Density resummation of perturbation series in a pion gas to leading order in chiral perturbation theory

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The mean field (MF) approximation for the pion matter, being equivalent to the leading order chiral perturbation theory, involves no dynamical loops and, if self-consistent, produces finite renormalizations only. The weight factor of the Haar measure of the pion fields, entering the path integral, generates an effective Lagrangian $\delta\mathcal{L}_H$ which is generally singular in the continuum limit. There exists only one parametrization of the pion fields, for which the weight factor is equal to unity and $\delta\mathcal{L}_H = 0$, respectively. This unique parametrization ensures self-consistency of the MF approximation. We use it to calculate thermal Green functions of the pion gas in the MF approximation as a power series over the density. The Borel transforms of thermal averages of a function $\mathcal{J}(\chi^\alpha\chi^\alpha)$ of the pion fields χ^α with respect to the scalar pion density are found to be $\frac{2}{\sqrt{\pi}}\mathcal{J}(4t)$. The perturbation series over the scalar pion density for basic characteristics of the pion matter, such as the pion propagator, the pion optical potential, the scalar quark condensate $\langle\bar{q}q\rangle$, the in-medium pion decay constant $\tilde{F}$, and the equation of state of pion matter, appear to be asymptotic ones. These series are summed up using the contour-improved Borel resummation method. The quark scalar condensate decreases smoothly until $T_{\max} \simeq 310$ MeV. The temperature $T_{\max}$ is the maximum temperature admissible for the thermalized nonlinear sigma model at zero pion chemical potentials. The estimate of $T_{\max}$ is above the chemical freeze-out temperature $T \simeq 170$ MeV in relativistic heavy-ion collisions and above the phase transition to two-flavor quark matter $T_c \simeq 175$ MeV, predicted by lattice gauge theories.

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I. INTRODUCTION

Ultrarelativistic heavy-ion collisions allow us to study phase diagrams of QCD at high temperatures and small chemical potentials. Under such conditions, lattice gauge theories (LGTs) predict a crossover or a first-order phase transition into the deconfined phase, accompanied by the restoration of chiral symmetry. In two- and three-flavor LGTs, the critical temperature is estimated to be $T_c \simeq 175$ MeV and $T_c \simeq 155$ MeV with a 5% systematic error [1–3]. These values are close to the chemical freeze-out temperature determined from statistical models by fitting particle yields in heavy-ion collisions [4]. The chemical freeze-out temperatures $T = 160 \div 174$ MeV and $T = 160 \div 166$ MeV are extracted [5] at top CERN SPS energies ($\sqrt{s_{NN}} = 17.3$ GeV) and top relativistic heavy-ion collider (RHIC) energies ($\sqrt{s_{NN}} = 200$ GeV), respectively. The boundary of the phase transition might apparently be already crossed.

In ultrarelativistic heavy-ion collisions, pions are the most abundant particles [4–6], so thermodynamic characteristics of a pion-dominated medium and in-medium pion properties are of current interest. At zero chemical potentials, light degrees of freedom, i.e. pions, give the dominant contribution to the pressure of hadron matter [7], while contributions of resonances like ρ and ω mesons with masses $M \gg T_c$ are suppressed as $\frac{T}{M}\exp(-\frac{M}{T}) \ll 1$. According to Gibbs' criterion, the balance of pressure determines the critical temperature of the phase transition at fixed chemical potentials. The QCD phase transition can therefore be driven by pions i.e. by the lightest QCD degrees of freedom.

The energy density of a resonance gas is proportional to the small parameter $\exp(-\frac{M}{T}) \ll 1$ which can, however, be compensated by an exponentially large amount of resonances, as has been conjectured by Hagedorn [8,9] and discussed recently e.g. in Ref. [10]. The contributions of resonances to the pressure are still suppressed in such models by a factor $\frac{T}{M} \ll 1$, where $M \sim 1$ GeV is a typical scale of resonance masses.

Pions are Goldstone particles and play a special role in the restoration of chiral symmetry. LGTs give evidence that deconfinement and chiral phase transitions occur at the same critical temperature for zero chemical potentials [11].

Chiral perturbation theory (ChPT) exploits the invariance of strong interactions under the $SU(2)_L \otimes SU(2)_R$ symmetry [12,13]. It has been proven to be highly successful in phenomenological descriptions of low-energy dynamics of pseudoscalar mesons [14,15]. The pion matter created in relativistic heavy-ion collisions is charge symmetric [4], so the pion chemical potentials can be set equal to zero. The ChPT expansion in the vacuum runs over inverse powers of the pion decay constant $F = 93$ MeV and additionally, in the medium, over the pion density (or temperature). Properties of the pion gas at finite temperatures, the associated chiral phase transition, and the in-medium modifications of pseudoscalar mesons within ChPT have been extensively studied [16–26].

The optical potential of an in-medium particle yields the self-energy operator to the first order in the density. It is determined by the forward two-body scattering amplitude on particles of the medium. ChPT is suited for calculation of the pion self-energy beyond the lowest order in density, since ChPT has all multipion scattering amplitudes fixed. The forward scattering amplitudes of a probing pion scattered off n thermalized pions describe $O(\rho^n)$ contributions to the in-medium pion self-energy operator. The leading ChPT amplitudes do not contain loops. This approximation is equivalent to the mean field (MF) approximation.

In this paper, we perform to the leading ChPT order the generalized Borel resummation of the asymptotic density series for the pion propagator, the pion optical potential, the scalar quark condensate $\langle\bar{q}q\rangle$, the in-medium pion decay constant $\tilde{F}$, and the equation of state (EOS) of pion matter.

The outline of the paper is as follows: In the next section, we find for the pion field a parametrization which provides a constant Lagrange measure for path integrals. The integration over the pion field variables is extended from $-\infty$ to $+\infty$ to convert path integrals to the Gaussian form. Such a procedure apparently does not affect the perturbative part of the Green functions. In Sec. III, a method for calculation of thermal averages within ChPT in the MF approximation is described. The thermal averages appear as inverse Borel transforms with respect to the nonrenormalized scalar pion density. The lowest order expansion coefficients are calculated for the pion effective mass and other quantities and compared to the earlier calculations. In Sec. IV, we show that the power series with respect to the density are asymptotic and hence divergent. They can, however, be summed up with the help of the contour-improved Borel resummation technique. In Sec. V, predictions of the nonlinear sigma model are compared to LGTs. The results obtained by the resummation are discussed in Sec. VI.

II. PION FIELD PARAMETRIZATION

The lowest order ChPT Lagrangian has the form

$$\mathcal{L} = \frac{F^2}{4}\,\mathrm{Tr}[\partial_\mu U \partial^\mu U^\dagger] + \frac{F^2 M_\pi^2}{4}\,\mathrm{Tr}[U^\dagger + U], \quad (2.1)$$

where M_π is the pion mass. The kinetic term for the matrix $U(x) \in SU(2)$ is invariant under the chiral transformations $U \to U' = RUL^+$, where $R, L \in SU(2)$. The second term in Eq. (2.1) breaks chiral symmetry explicitly. In what follows, we set $F = 1$.

The matrix $U(x)$ can be parametrized in various ways. The on-shell vacuum amplitudes do not depend on the choice of variables for pions [27,28]. The method proposed by Gasser and Leutwyler [12] establishes a connection between QCD Green functions and amplitudes of the effective chiral Lagrangian. Using this method, the QCD on- and off-shell amplitudes can be calculated in a way independent of the parametrization. The detailed studies of Refs. [29,30] demonstrate that the in-medium effective meson masses are independent of the choice of meson field variables up to next-to-leading order in ChPT and to first order in density, in accordance with the equivalence theorem. The in-medium off-shell behavior and going beyond the linear-density approximation are discussed in Refs. [30–32].

In order to have the S-matrix invariant with respect to a symmetry group, both the action functional and the Lagrange measure entering the path integral should be invariant. The chirally invariant measure $d\mu[U] = d\mu[RUL^+]$ entering the path integral over the generalized coordinates coincides with the Haar measure of the $SU(2)$ group. In the exponential parametrization,

$$U(\phi^\alpha) = e^{i\tau^\alpha \phi^\alpha} \quad (2.2)$$

and

$$d\mu[U] = \sin^2(\phi)\phi^{-2} d^3\phi \quad (2.3)$$

(see e.g. [33]). The leading order ChPT is equivalent to the $O(4)$ nonlinear sigma model due to the isomorphism of algebras $su(2)_L \oplus su(2)_R \sim so(4)$. The quantization of the $O(4)$ nonlinear sigma model [34] automatically yields the measure (2.3).

Any perturbation theory uses for dynamical fields an oscillator basis to convert path integrals into a Gaussian form. It is necessary to specify variables, χ^α, in terms of which the Lagrange measure is "flat":

$$d\mu[U] = d^3\chi. \quad (2.4)$$

The weight factor can always be exponentiated to generate an effective Lagrangian $\delta\mathcal{L}_H$, in which case $\chi^\alpha = \phi^\alpha$ provides the desired parametrization. The exponential parametrization (2.2) gives, in particular, $\delta\mathcal{L}_H = -\frac{1}{a^4} \times \log(\sin^2(\phi)\phi^{-2})$, where a is a lattice size. $\delta\mathcal{L}_H$ diverges in the continuum limit. The nonlinear sigma model is not a renormalizable theory, so divergences cannot be absorbed into a redefinition of F and M_π. Using the MF approximation, it is usually possible to keep renormalizations finite. The exponentiation of a variable weight factor breaks, in general, self-consistency of the MF approximation.

The divergences arising from $\delta\mathcal{L}_H$ could, however, be compensated by divergences coming from the higher order ChPT loops. From this point of view it is natural to attribute $\delta\mathcal{L}_H$ to higher order ChPT loop expansion starting from one loop. The MF approximation for the ChPT implies then that the tree-level approximation neglects $\delta\mathcal{L}_H$ from the start. It is hard to expect that such an approximation is relevant at the high temperature limit where the chiral invariance is supposed to be restored.

The interaction terms in the effective Lagrangian which appear due to the presence of the Haar measure have been discussed earlier in QCD [35,36]. The exponentiation of the weight factor gives consistent results due to renormalizability of QCD. The divergences appearing in the continuum limit from $\delta\mathcal{L}_H$ at tree level are compensated by one-loop gluon self-interaction diagrams.

The consistency of the MF approximation of the nonlinear sigma model survives with one parametrization only, which uses the stretched pion field variables $\chi^\alpha = \phi^\alpha\chi/\phi$ such that $\chi^2\chi' = \sin^2(\phi) \geq 0$ where $\chi = (\chi^\alpha\chi^\alpha)^{1/2}$ and

$$\chi^3 = \tfrac{3}{2}(\phi - \sin(\phi)\cos(\phi)). \tag{2.5}$$

The vacuum value $\phi^\alpha_{\rm vac} = 0$ corresponds to $\chi^\alpha_{\rm vac} = 0$; χ is a monotonously increasing function of ϕ. The value of $4\pi\chi^3/3$ has the meaning of a volume covered by a 3-dimensional surface of radius χ in a 4-dimensional space.

The parametrization based on the dilatation of ϕ^α gives $\delta\mathcal{L}_H = 0$, does not require the higher order ChPT loop expansion for the consistency, and allows us to work in the continuum limit with finite quantities only.

The $SU(2)$ group has a finite group volume. The magnitude of the χ^α fields is restricted by $\chi_{\max} = (3\pi)^{1/3}$. It is generally believed that, using the perturbation theory, one can extend the integrals over χ^α from $-\infty$ to $+\infty$. The modification of the result is connected to large field fluctuations of a nonperturbative nature, which apparently do not affect perturbation series. This conjecture is essential for the standard loop expansion in ChPT. A similar extension of the integration region is used in QCD [35,36].

The Weinberg parametrization [37] does not restrict the magnitude of the pion fields. Such a parametrization looks especially attractive as it simplifies conversion of the path integrals into the Gaussian form. Because of the variable Lagrange measure, it can be effective starting from one loop.

The propagators (heat kernels) of free particles moving on compact group manifolds have been analyzed in Refs. [38–42]. It was shown that the semiclassical approximation for the propagators is exact. An explicit form of the propagator for the group $SU(2)$ is given by Schulman [40] and Duru [41]. The problem of the variable Lagrange measure of the path integrals appears already at the quantum mechanical level. The path integral as it was recognized by Marinov and Terentev [39] is ill posed at tree level. As demonstrated by Baaquie [42], the divergent contributions coming from the variable Lagrange measure are canceled by loops generated by the effective Lagrangian. A system of coupled oscillators on a compact group manifold represents the unsolved problem.

The requirement of chiral invariance of the Lagrange measure $d\mu[U]$ being combined with the requirements that (i) the perturbation expansion is based on the oscillator basis and (ii) the MF approximation does not involve loops restricts the parametrizations of the pion fields to only one admissible parametrization.

III. THERMAL AVERAGES IN THE MF APPROXIMATION

The quadratic part of the Lagrangian with respect to the fields χ^α can be extracted from Eq. (2.1), the rest $\delta\mathcal{L}$ is treated as a perturbation:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\chi^\alpha\partial_\mu\chi^\alpha - \frac{\tilde{M}_\pi^2}{2}\chi^2 + \delta\mathcal{L}, \tag{3.1}$$

where $\tilde{M}_\pi$ is an effective pion mass to be determined self-consistently and

$$\delta\mathcal{L} = \frac{1}{2}\partial_\mu\chi^\alpha\partial_\mu\chi^\alpha\mathcal{J}_0 + \frac{1}{2}\partial_\mu\chi^\alpha\partial_\mu\chi^\beta\chi^\alpha\chi^\beta\chi^{-2}\mathcal{J}_1 + M_\pi^2(\mathcal{J}_2 - 1) + \frac{\tilde{M}_\pi^2}{2}\mathcal{J}_3, \tag{3.2}$$

with

$$\mathcal{J}_0 = \frac{\sin^2(\phi)}{\chi^2} - 1, \tag{3.3}$$

$$\mathcal{J}_1 = \frac{\chi^4}{\sin^4(\phi)} - \frac{\sin^2(\phi)}{\chi^2}, \tag{3.4}$$

$$\mathcal{J}_2 = \cos(\phi), \tag{3.5}$$

$$\mathcal{J}_3 = \chi^2. \tag{3.6}$$

The physical observables are expressed in terms of the thermal Green functions.

A. Thermal Green functions

The scalar pion density ρ in a thermal bath with temperature T is defined by

$$\delta^{\alpha\beta}2\rho = \langle\chi^\alpha\chi^\beta\rangle = \delta^{\alpha\beta}Z_\chi\int\frac{d\mathbf{k}}{(2\pi)^3}2n_0(\mathbf{k}), \tag{3.7}$$

where Z_χ is a renormalization constant of the pion χ field and

$$n_0(\mathbf{k}) = \frac{1}{2\omega(\mathbf{k})}\frac{1}{e^{\omega(\mathbf{k})/T} - 1}. \tag{3.8}$$

The average (3.7) involves the normally ordered χ^α operators. The energy $\omega(\mathbf{k}) = \sqrt{\tilde{M}_\pi^2 + \mathbf{k}^2}$ which enters the Bose-Einstein distribution contains the in-medium pion mass $\tilde{M}_\pi$. A similar situation occurs in the Bogoliubov model of a weakly interacting nonideal Bose gas. Also in Fermi liquid theory the momentum space distribution is determined by the in-medium dispersion law of quasiparticles (see e.g. [43]).

In terms of the thermal two-body Green function,

$$\mathcal{G}^{\alpha\beta}(\tau, \mathbf{x}) = -\langle \mathcal{T} \chi^{\alpha}(\tau, \mathbf{x})\chi^{\beta}(0, \mathbf{0})\rangle, \tag{3.9}$$

the momentum space distribution $n_0(\mathbf{k})$ is determined by the equation

$$\delta^{\alpha\beta} Z_{\chi}\left(2n_0(\mathbf{k}) + \frac{1}{2\omega(\mathbf{k})}\right) = -T \lim_{\epsilon \to -0} \sum_{s=-\infty}^{+\infty} \mathcal{G}^{\alpha\beta}(\omega_s, \mathbf{k}) \times e^{-i\omega_s \epsilon}, \tag{3.10}$$

where $\omega_s = 2\pi s T$ and

$$\mathcal{G}^{\alpha\beta}(\tau, \mathbf{x}) = T \sum_{s=-\infty}^{+\infty} \int \frac{d\mathbf{k}}{(2\pi)^3} \mathcal{G}^{\alpha\beta}(\omega_s, \mathbf{k}) e^{-i\omega_s \tau + i\mathbf{k}\mathbf{x}}. \tag{3.11}$$

The density ρ can be found from

$$\delta^{\alpha\beta}(2\rho + \rho_{\text{vac}}) = -\lim_{\epsilon \to -0} \mathcal{G}^{\alpha\beta}(\epsilon, \mathbf{0}), \tag{3.12}$$

where ρ_{vac} is the vacuum density of the zero-point field fluctuations,

$$\rho_{\text{vac}} = Z_{\chi} \int \frac{d\mathbf{k}}{2\omega(\mathbf{k})(2\pi)^3}.$$

The vacuum density can be treated self-consistently beyond the lowest order ChPT only. In the MF approximation it is assumed that zero-point field fluctuations are absorbed into a redefinition of physical parameters entering the Lagrangian. In our case, the pion mass and the pion decay constant receive these contributions.

We thus neglect the vacuum fluctuations. In what follows, ρ_{vac} is set equal to zero wherever it appears. This is equivalent to using the thermal part of the renormalized Green function within the loops:

$$\lim_{\epsilon \to -0} Z_{\chi}^{-1} \mathcal{G}^{\alpha\beta}(\epsilon, \mathbf{0}) \to \lim_{\epsilon \to -0} Z_{\chi}^{-1} \mathcal{G}^{\alpha\beta}(\epsilon, \mathbf{0}) - \lim_{\epsilon \to -0} \mathcal{G}^{\alpha\beta}(\epsilon, \mathbf{0})|_{T=0}. \tag{3.13}$$

Equation (3.12) is illustrated graphically in Fig. 1. Since we neglect the zero-point field fluctuations, we need not worry about the ordering of operators, entering thermal averages such as (3.7) and (3.14) and others, anymore.

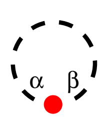

FIG. 1 (color online). Diagram representation of the thermal average $\langle \chi^{\alpha}\chi^{\beta}\rangle$ defined by Eqs. (3.7) and (3.12). The thick dashed line represents the dressed pion propagator.

B. In-medium pion mass and other observables

In the MF approximation, the self-energy operator can be calculated from

$$\delta^{\alpha\beta}\Sigma(x - y) = -\left\langle \frac{\delta^2}{\delta\chi^{\alpha}(x)\delta\chi^{\beta}(y)} \int \delta\mathcal{L}(z) d^4 z \right\rangle. \tag{3.14}$$

The equation $\langle \chi^{\alpha}\partial_{\mu}\chi^{\beta}\rangle = 0$ is a consequence of the symmetry of the pion matter with respect to isospin rotations. In the momentum representation, using equation

$$\langle \partial_{\mu}\chi^{\alpha}\partial_{\mu}\chi^{\beta}\rangle = \delta^{\alpha\beta}\tilde{M}_{\pi}^2 2\rho, \tag{3.15}$$

one gets

$$\Sigma(k^2) = k^2\Sigma_1 + M_{\pi}^2\Sigma_2 + \tilde{M}_{\pi}^2\Sigma_3, \tag{3.16}$$

where $k^2 = -\omega_s^2 - \mathbf{k}^2$ and

$$\Sigma_1 = -\langle \mathcal{J}_0 + \tfrac{1}{3}\mathcal{J}_1\rangle, \tag{3.17}$$

$$\Sigma_2 = \frac{1}{2\rho}\left\langle \left(1 - \frac{\chi^2}{6\rho}\right)\mathcal{J}_2\right\rangle, \tag{3.18}$$

$$\Sigma_3 = \frac{3}{2}\left\langle \left(1 - \frac{\chi^2}{6\rho}\right)\left(\mathcal{J}_0 + \frac{1}{3}\mathcal{J}_1\right)\right\rangle - 1. \tag{3.19}$$

The self-energy operator is presented graphically in Fig. 2.

Equation (3.15) can be derived as follows: The thermal average (3.15) can be rewritten in terms of the two-body Green function

$$\langle \partial_{\mu}\chi^{\alpha}\partial_{\mu}\chi^{\beta}\rangle = \lim_{\tau \to +0} \Box \mathcal{G}^{\alpha\beta}(\tau, \mathbf{0}), \tag{3.20}$$

where $\Box = -\frac{\partial^2}{\partial\tau^2} - \Delta$. The two-body Green function satisfies the equation

$$(\Box + \tilde{M}_{\pi}^2 + \Sigma(-\Box))\mathcal{G}^{\alpha\beta}(\tau, \mathbf{x}) = -\delta^{\alpha\beta}\delta(\tau)\delta^3(\mathbf{x}). \tag{3.21}$$

The perturbation $\delta\mathcal{L}(z)$ is quadratic with respect to the derivatives, so $\Sigma(k^2)$ in the MF approximation is a first-

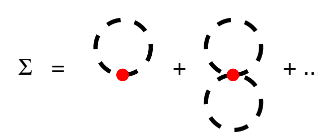

FIG. 2 (color online). Diagram representation of the self-energy operator in the MF approximation.

order polynomial. The expansion (3.16) makes this feature explicit. Equation (3.21) can be rewritten in the form

$$(\Box + \tilde{M}_\pi^2)\mathcal{G}^{\alpha\beta}(\tau, \mathbf{x}) = -Z_\chi \delta^{\alpha\beta}\delta(\tau)\delta^3(\mathbf{x}), \tag{3.22}$$

where Z_χ is the renormalization constant introduced earlier. Using $\lim_{\tau\to+0}\delta(\tau) = 0$ and Eqs. (3.12), (3.20), (3.22), and (A4), we arrive at Eq. (3.15) with 2ρ replaced by $2\rho + \rho_{\mathrm{vac}}$. The value of ρ_{vac} must be neglected, however.

The effective pion mass can be determined from the equation $\Sigma(\tilde{M}_\pi^2) = 0$:

$$\tilde{M}_\pi^2 = -M_\pi^2 \frac{\Sigma_2}{\Sigma_1 + \Sigma_3}, \tag{3.23}$$

while the renormalization constant is given by

$$Z_\chi^{-1} = 1 - \frac{\partial\Sigma(k^2)}{\partial k^2} = 1 - \Sigma_1. \tag{3.24}$$

The Green function in the MF approximation has the form

$$\mathcal{G}^{\alpha\beta}(\omega_s, \mathbf{k}) = -\delta^{\alpha\beta}\frac{Z_\chi}{\omega_s^2 + \mathbf{k}^2 + \tilde{M}_\pi^2}. \tag{3.25}$$

It is presented graphically in Fig. 3.

In what follows, we also need the functions

$$\mathcal{J}_4 = \frac{\sin(\phi)}{\chi}, \tag{3.26}$$

$$\mathcal{J}_5 = -\mathcal{J}_2(-\mathcal{J}_4 + (20\rho - 2\chi^2)\mathcal{J}_4' + 8\rho\chi^2\mathcal{J}_4'') + \mathcal{J}_4((12\rho - 2\chi^2)\mathcal{J}_2' + 8\rho\chi^2\mathcal{J}_2'') \tag{3.27}$$

which appear in calculations of the density dependent renormalization constant Z_π and the pion decay constant $\tilde{F}$ according to Eqs. (3.29) and (3.31). Here, $\mathcal{J}_i'$ and $\mathcal{J}_i''$ are first and second derivatives with respect to ϕ.

The thermal Green function of the pion is defined by $\Delta^{\alpha\beta}(x - y) = -\langle\mathcal{T}\pi^\alpha(x)\pi^\beta(y)\rangle$. The pion fields $\pi^\alpha(x) = \frac{1}{2}\,\mathrm{Tr}[\tau^\alpha U(x)]$ do not depend on derivatives of χ^α. In such a case, no additional k^2 dependence appears as compared to the χ propagator. Given the renormalization constant Z_χ, the renormalization constant Z_π of the pion propagator can be found from

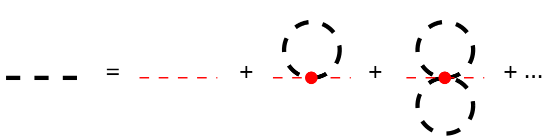

FIG. 3 (color online). Diagram representation of the dressed pion propagator (thick dashed lines) in terms of the scalar density and the bare pion propagators (thin dashed lines). The scalar density appears from loops formed by the dressed pion propagator according to Eq. (3.12).

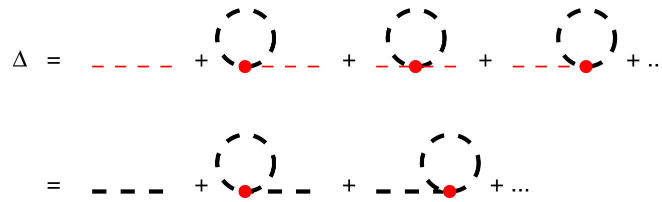

FIG. 4 (color online). Diagram representation of the pion propagator $\Delta^{\alpha\beta}(x - y)$. The dressed pion propagators are shown as thick dashed lines and the bare pion propagators are shown as thin dashed lines. The second line contains insertions into the ends of the dressed pion propagators only.

$$\delta^{\alpha\beta}Z_\pi^{1/2}Z_\chi^{-1/2} = \left\langle\frac{\partial\pi^\alpha}{\partial\chi^\beta}\right\rangle. \tag{3.28}$$

The pion propagator $\Delta^{\alpha\beta}(x - y)$ is shown graphically in Fig. 4.

The quark scalar condensate $\langle\bar{q}q\rangle$ is proportional to the expectation value of the scalar field $\sigma = \frac{1}{2}\,\mathrm{Tr}[U(x)]$. The calculation of the averages $\langle\partial\pi^\alpha/\partial\chi^\beta\rangle$ and $\langle\sigma\rangle$ leads, after factorization of isotopic indices, to the calculation of averages defined in terms of the functions (3.5) and (3.26):

$$Z_\pi^{1/2}Z_\chi^{-1/2} = \left\langle\frac{\chi^2}{6\rho}\mathcal{J}_4\right\rangle, \tag{3.29}$$

$$\langle\bar{q}q\rangle/\langle\bar{q}q\rangle_{\mathrm{vac}} = \langle\mathcal{J}_2\rangle. \tag{3.30}$$

In the pion gas, the spectral density of the axial two-point function $\langle\mathcal{T}A_\mu^\alpha(x)A_\nu^\beta(y)\rangle$ is characterized by two different pion decay constants [23]. In the limit $T = 0$, these constants coincide with F. Let us find an in-medium pion decay constant, $\tilde{F}$, appearing in the spectral density of the two-point function $\langle\mathcal{T}\partial_\mu A_\mu^\alpha(x)\partial_\nu A_\nu^\beta(y)\rangle$. This definition is equivalent to the equation $\partial_\mu A_\mu^\alpha = \tilde{F}\tilde{M}_\pi^2 Z_\pi^{-1/2}\pi^\alpha$, where the right side gives the physical pion $Z_\pi^{-1/2}\pi^\alpha$. The axial current has the form $A_\mu^\alpha = -\sigma\partial_\mu\pi^\alpha + (\partial_\mu\sigma)\pi^\alpha$. The calculation of the average $\langle\mathcal{T}A_\mu^\alpha(x)A_\nu^\beta(y)\rangle$ leads, after factorization of isotopic indices, to the calculation of an average defined in terms of the function (3.27). In this way we obtain

$$\tilde{F}Z_\chi^{-1/2} = \left\langle\frac{\chi^2}{6\rho}\mathcal{J}_5\right\rangle. \tag{3.31}$$

C. Thermal averages as inverse Borel transforms

Suppose we want to find the thermal average of a function $\mathcal{J}(\chi^2)$ which can be expanded in a power series of χ^2. The average values of powers of the operator $\chi^2 = \chi^\gamma\chi^\gamma$ can be calculated as follows:

$$\begin{aligned}\langle(\chi^\gamma\chi^\gamma)^n\rangle &= \left\langle \sum_{s_1s_2s_3}\frac{n!}{s_1!s_2!s_3!}(\chi^1)^{2s_1}(\chi^2)^{2s_2}(\chi^3)^{2s_3}\right\rangle = \sum_{s_1s_2s_3}\frac{n!}{s_1!s_2!s_3!}\langle(\chi^1)^{2s_1}\rangle\langle(\chi^2)^{2s_2}\rangle\langle(\chi^3)^{2s_3}\rangle\\ &= \sum_{s_1s_2s_3}\frac{n!}{s_1!s_2!s_3!}(2s_1-1)!!(2s_2-1)!!(2s_3-1)!!(2\rho)^n\\ &= \sum_{s_1s_2s_3}\frac{n!}{s_1!s_2!s_3!}\frac{1}{\pi^{3/2}}\Gamma\Big(s_1+\frac{1}{2}\Big)\Gamma\Big(s_2+\frac{1}{2}\Big)\Gamma\Big(s_3+\frac{1}{2}\Big)(4\rho)^n\\ &= \sum_{s_1s_2s_3}\frac{n!}{s_1!s_2!s_3!}\frac{1}{\pi^{3/2}}(4\rho)^n\int_0^\infty\int_0^\infty\int_0^\infty x_1^{s_1-1/2}x_2^{s_2-1/2}x_3^{s_3-1/2}e^{-(x_1+x_2+x_3)}dx_1dx_2dx_3\\ &= \frac{1}{\pi^{3/2}}(4\rho)^n\int_0^\infty\int_0^\infty\int_0^\infty\frac{(x_1+x_2+x_3)^n}{\sqrt{x_1x_2x_3}}e^{-(x_1+x_2+x_3)}dx_1dx_2dx_3 = \frac{4}{\sqrt{\pi}}\int_0^\infty(4\rho x^2)^n x^2e^{-x^2}dx\\ &= \frac{2}{\sqrt{\pi}}\int_0^\infty(4\rho t)^n t^{1/2}e^{-t}dt. \end{aligned}\tag{3.32}$$

The summations run for $s_1 + s_2 + s_3 = n$, where $s_i = 0, 1, \ldots$. In the first line, we use the multibinomial formula for $((\chi^1)^2 + (\chi^2)^2 + (\chi^3)^2)^n$. The factorization of the average is a consequence of the MF approximation according to which the averages are calculated over the free gas of quasiparticles, i.e., noninteracting effective pions in three isotopic states. In order to calculate, e.g., the average value $\langle(\chi^1)^{2s_1}\rangle$, one has to consider $(2s_1)!$ permutations of the operators χ^1 due to their possible pairings. Permutations inside of each pair are counted twice, whereas permutations of the pairs are counted $s_1!$ times. This gives the factor $(2s_1)!/(2^{s_1}s_1!) = (2s_1 - 1)!!$ in the second line of (3.32). To arrive at the final result one has to express $(2s_i - 1)!!$ in terms of Euler's gamma functions and use their integral representations to convert the 3-dimensional integral into the 1-dimensional integral.

The MF approximation we used consists of the neglection of dynamical loops, i.e., loops composed from more than one dressed pion propagator. The loops composed from one dressed pion propagator give the density (3.7) after neglecting $\rho_{\rm vac}$. The calculation of the $n = 1$ average value is graphically presented in Fig. 1 (the indices α and β have to be contracted with $\delta^{\alpha\beta}$). The calculation of the $n = 2$ is illustrated in Fig. 5. An alternative proof of Eq. (3.32), based on the thermal Green functions method, is given in Appendix A.

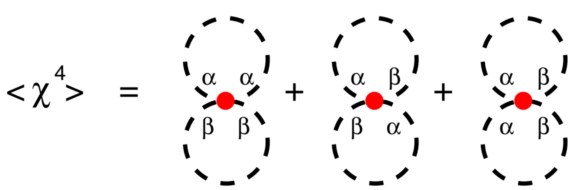

FIG. 5 (color online). Diagram representation of the thermal average $\langle\chi^4\rangle = \langle\chi^\alpha\chi^\alpha\chi^\beta\chi^\beta\rangle = (3\times 3 + 3 + 3)\times 4\rho^2$. Greek indices, e.g. α and β, within one loop designate $\delta^{\alpha\beta}$. The result $60\rho^2$ is in agreement with Eq. (3.32) for $n = 2$.

The average value of $\mathcal{J}(\chi^2)$ can be written in the form

$$\langle\mathcal{J}(\chi^2)\rangle = \frac{2}{\sqrt{\pi}}\int_0^\infty \mathcal{J}(4\rho t)t^{1/2}e^{-t}dt. \tag{3.33}$$

The thermal fluctuations are suppressed exponentially according to the value of χ^2 [see also Eq. (B1)]. For small fluctuations, χ coincides with ϕ.

The thermal average $\langle\mathcal{J}(\chi^2)\rangle$ to all orders in the pion density and to leading order in ChPT is given by the inverse Borel transform of the function $\frac{2}{\sqrt{\pi}}\mathcal{J}(4t)$.

Using the explicit form of $\mathcal{J}_i$ and reexpanding results in terms of the physical density $\rho_0 = Z_\chi^{-1}\rho$, we obtain for $\rho_0 \ll 1$

$$\tilde{M}_\pi^2/M_\pi^2 = 1 + \rho_0 - \frac{39}{10}\rho_0^2 - \frac{2601}{50}\rho_0^3 - \frac{614\,163}{1400}\rho_0^4 + \ldots, \tag{3.34}$$

$$Z_\chi^{-1} = 1 + \frac{12}{5}\rho_0^2 + \frac{368}{25}\rho_0^3 + \frac{80\,832}{875}\rho_0^4 + \ldots, \tag{3.35}$$

$$Z_\pi^{-1} = 1 + 2\rho_0 + \frac{46}{5}\rho_0^2 + \frac{1108}{25}\rho_0^3 + \frac{204\,202}{875}\rho_0^4 + \ldots, \tag{3.36}$$

$$\langle\bar{q}q\rangle/\langle\bar{q}q\rangle_{\rm vac} = 1 - 3\rho_0 - \frac{3}{2}\rho_0^2 + \frac{39}{10}\rho_0^3 + \frac{7803}{200}\rho_0^4 + \ldots, \tag{3.37}$$

$$\tilde{F}/F = 1 - 2\rho_0 + \frac{2}{5}\rho_0^2 + \frac{156}{25}\rho_0^3 + \frac{22\,674}{875}\rho_0^4 + \ldots. \tag{3.38}$$

For massless pions, $\rho_0 = \frac{1}{24}T^2$. In such a case, the power series expansions over density and temperature coincide.

The result for $\tilde{M}_\pi^2$ to order $O(\rho_0)$ is in agreement with [16,23,25]; the result for Z_π^{-1} to order $O(\rho_0)$ is in agreement with [25]; the result for $\langle\bar{q}q\rangle$ to order $O(\rho_0^2)$ is in agreement with [16,17,19,21,23]; the result for $\tilde{F}$ to order $O(\rho_0)$ is in agreement with [16,18,21,22].

Chiral invariance of the nonlinear sigma model in the MF approximation is commented in Appendix B.

IV. SUMMATION OF THE DENSITY SERIES

Most perturbation series in quantum field theory are believed to be asymptotic and hence divergent [44,45]. The integral representation (3.33) permits, by changing the variable $t \to t' = t/\rho$, an analytical continuation into the complex half-plane $\mathrm{Re}[\rho] > 0$, whereas at $\mathrm{Re}[\rho] < 0$ the integral diverges. The half-plane $\mathrm{Re}[\rho] < 0$ contains, in general, singularities, the character of which depends on properties of the function $\mathcal{J}(\chi^2)$. The convergence radius of the Taylor expansion is determined by the nearest singularity. One cannot exclude that the point $\rho = 0$ around which we make the expansion is a singular point.

Inspecting Eqs. (3.3), (3.6), (3.26), and (3.27), we observe that the Borel transforms involving $\mathcal{J}_1$ and $\mathcal{J}_5$ are singular at $\phi = k\pi$, where $k = 1, 2, \ldots$. The integral along the real half-axis of the Borel variable t is therefore not feasible. Series involving such functions are asymptotic, have zero convergence radii, and are furthermore not Borel summable [45–47].

In order to clarify the character of the power series over the physical density ρ_0, we calculate the higher order expansion coefficients. We write $O(\rho_0) = \sum_{k=0}^{\infty} c_n \rho_0^n$ for observables (3.34), (3.35), (3.36), (3.37), and (3.38) and plot in Fig. 6 the ratios c_{n+1}/c_n versus n up to $n = 50$ [48]. The apparent asymptotic regime starts at $n \sim 10$. The ratios c_{n+1}/c_n increase linearly with n, indicating clearly that the power series are asymptotic. The slopes of these ratios are approximately equal. One can expect that the nearest singularity $\phi = \pi$ that appears in the series expansion over ρ appears again in the series expansion over ρ_0 by virtue of $\rho = Z_\chi \rho_0$. In such a case, we would expect a singularity in the Borel plane at $4t = (\frac{3\pi}{2})^{2/3}$ as well. From the other side, for $c_n \sim n! a^n$ the Borel transform is singular at $t = 1/a$. The slope then equals $a = (\frac{16}{3\pi})^{2/3} \simeq 1.4$, in good agreement with results presented in Fig. 6. The power series (3.34), (3.35), (3.36), (3.37), and (3.38) with respect to the physical density ρ_0 have zero convergence radii and are also not Borel summable.

Generalizations of the Borel summation method have been proposed and effectively used in mathematics and atomic physics [46,47]. In our case, fortunately, the residues at $\phi = k\pi$ of the Borel transforms vanish. A contour-improved Borel resummation technique can therefore be applied which consists of a shift of the integration contour into the complex t plane. The result is stable against small variations of the contour around the positive real half-axis. The thermal averages, in particular, do not acquire imaginary parts and thus the expectation value, e.g., of the Hermitian operator $\bar{q}q$ remains real. The uniqueness of the result justifies the integrations by parts of the functions $\mathcal{J}_i$, performed to derive Eqs. (3.18), (3.19), (3.29), and (3.31).

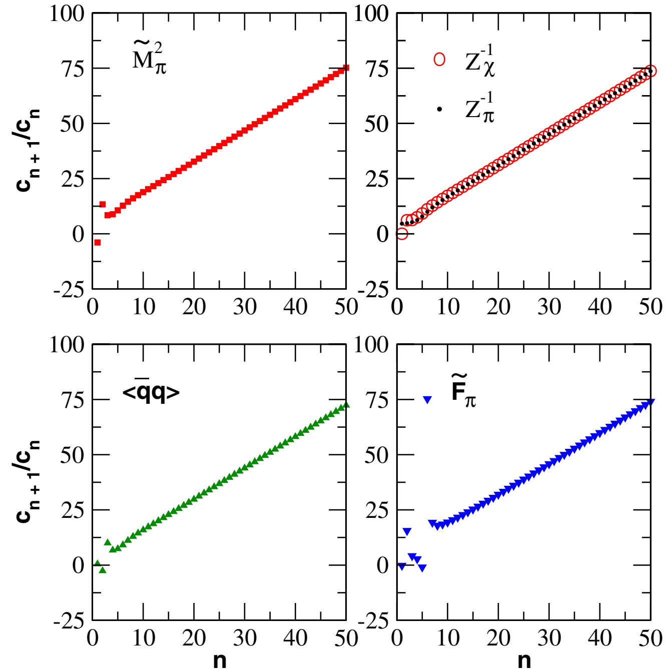

FIG. 6 (color online). The ratios c_{n+1}/c_n of the expansion coefficients of the power series $\sum_{k=0}^{\infty} c_n \rho_0^n$ over the pion density ρ_0 for (a) the in-medium pion mass $\tilde{M}_\pi^2$, (b) the propagator renormalization constants Z_χ^{-1} of the χ field and Z_π^{-1} of the pion field, (c) the scalar quark condensate $\langle\bar{q}q\rangle$, and (d) the pion decay constant $\tilde{F}$. The linear growth of the ratios beyond $n \sim 10$ indicates that all power series are asymptotic. The same sign of c_n for $n > 10$ indicates furthermore that the series are not Borel summable. The approximately equal slopes are related to a singularity of the Borel transforms on the real axis at $\phi = \pi$ (see text).

The power series (3.34), (3.35), (3.36), (3.37), and (3.38) become summable using the contour-improved Borel resummation method.

The EOS of pion matter can be determined by averaging the energy-momentum tensor constructed from the Lagrangian (3.1). The renormalized energy density $\varepsilon = Z_\chi^{-1} T_{00}$ and the pressure $p = Z_\chi^{-1} \sum_{i=1,3} T_{ii}/3$ are given by

$$\varepsilon = 6 \int \frac{d\mathbf{k}}{(2\pi)^3} \omega^2(\mathbf{k}) n_0(\mathbf{k}) - Z_\chi^{-1}(\tilde{M}_\pi^2 3\rho_0 + M_\pi^2(\sigma - 1)), \tag{4.1}$$

$$p = 2 \int \frac{d\mathbf{k}}{(2\pi)^3} \mathbf{k}^2 n_0(\mathbf{k}) + Z_\chi^{-1}(\tilde{M}_\pi^2 3\rho_0 + M_\pi^2(\sigma - 1)), \tag{4.2}$$

where $\sigma = \langle\bar{q}q\rangle/\langle\bar{q}q\rangle_{\mathrm{vac}}$. The results are plotted in Fig. 7.

There exists phenomenologically a broad interval of $0 < T < 2.48F \simeq 230$ MeV with $Z_\chi < 1$, covering the range of

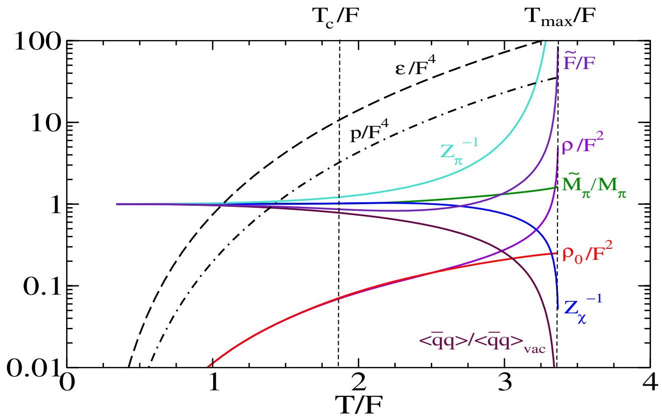

FIG. 7 (color online). The nonrenormalized scalar pion density ρ, the renormalized scalar pion density ρ_0, the ratio $\tilde{M}_\pi/M_\pi$ between the in-medium and the vacuum pion mass, the inverse renormalization constants Z_χ^{-1} and Z_π^{-1} of the pion χ field and the pion field $\pi^\alpha(x) = \frac{1}{2}\,\mathrm{Tr}[\tau^\alpha U(x)]$, respectively, the scalar quark condensate $\langle \bar{q}q \rangle$, and the in-medium pion decay constant $\tilde{F}$ versus temperature T are shown. The energy density ε (dashed curve) and the pressure p (dot-dashed curve) of the pion gas are also plotted. Solutions do not exist above $T_{\max} = 3.36F \simeq 310$ MeV (the right vertical line). The critical temperature $T_c \simeq 175$ MeV of the phase transition into the quark matter is shown by the left vertical line. All quantities are given in units of the pion decay constant $F = 93$ MeV.

temperatures specific for heavy-ion colliders. Above $T \simeq 230$ MeV, the renormalization constant $Z_\chi^{-1} < 1$. In the vacuum, one has $Z^{-1} > 1$ as a consequence of positive definiteness of the spectral density of two-point Green functions. In the medium, the spectral density of temperature Green functions of bosons is not positive definite (see e.g. [43], Chap. 17-3), so $Z_\chi^{-1} < 1$ apparently does not contradict unitarity. Although the product of ω and the pion spectral density cannot be negative, the sum rule for this product is unknown. The region of self-consistency of the MF approximation within perturbation theory over the density extends up to $T_{\max} = 3.36F \simeq 310$ MeV.

A finite-temperature extension of the Weinberg sum rule for the product of ω and the spectral density of vector mesons is discussed in [49,50]. In the MF approximation resonances do not exist, so that the constraints [49,50] apply starting from one-loop ChPT.

For $T < T_{\max} = 3.36F \simeq 310$ MeV the system has regular behavior. The scalar quark condensate decreases smoothly to zero. The effective pion mass increases with temperature in agreement with the partial restoration of chiral symmetry and approaches $\tilde{M}_{\pi,\max} = 1.61 M_\pi$. The pion optical potential $V_{\mathrm{opt}} = (\tilde{M}_\pi^2 - M_\pi^2)/(2M_\pi)$ increases also.

The pion optical potential V_{opt} is calculated to all orders in density. It corresponds to the summation of the forward scattering amplitudes of a probing pion scattered on n surrounding pions, with n running from 1 to infinity.

For $T \simeq T_{\max}$, $Z_\pi^{-1} \to \infty$. The pions disappear from the spectral density of the propagator $\langle \mathcal{T} \pi^\alpha(x) \pi^\beta(y) \rangle$. As elementary excitations, the pions do not disappear, however, from the energy spectrum, Eq. (4.1), and they contribute to the pressure, Eq. (4.2). Moreover, the pions contribute to the spectral density, e.g., of the two-point Green function $\langle \mathcal{T} \pi^\alpha(x) \chi^\beta(y) \rangle$.

The estimate of $T_{\max}$ is significantly above the chemical freeze-out temperature $T \simeq 170$ MeV at RHIC.

It is worthwhile to notice that, for massless pions, the scalar quark condensate vanishes at $T_{\max}(M_\pi = 0) = 2.46F \simeq 230$ MeV. The behavior of observables (3.34), (3.35), (3.36), (3.37), and (3.38) does not change qualitatively. The EOS becomes identical to the EOS of the ideal pion gas, as can be seen from Eqs. (4.1) and (4.2).

Let us check the accuracy of the presentation of the results by the truncated series. In Fig. 8, we show the ratio $\sigma = \langle \bar{q}q \rangle / \langle \bar{q}q \rangle_{\mathrm{vac}}$ calculated using the series expansion (3.37) truncated to orders $O(\rho_0^n)$ for $n = 1, 2, 3$, and 4. It is compared to the exact (numerical) summation of the perturbation series. We see that, in the hadron phase $T \leq T_c$, the first-order approximation is very precise. Noticeable, irregular deviations appear beyond T_c only. We expect, therefore, a 20% decrease of the scalar quark condensate at $T = T_c$.

The first-order approximation is sufficiently precise for $\tilde{M}_\pi^2$: At $T = T_c$, the first order of (3.34) gives the effective pion mass squared $\tilde{M}_\pi^2 \simeq 1.07 M_\pi^2$, while the exact summation gives $1.03 M_\pi^2$. The other observables display similar features.

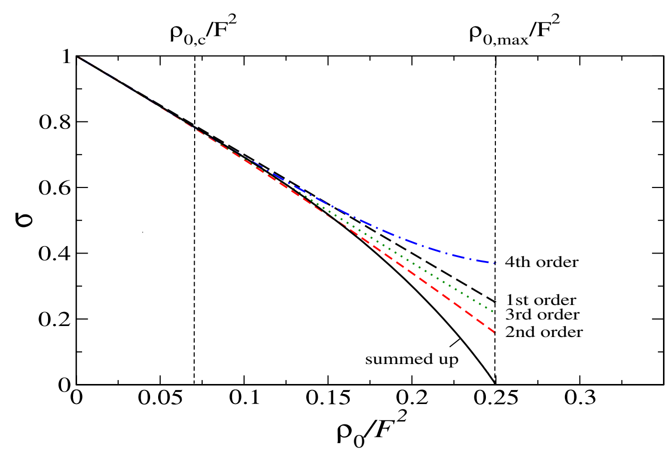

FIG. 8 (color online). The normalized scalar quark density $\sigma = \langle \bar{q}q \rangle / \langle \bar{q}q \rangle_{\mathrm{vac}}$ versus the pion scalar density ρ_0. The power series expansion (3.37) is truncated to orders $O(\rho_0^n)$ for $n = 1, 2, 3$, and 4 (long-dashed, dashed, dotted, and dot-dashed curves, respectively) and results are compared to the exact numerical summation of the series (3.37) using the contour-improved Borel resummation technique (solid curve). The value of $\rho_{0,c}$ is the critical scalar pion density for the phase transition into the quark matter. The value of $\rho_{0,\max}$ is the maximum scalar pion density admissible in the thermalized nonlinear sigma model.

A prescription for the approximate summation of the asymptotic series, which is effective if initial sequential terms of the asymptotic series decrease first before increasing, can be found in Ref. [51]. One should attempt to truncate the asymptotic series where two sequential terms are of the same order. The magnitude of the first neglected term gives the accuracy of the summation. Comparing $n = 1$ and $n = 2$ terms of (3.37), one may conclude that the accuracy of the series truncated at $n = 1$ is 100% for $\rho_0 = 2F^2$. One can expect that, for $\rho_{0,c} = 0.07F^2 \ll 2F^2$, the accuracy is very good. As we observed, this is the case. The same arguments require that ρ_0 be less than $0.25F^2$ for $\tilde{M}_\pi^2$. Such a requirement is satisfied also, however, with a lower precision.

V. COMPARISON WITH LATTICE GAUGE THEORIES

LGTs allow us to calculate QCD observables from first principles. The smallness of physical quark masses and the finite lattice spacing restrict the power of LGTs to temperatures above ~100 MeV. The nonlinear sigma model, on the other hand, is an effective theory of QCD at temperatures small compared to the pion mass. There are no intrinsic restrictions to extrapolate results of the nonlinear sigma model up to $T_{\max} \simeq 310$ MeV. However, above the pion mass, the nonlinear sigma model represents, in a strict sense, a model rather than an effective theory of QCD.

The phase transition to the quark matter appears in LGTs at $T_c \simeq 175$ MeV [1,2]. The domain of validity of the nonlinear sigma model is therefore limited to temperatures below T_c. The interval of temperatures from ~100 MeV up to T_c is suitable for a meaningful comparison of predictions from the nonlinear sigma model and LGTs.

Figure 9 shows the temperature dependence of the quark scalar condensate. The overall mass scale in LGTs is fixed assuming the string tension is flavor and quark mass independent. The agreement of the nonlinear sigma model and LGT [11] is not unreasonable with regard to the first 2–3 points with lowest temperatures. In the narrow deconfinement region the models strongly diverge, basically because the nonlinear sigma model does not expose quark-gluon degrees of freedom.

The energy density as a function of temperature is shown in Fig. 10 at zero chemical potentials. The solid line and the dashed line are predictions of the nonlinear sigma model. In addition, the lattice simulations from Refs. [1,11] are shown.

At a first-order phase transition, the energy density experiences a jump, while the pressure remains a smooth function. It is not quite clear from the LGT data presented in Fig. 10 whether we observe a crossover or a first-order phase transition. Usually, in the two-flavor case lattice simulations give evidence of a first-order phase transition, while pure gauge theory predicts a second-order transition

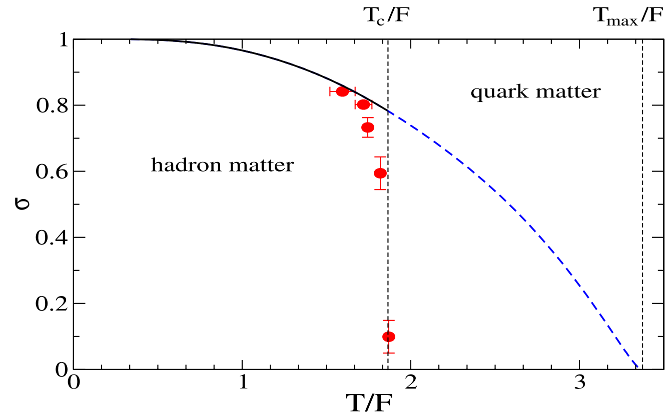

FIG. 9 (color online). The scalar quark condensate $\sigma = \langle\bar{q}q\rangle/\langle\bar{q}q\rangle_{\text{vac}}$ versus the temperature in units of $F = 93$ MeV obtained within the nonlinear sigma model using the MF approximation (solid line below T_c and dashed line above T_c) and from lattice gauge theory for two flavors [11].

and 2 + 1 flavor LGTs with physical quark masses predict a smooth crossover [3].

If the phase transition is of first order, the energy density in LGT appears to be several times greater than in the nonlinear sigma model. Such a divergence can be attributed to resonances like ρ and ω mesons which contribute to the energy density, but do not exist in the nonlinear sigma model in the MF approximation. An ideal resonance gas model which is in agreement with lattice simulations has e.g. been developed by Karsch *et al.* [10].

In the case of a first-order phase transition, the jump in the energy density can be as large as $\Delta\varepsilon_c/T_c^4 \sim 8$, in which case pions alone saturate the energy density. If the jump is smaller, Hagedorn's conjecture on the exponential growth of the number of resonances with mass might be appro-

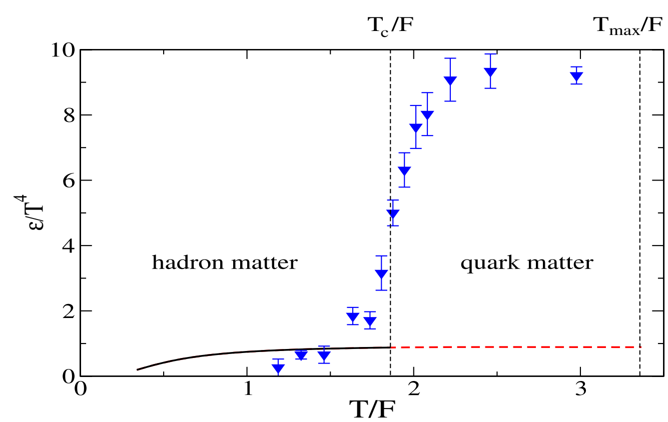

FIG. 10 (color online). The normalized energy density ε/T^4 versus the temperature T obtained within the MF approximation of the nonlinear sigma model (solid line below T_c and dashed line above T_c) and from lattice gauge theory (filled triangles) with two flavors [1,11]. The temperature is given in units of $F = 93$ MeV.

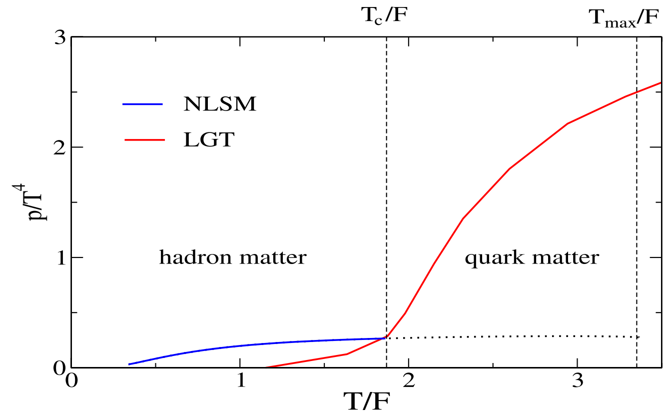

FIG. 11 (color online). The normalized pressure p/T^4 versus the temperature T in units of $F = 93$ MeV in the nonlinear sigma model (NLSM) using the MF approximation (solid line below T_c and dashed line above T_c) and in lattice gauge theory (LGT) with two flavors [1,11].

priate. The value of the jump is crucial in order to understand the role of higher resonances.

Modeling the first-order phase transition [7] within the framework of the MIT bag model [52] allows us to determine the critical temperature using the pressure balance $p_Q = p_H + B$, where $B = 57\ \mathrm{MeV/Fm^3}$ [53] is the vacuum pressure in $2 + 1$ flavor QCD; p_Q and p_H are pressures in quark and hadron phases, respectively, and determine the jump in the energy density $\Delta\varepsilon_c = \varepsilon_Q + B - \varepsilon_H$ where ε_Q and ε_H are energy densities of quark matter and hadron matter, respectively. For ideal gases of relativistic particles, $\Delta\varepsilon_c = 3p_Q + B - 3p_H = 4B$, and so $\Delta\varepsilon_c/T_c^4 \sim 3$. The vacuum pressure B is density and temperature dependent [54–59], which brings more uncertainties to $\Delta\varepsilon_c$.

The energy density predicted by LGTs is obviously underestimated around $T \sim 100$ MeV due to unphysical quark masses $m_Q/T = 0.4$ and restrictions from finite lattice size. The energy density and the pressure depend strongly on the quark masses and the pion mass. The horizontal part of the EOS at high T (dashed line and partially solid line) is close to the energy density of massless pions $\varepsilon/T^4 = \pi^2/10$.

The pressure is plotted in Fig. 11. The lattice predictions [1,11] and the nonlinear sigma model predictions are in agreement at the phase transition point, while at smaller temperatures the pressure in LGTs is significantly lower. The pressure of the pion gas approaches the pressure of the ideal gas of massless pions $p/T^4 = \pi^2/30$ at $T \to T_{\max}$. However, it does not fully reach the ultrarelativistic limit, so the effects from the finite pion mass remain.

According to Gibbs' criterion (see e.g. [60]) the phase of matter with the highest pressure at equal temperatures and chemical potentials is preferable. At $T \sim T_c$, heavy resonances are still slowly moving particles which contribute to the energy density $\delta\varepsilon$ mainly through their rest mass M and contribute to the pressure as $\delta p \sim \frac{T}{M}\delta\varepsilon \ll \delta\varepsilon$. For zero chemical potentials, the pressure in the hadron phase is dominated by light pions [7].

The nonlinear sigma model predicts the critical pressure p_c reasonably well. The critical temperature can be obtained phenomenologically from Fig. 11, applying Gibbs' criterion for the hadronic EOS derived using the nonlinear sigma model and the quark matter EOS derived from LGTs. The intersection of the two pressure curves then yields the temperature of the phase transition. The value obtained in such a way is in remarkable agreement with lattice estimates based on the position of the steep rise of the energy density seen in Fig. 10.

VI. CONCLUSION

It is generally believed that ChPT represents an adequate tool for studying the pion matter at low temperatures. The Haar measure appearing in the path integral is important to keep the chiral symmetry at high temperatures unbroken within the MF approximation. In an arbitrary parametrization of the pion fields, one faces a dilemma: Accounting for the effective potential $\delta\mathcal{L}_H$ arising due to exponentiating the weight factor of the path integral measure brings divergences which can be compensated by going beyond the MF approximation only, i.e., by including pion loops appearing in the higher order ChPT expansion. If $\delta\mathcal{L}_H$ is neglected, the MF approximation does not restore the chiral invariance with increasing the temperature. There is only one parametrization with $\delta\mathcal{L}_H = 0$, which makes the MF approximation self-consistent: The renormalizations are finite and the chiral symmetry is restored at high temperatures. This parametrization was used to study the in-medium modifications of pions and collective characteristics of the thermalized pion matter in the MF approximation.

We made essentially two approximations:

(a) The results are obtained by extending the integration region over the pion fields χ^α from $-\infty$ to $+\infty$.

(b) The MF approximation consists of the neglection of loops composed from more than one dressed pion propagator. Loops formed by one dressed pion propagator give the pion density (3.7), i.e., the thermal parts of loops are accounted for, whereas the vacuum contributions of the zero-point field fluctuations are systematically neglected. Accounting for the vacuum parts of the pion loops in a thermal bath is possible beyond the lowest ChPT order.

We constructed Borel transforms of the most important thermal averages. The corresponding power series with respect to the density are asymptotic and have zero convergence radius, but are summable using the contour-improved Borel resummation method. The pion gas does not exist above $T_{\max} \simeq 310$ MeV, whereas at $T < T_{\max}$ thermodynamic observables are smooth functions.

The deconfinement temperature $T_c = 1.86F$ and the corresponding scalar density $\rho_{0,c} = 0.07F^2$ appear to be small enough for approximate calculation of observables in the hadron phase using the truncated asymptotic series.

The region of validity of the method can be evaluated by requiring $\langle\chi^2\rangle \leq \chi^2_{\max}$. The corresponding restriction $T \leq \sqrt{8(3\pi)^{2/3}}F \approx 560$ MeV does not appear to be stringent.

The method of summation of the density series proposed in this work can be extended to higher orders of ChPT loop expansion.

ACKNOWLEDGMENTS

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APPENDIX A: MF APPROXIMATION FOR $\langle(\chi^\gamma\chi^\gamma)^n\rangle$

The order of the operators $\chi^\alpha = \chi^\alpha(0, \mathbf{0})$ entering the thermal average $\langle(\chi^\gamma\chi^\gamma)^n\rangle$ is not specified so far. As we shall see, it is not important. Let us rewrite $\langle(\chi^\gamma\chi^\gamma)^n\rangle$ with the help of the ordering operator $\mathcal{T}$. To make its action well defined, we set first arguments of the operators $\chi^\alpha(0, \mathbf{0})$ equal to small parameters ϵ_α, i.e., we make replacements $\chi^\alpha(0, \mathbf{0}) \to \chi^\alpha(\epsilon_\alpha, \mathbf{0})$. If the limit $\epsilon_\alpha \to 0$ exists and does not depend on the way the parameters ϵ_α approach zero, the thermal average $\langle(\chi^\gamma\chi^\gamma)^n\rangle$ can be expressed in terms of the Green function:

$$\langle(\chi^\gamma\chi^\gamma)^n\rangle = \lim_{\epsilon_\alpha\to 0}\langle\mathcal{T}\chi^{\alpha_1}\chi^{\alpha_1}\chi^{\alpha_2}\chi^{\alpha_2}\dots\chi^{\alpha_n}\chi^{\alpha_n}\rangle. \quad \text{(A1)}$$

In order to evaluate the Green function, we pass to the interaction representation,

$$\langle\mathcal{T}\chi^{\alpha_1}\chi^{\alpha_1}\chi^{\alpha_2}\chi^{\alpha_2}\dots\chi^{\alpha_n}\chi^{\alpha_n}\rangle = \left\langle\mathcal{T}\chi_0^{\alpha_1}\chi_0^{\alpha_1}\chi_0^{\alpha_2}\chi_0^{\alpha_2}\dots\chi_0^{\alpha_n}\chi_0^{\alpha_n}S\left(\frac{1}{T}, 0\right)\right\rangle\left\langle S\left(\frac{1}{T}, 0\right)\right\rangle^{-1}, \quad \text{(A2)}$$

where $\chi_0^\alpha(\tau, \mathbf{x}) = S(\tau, 0)\chi^\alpha(0, \mathbf{x})S^{-1}(\tau, 0)$ and $S(\tau_2, \tau_1)$ is the thermal S matrix (see e.g. [43]). Applying Wick's theorem to the $2n$-body Green function (A2) and neglecting all the diagrams except for those corresponding to the noninteracting gas of the dressed pions, we obtain

$$\mathcal{G}(\epsilon_1, \epsilon_2, \dots, \epsilon_{2n}) = \sum_{(\beta_1\beta_2\dots\beta_n),(\gamma_1\gamma_2\dots\gamma_n)}\langle\mathcal{T}\chi^{\beta_1}\chi^{\gamma_1}\rangle \times\langle\mathcal{T}\chi^{\beta_2}\chi^{\gamma_2}\rangle\dots\langle\mathcal{T}\chi^{\beta_n}\chi^{\gamma_n}\rangle, \quad \text{(A3)}$$

where $(\beta_1\beta_2\dots\beta_n)$ and $(\gamma_1\gamma_2\dots\gamma_n)$ are permutations of $(\alpha_1\alpha_2\dots\alpha_n)$. The summation runs over those permutations which occur according to the diagram decomposition of the $2n$-body Green function of a noninteracting gas. The pion propagators are dressed. The Green functions $\langle\mathcal{T}\chi^\alpha\chi^\beta\rangle$ therefore enter Eq. (A3) instead of $\langle\mathcal{T}\chi_0^\alpha\chi_0^\beta\rangle$.

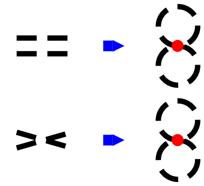

FIG. 12 (color online). In the limit $\epsilon_\alpha \to 0$, the disconnected diagrams of the four-body Green function produce no loops with more than one dressed pion propagator, thus being significant in the MF approximation.

Shown in Fig. 12 are diagrams of the four-body Green function corresponding to the noninteracting gas of the effective pions. Such diagrams contribute to thermal averages. The diagram shown in Fig. 13 which accounts for the pion rescattering should be discarded in the MF approximation. This feature holds for $2n$-body Green functions. The MF approximation does not involve terms except for those entering Eq. (A3).

The isotopic symmetry of the pion matter amounts to the two-body Green function $\mathcal{G}^{\alpha\beta}(\tau, \mathbf{k})$ which is symmetric under the permutation $\alpha \leftrightarrow \beta$ at $\tau = 0$. Moreover,

$$\lim_{\epsilon\to+0}\mathcal{G}^{\alpha\beta}(\epsilon, \mathbf{k}) = \lim_{\epsilon\to-0}\mathcal{G}^{\alpha\beta}(\epsilon, \mathbf{k}). \quad \text{(A4)}$$

Equations (A3) and (A4) imply that, after neglection by all terms proportional to ρ_{vac}, the limit $\epsilon_\alpha \to 0$ in Eq. (A1) indeed exists and, at least within the MF approximation, does not depend on the way the parameters ϵ_α approach zero. Loops formed by one pion propagator give the density (3.7), while loops involving more than one dressed pion propagator are neglected.

We therefore obtain

$$\langle(\chi^\gamma\chi^\gamma)^n\rangle = (2\rho)^n \sum_{(\beta_1\beta_2\dots\beta_n),(\gamma_1\gamma_2\dots\gamma_n)}\delta^{\beta_1\gamma_1}\delta^{\beta_2\gamma_2}\dots\delta^{\beta_n\gamma_n}. \quad \text{(A5)}$$

The combinatorial structure of Wick's decomposition is identical to that of the corresponding Gaussian integral, so one can write

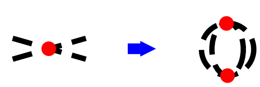

FIG. 13 (color online). In the limit $\epsilon_\alpha \to 0$, connected diagrams of the four-body Green function produce loops involving more than one pion propagator and therefore are not significant in the MF approximation. The only diagrams essential for the thermal average $\langle(\chi^\gamma\chi^\gamma)^n\rangle$ are those which correspond to the noninteracting gas of the dressed pions.

$$\sum_{(\beta_1\beta_2\ldots\beta_n),(\gamma_1\gamma_2\ldots\gamma_n)} \delta^{\beta_1\gamma_1}\delta^{\beta_2\gamma_2}\ldots\delta^{\beta_n\gamma_n} = \frac{1}{(2\pi)^{3/2}}\iiint x^{\alpha_1}x^{\alpha_1}x^{\alpha_2}x^{\alpha_2}\ldots x^{\alpha_n}x^{\alpha_n}\exp\left(-\frac{1}{2}((x^1)^2+(x^2)^2+(x^3)^2)\right)dx^1dx^2dx^3 = \frac{4\pi}{(2\pi)^{3/2}}\int_0^{+\infty} r^{2n+2}\exp\left(-\frac{1}{2}r^2\right)dr. \tag{A6}$$

Replacing $r \to \sqrt{2t}$, we arrive at Eq. (3.32).

APPENDIX B: CHIRAL INVARIANCE IN THE MF APPROXIMATION

As discussed in Sec. II, the MF approximation imposes restrictions on the pion field parametrizations. If the tree-level $\delta\mathcal{L}_H$ is included in a MF calculation, one gets divergences. On the other hand, the neglection by $\delta\mathcal{L}_H$ violates the chiral symmetry. Let us demonstrate it explicitly.

Equation (3.33) can be rewritten in more physical terms,

$$\langle\mathcal{J}(\chi^2)\rangle = \frac{1}{(4\pi\rho)^{3/2}}\int \mathcal{J}(\chi^2)e^{-\chi^2/(4\rho)}d^3\chi. \tag{B1}$$

The scalar product

$$\cos(\Theta) = \tfrac{1}{2}\mathrm{Tr}[U(\phi)U^\dagger(\phi')] \tag{B2}$$

is obviously chirally invariant. The value of $\Theta = \Theta(\phi, \phi')$ determines the angular distance between two sets of the pion fields. The parameter space is thus a metric space with an infinitesimal distance,

$$d\Theta^2 = d\phi^2 + \sin^2(\phi)(d\theta^2 + \sin^2(\theta)d\varphi^2), \tag{B3}$$

where θ and φ are polar and azimuthal angles of the vector ϕ^α.

Equation (B1) has a transparent physical meaning. Quantum fluctuations of the fields χ^α induce fluctuations of a function $\mathcal{J}(\chi^2)$. These fluctuations are suppressed by the exponential factor entering (B1).

According to Eq. (2.5), χ depends on ϕ which has the meaning of a metric distance

$$\phi = \Theta(\phi^\alpha, \phi^\alpha_{\mathrm{vac}}) \tag{B4}$$

between the vector ϕ^α and the vector $\phi^\alpha_{\mathrm{vac}} = 0$ which specifies the vacuum state on the 4-dimensional sphere, thereby making the chiral symmetry spontaneously broken. The only essential place in Eq. (B1) where the dependence on the $\phi^\alpha_{\mathrm{vac}} = 0$ shows up is the exponential factor. In the limit $T \to \infty$ the density increases, $\rho \to \infty$, and so the dependence on $\phi^\alpha_{\mathrm{vac}} = 0$ drops out. The exponential factor becomes a constant. This means that quantum fluctuations at all points of the 4-dimensional sphere contribute equally to thermal averages. This is in agreement with our expectations that the chiral symmetry is restored by increasing the temperature.

The use of the exponential parametrization with the Haar measure neglected would result in

$$\langle\mathcal{J}(\chi^2)\rangle \stackrel{?}{=} \frac{1}{(4\pi\rho)^{3/2}}\int \mathcal{J}(\chi^2)e^{-\phi^2/(4\rho)}d^3\phi. \tag{B5}$$

The Euclidean volume $d^3\phi$ explicitly keeps a reference to the vacuum vector $\phi^\alpha_{\mathrm{vac}} = 0$. This means that the high temperature limit does not restore the chiral invariance in the MF approximation provided the Haar measure is neglected.

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