rev. D* **73**, 013003 (2006).
- [30] G. Gabrielse, D. Hanneke, T. Kinoshita, M. Nio, and B. Odom, *Phys. Rev. Lett.* **97**, 030802 (2006).
- [31] P. Cladé *et al.*, *Phys. Rev. Lett.* **96**, 033001 (2006).
- [32] A. Wicht *et al.*, in *Proceedings of the 6th Symp. on Freq. Standards and Metrology* (World Scientific, Singapore, 2002), pp. 193–212; *Phys. Scripta T* **102**, 82 (2002).
- [33] V. Gerginov *et al.*, *Phys. Rev. A* **73**, 032504 (2006).
- [34] M.P. Bradley *et al.*, *Phys. Rev. Lett.* **83**, 4510 (1999).
- [35] S. Narison, *Phys. Lett. B* **513**, 53 (2001); **526**, 414(E) (2002).
- [36] M. A. Samuel, G. Li, and R. Mendel, *Phys. Rev. Lett.* **67**, 668 (1991); **69**, 995(E) (1992).
- [37] M. Passera, *Nucl. Phys. B, Proc. Suppl.* **155**, 365 (2006); M. Davier and W.J. Marciano, *Annu. Rev. Nucl. Part. Sci.* **54**, 115 (2004); M. Knecht, hep-ph/0307239; A. Nyffeler, *Acta Phys. Pol. B* **34**, 5197 (2003); K. Melnikov, *Int. J. Mod. Phys. A* **16**, 4591 (2001); V.W. Hughes and T. Kinoshita, *Rev. Mod. Phys.* **71**, S133 (1999); A. Czarnecki and W.J. Marciano, *Nucl. Phys. B, Proc. Suppl.* **76**, 245 (1999).
- [38] S. Friot, D. Greynat, and E. de Rafael, *Phys. Lett. B* **628**, 73 (2005).
- [39] T. Kinoshita and M. Nio, *Phys. Rev. D* **70**, 113001 (2004).
- [40] T. Kinoshita and M. Nio, *Phys. Rev. D* **73**, 053007 (2006).
- [41] A.L. Kataev, *Phys. Lett. B* **284**, 401 (1992); *Nucl. Phys. B, Proc. Suppl.* **155**, 369 (2006); hep-ph/0602098; A.L. Kataev and V. V. Starshenko, *Phys. Rev. D* **52**, 402 (1995).