Reparametrizing the Skyrme model using the lithium-6 nucleus

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The minimal-energy $B = 6$ solution of the Skyrme model is a static soliton with D_{4d} symmetry. The symmetries of the solution imply that the quantum numbers of the ground state are the same as those of the lithium-6 nucleus. This identification is considered further by obtaining expressions for the mean charge radius and quadrupole moment, dependent only on the Skyrme model parameters e (a dimensionless constant) and F_π (the pion decay constant). The optimal values of these parameters have often been deliberated upon, and we propose, for $B > 2$, changing them from those which are most commonly accepted. We obtain specific values for these parameters for $B = 6$, by matching with properties of the lithium-6 nucleus. We find further support for the new values by reconsidering the α-particle and deuteron as quantized $B = 4$ and $B = 2$ Skyrmions.

I. INTRODUCTION

A nonlinear theory was put forward by T. H. R. Skyrme in 1961 [1]. It sets out to describe a low-energy effective theory of QCD, and the Lagrangian of the theory is defined in terms of pion fields. The most successful and promising physical interpretation of the theory is as a description of atomic nuclei. The Skyrme model admits topological soliton solutions, falling into sectors labeled by an integer-valued topological degree. This quantity is identified with baryon number. A quantized Skyrmion of topological charge B is interpreted as a nucleus with baryon number B.

To date, relatively little work has been done on the electromagnetic properties of quantized Skyrmions and their comparison to known experimental results. Two papers by Braaten and Carson [2,3] consider the electrostatic properties of the $B = 2$ Skyrmion, and their comparison to the experimentally determined properties of the deuteron. This analysis was extended in [4] to account for the softest vibrational modes of the $B = 2$ Skyrmion, the quantization of which resulted in an accurate prediction of the mean charge radius of the deuteron.

In this paper we perform a similar analysis for the $B = 6$ Skyrmion. However, performing a quantization of all possible vibrational modes for $B > 2$ is technically very difficult. We therefore consider rescaling the Skyrme model parameters (introduced in the next section), while suppressing the vibrational modes, in order to calculate observables that more closely correspond to experiment.

The outline of the paper is as follows. In the following section we present the Skyrme model and, following [5], give an expression for the electromagnetic current. Section III provides an overview of the rational map ansatz for Skyrmions [6], which is used in later sections.

Section IV discusses the fermionic quantization of solitons, as applied to the Skyrme model, recalls the Finkelstein-Rubinstein (FR) constraints [7], and reviews Krusch's method of determining the FR constraints [8]. In Sec. V we recall the semiclassical method of soliton quantization, following [2], in which only the translational, rotational, and isorotational collective coordinates are considered. In Sec. VI this quantization of the $B = 6$ Skyrmion is performed, and we find that the symmetries of the minimal-energy classical solution, via the FR constraints, ensure that the lowest-lying quantized state has spin 1 and isospin 0, the quantum numbers of the lithium-6 nucleus. In Sec. VII we present an expression for the mean charge radius of the quantized $B = 6$ Skyrmion. Section VIII presents an expression for the quadrupole moment of the quantized $B = 6$ Skyrmion. The reparametrization of the Skyrme model in the $B = 6$ sector is considered in Sec. IX, and the mean charge radius and quadrupole moment of the quantized $B = 6$ Skyrmion are explicitly calculated and compared to experiment. In Sec. X we reconsider the $B = 2$ and $B = 4$ Skyrmions in the light of the reparametrization. We provide a conclusion in Sec. XI. The calculations presented in this paper support our interpretation of the lithium-6 nucleus as the lowest-lying quantum state of the $B = 6$ Skyrmion. They confirm that if one wishes to apply the Skyrme model to nuclei, then a reparametrization is desirable.

II. THE SKYRME MODEL

The Skyrme model is a nonlinear theory of pions in three spatial dimensions, defined in terms of an $SU(2)$-valued scalar, the Skyrme field [1,9]. It provides a low-energy effective theory of QCD, becoming exact as the number of quark colors becomes large [10,11]. Its topological soliton solutions, Skyrmions, can be interpreted as nucleons and nuclei.

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The $SU(2)$ Skyrme model has the Lagrangian density

$$\mathcal{L} = \frac{F_\pi^2}{16} \operatorname{Tr}\partial_\mu U \partial^\mu U^{-1} + \frac{1}{32e^2} \operatorname{Tr}[\partial_\mu U U^{-1}, \partial_\nu U U^{-1}]$$
$$\times [\partial^\mu U U^{-1}, \partial^\nu U U^{-1}] + \frac{1}{8} m_\pi^2 F_\pi^2 \operatorname{Tr}(U - 1_2), \quad (1)$$

where $U(t, \mathbf{x})$ is the Skyrme field, F_π is the pion decay constant, e is a dimensionless parameter and m_π is the pion mass.

It is helpful to use energy and length units of $F_\pi/4e$ and $2/eF_\pi$ respectively. In terms of these Skyrme units the Skyrme Lagrangian becomes

$$L = \int \left\{ -\frac{1}{2} \operatorname{Tr}(R_\mu R^\mu) + \frac{1}{16} \operatorname{Tr}([R_\mu, R_\nu][R^\mu, R^\nu]) \right.$$
$$\left. + m^2 \operatorname{Tr}(U - 1_2) \right\} d^3x, \quad (2)$$

where we have introduced the $\mathfrak{su}(2)$-valued current $R_\mu = (\partial_\mu U)U^{-1}$, and defined $m = 2m_\pi/eF_\pi$.

The scalar field U, at a fixed time, is a map from $\mathbb{R}^3$ into S^3, the group manifold of $SU(2)$. However, the boundary condition $U \to 1_2$ at spatial infinity implies a one-point compactification of space, so that topologically $U: S^3 \to S^3$, where the domain S^3 is identified with $\mathbb{R}^3 \cup \{\infty\}$. Thus, configurations U obeying this boundary condition fall into topological sectors labeled by their topological degree

$$B = \int B_0(\mathbf{x}) d^3x \quad (3)$$

where

$$B_\mu(x) = \frac{1}{24\pi^2} \epsilon_{\mu\nu\alpha\beta} \operatorname{Tr}\partial^\nu U U^{-1} \partial^\alpha U U^{-1} \partial^\beta U U^{-1}. \quad (4)$$

The degree B, which takes integer values, is identified with baryon number. We refer to B_0 as the baryon density. The minimal-energy, static solutions of the field equation for each nonzero baryon number we call Skyrmions.

The internal symmetry of the Skyrme model is the global isospin symmetry $U \to AUA^\dagger$ where A is an $SU(2)$ matrix. This is generated by the infinitesimal transformations $U \to U + i\epsilon[\tau_p, U]$, where τ_p: $p = 1, 2, 3$ are the Pauli matrices. To couple electromagnetism to the Skyrme model, the $U(1)$ symmetry generated by the third component of isospin is gauged, with the derivatives in (1) and (2) replaced by the covariant derivative $D_\mu U = \partial_\mu U - ie_0 A_\mu[Q, U]$, where $Q = \frac{1}{2}\tau_3 + \frac{1}{6}1_2$ is the charge matrix of quarks. In the presence of electromagnetism, the form of the baryon current density given previously is unsatisfactory as it is not gauge invariant under

$$U(x) \to U(x) + ie_0 \alpha(x)[Q, U], \qquad A_\mu \to A_\mu + \partial_\mu \alpha. \quad (5)$$

The gauge invariant, conserved generalization is given by [5]

$$B_\mu(x) = \frac{1}{24\pi^2} \epsilon_{\mu\nu\alpha\beta} \{ \operatorname{Tr} U^{-1}\partial^\nu U U^{-1}\partial^\alpha U U^{-1}\partial^\beta U + 3ie_0 A^\nu \operatorname{Tr} Q(U^{-1}\partial^\alpha U U^{-1}\partial^\beta U - \partial^\alpha U U^{-1}\partial^\beta U U^{-1})$$
$$+ 3ie_0 \partial^\nu A^\alpha \operatorname{Tr} Q(U^{-1}\partial^\beta U + \partial^\beta U U^{-1}) \}. \quad (6)$$

The electromagnetic current is determined as

$$J_\mu(x) = \frac{1}{16\pi^2} \epsilon_{\mu\nu\alpha\beta} \operatorname{Tr} Q(\partial^\nu U U^{-1}\partial^\alpha U U^{-1}\partial^\beta U U^{-1} + U^{-1}\partial^\nu U U^{-1}\partial^\alpha U U^{-1}\partial^\beta U)$$
$$+ \frac{ie_0}{4\pi^2} \epsilon_{\mu\nu\alpha\beta} \partial^\nu A^\alpha \operatorname{Tr} \left\{ Q^2 \partial^\beta U U^{-1} + Q^2 U^{-1}\partial^\beta U + \frac{1}{2} Q \partial^\beta U Q U^{-1} - \frac{1}{2} Q U Q \partial^\beta U^{-1} \right\} + I_\mu^3(x). \quad (7)$$

We are interested in static or slowly varying Skyrme fields, quantized to have isospin zero. In this case the electromagnetic current simplifies. It can be shown, assuming $I_0^3(x) = 0$, $A_0 \neq 0$, $\mathbf{A} = 0$ and $\partial_0 A_\mu = 0$, that

$$J_0 = \frac{1}{2} B_0 = \frac{1}{24\pi^2} \epsilon_{0ijk} \epsilon_{pqr} \gamma_i^p \gamma_j^q \gamma_k^r = \frac{1}{4\pi^2} \det\Gamma, \quad (8)$$

where the matrix Γ has elements γ_i^p defined by

$$(\partial_i U)U^{-1} = -i\gamma_i^p \tau_p. \quad (9)$$

Therefore, the electric charge density is half the original baryon density.

III. THE RATIONAL MAP ANSATZ

The rational map ansatz [6] applies the topological notion of suspension, using a rational map from $S^2 \to S^2$ to construct an approximate Skyrmion, a map from $\mathbb{R}^3 \to S^3$. It exhibits a nonlinear separation of variables, separating the angular and radial dependence of the Skyrme field. One identifies the domain S^2 with concentric spheres in $\mathbb{R}^3$, and the target S^2 with spheres of latitude on S^3. A point in $\mathbb{R}^3$ can be denoted by its coordinates (r, z), where r is the radial distance from the origin and z specifies the direction from the origin, a point on the unit sphere. Via stereographic projection, the complex coordinate z can be identified with conventional polar coordinates by

$z = \tan(\theta/2)e^{i\phi}$. Equivalently, the point z corresponds to the unit vector

$$\mathbf{n}_z = \frac{1}{1 + |z|^2}(z + \bar{z}, i(\bar{z} - z), 1 - |z|^2), \qquad (10)$$

and conversely

$$z = \frac{(\mathbf{n}_z)_1 + i(\mathbf{n}_z)_2}{1 + (\mathbf{n}_z)_3}. \qquad (11)$$

The ansatz for the Skyrme field depends on a rational map $R(z) = p(z)/q(z)$, where p and q are polynomials in z, and a radial profile function $f(r)$. The value of the rational map R is associated with the unit vector

$$\mathbf{n}_R = \frac{1}{1 + |R|^2}(R + \bar{R}, i(\bar{R} - R), 1 - |R|^2). \qquad (12)$$

The ansatz is

$$U(r, z) = \exp(if(r)\mathbf{n}_{R(z)} \cdot \boldsymbol{\tau}), \qquad (13)$$

with $f(r)$ satisfying $f(0) = \pi$ and $f(\infty) = 0$.

The degree of the rational map, N, is the greater of the algebraic degrees of the polynomials p and q. N is also equal to the topological degree of the map R viewed as a map from $S^2 \to S^2$. The baryon density of the ansatz is given by

$$B_0(\mathbf{x}) = \frac{-f'}{2\pi^2}\left(\frac{\sin f}{r}\right)^2\left(\frac{1 + |z|^2}{1 + |R|^2}\left|\frac{dR}{dz}\right|\right)^2, \qquad (14)$$

and so the baryon number is given by

$$B = \int \frac{-f'}{2\pi^2}\left(\frac{\sin f}{r}\right)^2\left(\frac{1 + |z|^2}{1 + |R|^2}\left|\frac{dR}{dz}\right|\right)^2 \frac{2idzd\bar{z}}{(1 + |z|^2)^2} r^2 dr, \qquad (15)$$

where $2idzd\bar{z}/(1 + |z|^2)^2$ is equivalent to the usual 2-sphere area element $\sin\theta d\theta d\phi$. It is straightforward to show that $B = N$.

An $SU(2)$ Möbius transformation M_1 of R corresponds to an isorotation; an $SU(2)$ Möbius transformation M_2 of z corresponds to a rotation in physical space. They induce the following transformation of the rational map $R(z)$:

$$R(z) \to \tilde{R}(z) = M_1(R(M_2(z))). \qquad (16)$$

The energy for a field of the form (13) is

$$E = \int \left\{ f'^2 + 2\frac{\sin^2 f}{r^2}(f'^2 + 1)\left(\frac{1 + |z|^2}{1 + |R|^2}\left|\frac{dR}{dz}\right|\right)^2 \right.$$
$$\left. + \frac{\sin^4 f}{r^4}\left(\frac{1 + |z|^2}{1 + |R|^2}\left|\frac{dR}{dz}\right|\right)^4 + 2m^2(1 - \cos f) \right\}$$
$$\times \frac{2idzd\bar{z}}{(1 + |z|^2)^2} r^2 dr, \qquad (17)$$

which can be simplified to

$$E = 4\pi \int_0^\infty \left(r^2 f'^2 + 2B\sin^2 f(f'^2 + 1) + I\frac{\sin^4 f}{r^2} \right.$$
$$\left. + 2m^2 r^2(1 - \cos f) \right) dr. \qquad (18)$$

Here I denotes the purely angular integral

$$I = \frac{1}{4\pi} \int \left(\frac{1 + |z|^2}{1 + |R|^2}\left|\frac{dR}{dz}\right|\right)^4 \frac{2idzd\bar{z}}{(1 + |z|^2)^2} \qquad (19)$$

which only depends on the rational map R. Both B and I, and hence the energy E, are invariant under the transformations (16).

To minimize E one first minimizes I over all maps of degree B. The profile function f may then be found by solving the second order ODE that is the Euler-Lagrange equation for the expression (18) with B and I as fixed parameters:

$$(r^2 + 2B\sin^2 f)f'' + 2rf' + \sin 2f\left(B(f'^2 - 1) - I\frac{\sin^2 f}{r^2}\right)$$
$$- m^2 r^2 \sin f = 0. \qquad (20)$$

Note that the inclusion of the pion mass term in the Lagrangian density has no effect on the rational map, but results in the profile function being slightly modified, leading to higher energies than in the massless case. The optimized rational map ansatz has been shown to give a good approximation to the true Skyrmion for baryon numbers up to $B = 7$ (and for a larger range of B in the massless pion case) [6,12]. For $B = 8$, 9 and beyond, and m of order 1, the structure of the minimal-energy Skyrmions differs qualitatively from that given by the rational map ansatz [13,14].

IV. FERMIONIC QUANTIZATION OF THE SKYRME MODEL

We recall that the configuration space $\mathcal{C}$ of the Skyrme model has connected components $\mathcal{C}_B$, which are the homotopy classes, labeled by baryon number. The fundamental group of each component satisfies $\pi_1(\mathcal{C}_B) = \mathbb{Z}_2$, and therefore all $\mathcal{C}_B$ admit double-valued functions. However, these double-valued functions can be defined as single-valued functions on $\tilde{\mathcal{C}}_B$, the universal covering space of $\mathcal{C}_B$.

We write $\tilde{q}$, $\tilde{q}'$ for different points of $\tilde{\mathcal{C}}_B$ covering the point $q \in \mathcal{C}_B$. The condition that $\tilde{q} \neq \tilde{q}'$ implies that a path from $\tilde{q}$ to $\tilde{q}'$ projects to a noncontractible loop in $\mathcal{C}_B$. Double-valued functions Ψ on $\mathcal{C}_B$ are thought of as functions on $\tilde{\mathcal{C}}_B$:

$$\Psi: \tilde{\mathcal{C}}_B \to \mathbb{C}, \qquad \Psi = \Psi(\tilde{q}). \qquad (21)$$

Fermionic quantization requires that the wavefunction is defined on $\tilde{\mathcal{C}}_B$ and satisfies

$$\Psi(\tilde{q}) = -\Psi(\tilde{q}'). \qquad (22)$$

Let a rational map R have the following symmetry, for some particular M_1 and M_2,

$$R(z) = M_1(R(M_2(z))), \qquad (23)$$

where M_1 corresponds to an isorotation by θ_1 around $\mathbf{n}_1$ and M_2 corresponds to a rotation by θ_2 around $\mathbf{n}_2$. For $\theta_2 \neq 2\pi k$ for $k \in \mathbb{Z}$, M_2 only leaves the antipodal points

$$z_{\mathbf{n}_2} = \frac{(\mathbf{n}_2)_1 + i(\mathbf{n}_2)_2}{1 + (\mathbf{n}_2)_3} \quad \text{and} \quad z_{-\mathbf{n}_2} = -\frac{(\mathbf{n}_2)_1 + i(\mathbf{n}_2)_2}{1 - (\mathbf{n}_2)_3} \qquad (24)$$

fixed. Similarly, M_1 only leaves $R_{\pm\mathbf{n}_1}$ fixed, where $R_{\pm\mathbf{n}_1}$ are defined similarly as above. Therefore, for the symmetry $R(z) = M_1(R(M_2(z)))$ to hold, we have

$$R(z_{-\mathbf{n}_2}) = R_{\pm\mathbf{n}_1}. \qquad (25)$$

By reversing the signs of $\mathbf{n}_1$ and θ_1 if necessary, we may set

$$R(z_{-\mathbf{n}_2}) = R_{\mathbf{n}_1}. \qquad (26)$$

Now consider the Skyrme field configuration defined by the rational map ansatz, using this rational map and some profile function f, and assume the quantum wavefunction is nonvanishing for this field configuration. The symmetry gives rise to a loop in configuration space (one thinks of this as a loop in $\mathcal{C}_B$ by letting the isorotation angle increase from 0 to θ_1, with the rotation angle increasing from 0 to θ_2). This leads to the following constraint on the wavefunction:

$$e^{i\theta_2\mathbf{n}_2\cdot\mathbf{L}}e^{i\theta_1\mathbf{n}_1\cdot\mathbf{K}}|\Psi\rangle = \chi_{\text{FR}}|\Psi\rangle, \qquad (27)$$

where $\mathbf{L}$ and $\mathbf{K}$ are the body-fixed spin and isospin operators, respectively, and the Finkelstein-Rubinstein (FR) phase χ_{FR} enforces the fermionic quantization condition:

$$\chi_{\text{FR}} = \begin{cases} +1 & \text{if the loop induced by the symmetry is contractible,} \\ -1 & \text{otherwise.} \end{cases} \qquad (28)$$

We note the following general result, proved in [8]:

The value of χ_{FR} for a given symmetry of a rational map only depends on the isorotation angle θ_1 and the rotation angle θ_2, where the angles are defined such that $R(z_{-\mathbf{n}_2}) = R_{\mathbf{n}_1}$, and is given by

$$\chi_{\text{FR}} = (-1)^{\mathcal{N}}, \quad \text{where } \mathcal{N} = \frac{B}{2\pi}(B\theta_2 - \theta_1). \qquad (29)$$

V. SEMICLASSICAL COLLECTIVE COORDINATE QUANTIZATION

The Skyrme Lagrangian is invariant under the Poincaré group of $(3 + 1)$-dimensional space, $SO(3)$ isorotations and some discrete parity transformations, which will not be considered here. Similarly, the space of static solutions, that is, configurations which minimize the energy functional, is invariant under the Euclidean group and isorotations, $\mathbb{E}_3 \times SO(3)^I$. By acting with the latter symmetry group on a static Skyrmion U_0 we generate a set of new solutions:

$$U(\mathbf{x}) = A_1 U_0(D(A_2)(\mathbf{x} - \mathbf{X}))A_1^\dagger, \qquad (30)$$

where A_1, A_2 are $SU(2)$ matrices and $D(A_2)_{ij} = \frac{1}{2}\text{Tr}(\tau_i A_2 \tau_j A_2^\dagger)$ is the associated $SO(3)$ rotation. The classical degeneracy of the above solutions is removed when the theory is quantized. We think of the parameters $\mathbf{X}(t)$, $A_1(t)$ and $A_2(t)$ as dynamical variables and then quantize the resulting dynamical system according to standard canonical methods [2]. This is the collective coordinate quantization of the Skyrmion U_0. As we shall only be concerned with the computation of spin and isospin, we shall ignore the translational degrees of freedom and quan-

tize the solitons in their zero-momentum frame. We shall also ignore possible deformations of the Skyrmion U_0, which lead to vibrational excitations.

Our dynamical ansatz is then $\hat{U}(\mathbf{x}, t) = A_1(t)U_0(D(A_2(t))\mathbf{x})A_1(t)^\dagger$. Inserting this into the Skyrme Lagrangian, one obtains the kinetic contribution to the total energy as

$$T = \frac{1}{2}a_i U_{ij}a_j - a_i W_{ij}b_j + \frac{1}{2}b_i V_{ij}b_j, \qquad (31)$$

where

$$a_j = -i\,\text{Tr}\tau_j A_1^\dagger \dot{A}_1, \qquad b_j = i\,\text{Tr}\tau_j \dot{A}_2 A_2^\dagger, \qquad (32)$$

and the inertia tensors U_{ij}, W_{ij} and V_{ij}, expressed as functionals of $U_0(\mathbf{x})$, are given by [8]:

$$U_{ij} = -\int \text{Tr}\left(T_i T_j + \frac{1}{4}[R_k, T_i][R_k, T_j]\right)d^3x, \qquad (33)$$

$$W_{ij} = \int \epsilon_{jlm}x_l\,\text{Tr}\left(T_i R_m + \frac{1}{4}[R_k, T_i][R_k, R_m]\right)d^3x, \qquad (34)$$

$$V_{ij} = -\int \epsilon_{ilm}\epsilon_{jnp}x_l x_n\,\text{Tr}\left(R_m R_p + \frac{1}{4}[R_k, R_m]\right.$$
$$\left. \times [R_k, R_p]\right)d^3x, \qquad (35)$$

where $R_k = (\partial_k U_0)U_0^{-1}$ is the right invariant $\mathfrak{su}(2)$ current defined previously and

$$T_i = \frac{i}{2}[\tau_i, U_0]U_0^{-1} \qquad (36)$$

is also an $\mathfrak{su}(2)$ current. The potential energy, in terms of

collective coordinates, is just a constant, the static mass of the Skyrmion.

The quantized momenta corresponding to b_i and a_i become the body-fixed spin and isospin angular momenta L_i and K_i satisfying the $\mathfrak{su}(2)$ commutation relations [2]. The usual space-fixed spin and isospin angular momenta J_i and I_i are related to the body-fixed operators by

$$J_i = -D(A_2)^T_{ij}L_j, \qquad I_i = -D(A_1)_{ij}K_j. \qquad (37)$$

We also have $\mathbf{J}^2 = \mathbf{L}^2$, $\mathbf{I}^2 = \mathbf{K}^2$. Thus the operators $\mathbf{J}$, $\mathbf{L}$, $\mathbf{I}$ and $\mathbf{K}$ form the Lie algebra of $SO(4)_{J,L} \otimes SO(4)_{I,K}$. Their action on A_1 and A_2 is given by

$$[J_i, A_2] = \frac{1}{2}A_2\tau_i, \qquad [I_i, A_1] = -\frac{1}{2}\tau_iA_1, \qquad (38)$$

$$[L_i, A_2] = -\frac{1}{2}\tau_iA_2, \qquad [K_i, A_1] = \frac{1}{2}A_1\tau_i, \qquad (39)$$

with all other commutators between momenta and coordinates zero.

A basis for the Hilbert space of states is given by $|J, J_3, L_3\rangle \otimes |I, I_3, K_3\rangle$, with $-J \leq J_3$, $L_3 \leq J$ and $-I \leq I_3$, $K_3 \leq I$. Concretely, $|J, J_3, L_3\rangle$ and $|I, I_3, K_3\rangle$ are Wigner functions of the Euler angles parametrizing A_2 and A_1 respectively. The ground states are the states with the lowest values of J and I that are compatible with the FR constraints arising from the symmetries of the given Skyrmion. The allowed values of L_3 and K_3 are also constrained by the symmetries of the Skyrmion. In what follows, the arbitrary third components of the space and isospace angular momenta J_3 and I_3 are omitted.

Recall that physical states satisfy [7]

$$e^{2\pi i\mathbf{n}\cdot\mathbf{L}}|\Psi\rangle = e^{2\pi i\mathbf{n}\cdot\mathbf{K}}|\Psi\rangle = (-1)^B|\Psi\rangle, \qquad (40)$$

so even B implies that the spin and isospin, J and I, are integral, and odd B implies that they are half-integral.

For general $B > 1$, the moments of inertia are larger for rotations than for isorotations. Since these appear in the denominator of the quantum Hamiltonian, the quantum states of lowest energy are those with minimal isospin, and spin excitations of Skyrmions require less energy than isospin excitations. In particular, for even B the ground state has zero isospin, but (because of the FR constraints) not necessarily zero spin. These observations match what is seen experimentally for a large range of nuclei up to baryon number about 40.

VI. QUANTIZATION OF THE $B = 6$ SKYRMION

The minimal-energy $B = 6$ Skyrmion, shown in Fig. 1,[1] has D_{4d} symmetry, and is well approximated by the rational map ansatz. We recall that the dihedral group D_4 is obtained from C_4, the cyclic group of order 4, by the

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[1]Thanks to P. M. Sutcliffe, University of Durham, UK, for providing the picture

FIG. 1. Baryon density isosurface (to scale) of the $B = 6$ Skyrmion.

addition of a C_2 axis which is orthogonal to the main C_4 symmetry axis. The group D_4 is extended to D_{4d} by including a reflection in a plane which contains the main symmetry axis and bisects a pair of the C_2 axes obtained by applying the C_4 symmetry to the generating C_2 axis. The optimized rational map, in a convenient orientation, is [6]

$$R(z) = \frac{z^4 + a}{z^2(az^4 + 1)}, \qquad a = 0.16i. \qquad (41)$$

The D_{4d} symmetry can be seen by considering the expression for the baryon density in (14). The baryon density vanishes where the Wronskian W of the map $R = p/q$, given by

$$W = qp' - pq' = -2z(az^8 + (3a^2 - 1)z^4 + a), \qquad (42)$$

vanishes. In addition to the nine zeros of W, the baryon density also vanishes at $z = \infty$ due to the factor of z^2 in the denominator of the rational map.

The D_4 subgroup is generated by two elements, a π rotation about the x_1-axis, and a π rotation about the $x_1 = x_2$ axis. The product of these is a $\pi/2$ rotation about the x_3-axis. The corresponding symmetries of the rational map are $R(z) = 1/R(1/z)$ and $R(z) = -1/R(i/z)$. Therefore, the solution is invariant under the following two symmetries:

(i) a π rotation about the x_1-axis combined with a π isorotation about the 1-axis, and

(ii) a π rotation about the $x_1 = x_2$ axis combined with a π isorotation about the 2-axis.

For the first symmetry, we have $\mathbf{n}_1 = \mathbf{n}_2 = (1, 0, 0)^T$, and $\theta_1 = \theta_2 = \pi$. As required, $R(z_{-\mathbf{n}_2}) = R_{\mathbf{n}_1} = 1$. We compute $\mathcal{N} = 15$ using the formula from the previous section, and deduce that $\chi_{FR} = -1$. For the second symmetry, we have $\mathbf{n}_1 = (0, 1, 0)^T$, $\mathbf{n}_2 = \frac{1}{\sqrt{2}}(1, 1, 0)^T$, and $\theta_1 = \theta_2 = \pi$. As required we have $R(z_{-\mathbf{n}_2}) = R_{\mathbf{n}_1} = i$. Again $\mathcal{N} = 15$, so $\chi_{FR} = -1$. Therefore the FR constraints reduce to

$$e^{i\pi L_1}e^{i\pi K_1}|\Psi\rangle = -|\Psi\rangle, \tag{43}$$

and

$$e^{i\pi(L_1+L_2)/\sqrt{2}}e^{i\pi K_2}|\Psi\rangle = -|\Psi\rangle. \tag{44}$$

We note that these constraints are equivalent to those obtained in [15] using an alternative method by which the contractibilities of the closed loops corresponding to the symmetry group elements were determined by continuous deformation of the minimal-energy solution into three well-separated $B = 2$ tori.

A basis for the Hilbert space of states is given by $|J, L_3\rangle \otimes |I, K_3\rangle$. We require the identities [16]:

$$e^{i\pi L_1}|J, L_3\rangle = \sum_{L_3'}|J, L_3'\rangle D^J_{L_3'L_3}(0, \pi, \pi) = (-1)^J|J, -L_3\rangle, \tag{45}$$

$$e^{i\pi K_1}|I, K_3\rangle = \sum_{K_3'}|I, K_3'\rangle D^I_{K_3'K_3}(0, \pi, \pi) = (-1)^I|I, -K_3\rangle, \tag{46}$$

$$e^{i\pi K_2}|I, K_3\rangle = \sum_{K_3'}|I, K_3'\rangle D^I_{K_3'K_3}(0, \pi, 0)$$
$$= (-1)^{I+K_3}|I, -K_3\rangle, \tag{47}$$

$$e^{i\pi(L_1+L_2)/\sqrt{2}}|J, L_3\rangle = \sum_{L_3'}|J, L_3'\rangle D^J_{L_3'L_3}(0, \pi, \pi/2)$$
$$= (-1)^{J+(1/2)L_3}|J, -L_3\rangle, \tag{48}$$

where $D^J_{L_3'L_3}$, for given J, is the matrix of Wigner functions representing the symmetry operation.

Seeking simultaneous solutions of the above FR constraints, we obtain the ground state as $|1, 0\rangle \otimes |0, 0\rangle$. The first excited state with $I = 0$ is $|3, 0\rangle \otimes |0, 0\rangle$, and the lowest state with $I = 1$ is $|0, 0\rangle \otimes |1, 0\rangle$ [15].

VII. MEAN CHARGE RADIUS OF THE QUANTIZED $B = 6$ SKYRMION

The state $|1, 0\rangle \otimes |0, 0\rangle$ of the $B = 6$ Skyrmion has the quantum numbers of the ground state of the lithium-6 nucleus. Let us calculate its mean charge radius $\langle r^2 \rangle^{1/2}$, defined as the square root of

$$\langle r^2 \rangle = \frac{1}{\langle\Psi| \int \hat{J}_0(\mathbf{x}, t)d^3x|\Psi\rangle}\left\langle \Psi \left| \int r^2\hat{J}_0(\mathbf{x}, t)d^3x \right| \Psi\right\rangle, \tag{49}$$

where $\hat{J}_0$ is the electric charge density operator and $|\Psi\rangle$ represents the state with any value of J_3. Only the isoscalar part of $\hat{J}_0(\mathbf{x}, t)$ contributes to the matrix elements, and so

the integrals are pure c numbers. The normalization factor is 3, the total electric charge of lithium-6. Therefore

$$\langle r^2 \rangle = \frac{1}{6} \int r^2 B_0(\mathbf{x})d^3x, \tag{50}$$

using (8).

Using the expression (14) for the baryon density in terms of the rational map ansatz, we obtain

$$\langle r^2 \rangle = \frac{1}{6}\frac{1}{4\pi} \int\left(\left(\frac{1 + |z|^2}{1 + |R|^2}\left|\frac{dR}{dz}\right|\right)\right)^2 \sin\theta d\theta d\phi$$
$$\times \frac{2}{\pi} \int(-f')r^2\sin^2 f dr, \tag{51}$$

$$= \frac{2}{\pi} \int(-f')r^2\sin^2 f dr. \tag{52}$$

The profile function f can be numerically determined using a shooting method,[2] for which the inputs are B, m and $\mathcal{I}$. Then the radial integral in (52) can be evaluated, and the mean charge radius in Skyrme units determined as the square root of this. $B = 6$ and $\mathcal{I} = 50.76$ for the rational map (41), but we defer discussion of m and the actual value of $\langle r^2 \rangle^{1/2}$ until Sec. IX.

VIII. QUADRUPOLE MOMENT OF THE QUANTIZED $B = 6$ SKYRMION

An additional static electromagnetic property that is known experimentally for the lithium-6 nucleus to quite high precision is its quadrupole moment. The classical quadrupole tensor of a Skyrmion is defined as

$$Q_{ab} = \int(3x_ax_b - r^2\delta_{ab})\hat{J}_0(\mathbf{x}, t)d^3x$$
$$= \frac{1}{2} \int(3x_ax_b - r^2\delta_{ab})B_0(\mathbf{x})d^3x, \tag{53}$$

using (8). It is traceless by definition. The D_{4d} symmetry of the $B = 6$ Skyrmion implies that the tensor Q_{ab} is diagonal, and further that $Q_{11} = Q_{22} = -Q_{33}/2$.

From (53) we find

$$Q_{33} = \frac{1}{2} \int r^2(3\cos^2\theta - 1)B_0(\mathbf{x})d^3x, \tag{54}$$

and therefore, using the rational map ansatz,

$$Q_{33} = \frac{1}{2}\frac{1}{4\pi} \int(3\cos^2\theta - 1)\left(\frac{1 + |z|^2}{1 + |R|^2}\left|\frac{dR}{dz}\right|\right)^2$$
$$\times \sin\theta d\theta d\phi \frac{2}{\pi} \int(-f')r^2\sin^2 f dr. \tag{55}$$

For the rational map given previously for the $B = 6$ Skyrmion,

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[2]Thanks to Bernard M. A. G. Piette, University of Durham, UK, for providing the C + + code

$$\left(\frac{1 + |z|^2}{1 + |R|^2} \left|\frac{dR}{dz}\right|\right)^2 = 4|z|^2(1 + |z|^2)^2 \left\{\frac{(az^8 + (3a^2 - 1)z^4 + a)(\bar{a}\bar{z}^8 + (3\bar{a}^2 - 1)\bar{z}^4 + \bar{a})}{(|a|^2|z|^{12} + |z|^8 + |z|^4(1 + az^4 + \bar{a}\bar{z}^4) + \bar{a}z^4 + a\bar{z}^4 + |a|^2)^2}\right\}, \tag{56}$$

and substituting $z = \tan(\theta/2)e^{i\phi}$, writing $a = i\alpha$, where $\alpha \in \mathbb{R}$, and setting $\beta = 1 + 3\alpha^2$, this becomes

$$\left(\frac{1 + |z|^2}{1 + |R|^2} \left|\frac{dR}{dz}\right|\right)^2 = 4\tan^2\frac{\theta}{2}\left(1 + \tan^2\frac{\theta}{2}\right)^2$$

$$\times \left\{\frac{\alpha^2\tan^{16}\frac{\theta}{2} + 2\alpha\beta\sin4\phi\tan^{12}\frac{\theta}{2} + (\beta^2 + 2\alpha^2\cos8\phi)\tan^8\frac{\theta}{2} - 2\alpha\beta\sin4\phi\tan^4\frac{\theta}{2} + \alpha^2}{(\alpha^2\tan^{12}\frac{\theta}{2} + (1 - 2\alpha\sin4\phi)\tan^8\frac{\theta}{2} + (1 + 2\alpha\sin4\phi)\tan^4\frac{\theta}{2} + \alpha^2)^2}\right\}. \tag{57}$$

A numerical integration technique can then be used to determine the angular integral in (55), for $\alpha = 0.16$. The result is

$$Q_{33} = 0.395 \cdot \frac{2}{\pi} \int (-f')r^2\sin^2 f\, dr, \tag{58}$$

where the radial integral is the squared charge radius (52). The accuracy of the numerical integration can be checked by considering the integral

$$\int \left(\frac{1 + |z|^2}{1 + |R|^2} \left|\frac{dR}{dz}\right|\right)^2 \sin\theta\, d\theta\, d\phi = 4\pi B. \tag{59}$$

The same procedure yields a result 75.40, which is equal to 24π.

Note that Q_{33} is positive, so the classical Skyrmion is prolate. It is helpful to consider upper and lower bounds on its value, for a given B and $\langle r^2 \rangle$. We observe that Q_{33} is maximal for baryon densities restricted to the x_3-axis and minimal for baryon densities restricted to the (x_1, x_2)-plane, hence

$$-\frac{B}{2}\langle r^2 \rangle \le Q_{33} \le B\langle r^2 \rangle. \tag{60}$$

For the $B = 6$ case,

$$-3 \le Q_{33}/\langle r^2 \rangle \le 6. \tag{61}$$

Within this range, the actual value $Q_{33}/\langle r^2 \rangle = 0.395$ is rather close to zero.

In the quantum state the Skyrmion occurs in all possible orientations. Substituting

$$\hat{U}(\mathbf{x}) = A_1 U_0(D(A_2)\mathbf{x})A_1^\dagger, \tag{62}$$

we obtain

$$\hat{Q}_{ij} = D(A_2)_{ia}^T Q_{ab} D(A_2)_{bj}, \tag{63}$$

for the rotated classical Skyrmion. We observe that $\hat{Q}_{ij}$ is independent of A_1. The symmetry relations reduce this quadrupole moment operator to the expression

$$\hat{Q}_{ij} = \frac{1}{2}Q_{33}(3D(A_2)_{i3}^T D(A_2)_{3j} - \delta_{ij}), \tag{64}$$

and thus

$$\hat{Q}_{33} = \frac{1}{2}Q_{33}(3\cos^2\theta - 1) = Q_{33}D_{00}^2(\phi, \theta, \psi), \tag{65}$$

where ϕ, θ, ψ now denote the Euler angles parametrizing A_2, and $D_{00}^2(\phi, \theta, \psi)$ is a Wigner function.

The wavefunction Ψ is the product of Wigner functions $D_{sm}^j(\phi, \theta, \psi) \otimes D_{tn}^i(\alpha, \beta, \gamma)$ on $SO(3)^J \times SO(3)^I$. For the ground state of the $B = 6$ Skyrmion, with $J = 1$ and $I = 0$, and $J_3 = m$,

$$\Psi = \sqrt{\frac{3}{8\pi^2}}D_{0m}^1(\phi, \theta, \psi). \tag{66}$$

The quadrupole moment of the quantized $B = 6$ Skyrmion is defined as the expectation value of $\hat{Q}_{33}$ in the state with $J_3 = 1$ (by convention the quadrupole moment is measured in the top spin state), i.e.

$$Q = \langle \Psi_{J_3=1}|\hat{Q}_{33}|\Psi_{J_3=1}\rangle$$

$$= \frac{3}{8\pi^2}Q_{33}\int D_{00}^2|D_{01}^1|^2 \sin\theta\, d\phi\, d\theta\, d\psi, \tag{67}$$

and using the well-known expression in terms of Wigner $3j$-symbols for the integral of the product of three Wigner functions,

$$\int D_{ab}^j D_{cd}^{j'} D_{ef}^{j''} \sin\theta\, d\phi\, d\theta\, d\psi = 8\pi^2 \begin{pmatrix} j & j' & j'' \\ a & c & e \end{pmatrix}$$

$$\times \begin{pmatrix} j & j' & j'' \\ b & d & f \end{pmatrix}, \tag{68}$$

we find that

$$Q = -\frac{3}{8\pi^2}Q_{33}\int D_{00}^2 D_{01}^1 D_{0-1}^1 \sin\theta\, d\theta\, d\phi\, d\psi$$

$$= -3Q_{33}\begin{pmatrix} 2 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & -1 \end{pmatrix}$$

$$= -3Q_{33}\sqrt{\frac{2}{15}}\sqrt{\frac{1}{30}}$$

$$= -\frac{1}{5}Q_{33}. \tag{69}$$

This is the standard result for an axisymmetric system in a $J = 1$ state. Note that a prolate classical shape (Q_{33} positive) gives an oblate quadrupole moment (Q negative),

and vice versa, because the prolate object can be regarded as spinning about its short axis aligned along the third axis in space when $J_3 = 1$. From (61) we deduce the inequalities

$$-1.2 \leq Q/\langle r^2 \rangle \leq 0.6. \tag{70}$$

We note that to obtain physically meaningful results for both the mean charge radius and the quadrupole moment, it is necessary to convert back to physical units, by reintroducing the Skyrme model parameters, and using the conversion factor $\hbar c = 197.3$ MeV fm. We do this in the next section.

IX. REPARAMETRIZING THE SKYRME MODEL

The parameters e and F_π can be fixed in a number of ways. It has been common practice to use the set of parameters given in [17], specifically

$$e = 4.84, \qquad F_\pi = 108 \text{ MeV} \quad \text{and} \quad m_\pi = 138 \text{ MeV}$$

$$\text{(which implies } m = 0.528\text{)} \tag{71}$$

for the Skyrme model with the physical pion mass taken into account. In [17], the values of e and F_π were tuned to reproduce the masses of the proton and the delta resonance. This parameter set was adjusted to optimize the predictions of the model in the $B = 1$ sector at the expense of the $B = 0$ sector, which requires $F_\pi = 186$ MeV. It is therefore not the optimal parameter set with which to describe the higher baryon number sectors of the Skyrme model. Nevertheless, in [2] this parameter set was used in a quantization of the collective coordinates of the toroidal $B = 2$ Skyrmion. The resulting quantum state is too small compared to the size of the deuteron, and it is too tightly bound. However, this discrepancy was dealt with by the quantization of selected vibrational modes of the $B = 2$ Skyrmion in [4]. We therefore do not propose to change e and F_π for $B = 1$ or $B = 2$. However, for higher B, it becomes technically very difficult to perform a quantization of the possible vibrational modes. Instead, we consider rescaling the parameters in order to fit the Skyrme model to experimental nuclear data, while suppressing the quantization of vibrational modes.

We have first determined $\langle r^2 \rangle^{1/2}$ and Q for the $B = 6$ Skyrmion using the set of parameters given in [17], and setting $a = 0.16i$ (the value of a that minimizes $\mathcal{I}$, as required by the rational map ansatz). After evaluating the radial integral (52), we find

$$\langle r^2 \rangle^{1/2} = 1.48 \text{ fm}, \qquad Q = -0.173 \text{ (fm)}^2. \tag{72}$$

In [18] the experimental root mean square radius of the charge density distribution of the lithium-6 nucleus is given as $\langle r^2 \rangle^{1/2} = 2.55$ fm, substantially larger than 1.48 fm, and its quadrupole moment is given as -0.82×10^{-3} barns $= -0.082$ (fm)^2 in [19]. We can change the length scale $2/eF_\pi$, to obtain a mean charge radius in close

agreement with experiment. This results in the dimensionless mass $m = 2m_\pi/eF_\pi$, with $m_\pi = 138$ MeV as before, being changed, and so the profile function and hence the radial integral in the expressions for the mean charge radius and quadrupole moment is also modified. By iteration, it is found that by setting $m = 1.125$, and hence $eF_\pi = 245.34$ MeV, we obtain $\langle r^2 \rangle^{1/2} = 2.55$ fm, which fits the data.

With this choice for eF_π, the theoretical prediction of the quadrupole moment becomes $Q = -0.512$ (fm)^2. Although the sign is right, this is unfortunately about 6 times the experimental value. One must bear in mind that the rational map ansatz does not generally determine exact solutions of the Skyrme field equation, but rather approximations to solutions. It is useful to allow a slight modification of the rational map (41), while preserving its symmetry and prolateness. If we take $a = 0.1933i$, rather than $0.16i$, and use the new value for eF_π, we obtain a quadrupole moment in very close agreement with experiment: $Q = -0.082$ (fm)^2. The parameter a is not a fundamental parameter of the Skyrme model. As it is just a parameter in the rational map approximation, it can be changed while preserving the D_{4d} symmetry of the solution.

We note that this modification of the rational map leads to the value of the integral $\mathcal{I}$ being slightly modified (from 50.76 to 51.49, i.e. not quite the minimum as required by the rational map ansatz), which subsequently leads to the profile function and hence the radial integral in both the expressions for the mean charge radius and the quadrupole moment being modified, but only slightly. In summary, using $eF_\pi = 245.34$ MeV, $m_\pi = 138$ MeV (which implies $m = 1.125$), and $a = 0.1933i$, we find

$$\langle r^2 \rangle^{1/2} = 2.55 \text{ fm}, \qquad Q = -0.082 \text{ (fm)}^2, \tag{73}$$

and hence $Q/\langle r^2 \rangle = -0.0127$. By virtue of (70), we consider the quadrupole moment of the lithium-6 nucleus to be only slightly oblate. The change we needed to make to the parameter a should not be regarded as a large change.

It remains to consider the energy scale $F_\pi/4e$ of the Skyrme model. Assuming negligible spin kinetic energy (it is expected to be in the region of 1 MeV), we may identify the static Skyrmion energy (18) with the mass of the lithium-6 nucleus, M, given by

$$M = 6m_N - E_{\text{binding}} = 5601 \text{ MeV}, \tag{74}$$

where $m_N = 939$ MeV is the average mass of a nucleon and $E_{\text{binding}} = 32$ MeV is the total binding energy of the nucleus [20]. The static Skyrmion energy is determined numerically, and we find for $m = 1.125$, after converting to physical units, that

$$E = \frac{F_\pi}{4e} \cdot 972. \tag{75}$$

Equating this to the mass of the nucleus implies that

$$F_\pi/4e = 5.76 \text{ MeV}, \tag{76}$$

and combined with $eF_\pi = 245.34$ MeV we obtain, finally

$$e = 3.263, \qquad F_\pi = 75.20 \text{ MeV} \quad \text{and} \quad m_\pi = 138 \text{ MeV}$$

$$\text{(which implies } m = 1.125\text{).} \tag{77}$$

We conclude that with this choice of parameters, and taking $a = 0.1933i$, the mass and static electromagnetic properties of the quantized $B = 6$ Skyrmion are in close agreement with those experimentally determined for the lithium-6 nucleus.

We recall that the Skyrmion is exponentially localized. This is because the equation for the profile function (20) has an asymptotic Yukawa-type solution. The classical solutions match the experimental pion tails of nuclei if we use the physical value of m_π of 138 MeV, and it is for this reason that we have chosen to keep m_π fixed at this value. As a result, our suggested value of F_π is considerably smaller than the experimental pion decay constant (186 MeV). This can be attributed to the simplicity of the model, and the fact that we are suppressing some quantum effects in the model, in particular, Skyrmion vibrations. Also, in earlier work F_π differs considerably from this experimental value [2].

X. RECONSIDERING THE $B = 2$ AND $B = 4$ SKYRMIONS

A. The quantized toroidal $B = 2$ Skyrmion

The minimal-energy solution in the $B = 2$ sector is axially symmetric, and has the shape of a torus. In [2], Braaten and Carson performed the collective coordinate quantization of the solution (see also [21]). Using the same notation as that used above for the $B = 6$ case, it was found that the axial symmetry of the solution requires

$$(2K_3 + L_3)|\Psi\rangle = 0. \tag{78}$$

In addition to this constraint, the FR constraint associated with the discrete rotations by π in the D_2 symmetry group is

$$e^{i\pi K_1}e^{i\pi L_1}|\Psi\rangle = -|\Psi\rangle. \tag{79}$$

The ground state is the $J = 1$, $I = 0$ state $|1, 0\rangle \otimes |0, 0\rangle$, which has the quantum numbers of the deuteron. The first excited state $|0, 0\rangle \otimes |1, 0\rangle$ may be identified with the isovector $^1S_0$ state of the two-nucleon system.

Further properties of the quantized Skyrmion were also determined, taking the Skyrme model parameters to be as in (71), and compared to experimental data on the deuteron. An expression equivalent to (50) is obtained for the mean charge radius of the $B = 2$ Skyrmion. The expression for the quantum quadrupole moment in this case is also given by $Q = -\frac{1}{5}Q_{33}$, where Q_{33} is negative here, because the classical Skyrmion is oblate. The quantity Q_{33} was evaluated in [2] using the numerical, exact classical

solution and its corresponding baryon number density. The predictions were:

$$\langle r^2 \rangle^{1/2} = 0.92 \text{ fm}, \tag{80}$$

$$Q = 0.082 \text{ (fm)}^2, \tag{81}$$

$$M = 1720 \text{ MeV}, \tag{82}$$

compared to experimental data of $\langle r^2 \rangle^{1/2} = 2.095$ fm [22], $Q = 0.2859$ (fm)^2 [23] and $M = 1876$ MeV. The mass of the nucleus (82) is given by the sum of the classical mass, equal to 1659 MeV, and a spin correction equal to $1/V_{11} = 61$ MeV. It was suggested in [2] that these discrepancies could be an artifact of the Skyrme model parameters, and that the disagreement with experiment could be assuaged by an adjustment of e and F_π such that eF_π is approximately halved and F_π/e is unchanged. The new parameter set which we propose is close to this, but we also take into account the corresponding change of m when fitting to experimental data.

B. Attractive channel Skyrmions and the deuteron

Because the deuteron appears too small and too tightly bound according to [2], it was suggested in [4] that the deuteron is better described as a quantum state on a ten-dimensional manifold of $B = 2$ Skyrme fields. This manifold includes the toroidal configurations of minimal energy (i.e. the eight-dimensional manifold considered in [2]), and soft deformations of these into configurations which are approximately the product of two separated $B = 1$ Skyrmions in the most attractive relative orientation. The ten coordinates of this manifold are the separation parameter, overall translations, rotations and isorotations. A unique bound quantum state is found with the quantum numbers of the deuteron. The values of e and F_π were taken to be 4.84 and 108 MeV, respectively, i.e. those of [2,17], but for technical reasons the pion mass was taken to be zero. In this model, the predictions are

$$\langle r^2 \rangle^{1/2} = 2.18 \text{ fm}, \tag{83}$$

$$Q = 0.83 \text{ (fm)}^2, \tag{84}$$

$$M = 1872 \text{ MeV}. \tag{85}$$

The calculated deuteron binding energy of 6 MeV is very close to the experimentally determined binding energy of the deuteron, and the deuteron mass (85) is simply the sum of the proton and neutron masses minus this binding energy. The prediction for the mean charge radius is in very close agreement with experiment. However, the deuteron's quadrupole moment has become much larger than the experimental value. This is explained in [4] as follows: The tensor force is responsible for the existence of a d-wave contribution to the deuteron wave function. In the

absence of a d-wave, Q would be zero. The size of Q is therefore a measure of the d-wave probability, which in turn indicates the strength of the tensor force. The truncation to the space of attractive channel fields systematically overestimates the strength of the tensor force. Therefore it is not surprising that the theoretical prediction for Q is rather large.

C. Is reparametrization helpful in the $B = 2$ sector?

Let us reconsider the collective coordinate quantization of [2], ignoring the attractive channel deformations into separated Skyrmions (described above), but using our new parameter set (77). To avoid recalculating the exact $B = 2$ Skyrmion solution for various values of the pion mass parameter m, we use the rational map ansatz. There is a unique map here, namely $R(z) = z^2$ [6]. This gives, for the quantized $B = 2$ Skyrmion,

$$\langle r^2 \rangle = \frac{1}{2} \frac{1}{4\pi} \int \left(\frac{1 + |z|^2}{1 + |R|^2} \left| \frac{dR}{dz} \right| \right)^2 \sin\theta d\theta d\phi$$
$$\times \frac{2}{\pi} \int (-f') r^2 \sin^2 f dr$$
$$= \frac{2}{\pi} \int (-f') r^2 \sin^2 f dr, \tag{86}$$

and

$$Q = -\frac{1}{5} \frac{1}{2} \frac{1}{4\pi} \int (3\cos^2\theta - 1) \left(\frac{1 + |z|^2}{1 + |R|^2} \left| \frac{dR}{dz} \right| \right)^2$$
$$\times \sin\theta d\theta d\phi \frac{2}{\pi} \int (-f') r^2 \sin^2 f dr. \tag{87}$$

It is straightforward to calculate the profile function f in this case, and to perform the numerical integrations. Converting back to physical units, and using first the traditional values of e, F_π and m given by (71), we obtain

$$\langle r^2 \rangle^{1/2} = 0.940 \text{ fm}, \tag{88}$$

$$Q = 0.102 \text{ (fm)}^2, \tag{89}$$

$$M = 1757 \text{ MeV}. \tag{90}$$

which are reasonably close to the values (80)–(82) in [2]. We have again assumed that the mass of the nucleus (90) can be equated to the sum of the static Skyrmion energy (1696 MeV) and the spin correction which we take to be the same as in [2] (61 MeV). Comparing this value of the static Skyrmion energy to the classical mass given in [2], we see that the rational map ansatz gives a very good approximation.

Using the new parameters (77), and their associated length and energy scales, we find after recalculating the radial integral (which we recall is a function of m)

$$\langle r^2 \rangle^{1/2} = 1.644 \text{ fm}, \tag{91}$$

$$Q = 0.311 \text{ (fm)}^2, \tag{92}$$

$$M = 1969 \text{ MeV}. \tag{93}$$

The deuteron mass (93) is again the static Skyrmion energy (1950 MeV) plus an estimated spin correction equal to 61 MeV times the ratio of the squares of the predicted mean charge radii (80) and (91), which allows for the increase in the moment of inertia. Comparing these with the experimental values, we conclude that the new parameters give quite a good description of the deuteron, better than those in [2], and provide an alternative to taking account of the soft deformations.

D. The $B = 4$ Skyrmion and the α-particle

The $B = 4$ Skyrmion is cubically symmetric, and was first quantized by Walhout [24], using the exact classical solution and the traditional parameter set. The rational map ansatz for this Skyrmion uses the unique map of degree 4 with cubic symmetry [6],

$$R(z) = \frac{z^4 + 2\sqrt{3}iz^2 + 1}{z^4 - 2\sqrt{3}iz^2 + 1}. \tag{94}$$

The FR constraints corresponding to the cubic symmetry were determined in [15] as

$$e^{(2\pi i/3\sqrt{3})(L_1+L_2+L_3)} e^{(2\pi i/3\sqrt{3})(K_1+K_2+K_3)} |\Psi\rangle = |\Psi\rangle, \tag{95}$$

$$e^{i(\pi/2)L_3} e^{i(\pi/\sqrt{2})(K_1-K_2)} |\Psi\rangle = |\Psi\rangle. \tag{96}$$

The ground state, in the basis introduced previously, is $|0, 0\rangle \otimes |0, 0\rangle$, which trivially satisfies the constraints. This state may be identified with the α-particle.

If vibrational modes are ignored, then the traditional parameters give a much too small and tightly bound α-particle. Walhout's rather complicated analysis takes into account a number of the vibrational modes of the $B = 4$ Skyrmion, obtaining $\langle r^2 \rangle^{1/2} = 1.58 \text{ fm}$ and $M = 3677 \text{ MeV}$, compared with the experimental values $\langle r^2 \rangle^{1/2} = 1.71 \text{ fm}$ [18] and $M = 3727 \text{ MeV}$.

We wish to ignore the vibrational modes, but in compensation, use our new parameter set (77). As the spin of the $B = 4$ Skyrmion is zero, there is no quadrupole moment to determine, just the mean charge radius. The mass of the quantized Skyrmion is equal to the classical mass, as there is no spin energy contribution. Numerically computing the profile function (with inputs $I = 20.65$ and $m = 1.125$), and numerically integrating the corresponding densities, we find

$$\langle r^2 \rangle^{1/2} = 2.13 \text{ fm}, \tag{97}$$

$$M = 3679 \text{ MeV}, \tag{98}$$

which agrees reasonably well with experiment.

XI. CONCLUSION

We conclude that in order to accurately predict certain properties of nuclei (with baryon numbers greater than two) using the Skyrme model, it is necessary to rescale the Skyrme model parameters, if the quantization of vibrational modes is not going to be considered in the calculation. We have in this paper explicitly determined the values of these parameters in the $B = 6$ case, and discussed how well they work when applied to the $B = 2$ and $B = 4$ cases. We note, however, that the deuteron is very weakly bound compared to other nuclei and therefore much more extended. The new parameters which we suggest are not ideal for the deuteron. It would be interesting to see how well the new parameters work for Skyrmions with other baryon numbers.

Our suggestion that the dimensionless pion mass m should be increased from 0.528 to 1.125 backs up the suggestion made in [25], in which it was found that as the pion mass is increased its effect becomes important for Skyrmions of all baryon numbers. In particular, the hollow, polyhedral shell-like solutions which exist for a large range of baryon numbers up to 20 or 30, and are stable for zero pion mass, become unstable for baryon number eight and above when m is of order unity. Instead, the stable solutions become more solid structures, some of which are related to chunks of the Skyrme crystal. This improves the qualitative fit of the Skyrme model to real nuclei. Also, in [13], low-energy Skyrmion solutions composed of charge four subunits were found for baryon numbers a multiple of four, with $m = 1$. This is the Skyrmion analogue of the α-particle model of nuclei.

We have worked with the standard $SU(2)$ Skyrme model. However, the introduction of explicit vector meson fields, allowing for the dispensation of the fourth order stabilizing term in the Lagrangian density (1), leads to an improved description of the short-range structure of the nucleons [26,27]. This could lead to further refinement of our discussion, and refine the modelling of nuclei, but since not much is known about Skyrmions with $B > 1$ in these extended models, we have decided to work with the original model. Moreover, an approximation to Skyrmion solutions, analogous to the rational map ansatz for the original model, has not been formulated for these extended models.

The standard values of the Skyrme parameters originate from [17], in which attention was restricted to the single nucleon. In recent years, the substantial increase in results available for Skyrmions over a range of baryon numbers enables us to fit the Skyrme parameters to experimental data for larger nuclei, with a view to modelling nuclei of all baryon numbers. The parameters we propose here should be regarded as provisional, since they rely on just a few basic properties of lithium-6, and depend on the rational map approximation. More work remains to be done within the Skyrme model to see if one set of Skyrme parameters can fit experimental data over a range of baryon numbers. Both static properties of nuclei, like their masses and electric charge distribution, and also the excitation energies of higher spin states, need to be considered further.

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