Matching the Hagedorn temperature in AdS/CFT correspondence

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We match the Hagedorn/deconfinement temperature of planar $\mathcal{N} = 4$ super Yang-Mills (SYM) on $\mathbb{R} \times S^3$ to the Hagedorn temperature of string theory on $AdS_5 \times S^5$. The match is done in a near-critical region where both gauge theory and string theory are weakly coupled. The near-critical region is near a point with zero temperature and critical chemical potential. On the gauge-theory side we are taking a decoupling limit found in Ref. [7] in which the physics of planar $\mathcal{N} = 4$ SYM is given exactly by the ferromagnetic $XXX_{1/2}$ Heisenberg spin chain. We find moreover a general relation between the Hagedorn/deconfinement temperature and the thermodynamics of the Heisenberg spin chain and we use this to compute it in two distinct regimes. On the string-theory side, we identify the dual limit for which the string tension and string coupling go to zero. This limit is taken of string theory on a maximally supersymmetric pp-wave background with a flat direction, obtained from a Penrose limit of $AdS_5 \times S^5$. We compute the Hagedorn temperature of the string theory and find agreement with the Hagedorn/deconfinement temperature computed on the gauge-theory side.

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I. INTRODUCTION AND SUMMARY

The AdS/CFT correspondence states that $SU(N)$ $\mathcal{N} = 4$ super Yang-Mills (SYM) on $\mathbb{R} \times S^3$ is equivalent to string theory on $AdS_5 \times S^5$ [1–3]. In particular, planar $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$ with 't Hooft coupling λ is conjectured to be equivalent to weakly-coupled string theory on $AdS_5 \times S^5$ with string tension $T_{\rm str}$, with the relation[1]

$$T_{\rm str} = \frac{1}{2}\sqrt{\lambda} \tag{1.1}$$

The most impressive checks on this correspondence have involved computing physical quantities on the gauge-theory side, such as the expectation value of Wilson loops [4,5] or the anomalous dimensions of gauge-theory operators [6], and extrapolating the results to strong coupling in order to compare with string theory.

In this paper we take a different route. We compute the Hagedorn/deconfinement temperature for planar $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$ at weak coupling $\lambda \ll 1$ in a certain near-critical region found in [7]. We match then this to the Hagedorn temperature computed in weakly-coupled string theory on $AdS_5 \times S^5$, in the corresponding dual near-critical region. Beyond this, we successfully match the low energy spectra of the gauge theory and the string theory in the near-critical region. The matching of the spectra and Hagedorn temperature is possible since we take a zero string tension limit on the string-theory side.

That the Hagedorn/deconfinement temperature of planar $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$ is dual to the Hagedorn temperature of string theory on $AdS_5 \times S^5$ was conjectured in [8–11]. This originated in the finding of a confinement/deconfinement phase transition in planar $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$ at weak coupling $\lambda \ll 1$ [8]. For large energies the theory has a Hagedorn density of states, with the Hagedorn temperature being equal to the deconfinement temperature [9–11].

On the string-theory side, it is unfortunately not possible to compute the Hagedorn temperature for string theory on $AdS_5 \times S^5$ since we do not know how to make a first quantization of string theory in this background. However, in certain Penrose limits, where the $AdS_5 \times S^5$ background becomes a maximally supersymmetric pp-wave background [6,12], one can find the string spectrum, and the computation of the Hagedorn temperature has been done [13–20].

From these facts it is clear that any successful matching of the Hagedorn/deconfinement temperature for the gauge theory with the Hagedorn temperature of string theory should be to the Hagedorn temperature of the maximally supersymmetric pp-wave background. Therefore, one should make the match for large R-charges/angular momenta.

However, if we consider the pp-wave/gauge-theory correspondence of [6] we encounter a problem. In [6] the gauge-theory states that are conjectured to correspond to string states on the pp-wave side are only a small subset of the possible gauge-theory states. But, at weak coupling $\lambda \ll 1$, all of these possible gauge-theory states are present. The crucial step of [6], in order to resolve this problem, is to consider a strong coupling limit $\lambda \to \infty$ on the gauge-theory side in which it is conjectured that most of the gauge-theory states will decouple, and only a small subset of the states, believed to be precisely the ones dual to the string states, should remain in this limit. More specifically, the ground state and zero modes of the pp-wave string theory are mapped to chiral primary states in

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[1]See main text for our conventions regarding $T_{\rm str}$ and λ.

$\mathcal{N} = 4$ SYM, and the surviving states in the large λ limit should then be the states that lie sufficiently close to the chiral primary states with respect to their anomalous dimensions. Thus, seemingly, we cannot match the Hagedorn/deconfinement temperature at weak coupling $\lambda \ll 1$ to the pp-wave Hagedorn temperature, since on the gauge-theory side we have many more states than the ones dual to the pp-wave string states.

In this paper we resolve this problem by employing a recently found decoupling limit of thermal $SU(N)$ $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$ [7]. Denoting the three R-charges for the $SU(4)$ R-symmetry as J_i, $i = 1, 2, 3$, and their corresponding chemical potentials as Ω_i, $i = 1, 2, 3$, and putting $(\Omega_1, \Omega_2, \Omega_3) = (\Omega, \Omega, 0)$, we can write the decoupling limit as [7]

$$T \to 0, \qquad \Omega \to 1, \qquad \lambda \to 0,$$
$$\tilde{T} \equiv \frac{T}{1 - \Omega} \text{ fixed}, \qquad \tilde{\lambda} \equiv \frac{\lambda}{1 - \Omega} \text{ fixed}, \qquad N \text{ fixed} \tag{1.2}$$

where T is the temperature for $\mathcal{N} = 4$ SYM. In this limit only the states in the $SU(2)$ sector survive, and $SU(N)$ $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$ reduces to a quantum mechanical theory with temperature $\tilde{T}$ and coupling $\tilde{\lambda}$. In the planar limit $N = \infty$, we have furthermore that in the limit (1.2) $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$ has the Hamiltonian $D_0 + \tilde{\lambda} D_2$, where D_0 is the bare scaling dimension and D_2 is the Hamiltonian for the ferromagnetic $XXX_{1/2}$ Heisenberg spin chain (without magnetic field). We see that the limit (1.2) includes taking a zero 't Hooft coupling limit $\lambda \to 0$, thus we are in weakly-coupled $\mathcal{N} = 4$ SYM after the limit.

The resolution to the above stated problem that there are too many states for $\lambda \ll 1$ is now as follows. Since $\tilde{\lambda}$ can be finite even though $\lambda \to 0$ we can consider, in particular, the $\tilde{\lambda} \gg 1$ region. In this region the low energy states for the D_2 Hamiltonian are the dominant states. These states are the vacua, plus the magnon states of the Heisenberg spin chain. The vacua precisely consist of the chiral primary sector of the $SU(2)$ sector. Therefore, by considering $\tilde{\lambda} \gg 1$ we can circumvent the apparent problem with matching the pp-wave spectrum to the spectrum of weakly-coupled gauge theory.

For planar $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$ in the decoupling limit (1.2) we find a direct connection between the Hagedorn/deconfinement temperature for finite $\tilde{\lambda}$ and the thermodynamics of the Heisenberg spin chain. If we denote t as the temperature and $-tV(t)$ as the thermodynamic limit of the free energy per site for the ferromagnetic Heisenberg chain with Hamiltonian D_2, then the Hagedorn temperature $\tilde{T} = \tilde{T}_H$ is given by

$$\tilde{T}_H = \frac{1}{V(\tilde{\lambda}^{-1} \tilde{T}_H)} \tag{1.3}$$

We use this to compute the Hagedorn temperature for small $\tilde{\lambda}$, in which case it corresponds to the high temperature limit of the Heisenberg chain. For large $\tilde{\lambda}$ the Hagedorn temperature is instead mapped to the low temperature limit of the Heisenberg chain, and we obtain in this limit the Hagedorn temperature

$$\tilde{T}_H = (2\pi)^{1/3} \left[\zeta\left(\frac{3}{2}\right) \right]^{-(2/3)} \tilde{\lambda}^{1/3} \tag{1.4}$$

where $\zeta(x)$ is the Riemann zeta function. Note that we have that the low energy behavior of the Heisenberg chain is tied to the large $\tilde{\lambda}$ limit, as we also stated above. In fact, the low energy spectrum consisting of the chiral primary vacua with the magnon spectrum gives rise to the Hagedorn temperature (1.4).

On the string-theory side, we find using the AdS/CFT duality the following decoupling limit of string theory on $AdS_5 \times S^5$, dual to the limit (1.2),

$$\epsilon \to 0, \qquad \tilde{H} \equiv \frac{E - J}{\epsilon} \text{ fixed}, \qquad \tilde{T}_{\rm str} \equiv \frac{T_{\rm str}}{\sqrt{\epsilon}} \text{ fixed},$$
$$\tilde{g}_s \equiv \frac{g_s}{\epsilon} \text{ fixed}, \qquad J_i \text{ fixed} \tag{1.5}$$

Here E is the energy of the strings, J_i, $i = 1, 2, 3$, are the angular momenta for the five-sphere, $J = J_1 + J_2$, and g_s is the string coupling. $\tilde{H}$ is the effective Hamiltonian for the strings in the decoupling limit. We see that both the string tension $T_{\rm str}$ and the string coupling g_s go to zero in this limit.

The next step is to consider a Penrose limit of the $AdS_5 \times S^5$ background, and to consider the string theory on the resulting pp-wave background. We note that the Penrose limit of [6] does not result in the right light-cone quantized string theory spectrum for our purposes. We need a pp-wave spectrum for which all states with $E = J$, $J = J_1 + J_2$, correspond to the string vacua. This is precisely what the Penrose limit of [12] provides. In more detail, on the gauge theory/spin chain side, $J_1 - J_2$ measures the total spin, and we have a vacuum for each value of the total spin. The dual manifestation of this is that in the pp-wave background that is obtained using the Penrose limit of [12] we have a flat direction, such that there is a vacuum for each value of the momentum along that direction, and that momentum is moreover dual to $J_1 - J_2$.

We implement then the decoupling limit (1.5) for the pp-wave background. This corresponds to a large μ limit of the pp-wave, with μ being a parameter in front of the square-well potential terms for six of the eight bosonic directions. We show that we get the same spectrum as that of the spectrum for large $\tilde{\lambda}$ and J of planar $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$ in the decoupling limit (1.2). Thus, we can match the spectrum of weakly-coupled string theory with weakly-coupled gauge theory in the decoupling limits.

We proceed to compute the Hagedorn temperature for string theory on the pp-wave background in the large μ limit in two different ways. The first way is to compute the Hagedorn temperature from the spectrum obtained by taking the large μ limit directly on the spectrum. The second way is to take the Hagedorn temperature for the full pp-wave spectrum, which was computed in [17], and take the large μ limit of that. The two ways of computing the Hagedorn temperature agree, which is a good check on the fact that most of the string states really do decouple in the large μ limit. Moreover, the resulting Hagedorn temperature can, via the AdS/CFT duality, be compared to the Hagedorn/deconfinement temperature (1.4) computed in weakly-coupled gauge theory, and they are shown to agree.

In summary, we match the Hagedorn/deconfinement temperature computed in weakly-coupled planar $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$, in the decoupling limit (1.2), to the Hagedorn temperature computed on a maximally supersymmetric pp-wave background in the dual decoupling limit (1.5). The fact that we are in a pp-wave background corresponds to being in the large J sector of string theory on $\mathrm{AdS}_5 \times S^5$. Moreover, we show that the low energy spectra of gauge theory and string theory in the decoupling limit are the same, which can be seen as the underlying reason for the matching of the Hagedorn temperature. In the Conclusions in Sec. VIII we discuss the matching in the larger framework of a decoupled sector of the AdS/CFT correspondence for which we have a spin chain/gauge-theory/string theory triality.

II. THE $SU(2)$ DECOUPLING LIMIT OF $\mathcal{N} = 4$ SYM ON $\mathbb{R} \times S^3$

In this section we review the decoupling limit of $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$ found in [7] in which $\mathcal{N} = 4$ SYM reduces to a quantum mechanical theory on the $SU(2)$ sector which becomes the ferromagnetic $XXX_{1/2}$ Heisenberg spin chain in the planar limit.

A. Thermal $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$ and the Hagedorn temperature

In [7] the thermal partition function of $SU(N)$ $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$ with nonzero chemical potentials is considered. We can write this in general as follows. Let D denote the dilatation operator giving the scaling dimension for a given operator (or energy of the corresponding state). Let J_i, $i = 1, 2, 3$, denote the three R-charges associated with the $SU(4)$ R-symmetry of $\mathcal{N} = 4$ SYM, and let Ω_i be the three chemical potentials corresponding to these charges. Then we can write the full partition function as

$$Z(\beta, \Omega_i) = \mathrm{Tr}(e^{-\beta D + \beta \sum_{i=1}^{3} \Omega_i J_i}) \tag{2.1}$$

where $\beta = 1/T$ is the inverse temperature. Here the trace is taken over all gauge-invariant states, corresponding to all the multitrace operators. When $\mathcal{N} = 4$ SYM is weakly coupled, we can expand the dilatation operator in powers of the 't Hooft coupling as follows [21,22]

$$D = D_0 + \sum_{n=2}^{\infty} \lambda^{n/2} D_n \tag{2.2}$$

where we have defined for our convenience the 't Hooft coupling as

$$\lambda = \frac{g_{\mathrm{YM}}^2 N}{4\pi^2} \tag{2.3}$$

g_{YM} being the Yang-Mills coupling of $\mathcal{N} = 4$ SYM.

For free $SU(N)$ $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$ in the planar limit $N = \infty$ the partition function exhibits a singularity at a certain temperature T_H [9–11]. The temperature T_H is a Hagedorn temperature for planar $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$ since the density of states goes like e^{E/T_H} for high energies $E \gg 1$ (we work in units with radius of the S^3 set to one). Moreover, we have that for $T < T_H$ the partition function is of order one, while for $T > T_H$ the partition function is of order N^2, and for large temperatures the partition function is like for free $SU(N)$ $\mathcal{N} = 4$ SYM on $\mathbb{R}^4$. Therefore we have a transition at T_H resembling the confinement/deconfinement phase transition in QCD, thus in this sense we can regard T_H as a deconfinement temperature for planar $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$.[2]

Turning on the coupling λ and the chemical potentials Ω_i the Hagedorn singularity for planar $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$ persists, at least for $\lambda \ll 1$ [7,24,25]. The Hagedorn temperature T_{H} is a function of λ and Ω_i, and it is known in certain limits. The first order correction in λ for $\Omega_i = 0$ was found in [24]. For $\lambda = 0$ and nonzero chemical potentials Ω_i the Hagedorn temperature was found in [7,25] while the one-loop correction was found in [7]. E.g. for weak coupling and small chemical potentials it is found that [7]

$$T_H = \frac{1}{\beta_0}\left(1 + \frac{\lambda}{2}\right) - \frac{1}{6\sqrt{3}}\left(1 - \frac{\lambda}{2}(11 - \beta_0\sqrt{3})\right)\sum_{i=1}^{3} \Omega_i^2 + \mathcal{O}(\lambda^2) + \mathcal{O}(\Omega_i^4) \tag{2.4}$$

with $\beta_0 \equiv -\log(7 - 4\sqrt{3})$. See [7] for the fourth order correction in the chemical potentials.

B. The $SU(2)$ decoupling limit

It was found in [7] that near the critical point $(T, \Omega_1, \Omega_2, \Omega_3) = (0, 1, 1, 0)$ most of the states of $\mathcal{N} = 4$ SYM decouple and we end up with a much simpler theory that we can regard as quantum mechanical. In order to write the decoupling limit we define the charge $J =$

[2] In [23] it was found for weakly-coupled large N pure Yang-Mills theory on $\mathbb{R} \times S^3$ that the deconfinement temperature is lower than the Hagedorn temperature, which means that this theory has a first order phase transition at the deconfinement temperature.

$J_1 + J_2$ and we define Ω as the corresponding chemical potential. In the following we are interested in the situation for which $\Omega_1 = \Omega_2 = \Omega$. We consider then the decoupling limit [7]

$$T \to 0, \qquad \Omega \to 1, \qquad \lambda \to 0, \qquad \tilde{T} \equiv \frac{T}{1-\Omega} \text{ fixed}, \qquad \tilde{\lambda} \equiv \frac{\lambda}{1-\Omega} \text{ fixed} \tag{2.5}$$

In this limit most of the states of $\mathcal{N} = 4$ SYM decouple. This is due to the fact that only the states with $D - J$ being of order $1 - \Omega$ survive. Therefore the states that survive are the ones with $D_0 = J$, i.e. with the bare scaling dimension equal to J. From this one can see that the total Hilbert space of the theory consists of all states corresponding to all the multitrace operators that one can write down from the two complex scalars Z and X, where Z and X have the R-charge weights $(1, 0, 0)$ and $(0, 1, 0)$, respectively. Thus, we have that our Hilbert space $\mathcal{H}$ consists of all possible linear combinations of the multitrace operators[3]

$$\mathrm{Tr}(A_1^{(1)} A_2^{(1)} \cdots A_{L_1}^{(1)}) \mathrm{Tr}(A_1^{(2)} A_2^{(2)} \cdots A_{L_2}^{(2)}) \cdots \times \mathrm{Tr}(A_1^{(k)} A_2^{(k)} \cdots A_{L_k}^{(k)}), \qquad A_j^{(i)} = Z, X \tag{2.6}$$

This is in fact the so-called $SU(2)$ sector of recent interest in the study of integrability of $\mathcal{N} = 4$ SYM [21,26–29]. We can view this as a quantum mechanical subset of $\mathcal{N} = 4$ SYM in the sense that all the states with covariant derivatives are decoupled, which can be interpreted to mean that the modes corresponding to moving around on the S^3 disappear, leaving us with only one point.

Furthermore, as we show in [7], the partition function (2.1) reduces in the decoupling limit (2.5) to the partition function

$$Z(\tilde{\beta}) = \mathrm{Tr}_{\mathcal{H}}(e^{-\tilde{\beta} H}) \tag{2.7}$$

with H being the Hamiltonian

$$H = D_0 + \tilde{\lambda} D_2 \tag{2.8}$$

Here $\tilde{\beta} = 1/\tilde{T}$, thus we see that $SU(N)$ $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$ in the limit (2.5) reduces to a quantum mechanical theory with Hilbert space $\mathcal{H}$ given by (2.6) and with Hamiltonian (2.8), with effective temperature $\tilde{T}$. Moreover, $\tilde{\lambda}$ can be regarded as the coupling of the theory, being a remnant of the 't Hooft coupling of $\mathcal{N} = 4$ SYM. It is very interesting to observe that we thus end up with a theory with two coupling constants: $\tilde{\lambda}$ and $1/N$, both of which we can choose freely. Indeed, since the D_2 term in (2.8) origins in the one-loop correction to the scaling dimension, we have full knowledge of the Hamiltonian (2.8) and we can in principle compute $Z(\tilde{\beta})$ for any value of $\tilde{\lambda}$ and N.

We can view the decoupling limit (2.5) from the alternative view point as a decoupling limit of nonthermal $SU(N)$ $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$. Then the decoupling limit is instead[4]

$$\epsilon \to 0, \qquad \frac{D-J}{\epsilon} \text{ fixed}, \qquad \tilde{\lambda} \equiv \frac{\lambda}{\epsilon} \text{ fixed}, \qquad J_i \text{ fixed}, \qquad N \text{ fixed} \tag{2.9}$$

Note then that the effective Hamiltonian is $\lim_{\epsilon\to 0} \frac{D-J}{\epsilon} = \tilde{\lambda} D_2$. We see that this is in accordance with the Hamiltonian (2.8) since we are restricting ourselves to be in a certain sector of fixed J. We see that this limit is remarkably different from pp-wave limits of $\mathcal{N} = 4$ SYM [6] in which one takes J and N to go to infinity and instead fixes $D - J$. However, as we shall see below we have an overlap between the two types of limits for a particular pp-wave limit found in [12].

C. The planar limit and the Heisenberg spin chain

If we consider the planar limit $N \to \infty$ of $SU(N)$ $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$, we know from large N factorization that the single-trace operators are decoupled from the multitrace operators. Therefore, in the planar limit, we can regard single-trace operators of a certain length as states for a spin chain where the letters correspond to the value of the spin [26]. In the $SU(2)$ sector, the single-trace operators are linear combinations of

$$\mathrm{Tr}(A_1 A_2 \cdots A_L), \qquad A_i = Z, X \tag{2.10}$$

If we write $S_z = (J_1 - J_2)/2$ we see that each Z has $S_z = 1/2$ and each X has $S_z = -1/2$, thus we get an $SU(2)$ spin chain. Furthermore, in the planar limit the D_2 term in (2.8) is given by [21,26]

$$D_2 = \frac{1}{2} \sum_{i=1}^{L} (I_{i,i+1} - P_{i,i+1}) \tag{2.11}$$

for a chain of length L, where $P_{i,i+1}$ is the permutation operator acting on letters at position i and $i + 1$. From this one can see that $\tilde{\lambda} D_2$ precisely is the Hamiltonian for a ferromagnetic $XXX_{1/2}$ Heisenberg spin chain of length L [26]. We can therefore write the single-trace partition function as [7]

$$Z_{\mathrm{ST}}(\tilde{\beta}) = \sum_{L=1}^{\infty} e^{-\tilde{\beta} L} Z_L^{(\mathrm{XXX})}(\tilde{\beta}) \tag{2.12}$$

where

[3]Here we will loosely refer to the single-trace or multitrace operators as states in a Hilbert space, the precise meaning being that any single-trace or multitrace operator $\mathcal{O}$ for $\mathcal{N} = 4$ SYM on $\mathbb{R}^4$ has a corresponding gauge-invariant state $|\mathcal{O}\rangle = \lim_{r\to 0} \mathcal{O}|0\rangle$ for $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$ (r being the radial coordinate of $\mathbb{R}^4$), and vice versa, by the state/operator correspondence.

[4]When we write that J_i is fixed we mean that all three R-charges J_1, J_2 and J_3 are fixed.

$$Z_L^{(\mathrm{XXX})}(\tilde{\beta}) = \mathrm{Tr}_L(e^{-\tilde{\beta}\tilde{\lambda}D_2}) \tag{2.13}$$

is the partition function for the ferromagnetic $XXX_{1/2}$ Heisenberg spin chain of length L with Hamiltonian $\tilde{\lambda}D_2$. Note that Tr_L here refers to the trace over single-trace operators with $J = L$ in the $SU(2)$ sector. The spin chain is required to be periodic and translationally invariant in accordance with the cyclic symmetry of single-trace operators. Using the standard relation between the single-trace and multitrace partition functions, we get that the full partition function of planar $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$ in the limit (2.5) is [7]

$$\log Z(\tilde{\beta}) = \sum_{n=1}^{\infty}\sum_{L=1}^{\infty}\frac{1}{n}e^{-\tilde{\beta}nL}Z_L^{(\mathrm{XXX})}(n\tilde{\beta}) \tag{2.14}$$

Therefore, the partition function of planar $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$ in the decoupling limit (2.5) is given exactly by (2.14) from the partition function $Z_L^{(\mathrm{XXX})}(\tilde{\beta})$ of the ferromagnetic $XXX_{1/2}$ Heisenberg spin chain [7].

III. GAUGE-THEORY SPECTRUM IN DECOUPLING LIMIT

In this section we find the large $\tilde{\lambda}$ and large L limit of the spectrum of planar $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$ in the decoupling limit (2.5).

From (2.8) we know that planar $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$ in the limit (2.5) has the Hamiltonian $L + \tilde{\lambda}D_2$, for single-traces of length L. Therefore, finding the spectrum of planar $\mathcal{N} = 4$ SYM in this decoupling limit is identical to the problem of finding the spectrum of the Heisenberg chain Hamiltonian $\tilde{\lambda}D_2$. The solution to this for low energies is well-known. Nevertheless, we rederive the spectrum in the following since we are interested in the case where we have a degeneracy of the vacuum with respect to the total spin. In our approach we employ a new way of putting in impurities which seems more natural for this situation. It also makes a direct construction of the eigenstates corresponding to the spectrum possible.

We begin by noting that the large $\tilde{\lambda}$ limit of the spectrum alternatively can be viewed as the low energy part of the spectrum for finite $\tilde{\lambda}$, since the interacting term in the Hamiltonian is $\tilde{\lambda}D_2$.

The low energy part of the spectrum of the ferromagnetic Heisenberg chain consists of the ferromagnetic vacuum states plus magnon excitations. The ferromagnetic vacua consist of all the states for which $P_{i,i+1}$ has eigenvalue one for any neighboring sites of the spin chain. One can make such a state for each possible value of the total spin S_z, here given by

$$S_z = \frac{1}{2}(J_1 - J_2) \tag{3.1}$$

In detail we have that the vacuum state for a given length L and total spin S_z is the totally symmetrized state [7]

$$|S_z\rangle_L \sim \mathrm{Tr}(\mathrm{sym}(Z^{J_1}X^{J_2})) \tag{3.2}$$

with $J_1 = \frac{1}{2}L + S_z$ and $J_2 = \frac{1}{2}L - S_z$. We see thus that we have $L + 1$ ferromagnetic vacua for a given length L. As observed in [7], the vacua (3.2) are precisely the chiral primary states with $D_0 = J$.

It will be useful below to have a more specific way of describing the vacuum states. To this end, define $A_{1/2} = Z$ and $A_{-1/2} = X$. Then we can write the basis of the $SU(2)$ sector as

$$\mathrm{Tr}\,(A_{s(1)}\cdots A_{s(L)}) \tag{3.3}$$

where $s(i) = \pm 1/2$ corresponds to having spin up or spin down. Write

$$Q = \left\{ s = (s(1), \ldots, s(L)) | \sum_{i=1}^{L} s(i) = S_z \right\} \tag{3.4}$$

Then we have that the vacuum for a given value of S_z and L is

$$|S_z\rangle_L \sim \sum_{s\in Q}\mathrm{Tr}(A_{s(1)}\cdots A_{s(L)}) \tag{3.5}$$

Turning to the magnons, which are the low energy excitations of the ferromagnetic vacua, we see that we cannot employ the usual Bethe ansatz technique of putting X impurities into a sea of Z's. This is due to the fact that we want to work in the limit in which the number of excitations is much less than L, and clearly it would take of order L impurities to describe excitations around vacua with $J_1 \ll J_2$. This difference to the usual approach basically comes in because the $J_1 = L$ vacuum $\mathrm{Tr}(Z^L)$ is not special, instead we have $L + 1$ vacua which are equally important.

Thus, we need a new way to put in impurities that does not change the value of S_z. The way to do this becomes clearer if we think of an impurity as the action of an operator on a particular site. In particular changing a Z at site number l into an X can be thought of as the action of S_- at site l. We instead want an operator in the $SU(2)$ group that commutes with the total spin S_z. Therefore, we propose that inserting an impurity corresponds to the action of S_z at a particular site l.[5]

Consider the insertion of two impurities. Define $S_{z,l}$ as the action of $\frac{1}{2}(Z\partial_Z - X\partial_X)$ on the site number l. We can then write the insertion of two impurities at sites l_1 and l_2 in the vacuum state $|S_z\rangle_L$ as

$$|l_1, l_2; S_z\rangle_L = S_{z,l_1}S_{z,l_2}|S_z\rangle_L \tag{3.6}$$

Using the form (3.5) for the vacuum states, we see that this corresponds to

$$|l_1, l_2; S_z\rangle_L \sim \sum_{s\in Q}s(l_1)s(l_2)\,\mathrm{Tr}(A_{s(1)}\cdots A_{s(L)}) \tag{3.7}$$

[5]This way of constructing magnons is inspired from the construction of gauge-theory states in [12].

We now want to find an eigenstate of the Hamiltonian $\tilde{\lambda} D_2$ with two impurities. Write

$$|\Psi\rangle = \sum_{1\leq l_1\leq l_2\leq L} \Psi(l_1, l_2)|l_1, l_2; S_z\rangle_L \tag{3.8}$$

The task is then to find $\Psi(l_1, l_2)$ such that

$$\tilde{\lambda} D_2|\Psi\rangle = \tilde{\lambda}\mathcal{E}|\Psi\rangle \tag{3.9}$$

To this end, we employ the Bethe ansatz

$$\Psi(l_1, l_2) = e^{ip_1 l_1 + ip_2 l_2} A_{12} + e^{ip_2 l_1 + ip_1 l_2} A_{21} \tag{3.10}$$

It is not hard to see that the eigenvalue Eq. (3.9) then gives

$$\mathcal{E} = 2\sum_{k=1}^{2} \sin^2\left(\frac{p_k}{2}\right), \qquad S(p_1, p_2) \equiv \frac{A_{12}}{A_{21}} = -\frac{1 + e^{i(p_1+p_2)} - 2e^{ip_1}}{1 + e^{i(p_1+p_2)} - 2e^{ip_2}} \tag{3.11}$$

Periodicity of the spin chain instead requires

$$e^{ip_1 L} = S(p_1, p_2), \qquad e^{ip_2 L} = S(p_2, p_1) \tag{3.12}$$

Furthermore, the cyclicity of the trace requires $p_1 + p_2 = 0$. Using these conditions, one can easily determine the spectrum for two impurities.

Considering the general case of inserting q impurities, we can use the integrability of the Heisenberg chain to find the spectrum, giving

$$\mathcal{E} = 2\sum_{i=1}^{q} \sin^2\left(\frac{p_i}{2}\right) \tag{3.13}$$

$$e^{ip_k L} = \prod_{j=1, j\neq k}^{q} S(p_k, p_j), \qquad S(p_k, p_j) = -\frac{1 + e^{i(p_k+p_j)} - 2e^{ip_k}}{1 + e^{i(p_k+p_j)} - 2e^{ip_j}} \tag{3.14}$$

$$\sum_{i=1}^{q} p_i = 0 \tag{3.15}$$

where (3.15) is due to the cyclicity of the trace. Taking the logarithm of (3.14) we have

$$p_k - \frac{2\pi n_k}{L} = -\frac{i}{L}\sum_{j=1, j\neq k}^{q} \log S(p_k, p_j) \tag{3.16}$$

where n_k is an integer. The leading order solution for large L is

$$p_k = \frac{2\pi n_k}{L} + \mathcal{O}(L^{-2}) \tag{3.17}$$

giving the spectrum

$$\mathcal{E} = \frac{2\pi^2}{L^2}\sum_{i=1}^{q} n_i^2, \qquad \sum_{i=1}^{q} n_i = 0 \tag{3.18}$$

Denoting the number of n_i which are equal to a particular integer k as M_k, we can write this spectrum as

$$\mathcal{E} = \frac{2\pi^2}{L^2}\sum_{n\neq 0} n^2 M_n, \qquad \sum_{n\neq 0} n M_n = 0 \tag{3.19}$$

This is the low energy spectrum of the Heisenberg spin chain with spin chain Hamiltonian D_2. Using now that for a single-trace operator of length L, the eigenvalue of the Hamiltonian $H = D_0 + \tilde{\lambda} D_2$ is $L + \tilde{\lambda}\mathcal{E}$, we see that the Hamiltonian H has the spectrum

$$H - L = \frac{2\pi^2 \tilde{\lambda}}{L^2}\sum_{n\neq 0} n^2 M_n, \qquad \sum_{n\neq 0} n M_n = 0 \tag{3.20}$$

This is the large $\tilde{\lambda}$ and large L limit of the spectrum of single-trace operators in planar $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$ in the decoupling limit (2.5).

We see that the spectrum (3.20) is stringlike, even though we are in weakly-coupled gauge theory. This is in contrast with previous approaches to find a stringlike spectrum in $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$, since those approaches rely on having λ large in order to decouple gauge-theory states which are not near the chiral primary states. Thus, in this sense, the spectrum (3.19) is the first example of a stringlike spectrum found in weakly-coupled $\mathcal{N} = 4$ SYM. As we shall see in Sec. VI, the resemblance to a string-spectrum is not accidental, and we can in fact map it to a spectrum of string states in a decoupling limit of strings on a pp-wave.

IV. GAUGE-THEORY HAGEDORN TEMPERATURE FROM THE HEISENBERG CHAIN

In this section we consider the Hagedorn temperature of planar $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$ in the decoupling limit (2.5) from a general perspective, and we find a relation between the Hagedorn temperature as function of $\tilde{\lambda}$ and the thermodynamics of the Heisenberg chain in the thermodynamic limit. We use this general connection to find the Hagedorn temperature for small and large $\tilde{\lambda}$.

A. General considerations

From (2.13) and (2.14) we have that the full partition function of planar $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$ in the decoupling limit (2.5) is

$$\log Z(\tilde{\beta}) = \sum_{n=1}^{\infty}\sum_{L=1}^{\infty} \frac{1}{n} e^{-\tilde{\beta} n L}\, \mathrm{Tr}_L(e^{-n\tilde{\beta}\tilde{\lambda} D_2}) \tag{4.1}$$

Define now the function $V(t)$ by

$$V(t) \equiv \lim_{L\to\infty} \frac{1}{L} \log[\mathrm{Tr}_L(e^{-t^{-1} D_2})] \tag{4.2}$$

This limit is well-defined since the thermodynamic limit of the free energy per site $f(t)$ at temperature t for the

Heisenberg chain is related to $V(t)$ by

$$f(t) = -tV(t) \tag{4.3}$$

Note that here the Hamiltonian of the ferromagnetic Heisenberg chain is D_2. We notice now that for large L

$$e^{-\tilde{\beta} nL}\,\mathrm{Tr}_L(e^{-n\tilde{\beta}\tilde{\lambda}D_2}) \simeq \exp(-nL\tilde{\beta} + LV[(n\tilde{\beta}\tilde{\lambda})^{-1}]) \tag{4.4}$$

Therefore, for $n = 1$ we see that we reach a singularity if $\tilde{\beta}$ decreases to $\tilde{\beta}_H$ given by[6]

$$\tilde{\beta}_H = V((\tilde{\beta}_H\tilde{\lambda})^{-1}) \tag{4.5}$$

This is the Hagedorn temperature for general $\tilde{\lambda}$. Thus, we have obtained a direct connection between the thermodynamics of the Heisenberg chain in the thermodynamic limit and the Hagedorn temperature.

We see now immediately from Eq. (4.5) that the Hagedorn temperature for $\tilde{\lambda} \ll 1$ is obtained from the high temperature limit $t \gg 1$ of the Heisenberg chain, while for $\tilde{\lambda} \gg 1$ the Hagedorn temperature is obtained from the low temperature limit $t \ll 1$. In the following we use this to obtain the Hagedorn temperature in these two regimes.

B. Hagedorn temperature for small $\tilde{\lambda}$

If we consider $t \to \infty$ in (4.2) we see that we can find the Hagedorn temperature from $\mathrm{Tr}_L(1)$. This corresponds to counting the number of independent single-trace operators of length L. This is less than 2^L but also bigger than $2^L/L$ since the cyclic symmetry of the trace can at most relate L states to each other. For large L we have therefore to leading order $\mathrm{Tr}_L(1) \simeq 2^L$. Inserting that in (4.2) we see that $V(t) \to \log 2$ for $t \to \infty$. This corresponds to $\tilde{\beta}_H = \log 2$ which is the correct Hagedorn temperature for the free $SU(2)$ sector.

We can also find the first correction to the Hagedorn temperature for small $\tilde{\lambda}$ in this fashion. For large t we see that

$$V(t) = \lim_{L\to\infty} \frac{1}{L}\left[\log \mathrm{Tr}_L(1) - t^{-1}\frac{\mathrm{Tr}_L(D_2)}{\mathrm{Tr}_L(1)}\right] \tag{4.6}$$

It is not hard to see that for large L

$$\frac{\mathrm{Tr}_L(D_2)}{\mathrm{Tr}_L(1)} \simeq \frac{L}{4} \tag{4.7}$$

Therefore, we get

$$V(t) = \log 2 - \frac{1}{4t} + \mathcal{O}(t^{-2}) \tag{4.8}$$

for large t. We see from (4.8) that $-tV(t)$ indeed is the previously computed high temperature limit of the free energy per site for the Heisenberg chain [30]. Inserting (4.8) into (4.5) we get

$$\tilde{T}_H = \frac{1}{\log 2} + \frac{1}{4\log 2}\tilde{\lambda} + \mathcal{O}(\tilde{\lambda}^2) \tag{4.9}$$

which precisely matches the Hagedorn temperature found previously in [7,24]. Note that the above computation of the Hagedorn temperature completely circumvents the somewhat complicated computation of the full single-trace partition function.

A much more powerful method of obtaining the high temperature behavior of the Heisenberg chain has been found in [31]. The result is that $V(t)$ as defined in (4.2) can be found from the integral equation

$$u(x) = 2 + \oint_C \frac{dy}{2\pi i}\left\{\frac{1}{x-y-2i}\exp\left[-\frac{2t^{-1}}{y(y+2i)}\right] + \frac{1}{x-y+2i}\exp\left[-\frac{2t^{-1}}{y(y-2i)}\right]\right\}\frac{1}{u(y)} \tag{4.10}$$

where C is a loop around the origin directed counterclockwise. $V(t)$ is then determined as

$$V(t) = \log[u(0)] \tag{4.11}$$

One can then make a systematic high energy expansion of $u(x)$ in powers of t^{-1} as

$$\log[u(x)] = \sum_{k=0}^{\infty} u_k(x)t^{-k} \tag{4.12}$$

Using (4.10) we can now determine $u(x)$ order by order in t^{-1}. This gives the high temperature expansion of $V(t)$ to order t^{-5}

$$V(t) = \log 2 - \frac{1}{4t} + \frac{3}{32t^2} - \frac{1}{64t^3} - \frac{5}{1024t^4} + \frac{3}{1024t^5} + \mathcal{O}(t^{-6}) \tag{4.13}$$

for large t. Inserting (4.13) into (4.5) we get[7]

$$\begin{aligned}\tilde{T}_H = {}& \frac{1}{\log 2} + \frac{1}{4\log 2}\tilde{\lambda} - \frac{3}{32}\tilde{\lambda}^2 + \left(\frac{3}{128} + \frac{\log 2}{64}\right)\tilde{\lambda}^3 \\ &+ \left(-\frac{3}{512} - \frac{17\log 2}{1024} + \frac{5(\log 2)^2}{1024}\right)\tilde{\lambda}^4 \\ &+ \left(\frac{3}{2048} + \frac{39\log 2}{4096} + \frac{3(\log 2)^2}{4096} - \frac{3(\log 2)^3}{1024}\right)\tilde{\lambda}^5 \\ &+ \mathcal{O}(\tilde{\lambda}^6)\end{aligned} \tag{4.14}$$

for small $\tilde{\lambda}$. It is straightforward to extend this to higher

[6]Note that there is a singularity for each value of n, but the $n = 1$ singularity is the first one that is reached as one decreases $\tilde{\beta}$ from infinity. This is seen using that $V(t)$ is a monotonically increasing function of t.

[7]Note that the $\tilde{\lambda}^2$ term matches the D_2^2 contribution to the λ^2 correction for the Hagedorn temperature in the $SU(2)$ sector found in [32].

orders in $\tilde{\lambda}$, e.g. from the results of [31] one can find $V(t)$ to order t^{-50} and thereby $\tilde{T}_H$ to order $\tilde{\lambda}^{50}$.

C. Hagedorn temperature for large $\tilde{\lambda}$

As stated above, we see from (4.5) that the Hagedorn temperature for large $\tilde{\lambda}$ is given from low temperature limit of the ferromagnetic Heisenberg chain. Therefore, to compute the Hagedorn temperature in this limit, we should use the low energy spectrum (3.19) of the Heisenberg chain to compute $V(t)$ for small t. Inserting the spectrum (3.19) in the partition function for the Heisenberg chain, we see that for large L and small t we have

$$\mathrm{Tr}_L(e^{-t^{-1}D_2}) = L\sum_{\{M_n\}}\int_{-1/2}^{1/2} du \exp\left(-\frac{2\pi^2}{tL^2}\sum_{n\neq 0} n^2 M_n + 2\pi i u \sum_{n\neq 0} n M_n\right) \quad (4.15)$$

where the integration over u is introduced to impose the cyclicity constraint in the spectrum (3.19). The L factor is due to the $L+1$ different vacua for a given L. Evaluating the sums over the M_n's (the sum range being from zero to infinity) we get

$$\mathrm{Tr}_L(e^{-t^{-1}D_2}) = L\int_{-1/2}^{1/2} du \prod_{n\neq 0}\left[1-\exp\left(-\frac{2\pi^2}{tL^2}n^2 + 2\pi i u n\right)\right]^{-1} = L\int_{-1/2}^{1/2} du \left|G\left(\frac{2\pi}{tL^2}, 2\pi u\right)\right|^2 \quad (4.16)$$

where $G(a, b)$ is the generating function defined by Eq. (A1) in the Appendix. We want to extract from (4.16) the part that diverges for $L \to \infty$. Using the analysis of the Appendix we get that the leading contribution to this divergence is from $u = 0$, which using Eq. (A9) is seen to give

$$\mathrm{Tr}_L(e^{-t^{-1}D_2}) \sim \exp\left\{L\zeta\left(\frac{3}{2}\right)\sqrt{\frac{t}{2\pi}}\right\} \quad (4.17)$$

for $L \to \infty$. Here $\zeta(x)$ is the Riemann zeta function. Inserting (4.17) into (4.2) and (4.16) we get

$$V(t) = \zeta\left(\frac{3}{2}\right)\sqrt{\frac{t}{2\pi}} \quad (4.18)$$

for $t \ll 1$. This result is the same as the analytically obtained result [30,33] for the low energy limit of the free energy $-tV(t)$ for the Heisenberg chain. As we discuss further below, it is also consistent with numerical calculations [34–37].

Applying now the result (4.18) to Eq. (4.5), we get the Hagedorn temperature

$$\tilde{T}_H = (2\pi)^{1/3}\left[\zeta\left(\frac{3}{2}\right)\right]^{-(2/3)}\tilde{\lambda}^{1/3} \quad (4.19)$$

for $\tilde{\lambda} \gg 1$. This is the Hagedorn temperature of planar $\mathcal{N} = 4$ SYM on $\mathbb{R}\times S^3$ in the decoupling limit (2.5) for large $\tilde{\lambda}$. We see that the Hagedorn temperature (4.19) goes to infinity for $\tilde{\lambda} \to \infty$. This is consistent with the fact that for $\tilde{\lambda} \to \infty$ all other states except the chiral primary states decouple, and the partition function ends up being a sum only over the chiral primaries, which means that we should not expect the presence of a Hagedorn singularity in this limit.

As stated above, the result (4.18) obtained for the low temperature limit of $V(t)$ is the same as that obtained for the ferromagnetic Heisenberg chain in [30,33], where also the next order of $V(t)$ has been computed

$$V(t) = \zeta\left(\frac{3}{2}\right)\sqrt{\frac{t}{2\pi}} - t + \mathcal{O}(t^{3/2}) \quad (4.20)$$

for $t \ll 1$. This result is consistent with numerical calculations, which reveals [34–37]

$$V(t) = 1.042\sqrt{t} - 1.00t + \mathcal{O}(t^{3/2}) \quad (4.21)$$

for $t \ll 1$. Using now (4.20) in (4.5) we find the following correction to the Hagedorn temperature

$$\tilde{T}_H = \frac{(2\pi)^{1/3}}{\zeta(\frac{3}{2})^{2/3}}\tilde{\lambda}^{1/3} + \frac{4\pi}{3\zeta(\frac{3}{2})^2} + \mathcal{O}(\tilde{\lambda}^{-1/3}) \quad (4.22)$$

for large $\tilde{\lambda}$.

V. DECOUPLING LIMIT OF STRING THEORY ON $\mathrm{AdS}_5 \times S^5$

As reviewed in Sec. II, thermal $\mathcal{N} = 4$ SYM on $\mathbb{R}\times S^3$ decouples to $SU(2)$ sector in the decoupling limit (2.5) [7]. We consider in this section the corresponding limit that one obtains for type IIB string theory on $\mathrm{AdS}_5 \times S^5$ by employing the AdS/CFT duality [1–3].

We consider type IIB string theory on the $\mathrm{AdS}_5 \times S^5$ background given by the metric

$$ds^2 = R^2[-\cosh^2\rho dt^2 + d\rho^2 + \sinh^2\rho d\Omega_3'^2 + d\theta^2 + \sin^2\theta d\alpha^2 + \cos^2\theta d\Omega_3^2] \quad (5.1)$$

and the fiveform Ramond-Ramond field strength

$$F_{(5)} = 2R^4(\cosh\rho\sinh^3\rho dt d\rho d\Omega_3' + \sin\theta\cos^3\theta d\theta d\alpha d\Omega_3) \quad (5.2)$$

The AdS/CFT correspondence then fixes that $R^4 = 4\pi g_s l_s^4 N$ and $g_{\mathrm{YM}}^2 = 4\pi g_s$, where g_s is the string coupling and l_s is the string length. g_{YM}^2 and N are the gauge coupling and rank of $SU(N)$ as defined in Sec. II. With this, we see that we have the following dictionary between the gauge-theory quantities λ and N, and the string theory quantities g_s, l_s and the AdS radius R

$$T_{\rm str} \equiv \frac{R^2}{4\pi l_s^2} = \frac{1}{2}\sqrt{\lambda}, \qquad g_s = \frac{\pi\lambda}{N} \tag{5.3}$$

where $T_{\rm str}$ is the string tension for a fundamental string in the $AdS_5 \times S^5$ background (5.1) and (5.2).

A. Decoupling limit for strings on $AdS_5 \times S^5$ and induced gauge/string duality

We can now translate the decoupling limit reviewed in Sec. II. We consider first the nonthermal version of the decoupling limit given by (2.9). This limit translates into the following limit of type IIB string theory on the $AdS_5 \times S^5$ background (5.1) and (5.2)

$$\epsilon \to 0, \qquad \tilde{H} \equiv \frac{E-J}{\epsilon} \text{ fixed}, \qquad \tilde{T}_{\rm str} \equiv \frac{T_{\rm str}}{\sqrt{\epsilon}} \text{ fixed},$$
$$\tilde{g}_s \equiv \frac{g_s}{\epsilon} \text{ fixed}, \qquad J_i \text{ fixed} \tag{5.4}$$

Here E is the energy of the string while J_i, $i = 1, 2, 3$, are the three angular momenta for the five-sphere corresponding to the three R-charges of $\mathcal{N} = 4$ SYM. The energy E for a string state is equal to the scaling dimension D of a gauge-theory state of $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$ since we set the radius of the three-sphere to one. Note furthermore that we have defined $J = J_1 + J_2$.

We see that in this limit we scale the energies in such a way that in free string theory ($g_s = 0$) only string states for which $E - J \sim T_{\rm str}^2$ as $T_{\rm str} \to 0$ can survive. As in the gauge theory, we can regard this as a limit in which we look at small excitations near the BPS states with $E = J$. Note that even for $g_s = 0$ the obtained tree-level string theory is nontrivial since we have an effective string tension $\tilde{T}_{\rm str}$.

It is interesting to observe that in the limit (5.4) the string coupling goes to zero. From this and the corresponding gauge-theory limit (2.9), we see that the AdS/CFT correspondence in this limit necessarily becomes a duality between weakly-coupled $\mathcal{N} = 4$ SYM and weakly-coupled string theory.

After taking the limit (2.9) of $\mathcal{N} = 4$ SYM and the limit (5.4) of string theory on $AdS_5 \times S^5$, the AdS/CFT duality induces a duality between the decoupled sectors on the gauge-theory and string-theory sides. From the two limits (2.9) and (5.4) we see that we obtain a dictionary for this induced duality relating the quantities we keep finite in the limits:

$$\tilde{T}_{\rm str} = \frac{1}{2}\sqrt{\tilde{\lambda}}, \qquad \tilde{g}_s = \frac{\pi\tilde{\lambda}}{N} \tag{5.5}$$

We see that this induced dictionary perfectly mirrors the original AdS/CFT dictionary (5.3).

Finally, we note also that the string tension $T_{\rm str}$ goes to zero. Zero tension limits of string theory on $AdS_5 \times S^5$ have previously been connected to higher-spin theories. However, here we know from the gauge-theory side that only a particular sector of the theory survives the limit.

B. Decoupling limit of thermal partition function for strings on $AdS_5 \times S^5$

If we consider instead a gas of strings in the $AdS_5 \times S^5$ background (5.1) and (5.2) we can write the general partition function as

$$Z(\beta, \Omega_i) = \mathrm{Tr}(e^{-\beta E + \beta \sum_{i=1}^{3} \Omega_i J_i}) \tag{5.6}$$

where J_i, $i = 1, 2, 3$, are the angular momenta and Ω_i, $i = 1, 2, 3$, are the corresponding angular velocities. Here we trace over all the multistring states. Just like on the gauge-theory side we consider here the only special case $(\Omega_1, \Omega_2, \Omega_3) = (\Omega, \Omega, 0)$. Therefore, the partition function can be written

$$Z(\beta, \Omega_i) = \mathrm{Tr}(e^{-\beta E + \beta \Omega J}) \tag{5.7}$$

where $J = J_1 + J_2$. We now want to consider the region close to the critical point $(T, \Omega) = (0, 1)$. We notice first that we can rewrite the weight factor in (5.7) as

$$e^{-\beta E + \beta \Omega J} = e^{-\beta(1-\Omega)J - \beta(1-\Omega)E-J/1-\Omega} \tag{5.8}$$

From the gauge-theory decoupling limit (2.5) and the string theory decoupling limit (5.4) it is then clear that the appropriate limit for a string gas is

$$T \to 0, \qquad \Omega \to 1, \qquad \tilde{T} = \frac{T}{1-\Omega} \text{ fixed},$$
$$\tilde{H} \equiv \frac{E-J}{1-\Omega} \text{ fixed} \qquad \tilde{T}_{\rm str} \equiv \frac{T_{\rm str}}{\sqrt{1-\Omega}} \text{ fixed}, \tag{5.9}$$
$$\tilde{g}_s \equiv \frac{g_s}{1-\Omega} \text{ fixed}, \qquad J_i \text{ fixed}$$

Using (5.7) and (5.8) the partition function for the string gas becomes

$$Z(\tilde{\beta}) = \mathrm{Tr}_{\mathcal{M}_s}(e^{-\tilde{\beta}(J+\tilde{H})}) \tag{5.10}$$

where $\mathcal{M}_s$ is defined as the set of all multistring states that survive the limit (5.9). We see that in the limit (5.9) we effectively end up with a theory for a string gas of temperature $\tilde{T}$ and energies given by $J + \tilde{H}$, and with a reduced set of string states compared to the full string theory on $AdS_5 \times S^5$.

VI. CONNECTION TO PP-WAVE WITH FLAT DIRECTION

In Sec. V we found a decoupling limit of string theory on $AdS_5 \times S^5$ which is dual to the $SU(2)$ decoupling limit of $\mathcal{N} = 4$ SYM reviewed in Sec. II. We do not know a first quantization of string theory on $AdS_5 \times S^5$. Therefore, we consider instead taking the decoupling limit (5.4) for string theory on a particular pp-wave background, obtained from $AdS_5 \times S^5$ by a Penrose limit. As we explain in the following, this pp-wave background is particularly well-suited for this limit, and we find indeed a successful match of the string theory and gauge-theory spectra.

A. Penrose limit for pp-wave with flat direction

We begin this section by employing a Penrose limit of $AdS_5 \times S^5$ found in [12] giving rise to a maximally supersymmetric pp-wave background with a flat direction. It is important to note that the Penrose limit is implemented in a slightly different manner here than in [12] in order to be consistent with the decoupling limit (5.4) for strings on $AdS_5 \times S^5$. We explain in Sec. VI B why the Penrose limit of [12] has the right features for the decoupling limit (5.4) that we are going to implement.

We begin by considering the $AdS_5 \times S^5$ background (5.1) and (5.2). We see from the decoupling limit (5.4) that the AdS radius R goes to zero like $\epsilon^{1/4}$ in the limit. We define therefore a rescaled AdS radius $\tilde{R}$ as follows

$$\tilde{R}^4 = \frac{R^4}{\epsilon} \tag{6.1}$$

Consider now the three-sphere Ω_3 part of the metric (5.1). Following [12], we can parameterize the three-sphere embedded in the five-sphere as

$$\begin{aligned} d\Omega_3^2 &= d\psi^2 + \sin^2\psi d\phi^2 + \cos^2\psi d\chi^2 \\ &= d\psi^2 + d\phi_-^2 + d\phi_+^2 + 2\cos(2\psi)d\phi_- d\phi_+ \end{aligned} \tag{6.2}$$

where we defined the angles $\phi_\pm$ as

$$\phi_\pm = \frac{\chi \pm \phi}{2} \tag{6.3}$$

Define now the coordinates x^+, x^-, x^1, x^2, r, $\tilde{r}$ by

$$x^- = \frac{1}{2}\mu\tilde{R}^2(t - \phi_+), \qquad x^+ = \frac{1}{2\mu}(t + \phi_+) \tag{6.4}$$

$$\begin{aligned} x^1 &= \tilde{R}\phi_-, \qquad x^2 = \tilde{R}\left(\psi - \frac{\pi}{4}\right), \\ r &= \tilde{R}\rho, \qquad \tilde{r} = \tilde{R}\theta \end{aligned} \tag{6.5}$$

Note that these coordinates are defined in terms of the rescaled AdS radius $\tilde{R}$. We then take the Penrose limit of the $AdS_5 \times S^5$ background (5.1) and (5.2) given by [12]

$$\tilde{R} \to \infty, x^+, x^-, x^1, x^2, r, \tilde{r}, \alpha \ \text{ fixed} \tag{6.6}$$

This gives the following pp-wave background with 32 supersymmetries

$$\begin{aligned} \frac{ds^2}{\sqrt{\epsilon}} &= -4dx^+dx^- - \mu^2 \sum_{I=3}^{8} x^I x^I (dx^+)^2 + \sum_{i=1}^{8} dx^i dx^i \\ &\quad - 4\mu x^2 dx^1 dx^+ \end{aligned} \tag{6.7}$$

$$\frac{F_{(5)}}{\epsilon} = 2\mu dx^+(dx^1dx^2dx^3dx^4 + dx^5dx^6dx^7dx^8) \tag{6.8}$$

This background was first found in [38].[8] Here x^3, x^4 are defined by $x^3 + ix^4 = \tilde{r}e^{i\alpha}$ and $x^5, \ldots, x^8$ are defined by $r^2 = \sum_{I=5}^8 (x^I)^2$ and $dr^2 + r^2 d\Omega_3'^2 = \sum_{I=5}^8 (dx^I)^2$. We see that the fact that we employed the rescaled AdS radius in the Penrose limit give rise to factors of ϵ in the metric and fiveform field strength. This will be important below.

It is important to note that the pp-wave background (6.7) and (6.8) has the special feature that x^1 is an explicit isometry of the pp-wave [12,38], hence we call this background a pp-wave with a flat direction.

In terms of the generators, we see that in the Penrose limit (6.6) we have

$$H_{\rm lc} = \sqrt{\epsilon}\mu(E - J), \qquad p^+ = \frac{E + J}{2\mu R^2}, \qquad p_1 = \frac{2S_z}{\tilde{R}} \tag{6.9}$$

where $H_{\rm lc}$ is the light-cone Hamiltonian, p^+ is the light-cone momentum and p_1 is the momentum along the x^1 direction. Here $J_1 = \frac{1}{2}J + S_z$ and $J_2 = \frac{1}{2}J - S_z$ are the angular momenta of the strings on the three-sphere (6.2).

From [12,38] we have that the strings can be quantized in the light-cone gauge with the following spectrum of the light-cone Hamiltonian $H_{\rm lc}$

$$\begin{aligned} \frac{l_s^2 p^+}{\sqrt{\epsilon}} H_{\rm lc} &= 2fN_0 + \sum_{n\neq 0}[(\omega_n + f)N_n + (\omega_n - f)M_n] \\ &\quad + \sum_{n\in\mathbb{Z}}\sum_{I=3}^{8}\omega_n N_n^{(I)} + \sum_{n\in\mathbb{Z}}\Bigg[\sum_{b=1}^{4}\left(\omega_n - \frac{1}{2}f\right)F_n^{(b)} \\ &\quad + \sum_{b=5}^{8}\left(\omega_n + \frac{1}{2}f\right)F_n^{(b)}\Bigg] \end{aligned} \tag{6.10}$$

with level matching condition

$$\sum_{n\neq 0} n\left[N_n + M_n + \sum_{I=3}^{8} N_n^{(I)} + \sum_{b=1}^{8} F_n^{(b)}\right] = 0 \tag{6.11}$$

and where we have defined

$$f = \mu l_s^2 p^+, \qquad \omega_n = \sqrt{n^2 + f^2} \tag{6.12}$$

Here $N_n^{(I)}$, $I = 3, \ldots, 8$ and $n \in Z$, are the number operators for bosonic excitations for the six directions $x^3, \ldots, x^8$, while N_n, $n \in \mathbb{Z}$, and M_n, $n \neq 0$, are the number operators for the two directions x^1 and x^2. $F_n^{(b)}$, $b = 1, \ldots, 8$ and $n \in$

[8] The pp-wave background (6.7) and (6.8) is related to the maximally supersymmetric pp-wave background of [6,39] by a coordinate transformation [12,38]. Even so, as we shall see in the following, the physics of this pp-wave is rather different, which basically origins in the fact that the coordinate transformation between them depends on x^+, i.e. it is time-dependent. See [12] for more comments on this.

$\mathbb{Z}$, are the number operators for the fermions. Note that the presence of the flat direction x^1 of the pp-wave is responsible for the fact that we only have seven bosonic zero modes N_0 and $N_0^{(3)}, \ldots, N_0^{(8)}$.

It is important to note that the vacua for the string spectrum are degenerate with respect to the eigenvalues of the momentum p_1 along the flat direction. I.e. we have a vacuum $|0, p_1, p^+\rangle$ for each value of p_1, and given any particular vacuum $|0, p_1, p^+\rangle$ we have the spectrum (6.10) of string excitations.

B. Decoupling limit of pp-wave spectrum and matching of spectra

We can now explain why the pp-wave background (6.7) and (6.8) is relevant for our decoupling limit (5.4) for strings on $AdS_5 \times S^5$. We see from (6.9) that the Penrose limit (6.6) corresponds to a limit in which $J = J_1 + J_2 \to \infty$ while $E - J$ is fixed. Thus, we keep all excitations that have a finite value of $E - J$. In particular, we keep any excitation which has a small $E - J$ and which is still present for large J.

Another argument why the pp-wave background (6.7) and (6.8) is suitable for our considerations is that the light-cone vacua $H_{lc} = 0$ correspond to $1/2$ BPS states with $E = J$. These $1/2$ BPS states are mapped to the chiral primary states of $\mathcal{N} = 4$ SYM with $D = J$, which precisely correspond to the vacua on the gauge-theory side.

We now implement the decoupling limit (5.4) on the pp-wave background (6.7) and (6.8). Notice first that we want to keep p^+ fixed in the decoupling limit. This gives us that $\mu\sqrt{\epsilon}$ should be held fixed. Using (6.9) we find that the decoupling limit (5.4) translates to the following decoupling limit on the pp-wave background (6.7) and (6.8)

$$\begin{gathered} \epsilon \to 0, \qquad \mu \to \infty, \qquad \tilde{\mu} \equiv \mu\sqrt{\epsilon} \ \text{fixed}, \\ \tilde{H}_{lc} \equiv \frac{H_{lc}}{\epsilon} \ \text{fixed}, \qquad \tilde{g}_s \equiv \frac{g_s}{\epsilon} \ \text{fixed}, \\ l_s, \qquad p^+ \ \text{fixed} \end{gathered} \tag{6.13}$$

Clearly this can be seen as a large μ limit of the pp-wave.

It is important to remark that the limit (6.13) is consistent with the Penrose limit (6.6) since the limit relies on having large $\tilde{R}$ and large J and these are both kept fixed in the limit (6.13). Furthermore, we see from (6.9) and (6.13) that we have

$$p^+ = \frac{J}{\tilde{\mu}\tilde{R}^2} \tag{6.14}$$

so having p^+ fixed is consistent with having large J and large $\tilde{R}$.

We consider now the spectrum of the light-cone Hamiltonian (6.10) and (6.11) in the limit (6.13). First we notice that $f \to \infty$, so $f^{-1}\omega_n \simeq 1 + n^2/(2f^2) + \mathcal{O}(f^{-4})$. Therefore, most of the excitations have $\epsilon^{-(1/2)} l_s^2 p^+ H_{lc}$ of order f. Such excitations do not survive the limit (6.13). It is easy to see that this means that $N_n = 0$, $N^{(I)} = 0$ and $F_n^{(b)} = 0$ for $n \in \mathbb{Z}$. Only the excitations connected to the number operator M_n have a chance of surviving since $\omega_n - f$ is not of order f when $f \to \infty$. Focusing on these excitations, we have

$$\frac{l_s^2 p^+}{\sqrt{\epsilon}} H_{lc} = \sum_{n \neq 0} (\omega_n - f) M_n \simeq \sum_{n \neq 0} \frac{n^2}{2f} M_n \tag{6.15}$$

We get therefore in the limit (6.13) the spectrum

$$\tilde{H}_{lc} = \frac{1}{2\tilde{\mu}(l_s^2 p^+)^2} \sum_{n \neq 0} n^2 M_n, \qquad \sum_{n \neq 0} n M_n = 0 \tag{6.16}$$

where we also included the level matching condition obtained from (6.11).

We now want to show that this spectrum indeed matches the spectrum (3.19) obtained in weakly-coupled $\mathcal{N} = 4$ SYM. First we notice that the fact that the string vacua are degenerate with respect to the momentum p_1 precisely fits with the fact that the gauge-theory vacua (3.5) are degenerate with respect to S_z, as one can see explicitly from (6.9).

As a next step, we see from (6.1) and (6.14) that

$$(\tilde{\mu} l_s^2 p^+)^2 = \frac{J^2}{4\pi^2 \tilde{\lambda}} \tag{6.17}$$

Thus, the Penrose limit (6.6) corresponds, in terms of the gauge theory, to the limit

$$\tilde{\lambda} \to \infty, \qquad J \to \infty, \qquad \frac{\tilde{\lambda}}{J^2} \ \text{fixed} \tag{6.18}$$

This fits perfectly with the fact that we want to match the spectrum (6.16) to the spectrum of planar $\mathcal{N} = 4$ SYM in the decoupling limit (2.5) for large $\tilde{\lambda}$ and large $J = L$. Employing now (6.18) we see that we can rewrite (6.16) as

$$\frac{1}{\tilde{\mu}} \tilde{H}_{lc} = \frac{2\pi^2 \tilde{\lambda}}{J^2} \sum_{n \neq 0} n^2 M_n, \qquad \sum_{n \neq 0} n M_n = 0 \tag{6.19}$$

This precisely matches the spectrum (3.19) of $\tilde{\lambda} D_2$ on the gauge-theory side, since we have $J = L$. Notice that the $1/\tilde{\mu}$ in (6.19) origins from (6.9), thus it is $\tilde{H}_{lc}/\tilde{\mu}$ and $\tilde{\lambda} D_2$ that one should match.

In conclusion, we have found that we can match the spectrum of weakly-coupled string theory in the pp-wave regime and in the pp-wave decoupling limit (6.13), with the spectrum of weakly-coupled planar $\mathcal{N} = 4$ SYM in the decoupling limit (2.5) for large $\tilde{\lambda}$ and large $J = L$. This gives a strong indication that the induced AdS/CFT correspondence suggested in Sec. V, between $\mathcal{N} = 4$ SYM in the decoupling limit (2.9) and string theory on $AdS_5 \times S^5$ in the dual decoupling limit (5.4), indeed is correct.

We note that there is a geometric picture of the large μ limit (6.13). Since the $x^3, \ldots, x^8$ directions have a square-well potential with μ as coefficient, it is clear that these directions decouple. Moreover, since only x^1 is a flat direction, while the other seven transverse directions are not, it is intuitively clear that only modes connected to the flat direction survive. Thus, we can see on a purely geometric level that it is the presence of a flat direction that enables us to perform a nontrivial large μ limit in which we have finite decoupled modes left. This is a more intuitive way to see why we are employing the pp-wave background with a flat direction (6.7) and (6.8) rather than the usual pp-wave background used in [6] in which there are no flat transverse directions.

Finally, we note that the limit (6.13) easily can be turned in to a decoupling limit for a gas of strings on the pp-wave background (6.7) and (6.8), implementing the limit (5.9) on the pp-wave. This is done by supplementing the limit (6.13) with

$$T \to 0, \qquad \Omega \to 1, \qquad \epsilon = 1 - \Omega, \qquad \tilde{T} \equiv \frac{T}{1 - \Omega} \text{ fixed} \tag{6.20}$$

in accordance with the limits (5.4) and (5.9).

C. Comments on matching of spectra

The result of Sec. VI B of the matching of the spectra of weakly-coupled gauge theory and string theory in their respective decoupling limits is a highly nontrivial result: We have matched the spectrum of gauge-theory states in weakly-coupled gauge-theory with the spectrum of free strings on a pp-wave. It is interesting to consider how it is possible that the spectra indeed can match. There are several underlying reasons for this:

(i) We can consider large $\tilde{\lambda}$ on the gauge-theory side even though we have $\lambda \to 0$ in the decoupling limit (2.5). This ensures that only the magnon states of the Heisenberg spin chain contribute. For $\lambda \ll 1$ with fixed chemical potentials there would be many more states present than the ones dual to pp-wave strings states, since this merely is a perturbation of the spectrum of free $\mathcal{N} = 4$ SYM.

(ii) That the limit involves $E - J \to 0$ means that we are expanding around the chiral primary states (3.2). Thus, we are matching states of the gauge theory and string theory which lie close to the chiral primaries.

(iii) On the gauge-theory side, the Hamiltonian truncates to $H = D_0 + \tilde{\lambda} D_2$. This enables us to compute the spectrum for large $\tilde{\lambda}$.

(iv) We have a pp-wave, being the pp-wave background (6.7) and (6.8), with the same vacuum structure as that of $\mathcal{N} = 4$ SYM in the decoupling limit (2.5). Furthermore, the pp-wave is a good approximation for large $\tilde{\lambda}$ and J, which precisely is the regime that we can match to the gauge-theory side.

(v) The pp-wave background (6.7) and (6.8) is a maximally supersymmetric background of type IIB supergravity, and is furthermore an α' exact background of type IIB string theory (see e.g. [40]). This makes the pp-wave spectrum (6.10) reliable in the decoupling limit (6.13).

In Sec. VII we match furthermore the Hagedorn temperature of gauge theory and string theory, in their respective decoupling limits. That this works can be seen as a direct consequence of the matching of the spectra.

VII. STRING THEORY HAGEDORN TEMPERATURE

In this section we compute the Hagedorn temperature for strings on the pp-wave background (6.7) and (6.8) in the decoupling limit (6.13) and (6.20) in two different ways. In Sec. VII A we compute the Hagedorn temperature directly from the reduced pp-wave spectrum (6.16). In Sec. VII B we instead take the decoupling limit (6.13) and (6.20) of the Hagedorn temperature for the full pp-wave spectrum (6.10). Both of these computations give the same result, which we show can be matched with the Hagedorn temperature (4.19) computed in weakly-coupled $\mathcal{N} = 4$ SYM.

A. Hagedorn temperature for reduced pp-wave spectrum

In this section we compute the Hagedorn temperature for the reduced pp-wave spectrum (6.16). This is the spectrum obtained for type IIB superstring theory in the pp-wave background (6.7) and (6.8) in the decoupling limit (6.13). We show that the result for the Hagedorn temperature coincides with the one of the dual gauge theory (4.19).

We consider first the multistring partition function

$$\log Z(\tilde{a}, \tilde{b}, \tilde{\mu}) = \sum_{n=1}^{\infty} \frac{1}{n} \mathrm{Tr}(e^{-\tilde{a} n \tilde{H}_{\mathrm{l.c.}} - \tilde{b} n p^+}) \tag{7.1}$$

where the trace is taken over single-string states with spectrum (6.16). The parameters $\tilde{a}$ and $\tilde{b}$ can be viewed as inverse temperature and chemical potential, respectively, for the pp-wave strings. We find the values for $\tilde{a}$ and $\tilde{b}$ in terms of the $\mathrm{AdS}_5 \times S^5$ parameters below. Note that we do not have fermions in the spectrum. The measure for the trace over p^+ is $\frac{l}{2\pi} \int_0^\infty dp^+$, where l is the (infinite) length of the 9'th dimension. We get

$$\log Z = -\sum_{n=1}^{\infty} \frac{l\tilde{\beta}}{8\pi^2 l_s^2} \int_0^\infty \frac{d\tau_2}{\tau_2^2} \int_{-(1/2)}^{1/2} d\tau_1 \sum_{M_m=0}^{\infty} e^{-(\tilde{b}n^2\tilde{\beta}/4\pi l_s^2 \tau_2) - (2\pi \tilde{a} \tau_2/\tilde{\beta}\tilde{f}) \sum_{m\neq 0} m^2 M_m + 2\pi i \tau_1 \sum_{m\neq 0} m M_m} \tag{7.2}$$

where the level matching condition is imposed by introducing an integration over the Lagrange multiplier τ_1 and we introduced the quantities

$$\tau_2 = \frac{n\tilde{\beta}}{4\pi l_s^2 p^+}, \qquad \tilde{f} = l_s^2 p^+ \tilde{\mu} = \frac{n\tilde{\beta}\,\tilde{\mu}}{4\pi\tau_2} \tag{7.3}$$

Summing over the occupation number we get

$$\log Z = -\sum_{n=1}^{\infty} \frac{l\tilde{\beta}}{8\pi^2 l_s^2} \int_0^{\infty} \frac{d\tau_2}{\tau_2^2} \times \int_{-(1/2)}^{1/2} d\tau_1 e^{-(\tilde{b}n^2\tilde{\beta}/4\pi l_s^2 \tau_2)} |G(\tau_1, \tau_2, \tilde{f})|^2 \tag{7.4}$$

where the generating function G is given by

$$G(\tau_1, \tau_2, \tilde{f}) = \prod_{m=1}^{\infty} \left(\frac{1}{1 - e^{-(2\pi\tilde{a}\tau_2/\tilde{\beta}\tilde{f})m^2 + 2\pi i \tau_1 m}} \right) \tag{7.5}$$

To see where the partition function diverges we need to estimate the asymptotic behavior of the function G. This is done in the Appendix were we show that it diverges in the limit $\tau_2 \to 0$. More precisely, one can show that for τ_2 that goes to zero, there is a divergence only if $\tau_1 = 0$ and the leading contribution is given by

$$G(0, \tau_2, \tilde{f}) \sim \exp\left(\zeta\left(\frac{3}{2}\right) \sqrt{\frac{\tilde{\beta}\tilde{f}}{8\tilde{a}\tau_2}} \right) = \exp\left(\zeta\left(\frac{3}{2}\right) \frac{\tilde{\beta}}{4\tau_2} \sqrt{\frac{n\tilde{\mu}}{2\pi\tilde{a}}} \right) \tag{7.6}$$

After substituting this result in the expression for the partition function (7.4) in the limit $\tau_2 \to 0$ we find that we have a Hagedorn singularity for[9]

$$\tilde{b}\sqrt{\tilde{a}} = l_s^2 \zeta\left(\frac{3}{2}\right) \sqrt{2\pi\tilde{\mu}} \tag{7.7}$$

where the relevant contribution is given by the $n = 1$ mode.

In order to compare (7.7) with the gauge-theory result (4.19) we have to express the parameters $\tilde{a}$ and $\tilde{b}$ in terms of the gauge-theory quantities [16]. Using Eqs. (5.5) and (6.9) it is not difficult to see that $\tilde{a}$ and $\tilde{b}$ should be identified in the following way

$$\tilde{a} = \frac{\tilde{\beta}}{\tilde{\mu}}, \qquad \tilde{b} = 4\pi l_s^2 \tilde{\beta}\,\tilde{\mu}\,\tilde{T}_{\rm str} = 2\pi l_s^2 \tilde{\beta}\,\tilde{\mu}\sqrt{\tilde{\lambda}} \tag{7.8}$$

[9]We note that to gain a better understanding of the behavior of the partition function (7.4) one should perform the integral over τ_1. This however would just produce a different power of τ_2 in the prefactor of the partition function and it would not modify the result (7.7) for the Hagedorn temperature.

With these identifications Eq. (7.7) gives

$$\tilde{T}_H = (8\pi)^{1/3} \left[\zeta\left(\frac{3}{2}\right) \right]^{-(2/3)} \tilde{T}_{\rm str}^{2/3} = (2\pi)^{1/3} \left[\zeta\left(\frac{3}{2}\right) \right]^{-(2/3)} \tilde{\lambda}^{1/3} \tag{7.9}$$

which precisely coincides with the result (4.19) obtained on the gauge-theory side.

We have thus shown that the Hagedorn temperature of type IIB string theory on $AdS_5 \times S^5$ in the decoupling limit (5.9) matches with the Hagedorn/deconfinement temperature (4.19) computed in weakly-coupled $\mathcal{N} = 4$ SYM in the dual decoupling limit (2.5). This is done in the regime of large $\tilde{\lambda}$. On the string side we obtained the Hagedorn temperature by considering the large $\tilde{\lambda}$ and J limit corresponding to strings on the pp-wave background (6.7) and (6.8) in the decoupling limit (6.13). The result means that in the sector of AdS/CFT defined by the decoupling limits (5.9) and (2.5) we can indeed show that the Hagedorn temperature for type IIB string theory on the $AdS_5 \times S^5$ background is mapped to the Hagedorn/deconfinement temperature of weakly-coupled planar $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$. Thus we have direct evidence that the confinement/deconfinement transition found in weakly-coupled planar $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$ is linked to a Hagedorn transition of string theory on $AdS_5 \times S^5$, as conjectured in [8–11].

Note that the matching of the Hagedorn temperature made above to some extent follows directly from the matching of the spectra made in Sec. VI. However, to check that the computation of the Hagedorn temperature indeed is consistent with taking the decoupling limit (6.13) and (6.20) of strings on the pp-wave background (6.7) and (6.8) we check in the following section that one can find the same Hagedorn temperature directly by taking the decoupling limit on the Hagedorn singularity for the full pp-wave spectrum (6.10).

B. Limit of Hagedorn temperature for full pp-wave spectrum

In this section we show that by computing the Hagedorn temperature using the full spectrum (6.10) and subsequently taking the limit (6.13) and (6.20) we obtain again the result (7.9) for the Hagedorn temperature.

We consider the multistring partition function

$$\log Z(a, b, \mu) = \sum_{n=1}^{\infty} \frac{1}{n} \mathrm{Tr}((-1)^{(n+1)\mathbf{F}} e^{-anH_{\rm l.c.} - bnp^+}) \tag{7.10}$$

where the trace is over single-string states with the spectrum (6.10), and $\mathbf{F}$ is the space-time fermion number. The computation of the partition function (7.10) is similar to that of the reduced spectrum done in Sec. VII A and it has been done in Ref. [17] for $b = 0$.[10] Generalizing the computation to nonzero b, we get that the Hagedorn singularity occurs for

$$b = 4l_s^2\mu \sum_{p=1}^{\infty} \frac{1}{p}\left[3 + \cosh(\mu a p) - 4(-1)^p \cosh\left(\frac{1}{2}\mu a p\right)\right] \times K_1(\mu a p) \tag{7.11}$$

where $K_\nu(x)$ is the modified Bessel function of the second kind. Using (6.9) we see that we should identify

$$a = \frac{\mu\tilde{\beta}}{\tilde{\mu}^2}, \qquad b = 4\pi\mu l_s^2 T_{\rm str}\tilde{\beta} \tag{7.12}$$

We now take the limit (6.13) and (6.20). The Bessel function can be approximated by its behavior for large values of the argument

$$K_1(x) \sim e^{-x}\sqrt{\frac{\pi}{2}}\left(\sqrt{\frac{1}{x}} + \mathcal{O}(x^{-3/2})\right) \tag{7.13}$$

It is easy to see that in this limit only the $\frac{1}{2}e^{\mu a p}$ term inside the $[\cdots]$ paranthesis in (7.11) survives. We note that this is precisely the contribution from the M_n oscillators in (6.10). To see that the other terms in (7.11) vanish it is enough to consider $p = 1$ since the higher p terms are exponentially suppressed. From the surviving term it is then straightforward to show that we again get the Hagedorn temperature (7.9), which matches the gauge-theory result (4.19).

We can conclude from the above that taking the decoupling limit (6.13) and (6.20) on the spectrum (6.10) on the pp-wave (6.7) and (6.8) is consistent with taking the decoupling limit of the Hagedorn singularity on the pp-wave. I.e. taking the decoupling limit before computing the Hagedorn temperature commutes with computing the Hagedorn temperature and then subsequently taking the decoupling limit. This is a good check on the consistency of the decoupling limit (6.13) and (6.20).

VIII. DISCUSSION AND CONCLUSIONS

The general idea of this paper is that by taking a certain decoupling limit we get a self-consistent decoupled sector of the AdS/CFT correspondence. On the gauge-theory side, we take the decoupling limit (2.5) of $SU(N)$ $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$. On the string-theory side, we take the decoupling limit (5.4) (see also (5.9)) of type IIB strings on $AdS_5 \times S^5$. In [7] it was shown that the sector of planar $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$ obtained in the decoupling limit (2.5) also is described by the ferromagnetic Heisenberg spin chain, as reviewed in Sec. II. On the string-theory side, the planar limit of $\mathcal{N} = 4$ SYM corresponds to free strings propagating on $AdS_5 \times S^5$. We have thus the spin chain/gauge theory/string theory triality depicted in Fig. 1. Since the Heisenberg chain is integrable, we get that both the gauge theory and the string theory should be integrable. In this sense we have found a solvable sector of AdS/CFT. One of the important features of the triality of Fig. 1 is that we are considering small 't Hooft coupling $\lambda \to 0$ on the gauge-theory side. On the string-theory side this corresponds to having a small string tension $T_{\rm str}$.

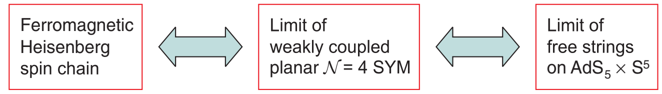

FIG. 1 (color online). A spin chain/gauge theory/string theory triality.

We have succeeded in this paper to show that the low energy spectrum (3.19) obtained on the spin chain/gauge-theory side matches the spectrum of free strings on a maximally supersymmetric pp-wave background. With this, we have shown that the low energy part of the spectrum of the gauge-theory and string-theory sides of the triality of Fig. 1 matches. This is a rather nontrivial result in that we have obtained a string theory spectrum, which is calculable on the string-theory side, directly in weakly-coupled gauge theory. Indeed, to our knowledge, this is the first nontrivial matching in AdS/CFT done between gauge theory and string theory in the $\lambda \ll 1$ regime.

Related to this result, we have shown that the Hagedorn/deconfinement temperature in weakly-coupled planar $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$ in the limit (2.5) matches the Hagedorn temperature of weakly-coupled string theory on a maximally supersymmetric pp-wave background (6.7) and (6.8) in the decoupling limit (6.13) and (6.20). This shows that the confinement/deconfinement transition found in weakly-coupled planar $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$ is linked to a Hagedorn transition of string theory on $AdS_5 \times S^5$, as conjectured in [8–11].

The mechanism behind these successful matches between string theory and gauge theory is the $SU(2)$ decoupling limit found in [7]. In this decoupling limit we consider the gauge-theory states lying very close to a certain chiral primary sector (defined by $D = J$). This enables us to decouple most of the gauge-theory states leaving only the $SU(2)$ sector, and the Hamiltonian truncates to (2.8), which has the consequence that we can study the decoupled sector for finite $\tilde{\lambda}$. On the string-theory side, we find that the Penrose limit [12] of $AdS_5 \times S^5$ leading to the pp-wave background (6.7) and (6.8) with a flat direction gives a pp-wave string spectrum for which the vacua precisely are dual to the chiral primary states expanded around on the gauge-theory side. Translating the dual decoupling limit for string on $AdS_5 \times S^5$ into a decoupling limit for the pp-wave enables us to study the decoupled

[10] In [17] the direction x^1 is compactified and it is shown that only the sector with zero winding number contributes to the partition function.

sector from the string-theory side. Unlike the usual gauge-theory/pp-wave correspondence we can match the gauge-theory and string-theory spectra for small 't Hooft coupling $\lambda \to 0$ since for finite $\tilde{\lambda}$ only the gauge-theory states in the $SU(2)$ sector close to the chiral primary states contribute at low energies.

Future directions

One of the most interesting extensions of the matching of the Hagedorn temperature between gauge theory and string theory of this paper would be to reproduce the $\tilde{\lambda}^{-1/3}$ correction from string-theory side. From the thermodynamics of the Heisenberg chain, we found the correction (4.22). On the string-theory side, computing this correction would involve going away from the large J limit. More generally, it would be highly interesting to match finite size corrections to the spectrum of the Heisenberg chain, to $1/J$ corrections to the pp-wave spectrum.

Another interesting class of corrections to consider would be to look at corrections coming from terms of order $\tilde{\lambda}\lambda$ in the Hamiltonian. I.e. in [7] we have that the leading correction for small λ to the Hamiltonian for the $SU(2)$ sector is

$$H = D_0 + \tilde{\lambda} D_2 + \tilde{\lambda}\lambda D_4 + \mathcal{O}(\tilde{\lambda}\lambda^2) \qquad (8.1)$$

In this regime one could be worried about corrections coming from the fact that states outside the $SU(2)$ sector are not completely decoupled. However, we do not expect that to be important, since such corrections appear nonperturbatively in terms of the expansion parameter $1 - \Omega$ [7].

Considering λ corrections could be very important for a better understanding of the three-loop discrepancy [29,41,42] between anomalous dimensions computed in $\mathcal{N} = 4$ SYM and string energies for strings on $AdS_5 \times S^5$. The reason for the three-loop discrepancy could very well be that there are interpolating functions in λ that one does not see when doing a naive large λ extrapolation of the gauge-theory results. For our decoupled sector we do not have any need for interpolating functions, since we are not extrapolating the anomalous dimensions to infinite λ. Therefore, it would be rather interesting in this light to see if there is a discrepancy for λ corrections to our decoupled sector.

One could furthermore consider other decoupling limits. In [7] we found a decoupling limit of planar $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$ in which it decouples to the $SU(2|3)$ spin chain, in a very similar way as that of the $SU(2)$ decoupling limit considered in this paper. We expect similar results for this sector. This could be interesting to work out since the spectrum is more complicated due to the presence of fermions. As mentioned in [7] it is moreover conceivable that there are other interesting decoupling limits of supersymmetric gauge theories with less supersymmetry, hence one could hope to match the spectrum and Hagedorn temperatures for such cases as well. In particular, it would be interesting to consider generalizing the $SU(2)$ decoupling limit of [7] used in this paper to $\mathcal{N} = 2$ quiver gauge theories dual to the pp-wave background (6.7) and (6.8) with x^1 compactified, following [12].

Finally, we note it would be very interesting to consider nonplanar corrections to the partition function on the gauge-theory side. In [7] the decoupling limit also works for finite N, thus it should be possible to gain more information about the Hagedorn/deconfinement phase transition, for example, whether it is a first order phase transition or not.[11]

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APPENDIX: ASYMPTOTIC BEHAVIOR OF THE GENERATING FUNCTION

In this appendix we will show how to estimate the asymptotic behavior of the function

$$G(a, b) = \prod_{n=1}^{\infty} \frac{1}{1 - e^{-an^2 + ibn}} \qquad (A1)$$

with a and b real and $a > 0$. The previous expression can be written as

$$G(a, b) = \exp\left\{\sum_{n=1}^{\infty} \sum_{p=1}^{\infty} \frac{e^{-apn^2 + ibpn}}{p}\right\} \qquad (A2)$$

We are interested in studying the $a \to 0$ limit.

Consider first the case $b \neq 0$. In the limit $a \to 0$ the sum over n in (A2) can be replaced by an integral and we have

$$G(a, b) \sim \exp\left\{\sum_{p=1}^{\infty} \int_1^{\infty} dx \frac{e^{-apx^2 + ibpx}}{p}\right\} = \exp\left\{\sum_{p=1}^{\infty} \sqrt{\frac{\pi}{a}} \frac{e^{-b^2 p/4a}}{2p} \mathrm{Erfc}\left(\sqrt{pa} - i\frac{b\sqrt{p}}{2\sqrt{a}}\right)\right\} \qquad (A3)$$

[11] In this connection one could also hope to get a better understanding of the small black hole in $AdS_5 \times S^5$ from the gauge-theory point of view [43–45].

where $\mathrm{Erfc}(x)$ is the complementary error function ($\mathrm{Erfc}(x) = 1 - \mathrm{erf}(x)$ where $\mathrm{erf}(x)$ is the error function). For $a \to 0$ and $b \neq 0$ the complementary error function can be approximated as

$$\mathrm{Erfc}\left(\sqrt{pa} - i\frac{b\sqrt{p}}{2\sqrt{a}}\right) \sim 2i\sqrt{\frac{a}{\pi p b^2}} e^{b^2 p/4a} \tag{A4}$$

so that the generating function becomes

$$G(a, b) \sim \exp\left\{\frac{i}{b}\zeta\left(\frac{3}{2}\right)\right\} \tag{A5}$$

where $\zeta(x)$ is the Riemann zeta function. We thus see that for $b \neq 0$ there is no divergent contribution.

To extract the divergent contribution we set $b = 0$ in (A1) so that

$$G(a, 0) = \prod_{n=1}^{\infty} \frac{1}{1 - e^{-an^2}} \sim \exp[F(a)] \tag{A6}$$

where we defined

$$F(a) \equiv -\int_1^\infty dx \log(1 - e^{-ax^2}) \tag{A7}$$

Here we have again approximated the sum over n by an integral. Introducing the new variable $y = x\sqrt{a}$ we have that

$$\lim_{a\to 0} \sqrt{a}\, F(a) = -\int_0^\infty dy \log(1 - e^{-y^2}) = \sum_{p=1}^{\infty} \int_0^\infty dy \frac{e^{-y^2 p}}{p} = \frac{\sqrt{\pi}}{2}\zeta\left(\frac{3}{2}\right) \tag{A8}$$

Thus, we see from this that for $b = 0$ there is a divergent contribution in (A1) in the $a \to 0$ limit, giving

$$G(a, 0) \sim \exp\left\{\zeta\left(\frac{3}{2}\right)\sqrt{\frac{\pi}{4a}}\right\} \tag{A9}$$

This is the leading asymptotic behavior of $G(a, 0)$ for $a \to 0$.

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