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Strong to weak coupling transitions of $SU(N)$ gauge theories in $2 + 1$ dimensions

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We find a strong-to-weak coupling crossover in $D = 2 + 1$ $SU(N)$ lattice gauge theories that appears to become a third-order phase transition at $N = \infty$, in a similar way to the Gross-Witten transition in the $D = 1 + 1$ $SU(N \to \infty)$ lattice gauge theory. There is, in addition, a peak in the specific heat at approximately the same coupling that increases with N, which is connected to Z_N monopoles (instantons), reminiscent of the first-order bulk transition that occurs in $D = 3 + 1$ lattice gauge theories for $N \geq 5$. Our calculations are not precise enough to determine whether this peak is due to a second-order phase transition at $N = \infty$ or to the third-order phase transition having a critical behavior different to that of the Gross-Witten transition. We show that as the lattice spacing is reduced, the $N = \infty$ gauge theory on a finite 3-torus appears to undergo a sequence of first-order Z_N symmetry breaking transitions associated with each of the tori (ordered by size). We discuss how these transitions can be understood in terms of a sequence of deconfining transitions on ever-more dimensionally reduced gauge theories. We investigate whether the trace of the Wilson loop has a nonanalyticity in the coupling at some critical area, but find no evidence for this. However we do find that, just as one can prove occurs in $D = 1 + 1$, the eigenvalue density of a Wilson loop forms a gap at $N = \infty$ at a critical value of its trace. We show that this gap formation is in fact a corollary of a remarkable similarity between the eigenvalue spectra of Wilson loops in $D = 1 + 1$ and $D = 2 + 1$ (and indeed $D = 3 + 1$): for the same value of the trace, the eigenvalue spectra are nearly identical. This holds for finite as well as infinite N; irrespective of the Wilson loop size in lattice units; and for Polyakov as well as Wilson loops.

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I. INTRODUCTION

A phase transition is associated with a singularity in the partition function, and so requires an infinite number of degrees of freedom. Usually that requires an infinite volume. One of the peculiarities of large-N field theories is that one can have phase transitions on finite, or even infinitesimal, volumes at $N = \infty$ because in this case we have an infinite number of degrees of freedom at each point in space. The classic example in the context of gauge field theories is the Gross-Witten transition [1] that occurs in the $D = 1 + 1$ $SU(\infty)$ lattice gauge theory (with the standard Wilson action). In this case the theory is analytically soluble and one finds a third-order phase transition at $N = \infty$ [1] at a value of the bare coupling that separates the strong and weak-coupling regions. The theoretical and practical interest of such phase transitions, particularly in $D = 3 + 1$, has recently been reviewed in [2].

In $D = 3 + 1$ $SU(N)$ gauge theories numerical studies reveal the existence for $N \geq 5$ of a first-order "bulk" transition separating the weak and strong coupling regions [3–5]. One also finds that the deconfinement transition, which is first order for $N \geq 3$, becomes sharper on smaller volumes as N increases suggesting [6] that here too one will have a phase transition on a finite volume at $N = \infty$. Indeed there appears to be a whole hierarchy of finite volume phase transitions at $N = \infty$ [2,7] which are, we shall argue below, related to the deconfinement transition.

These are all in some sense strong-to-weak coupling transitions, and this has led to the conjecture [2,8] that Wilson loops in general will show such $N = \infty$ transitions as the lattice spacing decreases, when the physical size of the loop passes some critical value. Such a transition in $D = 3 + 1$ could have interesting implications for dual string approaches to large-N gauge theories, as well as providing a natural explanation for the rapid crossover between perturbative and nonperturbative physics that is observed in the strong interactions [1,2,9]. In fact it is known [10,11] that in the $N = \infty$ $D = 1 + 1$ continuum theory the eigenvalue spectrum of a Wilson loop suffers a nonanalyticity for a critical area that is very similar to that of the plaquette at the Gross-Witten transition. However, in contrast to the Gross-Witten transition, there is no accompanying nonanalyticity in the trace of the loop and it is unclear what, if any, are its physical implications.

In this paper we investigate the existence of such phase transitions in $D = 2 + 1$ $SU(N)$ gauge theories, as a step towards a unified understanding of these phenomena in all dimensions.

In the next Section we briefly describe the $SU(N)$ lattice gauge theory and how we simulate it. There follows a longer section in which we review in more detail what is known about the large-N transitions and, in some cases, we extend the analysis. (We are interested in transitions that may be cross-overs or actual phase transitions, and when we refer to "transitions" in this paper it may be either one of these.) Having established the background, we move on to our detailed numerical results. Our conclusions contain a summary of our main results.

II. $SU(N)$ GAUGE THEORY ON THE LATTICE

We discretize Euclidean spacetime to a periodic cubic lattice with lattice spacing a and size $L_0 \times L_1 \times L_2$ in lattice units. We assign $SU(N)$ matrices, U_l, to the links l of the lattice. (We sometimes write U_l as $U_\mu(n)$ where the link l emanates in the positive μ direction from the site n.) We use the standard Wilson plaquette action

$$S = \beta \sum_p \left\{ 1 - \frac{1}{N} \mathrm{Re}\,\mathrm{Tr} U_p \right\} \tag{1}$$

where U_p is the ordered product of the $SU(N)$ link matrices around the boundary of the plaquette p. The partition function is

$$Z = \int \prod_l dU_l \exp(-S); \qquad \lim_{a\to 0} \beta = \frac{2N}{ag^2}. \tag{2}$$

Exactly the same expression defines the lattice gauge theory in $D = 1 + 1$ and $D = 3 + 1$ except that $\beta = 2N/a^2 g^2$ and $\beta = 2N/g^2$ respectively. Equation (2) also defines the finite temperature partition function, if we choose

$$T = \frac{1}{aL_0}; \qquad L_1, L_2 \gg L_0. \tag{3}$$

We simulate the above lattice theory using a conventional mixture of heat bath and over-relaxation steps applied to the $SU(2)$ subgroups of the $SU(N)$ link matrices.

It will sometimes be convenient to distinguish couplings, inverse bare couplings and (critical) temperatures in different spacetime dimensions, D, and we do so using subscripts or superscripts, e.g. g_D^2, β_D, T_c^D. Where there is no ambiguity we will often omit such subscripts.

We expect to obtain a smooth large N limit by keeping $g^2 N$ fixed [12]. It is therefore useful to define the bare 't Hooft coupling, λ, and the inverse bare 't Hooft coupling, γ,

$$\lambda = ag^2 N, \qquad \gamma = \frac{1}{\lambda} = \frac{\beta}{2N^2}. \tag{4}$$

Various numerical calculations have confirmed that a smooth $N \to \infty$ limit is indeed obtained by keeping $g^2 N$ fixed, both in $D = 2 + 1$ [13] and in $D = 3 + 1$ [4,6,14] and that to keep the cut-off a fixed as $N \to \infty$ one should keep γ fixed.

A useful order parameter for finite volume phase transitions is provided by taking the Polyakov loop, l_μ, which is the ordered product of link matrices around the μ-torus, and averaging it over the spacetime volume:

$$\bar{l}_\mu = c_\mu \sum_{n_{\nu\neq\mu}} \frac{1}{N} \mathrm{Tr} \left\{ \prod_{n_\mu = 1}^{n_\mu = L_\mu} U_\mu(n_0, n_1, n_2) \right\} \tag{5}$$

where the normalization is $c_\mu^{-1} = \prod_{\nu\neq\mu} L_\nu$. When the system develops a nonzero value for $\langle \bar{l}_\mu \rangle$ this indicates the spontaneous breaking of a global Z_N symmetry associated with the μ-torus. In particular such a symmetry breaking occurs at the deconfining temperature, if the μ-torus defines the temperature T.

III. BACKGROUND

A. The 'Gross-Witten' transition

By fixing gauge and making a change of variables, one can show [1] that the partition function of the $D = 1 + 1$ $SU(N)$ lattice gauge theory (with the Wilson plaquette action) factorizes into a product of integrals over $SU(N)$ matrices on the links and the theory can be explicitly solved. One then finds a crossover between weak and strong coupling that sharpens with increasing N into a third-order phase transition at $N = \infty$. In terms of the plaquette, $u_p = \mathrm{Re}\,\mathrm{Tr} U_p / N$, this shows up in a change of functional behavior

$$\langle u_p \rangle \overset{N\to\infty}{=} \begin{cases} \frac{1}{\lambda} & \lambda \geq 2, \\ 1 - \frac{\lambda}{4} & \lambda \leq 2. \end{cases} \tag{6}$$

More detailed information about the behavior of plaquettes and Wilson loops can be gained by considering not just their traces but their eigenvalues. The eigenvalues of an $SU(N)$ matrix are just phases, $\lambda = \exp\{i\alpha\}$, and are gauge-invariant. (We also use λ for the 't Hooft coupling: which is intended should be clear from the context.) As $\beta \to 0$ the eigenvalue distribution $\rho(\alpha)$ of a Wilson loop becomes uniform while as $\beta \to \infty$ it becomes increasingly peaked around $\alpha = 0$. As shown in [1], at the Gross-Witten transition a gap opens in the density of plaquette eigenvalues: in the strongly-coupled phase the eigenvalue density is nonzero for all angles $-\pi \leq \alpha \leq \pi$, but in the weakly-coupled phase it is only nonzero in the range $-\alpha_c \leq \alpha \leq \alpha_c$, where $\alpha_c < \pi$ [1].

In $D = 3 + 1$ it is known that at $N = \infty$ [3], and indeed for $N \geq 5$ [4,5], there is a strong first-order transition as β is varied from strong-to-weak coupling. Calculations in progress [15] suggest that the plaquette eigenvalue distribution does indeed show a gap formation at $N = \infty$ that is similar to the $D = 1 + 1$ Gross-Witten transition. However the first-order transition itself is usually believed to be part of the phase structure one finds with a mixed adjoint-fundamental action [16] and related to the condensation of Z_N monopoles and vortices [17]. This finite-N phase transition "conceals" any underlying $N = \infty$ Gross-Witten transition and makes the latter hard to identify unambiguously.

In $D = 2 + 1$ there has been, as far as we are aware, no systematic search for a Gross-Witten or "bulk" transition, and this is one of the gaps that the present work intends to fill.

B. Wilson loop transitions

The Gross-Witten transition involves the smallest possible Wilson loop, the plaquette. On the weak-coupling side the plaquette can be calculated in terms of usual weak-coupling perturbation theory; but this breaks down abruptly at the Gross-Witten transition, beyond which a strong coupling expansion becomes appropriate [1]. The coupling is the bare coupling and hence a coupling on the length scale of the plaquette. Thus one might interpret the transition as saying that as one increases the length scale, there is a critical scale at which perturbation theory in the running coupling will suddenly break down.

One might imagine that this generalizes to other Wilson loops: i.e. when we scale up a Wilson loop, at some critical size, in "physical units", there is a nonanalyticity. In fact precisely such a scenario has been conjectured for $SU(N \to \infty)$ gauge theories in $D = 3 + 1$ [2,8]. Unlike the lattice Gross-Witten transition, this would be a property of the continuum theory.

Such a nonanalyticity does in fact occur for the $SU(N \to \infty)$ continuum theory in $D = 1 + 1$ [10,11]. The transition occurs at a fixed physical area

$$A_{\text{crit}} = \frac{8}{g^2 N}. \tag{7}$$

Very much larger Wilson loops have a flat eigenvalue spectrum $\rho(\alpha)$ which becomes peaked as $A \to A_{\text{crit}}^+$. As A decreases through A_{crit} a gap appears in the spectrum near the extreme phases $\alpha = \pm\pi$. So for loops with $A < A_{\text{crit}}^+$ the eigenvalue density is only nonzero for $-\alpha_c \leq \alpha \leq \alpha_c$, where $\alpha_c < \pi$, and $\alpha_c \to 0$ as $A \to 0$. The nonanalyticity at $A = A_{\text{crit}}$ appears at first sight to be more singular than for the Gross-Witten transition, in that the derivative $\partial\rho/\partial\alpha$ diverges at $\alpha_c = \pm\pi$. However, unlike the Gross-Witten transition this is not a phase transition: the partition function is analytic. Moreover the trace of the Wilson loop, and the traces of all powers of the Wilson loop, remain analytic in the coupling. Thus it is unclear what if any is the physical significance of this nonanalyticity.

In this paper we shall investigate whether such a nonanalyticity develops in $D = 2 + 1$ $SU(N)$ gauge theories and whether it is accompanied by any nonanalyticity of the trace. The implications could be very interesting [9] and this makes a search in $D = 2 + 1$ (and even more so in $D = 3 + 1$ [15]) well worth while.

C. Finite volume transitions

Consider a $D = 3 + 1$ $SU(N)$ gauge theory on a $L_0L_1L_2L_3$ lattice with $L_0 \ll L_1 \ll L_2 \ll L_3$. As we increase β there will be a deconfining transition at $\beta = \beta_{c_0}$ where

$$a(\beta_{c_0})L_0 = 1/T_c^{D=4}. \tag{8}$$

This transition is first order for $N \geq 3$ [5,6]. A convenient order parameter is the Polyakov loop, $\langle \bar{l}_{\mu=0} \rangle$, which acquires a nonzero expectation value in the deconfined phase (related to the spontaneous breaking of a corresponding global Z_N symmetry). On our finite volume this is a crossover, but will sharpen to a true phase transition at $N = \infty$, even for $L_i = L_0 + \epsilon$ with ϵ arbitrarily small [5,6].

As we increase β and T further we will have the usual dimensional reduction to an effective $D = 2 + 1$ $SU(N)$ gauge theory coupled to adjoint scalars ϕ that are the remnants of the A_0 gauge field [18]. To leading order the gauge coupling and mass of the scalar of the effective $D = 2 + 1$ gauge-scalar theory are [18]

$$g_3^2 = g_4^2(T)T; \qquad m_a^2 \propto g_4^2(T)T^2 \tag{9}$$

so $m_a/g_3^2 = O(1/g_4(T))$ and at high enough T the $D = 3 + 1$ gauge theory reduces to the $SU(N)$ gauge theory in $D = 2 + 1$ on a $L_1 \ll L_2, L_3$ lattice. As we increase $\beta_4 \equiv \beta$ we simultaneously increase $\beta_3 \equiv 2N/ag_3^2 \simeq \beta_4 L_0$ (neglecting the difference between $g_4^2(a^{-1})$ and $g_4^2(T)$), Now this $D = 2 + 1$ gauge theory will deconfine at

$$a(\beta_{c_1})L_1 = 1/T_c^{D=3} \tag{10}$$

at which point $\langle l_{\mu=1} \rangle$ acquires a nonzero vacuum expectation value. We estimate the corresponding critical value of $\beta(\equiv \beta_4)$ to be

$$\beta_{c_1} \sim 0.36 N^2 \frac{L_1}{L_0}. \tag{11}$$

using $(T_c/\sqrt{\sigma})_{D=3} \sim 0.9$ [19] and $\sqrt{\sigma}/g_3^2 N \simeq 0.198$ [13], together with Eq. (9). For finite N this will be a crossover, but we expect (for the same reasons as in one higher dimension) that as $N \to \infty$ one will have a phase transition on any volume where L_2, $L_3 = L_1 + \epsilon$, for any fixed ϵ however small.

As we increase β beyond β_{c_1} on our $L_0 \ll L_1 \ll L_2 \ll L_3$ lattice, $T^{D=3} = 1/aL_1$ will become ever larger, and eventually the system will undergo a further dimensional reduction to a $D = 1 + 1$ $SU(N)$ gauge theory with $g_2^2 = g_3^2 T^{D=3}$ and with adjoint scalars which, however, do not decouple at high $T^{D=3}$ [20]. Thus this $D = 1 + 1$ theory, unlike the pure gauge theory, is a nontrivial confining field theory which we can expect to deconfine at some $T_c^{D=2} = 1/a(\beta_{c_2})L_2$. We estimate the corresponding critical value of the coupling $\beta(\equiv \beta_4)$ to be

$$\beta_{c_2} \sim 0.43 r^2 N^2 \frac{L_2^2}{L_0 L_1}; \qquad r = \frac{T_c^{D=2}}{\sqrt{\sigma}} \tag{12}$$

using the value $\sigma \sim (0.8)^2 g_3^2 TN/3$ extracted from [20]. Because we are in one spatial dimension the high-T phase cannot have a true nonzero expectation value for $\langle l_{\mu=2} \rangle$ but, as we shall see when we discuss our results in Section IV D 2, there can be a real phase transition at $N = \infty$.

If we increase β further we come to consider a field theory with a finite Euclidean time extent given by aL_3 living in an infinitesimal spatial volume $a^3 L_0 L_1 L_2$. Such systems can in principle have deconfining phase transitions

[21] although whether this one does or not we do not attempt to make plausible by a simple argument. If it does exist then it would provide the final step in our cascade of $N = \infty$ phase transitions $\beta_{c_0} \ll \beta_{c_1} \ll \beta_{c_2} \ll \beta_{c_3}$ on our $L_0 \ll L_1 \ll L_2 \ll L_3$ lattices.

We recall [7,8] that as one increases β on symmetric L^4 lattices one finds at $N = \infty$ a sequence of phase transitions, where Polyakov loops, $\langle l_\nu \rangle$ with $\nu \neq \mu$ chosen at random, acquire nonzero expectation values. It is natural to see this as a continuation in volume of the above sequence of deconfining transitions.

The above discussion has taken the $D = 3 + 1$ $SU(N)$ gauge theory as its starting point. It is obvious that we could equally well have started with a $L_0 \ll L_1 \ll L_2$ $D = 2 + 1$ $SU(N)$ gauge theory and followed that through a cascade of deconfining $N = \infty$ transitions. This is the case we shall later explore numerically.

IV. RESULTS

A. Preliminaries

1. Phase transitions

At a phase transition appropriate derivatives of $\frac{1}{V}\log Z$, where V is the volume and Z is the partition function, will diverge or be discontinuous as $V \to \infty$. The lowest order of such a singular derivative determines the order of the phase transition. For Z or its derivatives to be singular, we require an infinite number of degrees of freedom, and this usually demands an infinite volume, with a crossover at finite V sharpening to the appropriate singularity as $V \to \infty$. As $N \to \infty$ we have the possibility of a new kind of phase transition that takes place in a finite volume with the infinite number of degrees of freedom being provided by N.

With the standard plaquette action, a conventional first-order transition has a discontinuity at $V = \infty$ in the average plaquette,

$$\langle u_p \rangle = N_p^{-1} \partial \log Z / \partial \beta \qquad (13)$$

where N_p is the number of plaquettes. At finite V this discontinuity is a rapid crossover so that the specific heat $C \equiv N_p^{-1} \partial^2 \log Z / \partial \beta^2 = \partial \langle u_p \rangle / \partial \beta$ diverges linearly at the critical coupling $\beta = \beta_c$ as $N_p \to \infty$.

A conventional second-order transition has a continuous first derivative of Z but a diverging second derivative and a specific heat $C \to \infty$ as $V \to \infty$. Defining $\overline{u_p}$ to be the average value of u_p over the spacetime volume for a single lattice field, we readily see that the specific heat can be written as a correlation function:

$$C = N_p \langle (\overline{u_p} - \langle \overline{u_p} \rangle)^2 \rangle = N_p (\langle \overline{u_p}^2 \rangle - \langle \overline{u_p} \rangle^2) = \sum_p \langle (u_p - \langle u_p \rangle)(u_{p_0} - \langle u_p \rangle) \rangle \qquad (14)$$

where p_0 is some arbitrary reference plaquette. It is clear from eqn (15) that the divergence of C as $N_p \to \infty$ implies that there is a diverging correlation length-the standard signal of a second-order phase transition.

A conventional third-order transition has continuous first and second derivatives but a singular third derivative, $C' \equiv N_p^{-1} \partial^3 \log Z / \partial \beta^3$, at $V = \infty$. This may be written as

$$C' = \frac{\partial C}{\partial \beta} = N_p^2 \langle (\overline{u_p} - \langle \overline{u_p} \rangle)^3 \rangle = N_p^2 (\langle \overline{u_p}^3 \rangle - 3 \langle \overline{u_p} \rangle \langle \overline{u_p}^2 \rangle + 2 \langle \overline{u_p} \rangle^3). \qquad (15)$$

It should be clear that the higher the order of the transition, the greater is the statistics needed to determine its properties to a given precision. In particular, identifying third-order transitions is already a formidable numerical challenge, and we do not attempt to look for transitions that are of yet higher order.

Since we are particularly interested in transitions that develop as $N \to \infty$ and since we know that, in general, fluctuations in the pure gauge theory decrease by powers of N in the large-N limit [12,22] it is convenient to define the rescaled quantities

$$C_2 = N^2 \times C; \qquad C_3 = N^4 \times C' \qquad (16)$$

which one expects generically to have finite nonzero limits when $N \to \infty$. The signature of a phase transition which is only present for $N = \infty$ will be a crossover for finite N at which fluctuations decrease more slowly than the naive power of $1/N^2$. If, therefore, we find a crossover in C_2 or C_3 which does not sharpen with increasing volume at fixed N, but rather becomes a divergence or a discontinuity only in the large-N limit, then this will indicate a second- or third-order $N = \infty$ phase transition, respectively. (Assuming $\langle u_p \rangle$ to be continuous.)

Large-N phase transitions can have an unconventional behavior. Consider, for example, a second-order transition characterized by a value of C_2 that diverges at some $\lambda = \lambda_c$ as $N \to \infty$. This may indeed be due to a correlation length ξ that diverges (in lattice units) as $N \to \infty$: $\xi(\lambda_c) \propto N^\gamma$; $\gamma > 0$. However there is another, less conventional, possibility: the correlation length may be finite and it may be that local plaquette fluctuations have an anomalous N-dependence at the critical point: $\langle u_p^2 \rangle / \langle u_p \rangle^2 - 1 \propto N^{\gamma - 2}$; $\gamma > 0$.

Since a large-N phase transition can arise from fluctuations that are completely local-as in $D = 1 + 1$ where the lattice partition function factorizes-it is also useful to consider local versions of the quantities C_2 and C_3, where we replace $\overline{u_p}$ by u_p, and which we call P_2 and P_3 respectively. Calculations of P_2 and P_3 are statistically more accurate than those of C_2 and C_3, so they will be particularly useful at the largest values of N.

The eigenvalues $\lambda_j = \exp\{i\alpha_j\}$ of an $SU(N)$ matrix such as the plaquette provide additional gauge-invariant observables. At the $D = 1 + 1$ $N = \infty$ Gross-Witten transition a gap opens at $\pm\pi$ in the eigenvalue density of the plaquette [1] and the fluctuations of the extreme eigenvalues diverge when rescaled with N.

2. Wilson loop nonanalyticities

To investigate the possibility that Wilson loops undergo some analogous nonanalyticity as their area passes through some critical value, A_{crit}, we calculate Wilson loops of a fixed size, $n_1 \times n_2$, in lattice units and increase β so as to decrease the lattice spacing a and hence the area, $A = an_1 \times an_2$, in physical units. If there is a nonanalyticity at $A(\beta_c(n_1, n_2))$ we can then vary n_1, n_2 so as to check whether the transition occurs at a fixed area in the continuum limit, when expressed in units of say g^2N, i.e. whether

$$\frac{A_{\text{crit}}}{(g^2N)^2} = \lim_{a\to 0}\left(\frac{\beta_c}{2N^2}\right)^2 A(\beta_c) \tag{17}$$

is finite and nonzero. Since all the evidence is that the $D = 2 + 1$ $SU(N)$ lattice gauge theory has no phase transition, at zero temperature, once λ is on the weak-coupling side of the bulk transition, we expect any Wilson loop nonanalyticity not to correspond to a phase transition of the whole system. This will be an important constraint on what are the important observables to calculate. We also expect that any such transitions will be cross-overs at finite N, becoming real nonanalyticities only at $N = \infty$. This is because we can imagine that they are driven by the degrees of freedom close to the critical length scale, and that we need these to be infinite in number for a real nonanalyticity.

We remarked in Sec. III B that a nonanalyticity in the eigenvalue spectrum of the Wilson loop is known to occur [10,11] in the $D = 1 + 1$ $N = \infty$ gauge theory at the critical area given in Eq. (7). We have performed numerical lattice calculations in this theory for large N and find that the lattice critical area is very close to the continuum one for Wilson loops that are 2×2 or larger. To be more precise let us denote the product of link matrices around a square $n \times n$ Wilson loop by $U_w^{n\times n}$ and its trace by $u_w^{n\times n} = \frac{1}{N}\,\text{Re}\,\text{Tr}\{U_w^{n\times n}\}$ which we generically write as u_w. Then we find that the nonanalyticity occurs when u_w reaches a particular value

$$\langle u_w\rangle \simeq e^{-2}. \tag{18}$$

As an example we show in Fig. 1 the eigenvalue spectrum of a 3×3 Wilson loop in $D = 1 + 1$ for $N = 48$ at $\lambda = 0.7971$ where the trace satisfies Eq. (18) and we compare it to the continuum expression obtained from [10,11] We clearly have a very good match (apart from the $N = 48$ bumps that arise from the eigenvalue repulsion in the Haar measure). Now we know that in $D = 1 + 1$ the Wilson loop factorizes into a product of plaquettes

$$\langle u_w\rangle = \langle u_p\rangle^{A/a^2} \tag{19}$$

and that $\langle u_p\rangle = 1 - \lambda/4$ at $N = \infty$ [1]. Putting all this together, we have

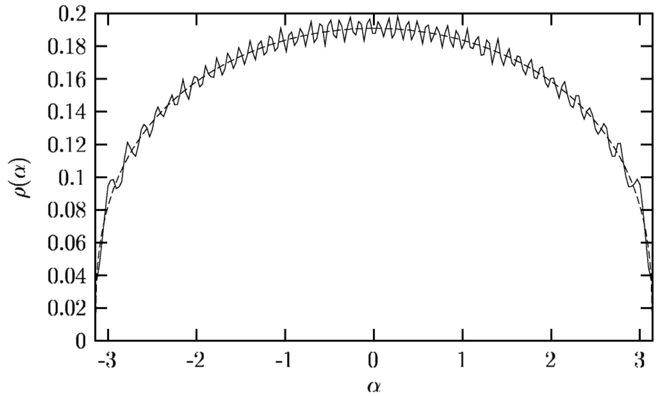

FIG. 1. The spectrum of eigenvalues, $e^{i\alpha}$, of a 3×3 Wilson loop for SU(48) in $D = 1 + 1$ at the critical coupling $\gamma = 1/\lambda = 1.255$, together with the continuum spectrum ($- - -$).

$$\left(1 - \frac{\lambda}{4}\right)^{A/a^2} = e^{A/a^2 \ln(1-(\lambda/4))} \simeq^{a\to 0} e^{-(A\lambda/4a^2)} \simeq e^{-2} \tag{20}$$

which we observe is nothing but the continuum relation in Eq. (7). These numerical calculations show that lattice corrections are small except for loops smaller than 2×2, such as the plaquette that has its nonanalyticity at the Gross-Witten transition where $\lambda = 2$. Because of the factorization in Eq. (19) the trace of u_w will be analytic in the (bare) coupling when this gap in the eigenvalue spectrum forms (except for the very smallest loops where it occurs at the Gross-Witten transition) and so it is not immediately obvious what is the significance of this gap formation. What this tells us, nonetheless, is that we should not only search in $D = 2 + 1$ for nonanalyticities of traces of Wilson loops, but also for such eigenvalue gap formation.

We shall search for nonanalyticities in $\langle u_w\rangle$ and its derivatives, such as $\partial\langle u_w\rangle/\partial\beta$, which can be expressed as correlators. We shall also calculate "local" versions of the latter, just as we do for $\langle u_p\rangle$, and various moments of the Wilson loops. Finally, we shall also calculate and analyze their eigenvalue spectra.

B. Bulk transition

In 3 + 1 dimensions the bulk transition is easily visible as a large discontinuity in the action for $N \geq 5$ (where the transition is first order) and as a (finite) peak in the specific heat for $N \leq 4$ (where the transition is a crossover). We have searched for an analogous jump or rapid crossover in 2 + 1 dimensional SU(6), SU(12), SU(24) and SU(48) gauge theories, in particular around $\gamma \equiv \beta/2N^2 \sim 1/2$. At $\gamma \sim 1/2$ our typical $L = 6$ lattice has a size $La\sqrt{\sigma} \sim 3$ and so is large enough that it should display a very sharp crossover for a conventional first-order transition. This should be more so as $N\uparrow$ and (most) finite volume effects disappear. (As a check we have repeated our calculations on 12^3 lattices for SU(6) and have indeed found no volume dependence.) What we see is that the action appears to be approaching a smooth crossover in the large-N limit, with no evidence for a first-order phase transition either at finite N or at $N = \infty$.

Our results for the specific heat C_2 for SU(6) and SU(12) are shown in Fig. 2. (For larger N errors dominate C_2.) There is a clear peak around $\gamma \simeq 0.42$ which appears to be growing stronger with increasing N. For SU(6) we repeated our calculations on 12^3 lattices and found no volume dependence. This tells us that we are not seeing a conventional second-order phase transition at fixed N for which the specific heat peak grows as the volume increases (since a larger volume can better accommodate the diverging correlation length). So if there is a second–order phase transition, it would appear to be not at finite N, but only at $N = \infty$.

To search for a possible third-order transition we have calculated C_3, but our calculations are not accurate enough to produce anything significant, even for SU(6).

P_2, the "local" version of C_2, is much more accurate, as we see in Fig. 3, where we show its values for SU(6), SU(12), SU(24) and SU(48). There is no significant evidence for a peak in P_2 which indicates that if there is a second-order transition at $N = \infty$, as suggested by the peak in C_2, it will primarily involve correlations between different plaquettes rather than arising from the fluctuations of individual plaquettes. However, what we do see in P_2 is definite evidence for a cusp developing with increasing N, at $\gamma \simeq 0.43$, where the derivative of P_2 will suffer a discontinuity. This corresponds to a third-order transition at $N = \infty$, just like the $D = 1 + 1$ Gross-Witten transition [1]. For comparison we have numerically calculated values of P_2 in $D = 1 + 1$ $SU(N)$ gauge theories (where $P_2 = C_2$) and find that, apart from a small relative shift in γ, the results for $D = 2 + 1$ and $D = 1 + 1$ are remarkably similar. This strengthens the evidence for a third-order $N = \infty$ transition.

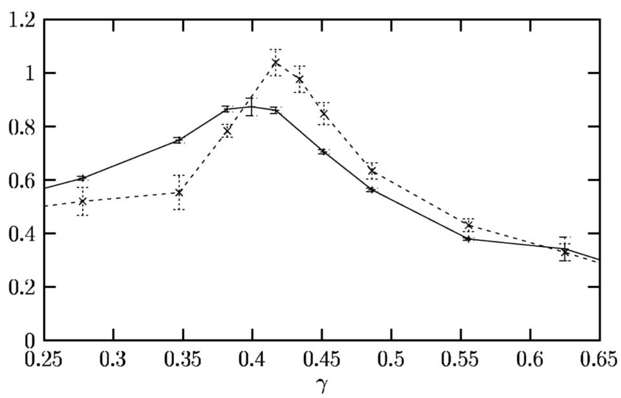

FIG. 2. The specific heat, C_2, as a function of $\gamma = \frac{1}{ag^2N} = \frac{\beta}{2N^2}$ for SU(6) (solid line) and SU(12) (dashed line).

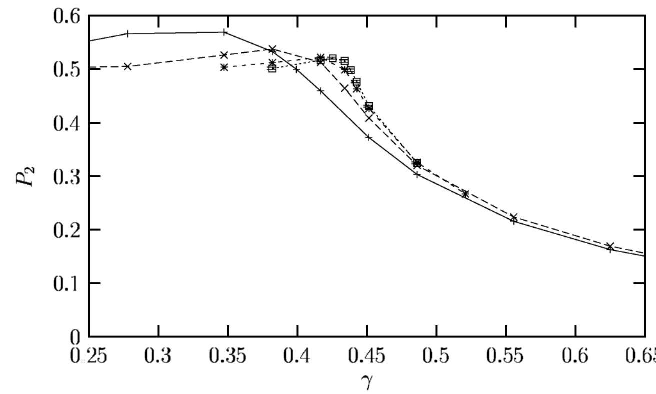

FIG. 3. The "local" specific heat, P_2, as a function of $\gamma = \frac{\beta}{2N^2}$ for SU(6) (+), SU(12) (×), SU(24) (∗) and SU(48) (□).

To investigate this further, we show in Fig. 4 our results for P_3 (the "local" version of C_3) for SU(6), SU(12), SU(24) and SU(48) in $D = 2 + 1$. There is clearly an increasingly sharp transition around $\gamma \simeq 0.43$ as N increases. For comparison we show in Fig. 5 corresponding numerical results for $D = 1 + 1$ (where $C_3 = P_3$) together with the analytic result for $SU(\infty)$ [1]:

$$C_3 = \begin{cases} 0, & \gamma \leq 0.5 \\ -\frac{1}{8\gamma^3}, & \gamma \geq 0.5. \end{cases} \tag{21}$$

which has a discontinuity at the Gross-Witten transition at $\gamma = 1/2$. It is clear that once again the behavior in $D = 2 + 1$ is remarkably similar to that in $1 + 1$ dimensions.

We see further evidence for a Gross-Witten-like transition in the plaquette eigenvalue spectra. The fluctuations of the extreme eigenvalues, normalised to those of eigenvalues in the "bulk" of the spectrum, show a clear peak around $\gamma \simeq 0.43$ whose height increases rapidly with N, just as in $D = 1 + 1$ at the Gross-Witten transition.

Finally if we compare the $D = 2 + 1$ and $D = 1 + 1$ transitions directly, by comparing the plaquette eigenvalue

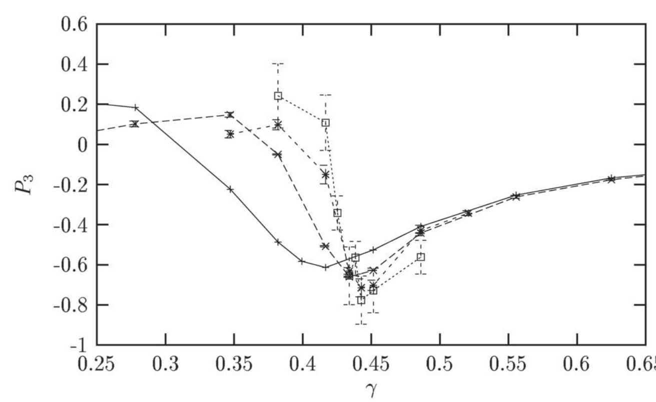

FIG. 4. The cubic local plaquette correlator, P_3, as a function of $\gamma = \frac{\beta}{2N^2}$ for SU(6) (+), SU(12) (×), SU(24) (∗) and SU(48) (□).

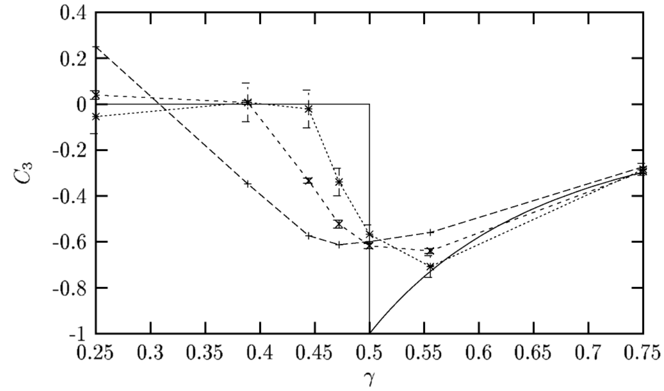

FIG. 5. The cubic plaquette correlator, C_3 (equal to P_3 here) as a function of $\gamma = \frac{\beta}{2N^2}$ in 1 + 1 dimensions for SU(6) (long dashes), SU(12) (short dashes), for SU(6) (+), SU(12) (×), SU(24) (∗) and the analytic result for SU(∞) (solid line).

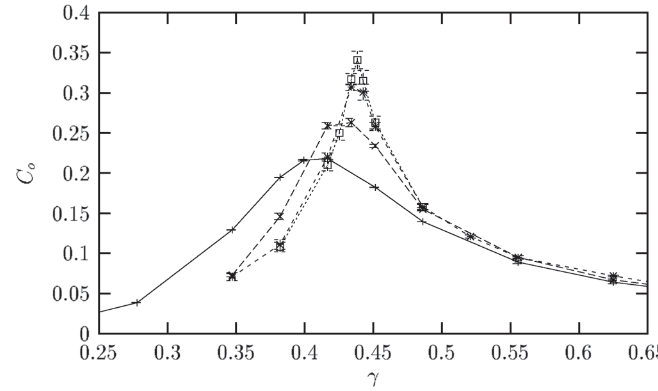

FIG. 6. The plaquette correlator, C_o, as a function of $\gamma = \frac{\beta}{2N^2}$ for SU(6) (+), SU(12) (×), SU(24) (∗) and SU(48) (□).

densities across the transition, we find that they are very similar both below and above the transition.

All the above suggests that $D = 2 + 1$ $SU(N)$ gauge theories possess an $N = \infty$ third-order strong-to-weak coupling transition that is remarkably similar to the $D = 1 + 1$ Gross-Witten transition.

Despite this striking similarity, when we look in more detail we also observe significant differences between the bulk transition in 2 + 1 dimensions and the Gross-Witten transition. In particular we have seen in Fig. 2 that there is a peak in the specific heat in $D = 2 + 1$ which is simply not present in $D = 1 + 1$. From Fig. 3 it is clear that this peak does not primarily come from fluctuations of individual plaquettes, but rather from correlations between different plaquettes. To investigate this we consider the following particular contributions to the specific heat C_2: the contribution from correlations between a plaquette and its neighbors in the same plane, which we label C_i; the contribution from correlations between a plaquette and its neighbors which share an edge but are not in the same plane, C_o; and finally C_f, the contribution from correlations between a plaquette and the plaquettes facing it across an elementary cube. We include a factor N^2, as for C_2. We find a clear peak, growing with N, in our results for C_o, plotted in Fig. 6. The peak accounts for about half of the difference between C_2 and P_2. There is also a much weaker peak in C_i, approximately a factor of 15 times smaller, which also clearly grows with N, at least up to $N = 24$. (The weakness of the signal means that we lose statistical significance for larger N.) For C_f, where we happen to have results only for SU(6) and SU(12), we see in both cases a clear peak. This is almost exactly a factor of 4 lower than the corresponding peak for C_o. Since each plaquette has 4 times as many out-of-plane neighbors as it has neighbors facing it across an elementary cube, this shows the correlation of a plaquette with its individual out-of-plane neighbors is in fact the same as with a facing plaquette. By contrast the correlation with the "nearer" neighboring plaquettes that are in the same plane (as measured by C_i) is very much weaker. This pattern is precisely what one would expect if the correlations were due to a flux emerging from the cube symmetrically through every face, i.e. due to the presence of monopole-instantons.

If such monopoles are present, we would expect the correlation of the plaquette with itself, P_2, to be also affected. These correlations of the plaquette with itself should be as large as with each of its eight out-of-plane neighbors, so this contribution to P_2 should be about one eighth of C_o. Even for $SU(48)$ this is only ~0.04, which is easily consistent with our results in Fig. 3.

There are several scenarios for what happens at $N = \infty$ that are consistent with our results. One possibility is that there is a third-order phase transition with critical exponent α different from -1. Such phase transitions occur in three-link and four-link chiral chain models (the latter is equivalent to $SU(N)$ gauge theory on a tetrahedron), which have critical exponents $\alpha = -\frac{1}{2}$ and $\alpha = 0^-$ respectively [23]. In this case the specific heat C_2 will asymptote to a cusp, and C_0 and P_2 will remain finite. Alternatively there could be a second-order phase transition at $N = \infty$, driven either by local fluctuations, in which case C_0 and P_2 would eventually diverge, or by the correlation length diverging (or both). Our results cannot distinguish between these scenarios, but the slow growth in C_0 seen in Fig. 6 suggests that, if there is a second-order transition driven by local fluctuations diverging, these fluctuations will only become dominant at very large values of N.

To search for the possibility of a diverging correlation length, we measured the mass of the lightest particle that couples to the plaquette, in both SU(6) and SU(12). Our results show a modest dip in the masses near the transition, which becomes more significant as we increase N. While this is certainly consistent with a second-order crossover, the masses are large (am is greater than 2.5), and if the correlation length is going to show any sign of diverging it will be at much larger values of N than are accessible to our calculations.

It is interesting to note what happens when we study the bulk transition on anisotropic lattices where the timelike and spacelike lattice spacings are different. We find that as we turn on the anisotropy the discontinuity in P_3 splits into two, with one jump occuring for the timelike plaquettes at a lower value of γ, and one for the spacelike plaquettes at a higher value of γ, so there are two apparently third-order transitions. However, there is only one peak in the specific heat, which always remains at the same coupling as the spacelike third-order transition.

C. Wilson loops

1. Traces and correlators

When we calculate how $\langle u_w \rangle$ varies with λ we see no sign of any singularity developing in this quantity, or in our simultaneous calculations of $\partial\langle u_w \rangle/\partial\lambda$, in contrast to the growing peak we saw for $C_2 \propto \partial\langle u_p \rangle/\partial\lambda$ in Fig. 2. The more accurately calculated local version of the correlator that is equivalent to the derivative, also shows no evidence of developing the sort of cusp that might suggest an $N = \infty$ singularity in the second derivative. Thus, at this level of accuracy, we see no evidence for any $N = \infty$ nonanalyticity in the variation of $\langle u_w \rangle$ as a function of the coupling λ.

Given our uncertainty in the type of nonanalyticity that might occur we have also looked at quantities analogous to P_2 and P_3 for the plaquette. The variation of these quantities with γ, for SU(6), SU(12), SU(24) and SU(48), does not become sharper with N, in contrast to the behavior in Fig. 3.

All our results are in fact essentially identical to those we obtain in similar calculations in $D = 1 + 1$, where we know that the $\langle u_w \rangle$ is analytic in λ except at the Gross-Witten transition.

Finally we recall that for the plaquette the Gross-Witten transition is characterized by a divergence in the relative fluctuation of extremal eigenvalues. For $n \times n$ Wilson loops the analogous quantity in, for example, SU(6), SU(12) and SU(24) shows no such behavior. Again this parallels what one finds in $D = 1 + 1$.

2. Matching eigenvalue spectra

Although we have found no evidence that the trace of a Wilson loop is nonanalytic in λ at some critical area, it is possible that there are more subtle nonanalyticities of the kind that exist in $D = 1 + 1$ and which are associated with a gap forming in the eigenvalue spectrum.

To determine numerically whether at some given λ the spectrum $\rho(\alpha)$ in some region close to $\alpha = \pm\pi$ will extrapolate exactly to zero when $N \to \infty$ is clearly a delicate matter, given that the values at finite N from which we extrapolate are already extremely small.

So to search for such nonanalytic behavior we explore the strategy of directly comparing Wilson loop eigenvalue spectra in $1 + 1$ and $2 + 1$ dimensions. We first evaluate the spectrum in $1 + 1$ dimensions at the critical coupling at which the gap forms. A true gap only forms at $N = \infty$; for finite N we use the same value of the critical 't Hooft coupling, [10,11]

$$\lambda_c = \frac{1}{\gamma_c} = 4(1 - e^{(-2a^2/A)}), \tag{22}$$

where A is the area of the Wilson loop in physical units. At this coupling the expectation value of the trace of the Wilson loop is, using Eq. (6) [10,11],

$$\langle u_w \rangle = \{\langle u_p \rangle\}^{A/a^2} N \to \infty = \left(1 - \frac{\lambda}{4}\right)^{A/a^2} = e^{-2}, \tag{23}$$

which is the same value as at the critical coupling in the continuum limit. Note also that as $a \to 0$ and $A/a^2 \to \infty$, Eq. (22) reduces to Eq. (7) as it should. Having obtained the spectrum (numerically) in $D = 1 + 1$ for a given size Wilson loop (in lattice units) and for a given value of N, we then calculate the eigenvalue spectrum in $D = 2 + 1$ for the same size loop and for the same N, varying the coupling to a value where the two eigenvalue spectra match.

We find that it is always possible to achieve such a match, for any N and for any size of Wilson loop. We show an example in Fig. 7, where we compare the eigenvalue density of the 3×3 Wilson loop in SU(12) in $1 + 1$ dimensions to the density in $2 + 1$ dimensions, at a coupling chosen to give the best match. In Fig. 7 the coupling in $D = 1 + 1$ is λ_c, the coupling at which the gap forms. The spectra are clearly very similar and indeed indistinguishable on this plot. We also find that the spectra can be matched when they are away from the critical coupling. In Fig. 7 we also plot the analytically known spectrum [10,11] in the continuum limit of the $N = \infty$ theory in $D = 1 + 1$ at the corresponding coupling. This clearly matches the corresponding finite-N spectra very well, except in two respects: the latter have N bumps which arise from the

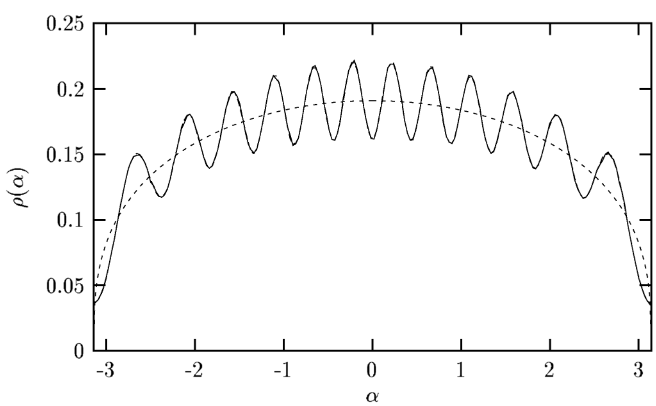

FIG. 7. 3×3 Wilson loop eigenvalue density, $e^{i\alpha}$, for SU(12) in $1 + 1$ dimensions at $\gamma = \frac{\beta}{2N^2} = 1.255$ (solid line) and in $2 + 1$ dimensions at $\gamma = 0.722$ (long dashes), and the continuum large-N distribution in $1 + 1$ dimensions at $A = A_{\text{crit}}$ (short dashes).

eigenvalue repulsion that is a well-known characteristic of the Haar measure, and the finite-N spectrum is not precisely zero in the region of the ‘‘gap‘‘.

The fact that at finite but large N we can match so precisely the $D = 1 + 1$ and $D = 2 + 1$ eigenvalue spectra for couplings at and above the $D = 1 + 1$ transition, provides convincing evidence that the Wilson loops in the $D = 2 + 1$ $N = \infty$ theory also undergo a transition involving the formation of a gap in the eigenvalue spectrum.

In Fig. 8 we plot the eigenvalue spectra of 2×2, 3×3, and 4×4 loops in SU(6) in $2 + 1$ dimensions. The three couplings have been chosen so as to give the best match to the eigenvalue spectra of Wilson loops of the same size in $D = 1 + 1$ at λ_c. We see that the three spectra are essentially identical. Moreover the critical value of the $D = 2 + 1$ coupling $\gamma_c = 1/\lambda_c$ appears to grow linearly with the size of the $L \times L$ loop, suggesting that there is a finite critical area for gap formation in the continuum limit: [10,11]

$$\lambda_c^2 L^2 \overset{a \to 0}{=} (ag^2N)^2 L^2 = (g^2N)^2 A_{\rm crit}. \tag{24}$$

As we shall see below, in Sec. IV C 4, this is nearly but not quite the case.

It turns out that all the above is an immediate corollary of a much stronger and rather surprising result concerning the matching of Wilson loop eigenvalue spectra in $1 + 1$ and $2 + 1$ (and indeed $3 + 1$) dimensions.

The general statement is that if we take an $n \times n$ Wilson loop $U_w^{n\times n}$ in the $SU(N)$ gauge theory and calculate the eigenvalue spectra in D and D' dimensions, we find that the eigenvalue spectra match at the couplings λ_D and $\lambda_{D'}$ at which the averages of the traces $u_w^{n\times n} = \frac{1}{N}\,\mathrm{Re\,Tr}\{U_w^{n\times n}\}$ are equal:

$$\langle u_w^{n\times n}(\lambda_D)\rangle_D = \langle u_w^{n\times n}(\lambda_{D'})\rangle_{D'}. \tag{25}$$

We have tested this matching for $D = 1 + 1$ and $D = 2 + 1$ over groups in the range $N = 2$ to $N = 48$ and for Wilson loops ranging in size from 1×1 (the plaquette) to 8×8 and, in $2 + 1$ dimensions, for couplings from $\lambda = 4.0$ to $\lambda = 0.40$. We have in addition tested it in the deconfined as well as in the confined phase. Some sample calculations in $D = 3 + 1$ have also been performed [15] strongly suggesting that the same is true there.

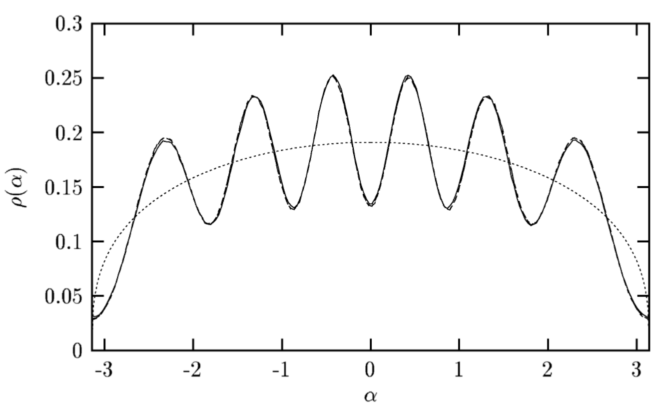

FIG. 8. Eigenvalue density in SU(6) in $2 + 1$ dimensions for the 2×2 loop at $\gamma = 0.483$ (solid line), the 3×3 loop at $\gamma = 0.719$ (long dashes), the 4×4 loop at $\gamma = 0.965$ (short dashes) and the continuum large-N distribution in $1 + 1$ dimensions at $A = A_{\rm crit}$ (dots).

The fact that such a precise matching is possible implies that the eigenvalue spectrum is completely determined by N, the size of the loop, and its trace. Hence the eigenvalues are not really independent degrees of freedom, which is unexpected. Moreover we have seen in Fig. 8 a demonstration of the fact that the spectra of Wilson loops that are 2×2 and larger can also be matched with each other. The matching occurs at values of the traces that are the same as those in $D = 1 + 1$ where they are calculable. In this sense, the size of the Wilson loop is not really an extra variable here. Finally, the N dependence is weak, and consists mainly of the two differences noted earlier.

Finally we remark that our results at this stage rely on a comparison that is visual and impressionistic. Ideally one would like to match the spectra by varying λ continuously and this can be done, from nearby calculated values of the coupling, by standard reweighting techniques. In addition it would be useful to quantify any differences (which must be very small) with a standard error analysis. We intend to provide such analyses elsewhere [15].

3. Polyakov loops

We have also investigated the eigenvalue spectra of Polyakov loops as defined in Sec. II. These are products of link matrices that wrap around one of the spacetime tori (and are of minimal length unless specified otherwise) i.e. they can be thought of as noncontractible Wilson loops. They provide the conventional order parameter for the deconfinement phase transition. As one crosses this transition the Polyakov loop that winds around the time (temperature) torus acquires a nonzero expectation value. This corresponds to the spontaneous breaking of a global center symmetry in the Euclidean system. To simulate the system at temperature T we use a $L_s^2 L_0$ lattice with $L_s \gg L_0$ so that $T = 1/aL_0$. As N grows one can weaken the inequality, so that one can take $L_s \to L_0$ as $N \to \infty$ while still maintaining the thermodynamic interpretation and the sharp phase transition.

We calculated the eigenvalue spectra of timelike Polyakov loops in SU(12) on $L_s^2 L_0$ lattices. We found that it is always possible to match the Polyakov loop eigenvalue spectra to those of Wilson loops in $1 + 1$ dimensions (and hence also to Wilson loops in $2 + 1$ dimensions) by choosing couplings at which the trace of the Polyakov loop equals that of the Wilson loop

$$|\langle \bar{l}_{\mu=0}\rangle| = \langle u_w\rangle \tag{26}$$

where, as we have seen, the size of the Wilson loop does not matter to a very good approximation. (We take the modulus because the Polyakov loop is proportional to some element of the center in the deconfined phase and the modulus effectively rotates that element to unity. The eigenvalue spectrum also needs to be rotated by the same center element.) This matching has the corollary that the Polyakov loop eigenvalue spectrum will develop a gap at $N = \infty$ when its trace crosses the critical value $|\langle \bar{l}_{\mu=0}\rangle| = e^{-2}$. For $N > 4$ the deconfining transition at $T = T_c$ is strongly first order and the value of $|\langle \bar{l}_{\mu=0}\rangle|$ will jump from $|\langle \bar{l}_{\mu=0}\rangle| = 0$ at $T < T_c$ to some nonzero value for $T = T_c^+$. The latter value will typically be greater than e^{-2} for small L_0, i.e. for coarse lattice spacings, and will $\rightarrow 0$ as $a \rightarrow 0$ and hence $L_0 \rightarrow \infty$. Moreover for fixed L_0 the trace increases with increasing T. (See Section IV C 4 for why this is so.) Thus for coarse lattice spacings we expect the gap formation to occur at the phase transition, $T = T_c$, while for larger L_0 it will not coincide with the deconfining transition; instead it will occur at some $T > T_c$. The critical value turns out to be $L_0 = 7$. Thus in the continuum limit the gap formation in timelike Polyakov loops does not occur at $T = T_c$ but rather at $T = \infty$.

As a numerical example of the eigenvalue matching we show in Fig. 10, the eigenvalue spectra of the timelike Polyakov loop just below and just above the deconfinement transition for $L_0 = 4$, together with a 3×3 Wilson loop spectrum in $1 + 1$ dimensions at a coupling chosen to match the spectrum of the deconfined Polyakov loop. The spectra clearly match closely. Since the $1 + 1$ dimensional γ is above γ_c for the 3×3 loop, the Wilson loop will develop a gap at this coupling in the large-N limit. Hence the Polyakov loop will presumably also develop a gap.

Finally we recall that as $N \rightarrow \infty$ the deconfining transition occurs on smaller spatial volumes $L_s \rightarrow L_0$ so that at $N = \infty$ one can discuss the transition on a L^3 lattice. Taking into account the fact that our preliminary results [15] indicate that all the above carries over to Wilson and Polyakov loops in $D = 3 + 1$, we can make direct contact with the observation in [7,8] that on an L^4 lattice the Polyakov loop develops a gap when it develops a nonzero expectation value.

4. Theoretical interpretation

The fact that at $N = \infty$ there is a gap at weak coupling in the eigenvalue spectra of Wilson loops, has a simple explanation in the theory of Random Matrices. (See e.g. [24] for a recent review.) At $N = \infty$ the Gaussian Unitary Ensemble (GUE) of complex Hermitian $N \times N$ matrices generates an eigenvalue spectrum that is the well-known Wigner semicircle

$$\rho(\lambda)N \rightarrow \infty \propto \left(1 - \frac{\lambda^2}{4}\right)^{1/2}. \qquad (27)$$

In weak coupling, when $\beta \rightarrow \infty$, the $SU(N)$ link matrices can be expanded in terms of the Hermitian gauge potentials and it is very plausible that the averages involved in the calculation of Wilson loops fall into the same "universality class" as the GUE. That is to say, once the eigenvalues of Wilson loops are clustered close to unity, the fact that the phases are on a circle rather than on the line becomes irrelevant and the phases (suitably rescaled by the coupling) should be distributed according to the semicircle in Eq. (27). In fact this is precisely what we find. Thus the existence of a gap in the eigenvalue spectrum at weak coupling has a rather general origin in terms of Random Matrix Theory.

On the other hand we know that in a confining theory

$$\langle u_w \rangle \propto e^{-\sigma A} \stackrel{A\rightarrow\infty}{\longrightarrow} 0 \qquad (28)$$

which requires a nearly flat eigenvalue spectrum in $[-\pi, +\pi]$. Thus as we decrease the lattice spacing, the eigenvalue spectrum of a $L \times L$ Wilson loop must change from being nearly uniform to eventually having a Wigner semicircle gap. Thus at some bare coupling it must pass through a transition where the gap forms.

For this gap to be physically significant, it must occur at a fixed physical area in the continuum limit. However, as we shall now see, this is not the case for either $2 + 1$ or $3 + 1$ dimensions (in contrast to $D = 1 + 1$). The reason is the perturbative self-energy of the sources whose propagators are the straightline sections of the Wilson loop. (Often referred to as the "perimeter term".) The leading correction is given by the Coulomb potential $V_c(r)$ at the "cut off" $r = a$. For a Wilson loop whose size is $l \times l = aL \times aL$ in physical units, this correction is

$$\delta \log\langle u_w \rangle \propto lV_c(a) \propto \begin{cases} \lambda L \log a & D = 2 + 1 \\ \lambda L & D = 3 + 1 \end{cases} \qquad (29)$$

using the fact that $V_c(r) \propto g^2 N \log r$, g^2N/r and $\lambda = ag^2N$, g^2N in $D = 2 + 1$, $3 + 1$ respectively. Let us, for illustrative purposes, assume that the full potential is given by this self-energy and the area piece, $\sigma A = a^2\sigma L^2$, that comes from linear confinement. Then we have

$$\langle u_w \rangle \propto \exp\{c\lambda L \log\lambda - c'\lambda^2 L^2\}: D = 2 + 1 \qquad (30)$$

using the fact that $a^2\sigma \propto (ag^2N)^2 = \lambda^2$ and $\log a = \log\lambda + \cdots$ in $D = 2 + 1$, and

$$\langle u_w \rangle \propto \exp\{c\lambda L - c' e^{-(c_r/\lambda)} L^2\}: D = 3 + 1 \qquad (31)$$

using the fact that $a^2\sigma \propto \exp\{-c_r/g^2N\}$ in $D = 3 + 1$, where c_r is given by the coefficients of the 2-loop renormalisation group equation.

Consider first the $D = 2 + 1$ case in Eq. (30). Since $\lambda L = ag^2NL = g^2Nl$ is the length scale in physical units, we see that if it were not for the weakly varying $\log\lambda$ term in Eq. (30), the Wilson loop trace would be the same on the lattice and in the continuum (up to the usual $O(a^2)$ lattice corrections). That is to say, we expect that as we approach the continuum limit, $\lambda \rightarrow 0$, the critical area for gap formation will vanish

$$A_{\rm crit} \propto \frac{1}{(\log\lambda)^2} \stackrel{a\to 0}{\longrightarrow} 0 \qquad (32)$$

rather than tending to some finite limit. At coarse a the logarithmic correction will be weak and one might well be tempted to perform an extrapolation to the continuum limit that does not include it. We illustrate all this with a numerical calculation of the coupling, and hence lattice spacing, at which $L \times L$ loops develop a gap. We define the appearance of a "gap" in our finite-N calculations as the coupling at which the spectrum is closest to the spectrum of the $L \times L$ loop in $1 + 1$ dimensions at the coupling γ_c in Eq. (22), for the same N. For SU(2) we calculated this coupling for L up to 8 on 16^3 lattices. For SU(6) we calculated up to $L = 4$ on 6^3 lattices. We show our results in Fig. 9, together with a best fit to the SU(2) data which has the asymptotic behavior in Eq. (32). The numerical data shows deviations from linearity which could either be interpreted as low L corrections to an asymptotic scaling behavior $\gamma = \lambda^{-1} \propto L$, or as a logarithmic violation of this asymptotic scaling. From our above analysis we know the latter to be the correct interpretation.

In contrast to the anomalous behavior we see when taking the continuum limit of $\lambda_c(A)$, the large-N limit is achieved rapidly and smoothly. To illustrate this we list in Table I the coupling for which the 3×3 loop develops a gap for $N \in [2, 48]$. The critical coupling is essentially constant from SU(6) onwards, showing that we are in the large-N limit. Indeed, even for SU(2) the corrections are small.

In the case of $D = 3 + 1$ the self-energy diverges linearly and will normally dominate the trace for all λ in the weak-coupling region. Thus we expect $A_{\rm crit} \propto a^2$ up to logarithmic corrections from the running coupling, so that the gap formation occurs in the deep ultraviolet as we approach the continuum limit. In contrast, in $D = 1 + 1$ where the Coulomb potential is linear $V_c(r) \propto g^2 N r$, the self-energy term contributes at most a mere lattice spacing correction that vanishes in the continuum limit.

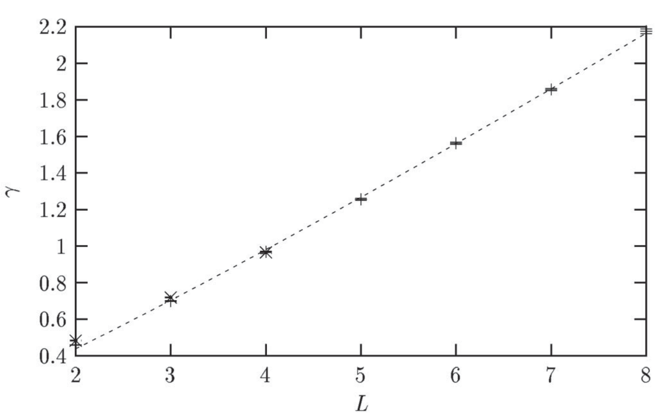

FIG. 9. Couplings for which the gap forms for $L \times L$ Wilson loops in SU(2) (+) and SU(6) ($*$), and fit to SU(2) data (dashed line).

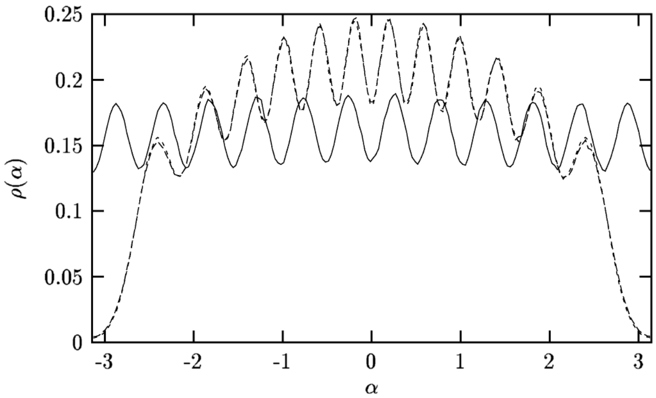

FIG. 10. Polyakov loop eigenvalue density in SU(12) in $2 + 1$ dimensions in the confined phase at $\gamma = 0.764$ (solid line) and in the deconfined phase at $\gamma = 0.833$ (long dashes), and the 3×3 Wilson loop in $1 + 1$ dimensions at $\gamma = 1.684$ (dashes).

From the above discussion we see that the anomalous behavior of $A_{\rm crit}$ as $a \to 0$ arises from divergent self-energy contributions. If the source had a finite mass, so that the propagator was smeared over some range $\delta r \sim 1/\mu$, we would evaluate the Coulomb self-interaction at $r = 1/\mu$ rather than at $r = a$ and hence would replace $\lambda L \log\lambda \to \lambda L \log\mu$ in Eq. (30), and $\lambda L \to \lambda/\mu$ in Eq. (31). Assuming the univerality of the gap formation persists for such loops, we would then expect them to form a gap at a value of $A_{\rm crit}$ that is finite in the continuum limit if we have chosen μ to be finite in physical units. The value of $A_{\rm crit}$ will of course depend on the value of μ.

Similar considerations apply to Polyakov loops.

While the above considerations make plausible the universality aspect of the gap formation in Wilson and Polyakov loops, they do not explain our most striking result which is that the complete eigenvalue spectra can be matched across spacetime dimension and loop size by merely matching traces.

Finally, whether the gap formation, and the associated nonanalyticity, has any significant physical implications is unclear. For that to be so one would require that the gap should form at a fixed physical area $A_{\rm crit}$ in the continuum limit. As we have seen that is not the case in $D = 2 + 1$ or in $D = 3 + 1$ and is only the case in $D = 1 + 1$, where

TABLE I. Inverse coupling at which gap forms for 3×3 loops in $SU(N)$.

N	γ_c
2	0.700(3)
6	0.719(3)
12	0.722(2)
24	0.722(1)
48	0.722(2)

there are no propagating degrees of freedom and so no "physics" in the usual sense. One can imagine regularizing the divergent self-energies so that $A_{\rm crit}$ is finite and nonzero in the continuum limit, but then it would appear to depend on the regularization mass scale μ used.

D. Finite volume

In Sec. III C we argued that $D = 3 + 1$ $SU(N)$ gauge theories on $L_0 \ll L_1 \ll L_2 \ll L_3$ lattices undergo a series of 3 (or possibly 4) $N = \infty$ phase transitions at $\beta_{c_0} \ll \beta_{c_1} \ll \beta_{c_2}$ ($\ll \beta_{c_3}$). These phase transitions are essentially deconfining transitions, $a(\beta_{c_i})L_i = 1/T_c^{D=4-i}$, in a sequence of ever-more dimensionally reduced theories. We conjecture that continuity, and vanishing finite size corrections at large N, link at least some, and perhaps all, of these transitions to the $N = \infty$ phase transitions on L^4 lattices that have been discussed in [2,7,8].

The same argument clearly holds for $SU(N)$ gauge theories on $L_0 \ll L_1 \ll L_2$ lattices in $D = 2 + 1$. Here we provide some (very) exploratory numerical results in support of this scenario. For practical reasons we do so on lattices with a less than asymptotic ordering, $L_0 < L_1 < L_2$.

1. First transition

The first transition is the usual deconfining phase transition when $L_1, L_2 \to \infty$. It is second order for SU(2) and SU(3), either second or first order for SU(4), and first order for $N \geq 5$ [19]. Because the latent heat for $N \geq 5$ is $\propto N^2$, the crossover on a finite $L_0 < L_1, L_2$ lattice will become a first-order phase transition at $N = \infty$. All this is well-established and does not require further numerical confirmation in this paper.

2. Second transition

To search for the second transition we simulate $SU(12)$ gauge fields on a $L_0L_1L_2 = 2 \times 4 \times 40$ lattice over a large range of $\gamma = \beta/2N^2$. We calculate the Polyakov loop around the $\mu = 1$ torus, average it over the given lattice field, and take the modulus: $|\bar{l}_1|$. This provides the conventional order parameter for a deconfining transition with the $L_1 = 4$ torus providing the (inverse) temperature. We plot results for the average of this, $\langle|\bar{l}_{\mu=1}|\rangle$, at each value of γ in Fig. 11 and of the plaquette difference, $\langle(u_{01} - u_{02})\rangle$, which should also reflect such a transition. We see in Fig. 11 a very clear signal for a transition at $\gamma \sim 3.2$ in both quantities. This occurs at a temperature $T_{c_1} \equiv T_c^{D=1+1} = 1/4a(\gamma \sim 3.2)$ in the reduced theory. In units of the usual deconfining temperature of the $D = 2 + 1$ gauge theory, $T_{c_0} = T_c^{D=2+1}$, this amounts to $T_{c_1} \sim 3T_{c_0}$.

The rapid, steep crossover suggests that the transition is first order. When we plot a histogram of $|\bar{l}_1|$ in the crossover region we indeed see a clear double-peak structure that is typical of a first-order deconfining transition.

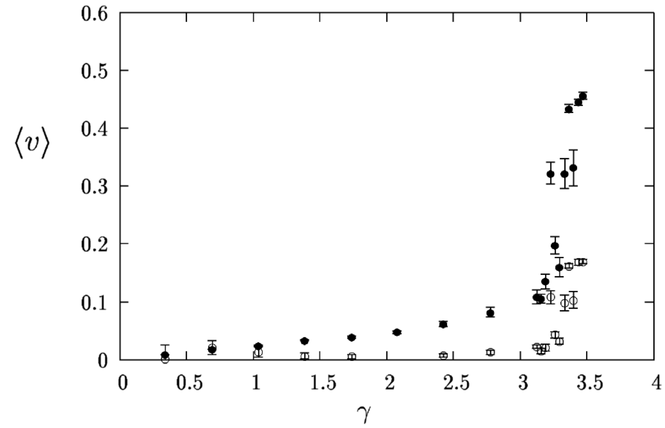

FIG. 11. Values of the shorter "spatial" Polyakov loop $\langle|\bar{l}_{\mu=1}|\rangle$, •, and the plaquette difference $500 \times \langle(u_{01} - u_{02})\rangle$, ∘, on a $2 \times 4 \times 40$ lattice in SU(12) versus the bare inverse 't Hooft coupling, $\gamma = \beta/2N^2$.

The next question is whether this crossover will sharpen into an actual phase transition in the two interesting limits: when we increase the spatial volume (here just aL_2) at fixed N; or when we increase N at fixed volume. In addressing the former question we need to remark upon some special features of first-order transitions in the effective $D = 1 + 1$ theory that we are discussing here. The high temperature deconfined phase is normally characterized by a center symmetry breaking so that $\bar{l} \sim c(\beta) \times \exp\{i2\pi n/N\}$ where $c(\beta)$ is a self-energy renormalisation factor. Two such phases, characterized by n and n' say, can coexist and will be separated by a domain wall whose tension we expect [25], for large T, to be

$$\sigma_k \propto k(N-k)\frac{T^2}{\sqrt{g_2^2 N}}; \qquad k = |n - n'|, \qquad T = \frac{1}{aL_1}. \tag{33}$$

In one spatial dimension the domain wall is just a "point" and so the usual energy/entropy arguments tell us that at $T = 1/aL_1 \geq T_{c_1}$ the field will break up into domains of typical size

$$\Delta r \propto \exp\left\{+\frac{\sigma_k}{T}\right\} \tag{34}$$

Thus at any T if we take $L_2 \to \infty$ the volume will consist of a "gas" of domain "walls", and hence domains, and on the average these will be equally distributed amongst all the center phases, so that $\langle|\bar{l}_{\mu=1}|\rangle \to 0$. However on volumes that satisfy $L_1 \ll L_2 \ll \Delta r$ we will typically be in one domain and will thus have the usual deconfining signal of a nonzero value for $|\bar{l}_{\mu=1}|$. In addition it is clear that the lightest mass, m_p, coupling to the $\mu = 1$ Polyakov loop will not vanish at $T > T_{c_1}$ but will approximately satisfy $m_p \propto \exp\{-cNT/\sqrt{\lambda}\}$. Note that this mass decreases with increasing $L_1 = 1/aT_1$ in contrast to the stringy behavior, $m_p \propto L_1$, in the confining phase. It is clear from the above that our conventional signals for being in a deconfined phase become more complicated to interpret in $D = 1 + 1$.

Similar considerations apply to the deconfining transition itself. Suppose that there are confining and deconfining phases that differ by a free energy density f. At $T = T_{c_1}$ we have $f = 0$ so that the typical field will consist of a "gas" of domain walls of typical size $\Delta r \propto \exp\{+\sigma_{cd}/T\}$ where σ_{cd} is the free energy of the confining-deconfining interface. This is the essential difference with higher dimensions. For large enough volume ($= L_2$) half the domains will be confining and half will be deconfining. Let us now increase the temperature T a little above T_{c_1}. Then $f = \epsilon_0(T - T_{c_1})$ near T_{c_1}, where $\epsilon_0 = [m]^0$ in $D = 1 + 1$. If $T - T_{c_1}$ is small enough, then $\Delta r f(T)/T \ll 1$ and the fraction of the volume that is still in the confined phase will be $\propto \exp\{-\Delta r f(T)/T\} \sim O(1)$. That is to say, the transition will take place over a range of temperatures ΔT that is no smaller than

$$\frac{\Delta T}{T_{c_1}} \propto \epsilon_0^{-1} \exp\{-\sigma_{cd}/T_{c_1}\} \qquad (35)$$

and this remains nonzero in the infinite volume limit. This implies that in $D = 1 + 1$, for any finite N, there cannot be an infinitely sharp first-order transition even in the large volume limit. However, because both σ_{cd} (probably) and f (certainly) grow $\propto N^2$, there can be a phase transition at $N = \infty$, and this can occur at finite volume.

Returning to our numerical results, we begin with SU(12) and show in Fig. 12 how the average plaquette difference $\langle(u_{01} - u_{02})\rangle$ varies across the transition when we vary the "spatial" volume, L_2. (We expect the plaquette difference to be less sensitive to domain formation than the Polyakov loop.) It is clear that the transition does become much sharper when we pass from $L_2 = 10$ to $L_2 = 40$ although the nature of the change between $L_2 = 40$ and $L_2 = 80$ is less clear. The evidence is for a would-be first-order transition inhibited by the domain formation described in the previous paragraph.

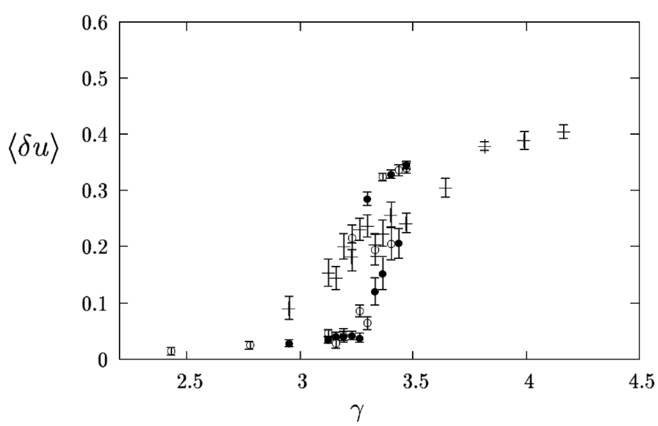

FIG. 12. The average plaquette difference $\langle\delta u\rangle = 10^3 \times \langle(u_{01} - u_{02})\rangle$ in SU(12) on $2 \times 4 \times L_2$ lattices with $L_2 = 10$ (+), $L_2 = 40$ ($\circ$), and $L_2 = 80$ ($\bullet$), versus $\gamma = \beta/2N^2$.

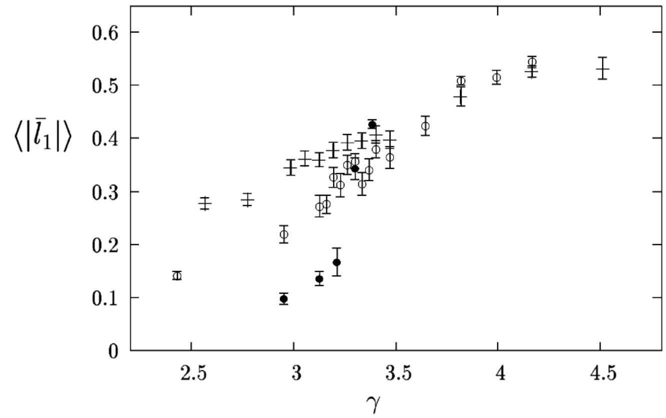

FIG. 13. The average $\mu = 1$ Polyakov loop for SU(6) (+), SU(12) ($\circ$), and SU(24) ($\bullet$) versus the inverse bare 't Hooft coupling $\gamma = \beta/2N^2$, all on $2 \times 4 \times 10$ lattices.

Turning now to the N-dependence of the transition, we show in Fig. 13 how $\langle|\bar{l}_{\mu=1}|\rangle$ varies with γ on a $2 \times 4 \times 10$ lattice for SU(6), SU(12) and SU(24) gauge theories. We see a rapid sharpening of the transition with increasing N which leaves little doubt that there is a first-order transition at $N = \infty$ at $L_2 = 10$, and presumably at other values of L_2 as well.

3. Third transition

To search for a third transition, characterized by a nonzero expectation value for $|\bar{l}_{\mu=2}|$, we take our SU(12) gauge theory on an $2 \times 4 \times 10$ lattice and increase γ beyond the values associated with the transitions discussed above. When we plot the resulting values of $\langle|\bar{l}_{\mu=2}|\rangle$ versus γ, we see a transition of the kind that we are looking for, but one which is very smooth. Increasing N to $N = 24$ we see what appears to be a significant sharpening of the transition, suggesting that it might become an actual phase transition at $N = \infty$.

Plotting a histogram of the values of $|\bar{l}_{\mu=2}|$ obtained in SU(24) on a $2 \times 4 \times 10$ lattice at $\gamma = 156.25$ (in the crossover region) shows a clear peak at low values that one naturally interprets as belonging to the confined phase, and a further peak (or peaks) at larger values that one naturally associates with the deconfined phase. This suggests that if this is a phase transition at $N = \infty$ then it is first order.

V. CONCLUSIONS

We have shown that there is a very close match for a number of observables between the bulk transition that separates strong and weak coupling in 2 + 1 dimensions, and the Gross-Witten transition in 1 + 1 dimensions. In particular the third derivative of the partition function, $C_3 \propto N^4 \partial^3 \log Z/\partial\beta^3$, appears to develop a discontinuity as $N \longrightarrow \infty$, just as it does across the Gross-Witten transition, providing strong evidence for a third-order transition in the large-N limit of $D = 2 + 1$ $SU(N)$ gauge theories.

When we expressed $\partial^3 \log Z/\partial \beta^3$ as a cubic correlator of plaquettes, we saw that some of the discontinuity arose from fluctuations of plaquettes at the same position. This is consistent with a genuine $N = \infty$ phase transition that arises from the $N^2 \to \infty$ degrees of freedom on each plaquette rather than from the collective behavior of a large number of separated plaquettes. This motivated us to study the eigenvalue spectrum of the plaquette. We found that at the critical inverse 't Hooft coupling, $\gamma = \gamma_c$, the spectrum develops a gap at the boundary of its range $e^{i\alpha} = \pm 1$ and this gap grows as γ increases. While the gap formation does not, in itself, lead to a nonanalyticity in Z, it possesses a feature that does. At $\gamma = \gamma_c$ and for $N \to \infty$ the spectrum $\rho(\alpha)$ approaches its end-points with a vanishing derivative. This means that the extreme eigenvalues possess fluctuations that diverge compared to the $O(1/N)$ fluctuations of the eigenvalues in the bulk of the spectrum, and this is directly related to the singularity in P_3, the local part of the third derivative of Z. All these features are exactly the same as in the $D = 1 + 1$ Gross-Witten transition. In addition we find that there is a very close match in the behavior of the plaquette eigenvalue density and in the ratio of plaquette eigenvalue fluctuations when we compare the transitions in $D = 2 + 1$ and in $D = 1 + 1$. Thus it would appear that the bulk transition in $2 + 1$ dimensions is very much like the Gross-Witten transition.

However, there is clearly more than this going on. The Gross-Witten transition has no peak in the specific heat, but we see in $2 + 1$ dimensions a clear peak that coincides with (or is very close to) the third-order transition. The contribution from neighboring or nearly neighboring plaquettes appears to grow with N, indicating a possible second-order phase transition driven by local fluctuations decreasing more slowly than $1/N^2$ at the critical point. An alternative is that there is a third-order phase transition, but that it has a critical exponent $\alpha \neq -1$, unlike the Gross-Witten transition. It is also possible that there is a second-order transition due to a correlation length that diverges as $N \to \infty$ (we see a slight decrease in the lightest mass that couples to the plaquette when we go from SU(6) to SU(12)). Numerical calculations that are both more accurate and extend to larger N are clearly needed here. In any case, the fact that the correlations between nearby plaquettes behave as if due to a flux emerging from an elementary cube, suggests that the transition may be due to center monopole(-instanton) and vortex condensation. It is therefore plausible that this (possible) second-order phase transition is connected to the line of specific heat peaks in the fundamental-adjoint plane found in SU(2) [26], which may also become a line of second-order phase transitions in the large-N limit, and which, just as in $D = 3 + 1$ [17], can be understood in terms of condensation of Z_N monopoles and vortices. In $D = 3 + 1$ this phase structure is believed to lead to the observed first-order bulk transition. The $N = \infty$ specific heat peak would appear to be a manifestation of the same dynamics, but in one lower dimension.

We have also investigated the sequence of finite volume transitions that occurs with increasing β on $L_0L_1L_2$ lattices. We argued that when the tori are strongly ordered, $L_0 \ll L_1 \ll L_2$, these can be understood in terms of deconfinement, followed by high-T dimensional reduction as β is increased, followed by deconfinement in the reduced system, and so on. The first transition is first-order for $N \geq 5$ and, for $N \to \infty$, will occur at any fixed, finite spatial volume L_1, $L_2 > L_0$. As we increase β, and hence T, the system will eventually be dimensionally reduced, $L_0L_1L_2 \to L_1L_2$. This $L_1 \ll L_2$ system will itself undergo a deconfining transition at some higher value of β. Because of the fragmentation of the high-T phase into domains (a feature of 1 spatial dimension) this transition is only a crossover at finite N, even in an infinite volume. But as $N \to \infty$ the domain "wall" tension will diverge (presumably as N^2), so that the domain structure is suppressed, and the crossover becomes a genuine first-order transition, even at finite L_2, as indicated by our simulations. At higher β we can again expect dimensional reduction to occur, $L_1L_2 \to L_2$, and there is some limited numerical evidence that the "infinitesimal" $L_0 \times L_1$ system may undergo a $N = \infty$ transition. These arguments can trivially be lifted to $SU(N)$ gauge theories in $3 + 1$ dimensions. Here they clearly have some relation, by continuation, to the $N = \infty$ finite volume transitions on L^4 lattices discussed in [2,7,8].

Motivated originally by conjectures [2,7,8] that Wilson loops in $D = 3 + 1$ may undergo $N = \infty$ nonanalyticities, when their area, in physical units, reaches a critical value, we have analyzed the behavior of Wilson loops in $D = 2 + 1$ $SU(N)$ gauge theories. Our results show a remarkable match between the behavior of Wilson loops in $D = 2 + 1$ and in $D = 1 + 1$. We find that the eigenvalue spectra of Wilson loops (and indeed Polyakov loops) in $D = 2 + 1$ match those of Wilson loops in $D = 1 + 1$ when the traces are equal. Moreover the spectra of Wilson loops of any size (in lattice units and when larger than about 2×2) also match if the couplings are tuned to values where their traces are equal. This is true for any fixed N. As a corollary, it immediately follows that in $D = 2 + 1$ at $N = \infty$ a gap will form in the eigenvalue spectrum of a Wilson loop at a critical coupling that depends on the size of the loop, just as it is known to do in $D = 1 + 1$ in both the lattice and continuum theories [10,11]. However because of a logarithmically divergent self-energy piece, this nonanalyticity in the spectrum will not occur at a finite nonzero value of the area in the continuum limit. This is in contrast to the case in $D = 1 + 1$. We have preliminary evidence [15] for a similar matching between Wilson loops in $D = 3 + 1$ and those with the same trace in lower dimensions. Here the self-energy divergence is even more severe and the gap forms deep in the ultraviolet. As in $D = 2 + 1$ one can imagine regularizing this self-energy by using finite mass sources in constructing the 'Wilson/Polyakov loops', so as to obtain a gap formation at a fixed physical area.

The appearance of the gap at $N = \infty$ follows quite generally if we make plausible connections with Random Matrix Theory. The spectrum of an $l \times l = aL \times aL$ Wilson loop should be flat at large a, since linear confinement demands its trace to be $\propto \exp\{-\sigma l^2\} \sim 0$, while at sufficiently small a we expect to find the Wigner semi-circle of the $N = \infty$ Guassian Unitary Ensemble. Somewhere in between a gap must form. Because the derivative of the spectrum diverges at its end-point, in contrast to that of the plaquette at the bulk transition, there are no anomalous fluctations of the extreme eigenvalues and no nonanalytic behavior in the correlators that are related to derivatives of the Wilson loop with respect to the coupling. And indeed we find the traces of Wilson loops to be analytic in the coupling just as they are in $D = 1 + 1$. Thus the physical implications of this nonanalyticity in the eigenvalue spectrum remain unclear.

The remarkable similarity between the eigenvalue spectra of Wilson loops in different dimensions does not appear to have a simple explanation within Random Matrix Theory and merits a more careful and quantitative investigation than the one provided in this paper.

In summary, the $D = 2 + 1$ large-N phase structure that we have investigated in this paper can be understood, as we have argued above, in terms that appear to allow a unified understanding of these phase transitions in $D = 1 + 1$, $D = 2 + 1$ and $D = 3 + 1$ $SU(N)$ gauge theories.

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Note added in proof.—This revised version arose from our discovery, immediately after sending the original version to the archive, of the papers [10,11] which then motivated our revised and extended study of Wilson loops in this paper. As this revision was in progress an interesting paper [27] on gap formation in smeared Wilson loops in $D = 3 + 1$ has appeared.

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