Quantum mechanics and the generalized uncertainty principle

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The generalized uncertainty principle has been described as a general consequence of incorporating a minimal length from a theory of quantum gravity. We consider a simple quantum mechanical model where the operator corresponding to position has discrete eigenvalues and show how the generalized uncertainty principle results for minimum uncertainty wave packets.

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I. INTRODUCTION AND MOTIVATION

There is reason to believe in the existence of a minimal length that can in principle be measured. This viewpoint stems in part from the realization that, if enough mass energy is confined to a small region of space, a black hole must form. For example, if one increases the energy of colliding particles beyond the Planck energy, one expects the short distance effects to be hidden behind an event horizon. In fact, as the energy is increased, the size of this event horizon increases. One way an effective minimal length might arise is through the discretization of spacetime. One might also ask the question whether the generic appearance of a minimal length in a low-energy effective theory of quantum gravity survives in the full theory when its ultraviolet sector is completed.

In a theory of quantum gravity, a fundamental distance scale is expected to be of order the Planck length L_P. The existence of a minimal length invites the possibility that there are corrections to the usual Heisenberg uncertainty principle such that it becomes what is known as a generalized uncertainty principle (GUP):

$$\Delta x \geq \frac{1}{2\Delta p} + \alpha L_P^2 \Delta p + \cdots. \quad (1)$$

We have set $\hbar = 1$. The terms in the ellipsis represent higher order contributions which should generically be present. They may involve higher order powers of Δp, but might also involve expectation values of higher powers of the momentum. We shall sometimes refer to the expression in Eq. (1) minus these extra terms as the truncated GUP. The minimal length in the theory is governed by the parameter α, and the generalized uncertainty principle in Eq. (1) shows that there is a minimum dispersion Δx for any value of Δp at least as long as the first two terms on the right-hand side are considered.

Interest in a minimum length or the generalized uncertainty principle has been motivated by studies of the short distance behavior of strings [1–4], considerations regarding the properties of black holes [5], and de Sitter space [6]. In recent years, the generalized uncertainty principle has been studied extensively in the literature [7–12]. Most of the research does not attempt to derive the uncertainty principle from quantum gravity explicitly. Rather the modifications to the uncertainty relation are motivated by what is a general property on any theory of quantum gravity, and the implications of the GUP are analyzed phenomenologically. Other approaches have involved trying to understand the quantum mechanical basis for the uncertainty principle in Eq. (1) often by considering a truncation of the full series of terms on the right-hand side. Commutation relations can be postulated which give rise to this truncated series and the associated algebra has been studied. In this paper we try to understand how the GUP can arise in a simple quantum mechanical model (which is well understood), and how the GUP is in fact completed when all the relevant terms are included. This means we derive all the terms in the ellipsis in Eq. (1). The model contains an operator with discrete eigenvalues in the familiar way: because the phase space is compactified.

The model is a quantum theory with a position operator that has discrete eigenvalues, so it may be considered a theory with a minimum length associated with the difference between eigenvalues. However, we do not *a priori* exclude the possibility that a physical state can be an eigenstate of this operator. That leads us to consider quantum mechanics on a circle. Usually one is concerned with the problem of a spatial direction compactified such that the eigenvalues of the momentum operator are quantized. Here we (eventually) consider the somewhat more unusual situation where one considers the *momentum* to be compactified, thus resulting in a quantum mechanics with discrete eigenvalues for the operator associated with a spatial coordinate. Quantum mechanics on a circle is a simple physical problem which has a long history, and does not contain gravity explicitly. The compactification of momentum does provide a discrete spectrum for the position coordinate which may be a feature of a full theory of quantum gravity. We are able to derive the generalized uncertainty relation in Eq. (1) and calculate the coefficient αL_P^2 in terms of the compactification "radius" of the momentum circle. In fact we are able to obtain straightforwardly the whole series of higher order terms (i.e.

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higher powers of Δp and factors like $\langle p^n \rangle$) on the right-hand side of the relation for minimum uncertainty wave packets. Since the terms explicitly expressed in Eq. (1) should represent only an approximation to a full expression, this may shed some light (at least as far as the simple model captures the necessary features of a full theory of gravity) on how the GUP fits into a complete theory of discretized spacetime. We believe this derivation of a generalized uncertainty principle in an elementary context may be useful in understanding what features of a discrete spacetime may be needed to obtain the uncertainty principle in a more realistic context. Quantum mechanics on a circle has been chosen simply for simplicity and because it is well studied in other contexts. It makes the presentation particularly elementary since modified Bessel functions are needed to describe the wave functions and the coefficients of the eigenfunctions of the minimum uncertainty wave packets.

The study of the proper way to treat quantum mechanics on a compact space goes back to the earliest days of the subject [13,14]. A history of the subject can be found in Ref. [15]. Therefore it has been known for a long time that the uncertainty principle as derived for a circular coordinate (phase) operator and the angular momentum operator is inadequate because the angular phase variable is periodic. More recent attempts to rectify this problem led to various definitions for the phase dispersion. Judge defined [16]

$$\Delta\theta^2 = \min_{-\pi\leq\gamma\leq\pi} \int_{-\pi}^{\pi} \theta^2 |\psi(\gamma + \theta)|^2 d\theta, \qquad (2)$$

for which he conjectured the uncertainty relation

$$\Delta L \frac{\Delta\theta}{1 - (3/\pi^2)\Delta\theta^2} \geq \frac{1}{2}. \qquad (3)$$

This conjecture was later proved by Bouten, Maene, and Van Leuven [17], and discussed further in Ref. [18]. The form of the uncertainty relation in Eq. (3) allows for an angular momentum eigenstate (for which $\Delta L = 0$) to satisfy the inequality because the bounded uncertainty in $\Delta\theta^2$ causes the denominator to vanish for these eigenstates. The uncertainty relation in Eq. (3) allows for a treatment of the case of an angular momentum eigenstate for which $\Delta L = 0$ whereas the dispersion in the angle is bounded $\Delta\theta = \pi^2/3$.

The subtle issue of an uncertainty principle involving the phase operator was subsequently discussed in a classic paper, Ref. [19]. There, Hermitian cosine and sine operators were defined and various uncertainty principles were suggested. The crucial realization is that the angular coordinate variable θ is not suitable for quantization, and one should use a phase operator $e^{i\theta}$ (or alternatively the cosine and sine operators). This early important work is developed and reviewed in Ref. [20]. This has been discussed and reviewed further and applications to more general situations have appeared in Refs. [21–23]. In the formulation of Ohnuki and Kitakado [24,25], it was shown that there are in fact an infinite number of representations of the algebra of operators which can be understood in terms of a certain gauge field. These representations are classified by the value of a parameter $\alpha \in [0, 1)$ which specifies the gauge inequivalent representations and interpolates between the discrete eigenvalues in the operator spectrum. Finally, in Ref. [25] the minimum uncertainty wave packets on the circle were shown to reduce in the large radius limit to the usual Gaussian wave packet. It is this limit of a large radius that will most concern us here, as we will show how the usual Heisenberg uncertainty principle expressed in terms of Δx and Δp on the line get modified in the large radius limit in which the corrections should be small. One potential benefit of identifying the generalized uncertainty principle in this limit is that one can extend the results to cases that are not approximately quantum mechanics on the line, and study how the minimum uncertainty wave packets on the circle interpolate between the Gaussian wave packet on the line to the (angular) momentum eigenstate on the circle. This extends the generalized uncertainty principle in Eq. (1), at least for this simple case, to physical situations that do not require the "extra" terms to be subleading in magnitude.

Finally, we note that the minimum uncertainty wave packets we examine are well known in the field of quantum optics where they are sometimes called (circular) squeezed states. This is a rich and well-developed subject and we refer the reader to Ref. [15] for an overview. Coherent and squeezed states have been developed for general Lie symmetries. Important examples are the Barut-Girardello coherent states [26]. See also Refs. [27–33]. Our emphasis is different, however, in that we are primarily concerned with the properties of the uncertainty relations for a compactified phase space in the large radius limit. It is in this limit that the corrections to the usual Heisenberg uncertainty relation involving a position operator x and a momentum operator p can be understood to be small. In a more general context, one is interested in how to quantize a system with some classical phase space. Of particular importance in understanding the correct algebra for the quantum system is the group theoretic quantization described in detail in Ref. [34]. Quantum mechanics on a circle serves as a simple example of this geometric quantization.

This paper is organized as follows. In Sec. II we review the quantum mechanics formulated on a circle as formulated by Ohnuki and Tanimura emphasizing the features of importance to us. In Sec. III we derive a generalization of the Heisenberg uncertainty involving Δx and Δp for the case where the space is compactified on a circle and the eigenvalues of the momentum operator are discrete. We show how all the terms in an infinite series are known. In Sec. IV we imagine that the momentum is compactified, so that the roles of the position and momentum operators are interchanged. In this case the usual form of the GUP results. In Sec. V we give some explicit numerical results for the minimum uncertainty wave packet and briefly examine for which states the generalized uncertainty principle as expanded in Eq. (1) is a reasonable expansion with decreasing terms. We summarize in Sec. V.

II. QUANTUM MECHANICS ON THE CIRCLE

We first review quantum mechanics on a circle. We use the particular formulation given by Ohnuki and Kitakado [24] and explored further in Ref. [25]. There is a Hermitian operator G and a unitary operator W satisfying the commutation relation

$$[G, W] = W. \tag{4}$$

The operators G, W, and $W^\dagger$ form the fundamental algebra of quantum mechanics on the circle. The algebra indicates that the operators W and $W^\dagger$ act as raising and lowering operators on the eigenstates of G. If $|\alpha\rangle$ is an eigenstate of the operator G,

$$G|\alpha\rangle = \alpha|\alpha\rangle, \tag{5}$$

then

$$GW|\alpha\rangle = (\alpha + 1)W|\alpha\rangle \qquad GW^\dagger|\alpha\rangle = (\alpha - 1)W^\dagger|\alpha\rangle. \tag{6}$$

The operator W can be called the phase operator because it has the eigenvalue solution

$$W|\theta\rangle = e^{i\theta}|\theta\rangle. \tag{7}$$

The solution is

$$|\theta\rangle = \kappa(\theta) \sum_{n=-\infty}^{+\infty} e^{-in\theta}|n + \alpha\rangle, \tag{8}$$

where $\kappa(\theta)$ is a periodic phase, i.e. $|\kappa(\theta)| = 1$ and $\kappa(\theta + 2\pi) = \kappa(\theta)$. It can be shown [24,25] that one may choose $\kappa(\theta) = 1$ which is a kind of gauge choice. Finally, there is the representation of action of the operator W on a wave function,

$$\langle\theta|W|\psi\rangle = e^{i\theta}\psi(\theta). \tag{9}$$

One can form the Susskind-Glogower operators

$$C = \frac{1}{2}(W + W^\dagger), \qquad S = \frac{1}{2i}(W - W^\dagger), \tag{10}$$

which are Hermitian. Carruthers and Nieto [20] studied uncertainty relations involving the operators C and S in a classic paper. Here we do not try to devise uncertainty relations involving G, W, and $W^\dagger$ (or equivalently G, C, and S), but rather make an identification between quantum mechanics on the circle when the large radius limit is taken and quantum mechanics on the line.

Applying the Schwarz inequality to the states $\Delta G|\psi\rangle$ and $\Delta W|\psi\rangle$, one obtains the uncertainty relation

$$\langle\Delta G^2\rangle\langle\Delta W^\dagger\Delta W\rangle \geq |\langle\Delta G\Delta W\rangle|^2. \tag{11}$$

The minimum uncertainty wave packet expressed in the notation of Ref. [25] is

$$\psi(\theta) = \frac{1}{\sqrt{I_0(2\beta)}}\exp[\beta e^{i(\theta-\phi)} + i\nu\theta]. \tag{12}$$

This state saturates the uncertainty relation, Eq. (11), and is a state peaked at $\theta = \phi$. The normalization convention adopted here is

$$\langle\psi|\psi\rangle = \frac{1}{I_0(2\beta)}\int_0^{2\pi}\frac{d\theta}{2\pi}e^{2\beta\cos(\theta-\phi)} = 1. \tag{13}$$

The parameter ν must take on an integer value so that $\psi(\theta)$ is single valued. The variances are

$$\langle\Delta W\Delta W^\dagger\rangle = 1 - \rho^2, \qquad \langle\Delta G^2\rangle = \beta^2(1 - \rho^2), \tag{14}$$

so that the uncertainty relation is expressed as

$$\langle\Delta G^2\rangle\langle\Delta W\Delta W^\dagger\rangle = \beta^2(1 - \rho^2)^2, \tag{15}$$

where

$$\rho = \frac{I_1(2\beta)}{I_0(2\beta)}, \tag{16}$$

and $I_n(z)$ are the modified Bessel functions. The right-hand side of this relation is plotted in Fig. 1. The notable features of this expression are that in the $\beta \to \infty$ ($\rho \to 1$) limit, it approaches $1/4$ as expected in the large radius limit. In the other limit $\beta \to 0$ ($\rho \to 0$), it goes to zero as shown in Fig. 1. This is the well-known case where the wave function is an eigenstate of (angular) momentum, so that there is zero dispersion. One also has

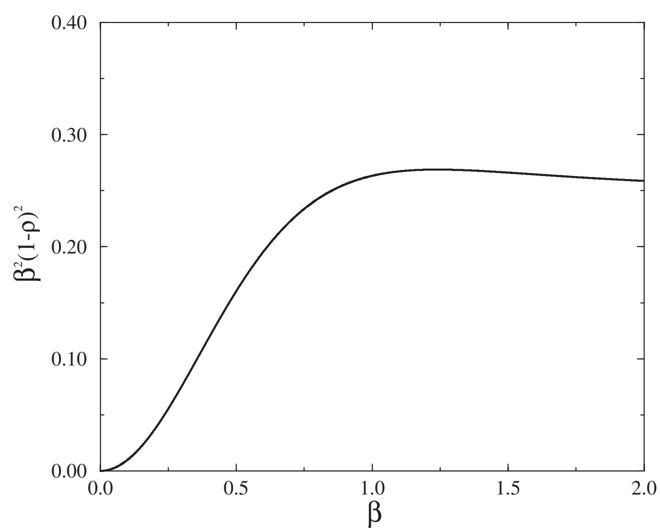

FIG. 1. The right-hand side of the uncertainty relation as a function of β. It approaches $1/4$ as $\beta \to \infty$ as expected, and goes to 0 for $\beta = 0$ which corresponds to an angular momentum eigenstate.

$$\beta^2(1-\rho^2)^2 = \frac{1}{4} + \mathcal{O}\left(\frac{1}{\beta^2}\right), \tag{17}$$

in the large β limit. This indicates that in the large β limit the variances for the operators G and W can be interpreted in terms of the usual operators on the line, namely x and p, and that the Heisenberg uncertainty relation involving x and p can be obtained in that limit.

On the circle the relevant operator is the phase operator W, and one encounters the well-known problems (briefly described in the introduction) if one insists on using the angle as a operator. However, in the large radius limit one can express an effective relationship between the dispersion in θ and the dispersion in momentum. In this limit the GUP appears as the first terms in an infinite expansion.

The expectation values and the variances for the operators in question were derived in Ref. [25]. The expectation values are

$$\langle W\rangle = \rho e^{i\phi}, \qquad \langle G\rangle = \nu + \alpha + \beta\rho. \tag{18}$$

The parameter $0 \leq \alpha < 1$ labels the inequivalent representations of the quantum mechanics algebra in Eq. (30). It can be viewed as an interpolation between the discrete eigenvalues of the operator G.

If one defines the variables x, $\langle x\rangle$, and $\langle p\rangle$ as

$$x = r\theta, \qquad \langle x\rangle = r\phi, \qquad \langle p\rangle = \frac{1}{r}\langle G\rangle, \tag{19}$$

then in the large radius limit one recovers the (minimum uncertainty) Gaussian wave packet [25]

$$\Psi(x) = \left(\frac{1}{2\pi d^2}\right)^{1/4} \exp\left[-\frac{1}{4d^2}(x-\langle x\rangle)^2 + i\langle p\rangle(x-\langle x\rangle)\right], \tag{20}$$

when one makes the identification

$$\frac{\beta}{r^2} = \frac{1}{2d^2}, \tag{21}$$

and defines the normalization condition

$$|\psi(\theta)|^2\frac{d\theta}{2\pi} = |\Psi(x)|^2 dx. \tag{22}$$

One can calculate the first order corrections (in $1/\beta$ or $1/r^2$) to find the modifications the finite size of the circle give to the Gaussian packet. Given a compactification radius r, then the degree the minimum uncertainty wave packet is localized is determined by either the parameter β or the parameter $d = r/\sqrt{2\beta}$ according to Eq. (21). These modifications in fact are of the form suggested by the GUP.

The momentum eigenstates are quantized so that the consecutive eigenvalues of the operator G differ by unity. So the momentum eigenvalues are separated by $1/r$ according to Eq. (20).

III. DERIVATION OF A GENERALIZED UNCERTAINTY PRINCIPLE

One expects on general grounds in any theory of quantum gravity that a GUP may apply when the momenta are of order the Planck scale and gravitational effects become important. In this section we demonstrate how in a simple quantum mechanical model with discrete space eigenvalues the GUP can be simply derived. Since our model does not contain gravity explicitly, the GUP will be seen to arise as a consequence of the discretization of space which may or may not be a property of quantum gravity. Since the toy model we utilize can be solved, the completion of the GUP in the ultraviolet limit can be derived and in fact the entire expansion of higher order terms in the GUP can be explicitly derived. This "derivation" of the GUP in a simple model may be useful in understanding how the GUP arises in more realistic physical situations.

The dispersion that we are interested in involves a parameter $x = r\theta$ where r is the compactification radius. The coordinate x is periodic as is θ, but for sufficiently localized wave packets ($d \ll r$), it can be used as an effective coordinate. The uncertainty principle for the operators W and G can be expressed in terms of a series expansion involving the expectation values of the angular parameter θ. For a sufficiently localized wave packet ($d \ll r$ or $\beta \gg 1$) the expansion will involve increasing smaller terms. We have

$$\begin{aligned}\langle W\rangle &= \int_0^{2\pi}\frac{d\theta}{2\pi}\psi^*\psi e^{i\theta} \\ &= \int_0^{2\pi}\frac{d\theta}{2\pi}\psi^*\psi\left(1 + i\theta - \frac{\theta^2}{2} - i\frac{\theta^3}{6} + \cdots\right) \\ &= 1 + i\langle\theta\rangle - \frac{\langle\theta^2\rangle}{2} - i\frac{\langle\theta^3\rangle}{6} + \cdots.\end{aligned} \tag{23}$$

and

$$\begin{aligned}\langle W^\dagger\rangle &= \int_0^{2\pi}\frac{d\theta}{2\pi}\psi^*\psi e^{-i\theta} \\ &= \int_0^{2\pi}\frac{d\theta}{2\pi}\psi^*\psi\left(1 - i\theta - \frac{\theta^2}{2} + i\frac{\theta^3}{6} + \cdots\right) \\ &= 1 - i\langle\theta\rangle - \frac{\langle\theta^2\rangle}{2} + i\frac{\langle\theta^3\rangle}{6} + \cdots.\end{aligned} \tag{24}$$

For a localized wave packet, these series contain terms that are increasingly smaller [35]. Then

$$\begin{aligned}1 - \langle W\rangle\langle W^\dagger\rangle &= \langle\theta^2\rangle - \langle\theta\rangle^2 + \cdots \\ &= \frac{1}{r^2}(\langle x^2\rangle - \langle x\rangle^2) + \cdots = \frac{1}{r^2}\Delta x^2 + \cdots,\end{aligned} \tag{25}$$

where the omitted terms involve terms which are smaller such as $\langle\theta^4\rangle$ and $\langle\theta^2\rangle^2$. In fact we calculate

$$1 - \langle W\rangle\langle W^\dagger\rangle = \frac{1}{r^2}(\langle x^2\rangle - \langle x\rangle^2) + \frac{1}{r^4}\left(-\frac{1}{12}\langle x^4\rangle + \frac{1}{3}\langle x^3\rangle\langle x\rangle - \frac{1}{4}\langle x^2\rangle^2\right) + \mathcal{O}\left(\frac{1}{r^6}\right). \quad (26)$$

Using also $\langle\Delta p^2\rangle = \frac{1}{r^2}\Delta G^2$, we get an uncertainty relation from Eq. (11),

$$\Delta p^2\left[(\langle x^2\rangle - \langle x\rangle^2) + \frac{1}{r^2}\left(-\frac{1}{12}\langle x^4\rangle + \frac{1}{3}\langle x^3\rangle\langle x\rangle - \frac{1}{4}\langle x^2\rangle^2\right) + \mathcal{O}\left(\frac{1}{r^4}\right)\right] = \beta^2(1-\rho^2)^2. \quad (27)$$

We can without loss of generality choose our coordinate system so that $\langle x\rangle = 0$. This is equivalent to taking $\phi = 0$ in Eq. (31), and centers the wave packet at $x = 0$, so that $\Delta x^2 = \langle x^2\rangle$. One then obtains

$$\Delta p^2\Delta x^2\left[1 - \frac{1}{4r^2}(\Delta x^2) - \frac{1}{12r^2}\frac{\langle x^4\rangle}{\Delta x^2} + \cdots\right] = \beta^2(1-\rho^2)^2, \quad (28)$$

where the ratio $\langle x^4\rangle/\Delta x^2$ is a calculable constant. For the approximation we are contemplating ($\Delta x^2/r^2 \ll 1$), one gets

$$\Delta p^2\Delta x^2 = \frac{1}{4}\left(1 + \frac{1}{4r^2}(\Delta x^2) + \frac{1}{12r^2}\frac{\langle x^4\rangle}{\Delta x^2} + \cdots\right), \quad (29)$$

where we have made use of Eq. (17). It should be noted that the complete generalization of the uncertainty principle involves higher expectation values such as $\langle x^4\rangle$, and that the GUP is only an approximation to this more complete expression.

IV. DISCRETIZED POSITION EIGENSTATES

The derivation in the previous section was performed in the "usual" case where space is compactified on a circle of radius r, and the above expansion should be valid when r is large compared to the dispersion in the state Δx^2. By interchanging the roles of configuration space and momentum space, one obtains a discrete spectrum for position space. We define an algebra as

$$[G_p, W_p] = W_p \quad (30)$$

for Hermitian G_p and unitary W_p. The minimum uncertainty wave packet in momentum space,

$$\psi_p(\theta_p) = \frac{1}{\sqrt{I_0(2\beta_p)}}\exp[\beta_p e^{i(\theta_p - \phi_p)} + i\nu_p\theta_p], \quad (31)$$

is centered at ϕ_p, and in the limit $\beta_p \to \infty$ one gets the Gaussian. It is important to note that ψ_p is the momentum space wave function rather than the position space wave function ψ considered earlier. The expectation values are the same as before:

$$\langle W_p\rangle = \rho_p e^{i\phi_p}, \qquad \langle G_p\rangle = \nu_p + \alpha_p + \beta_p\rho_p, \quad (32)$$

with

$$\rho_p = \frac{I_1(2\beta_p)}{I_0(2\beta_p)}. \quad (33)$$

Since the roles of position and momentum have been interchanged, we define the variables p, $\langle p\rangle$, and $\langle x\rangle$ as

$$p = r_p\theta_p, \qquad \langle p\rangle = r_p\phi_p, \qquad \langle x\rangle = \frac{1}{r_p}\langle G_p\rangle. \quad (34)$$

One obtains a discrete spectrum for G_p with consecutive eigenvalues separated by $1/r_p$ so that position space is discretized. Repeating the steps in the previous section with the roles of position and momentum interchanged, one arrives at an analogous expression expansion which has the form of a GUP as in Eq. (1),

$$\Delta x^2\Delta p^2 = \frac{1}{4}\left(1 + \frac{1}{4r_p^2}\Delta p^2 + \frac{1}{12r_p^2}\frac{\langle p^4\rangle}{\Delta p^2} + \cdots\right), \quad (35)$$

where r_p is the compactification radius of momentum space. If we want the discretization of configuration space to be at a certain scale, say L_P (which results when we require the parameter ν_p to be an integer), then r_p is determined in terms of L_P.

A gauge parameter α_p interpolates between discretizations. It takes on values in the range $\alpha_p \in [0, 1)$. The operator G_p is represented by

$$\langle\theta_p|G_p|\psi_p\rangle = \left[-i\frac{\partial}{\partial\theta_p} - i\kappa_p^*(\theta_p)\frac{\partial\kappa_p(\theta_p)}{\partial\theta_p} + \alpha_p\right]\psi_p \equiv \left[-i\frac{\partial}{\partial\theta_p} + A_p(\theta_p)\right]\psi_p, \quad (36)$$

so we can understand the parameter α_p as a quantity that interpolates between the discrete eigenvalues of G_p as shown in Fig. 2. Choosing α_p to be an integer represents a shift in the lattice and is physically equivalent to the case $\alpha_p = 0$. The periodic function $\kappa_p(\theta_p)$ represents a gauge choice, and for simplicity it can be chosen to be equal to one in which case it disappears from Eq. (36). It can be shown [25] that any periodic function $A_p(\theta_p) = A_p(\theta_p + 2\pi)$ is gauge equivalent to a constant function α_p, and that two constant functions are gauge equivalent if they differ

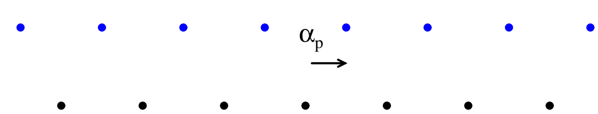

FIG. 2 (color online). The parameter α_p characterizes the inequivalent discretizations of the position operator G_p. It takes values in the range $\alpha_p \in [0, 1)$. The separation of eigenvalues of G_p is given in terms of the compactification radius as $1/r_p$ ($\hbar = 1$).

by an integer. This has an interesting interpretation for a discretized space as we are considering here where it can be viewed as parameterizing the ways to populate equally spaced eigenvalues on the line.

V. EXPECTATION VALUES FOR THE MINIMUM UNCERTAINTY WAVE PACKETS

In this section we calculate expectation values for the minimal uncertainty wave packets that appear in the uncertainty relation in Eq. (35). In particular, we show how the usual minimum uncertainty wave packet approaches the limit where the radius become very small. We also give examples of states which do not have any analog with respect to the usual Heisenberg uncertainty principle (because they are localized at discrete eigenvalues of the position operator).

The dispersion can be calculated for the minimum uncertainty wave packet in Eq. (31). One finds

$$\langle \theta_p^2 \rangle = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2}(-1)^n \frac{I_n(2\beta_p)}{I_0(2\beta_p)}, \tag{37}$$

and

$$\langle \theta_p^4 \rangle = \frac{\pi^4}{5} + \sum_{n=1}^{\infty} \left[\frac{8\pi^2}{n^2} - \frac{48}{n^4}\right](-1)^n \frac{I_n(2\beta_p)}{I_0(2\beta_p)}. \tag{38}$$

These quantities decrease from $\pi^2/3$ and $\pi^4/5$ to zero as β_p goes from zero to infinity. This merely reflects the fact that the minimum uncertainty wave packets become more localized (squeezed) in the angular parameter θ_p. Multiplying by the appropriate power of β_p results in functions that asymptote to nonzero values for large β_p as shown in Fig. 3. It is clear that these two expectation values give comparable contributions to the right-hand side of Eq. (35).

For the large radius limit, one can calculate

$$\beta_p \langle \theta_p^2 \rangle = \tfrac{1}{2}, \qquad \beta_p^2 \langle \theta_p^4 \rangle = \tfrac{3}{4}, \tag{39}$$

by using the Gaussian form. These formulas receive $\mathcal{O}(1/\beta_p)$ corrections for large β_p. The approach to the asymptotic limits are shown in Fig. 3. From Eq. (39) one has $\langle p^4 \rangle = 3\langle p^2 \rangle^2 = 3(\Delta p^2)^2$ for the minimum uncertainty wave packet and the equality in Eq. (35) becomes

$$\Delta x^2 \Delta p^2 = \frac{1}{4}\left(1 + \frac{1}{2r_p^2}\Delta p^2 + \cdots\right), \tag{40}$$

where the ellipsis refers to terms of order $1/r_p^4$ and higher. For a more general state that does not necessarily saturate the uncertainty relation, one does not expect the ratio $\langle p^4 \rangle / \langle p^2 \rangle^2$ to be a fixed constant (for the minimum uncertainty wave packet it is 3).

The GUP in Eq. (40) describes the relationship between the dispersion in x and p in the limit where the discrete eigenvalues of the position operator G_p are very finely spaced. As β_p is decreased from some large value, the finite size of the circle allows the wave function to attain some nonzero value for all values of the angular parameter θ_p (the wave function can "see" itself around the circle). When $\beta_p \to 0$, the Gaussian wave packet has become the state with equal probability for each point on the circle. This is the eigenstate of operator G_p corresponding to position. Since θ_p is bounded by the periodicity of the circle, the right-hand side of the uncertainty relation goes to zero. In our interpretation of the discrete space eigenvalue operator, the Gaussian which corresponds to the $\beta_p \to \infty$ case is smoothly interpolated to an eigenstate of the position operator as $\beta_p \to 0$. During this interpolation the initial corrections to the right-hand side of the uncertainty principle are positive as expected. Eventually when β_p becomes small enough, the expansion in terms of Δp is no longer adequate to approximate the right-hand side of the uncertainty principle, Eq. (15). When β_p reaches zero, one obtains an eigenstate of the position operator G_p:

$$\psi_p(\theta_p) = \exp(i\nu_p \theta_p), \tag{41}$$

and is characterized by a uniform distribution in (compactified) momentum space.

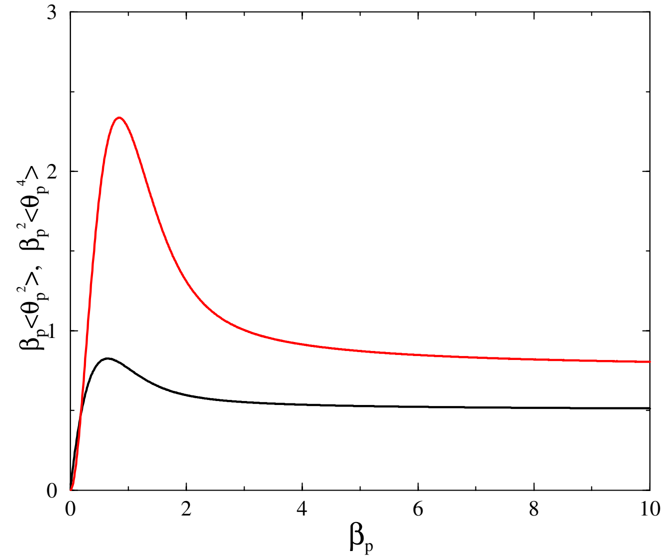

FIG. 3 (color online). The quantities $\beta_p\langle\theta_p^2\rangle$ (lower curve) and $\beta_p^2\langle\theta_p^4\rangle$ (upper curve) for the minimum uncertainty wave packet in Eq. (31) as a function of β_p. For large β_p they approach the expected values of $1/2$ and $3/4$, respectively.

VI. SUMMARY AND CONCLUSIONS

We have derived the generalized uncertainty principle from a toy model of discretized space by considering quantum mechanics on a circle where the compactification involves the *momentum*. This model may be useful in exploring how the ultraviolet limit is approached in more realistic models of discrete spacetime or models of quantum gravity with a fundamental or minimum length.

A GUP contains not only an infinite series of higher powers of the momentum dispersion, but also typically involves contributions from higher order quantities such as $\langle p^4 \rangle$. In fact this quantity may be of the same order as the one usually encountered in the definition of the GUP. It is probably worthwhile to pursue some models that fully capture this infinite series in addition to the more common procedure of investigating the implications of the truncated GUP in terms of deformed algebra. From our perspective, the defining algebra in Eq. (30) leads to the uncertainty principle in Eq. (11), and the truncated GUP arises as the appropriate formula in a certain physically interesting limit.

The large radius limit of the defining algebra yields a discretized position operator with finely spaced eigenvalues. It seems that the simple discretization of space implied by a compactified momentum is enough to obtain the leading order terms in the generalized uncertainty principle in a natural way. We find it interesting that a detailed incorporation of gravity does not seem to be necessary to obtain the GUP. In the quantum mechanics on the circle, the uncertainty relation is known for all values of the momentum compactification radius, so physically what happens to the variances of physical operators is known completely from one limit (an almost-Gaussian localized to a small region of the momentum circle) to the other (an eigenstate of the position operator). This may result in an improved understanding of the origin of the generalized uncertainty principle in theories of quantum gravity.

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[35] Note that $\langle\theta^2\rangle$ coincides with the definition of Judge [16] if coordinates are chosen so that $\langle\theta\rangle = 0$.