Leading quantum gravitational corrections to QED

M. S. Butt*

The Niels Bohr Institute, Blegdamsvej 17, DK-2100 Copenhagen, Denmark

(Received 30 May 2006; published 13 December 2006)

We treat the coupled Dirac-Einstein system in the framework of effective field theories in order to quantize the gravitational field at long distances in a consistent manner. In the Dirac-Einstein system we consider the leading post-Newtonian and quantum corrections to the nonrelativistic scattering amplitude of charged spin-$\frac{1}{2}$ fermions. We extract the relevant vertex rules from the action appropriate to the one-loop level calculations and find the nonrelativistic scattering matrix potential for two massive charged spin-$\frac{1}{2}$ fermions. Our focus is kept only on the nonanalytic parts contributing to the potential which are known to generate the long-range interactions.

DOI: 10.1103/PhysRevD.74.125007 PACS numbers: 11.10.Ef, 04.60.−m

I. INTRODUCTION

On the search for a theory of quantum gravity, Donoghue [1] proposed 12 years ago an interesting new way to look at general relativity. He suggested that, when treating general relativity as an effective field theory [2] using the background field method [3,4], reliable quantum predictions at low energies could be made. Similarly, chiral perturbation theory is viewed as the low energy effective field theory of QCD. It is also well known that a field theory need not necessarily be strictly renormalizable in order to be able to yield quantum predictions at low energies. However, a fundamental quantum theory of gravity does not appear in this way, but this method nevertheless makes it possible to calculate quantum corrections order by order in a momentum expansion.

Having laid the foundations for this new approach, Donoghue and collaborators turned their attention to the practical applications of this idea. A number of interesting calculations have been made involving quantum gravitational corrections to various quantities [1,5–8].

Prior to the effective field theoretical description of general relativity, attempts had been made to find a quantum theory for gravity. In particular, many proved that general relativity was indeed not renormalizable, be it pure general relativity or general relativity coupled to bosonic or fermionic matter; see e.g. [9–12]. Of course, it is a well-known fact that general relativity is a *non-renormalizable* theory *per se*, and these authors succeeded in exactly confirming that gravity indeed is explicitly nonrenormalizable, with or without matter. However, when looked at in the framework of an effective field theory, these theories do become order by order renormalizable in the low energy limit. Many interesting results have been found from this procedure. The most interesting from the point of view of this paper is the bosonic quantum corrections to the Newtonian/Coulomb potentials [6].

In Ref. [6] the post-Newtonian as well as the quantum corrections that were generated to the Newtonian and Coulomb potentials were explicitly found. We wish to repeat this calculation, but now in terms of couplings to fermions. Previously, Nieves and Pal have considered the gravitational couplings of neutrinos [13] and charged leptons [14] in a medium. We wish, in particular, to see explicitly if the post-Newtonian as well as the quantum gravitational corrections generated are identical to [6] or not.

We will more or less follow a similar procedure as in [6] mostly to avoid confusion about conventions and to make it easy to compare the results at the end. However, it is by no means a straightforward task to complete a similar calculation in terms of fermions. Some obstacles have to be overcome compared to bosonic matter in curved space, e.g. the issue of introducing fermionic matter into curved space-time. Luckily, this issue has been dealt with before [9,10,15–19]. The additional formalism is resolved by introducing the vierbein formalism and deriving the proper covariant derivative for the spinor fields.

Donoghue devised a particularly elegant way to extract relevant information in terms of analytical and nonanalytical contributions to the scattering matrix [1]. This was realized through the integrals occurring in the calculations, and propagation of massless particles. Since it is possible to fully determine the post-Newtonian and quantum corrections by the nonanalytical pieces of the 1-loop amplitude generated by the lowest order Einstein action alone, it becomes possible to perform this calculation completely by merely focusing on these contributions. We will also only consider 1-loop effects in this paper. We will extract the nonanalytical parts of the full set of 1-loop diagrams needed for the 1-loop scattering matrix in the combined quantum theory of general relativity and QED. As we will see in this paper, and as can also be seen in [1,5–8], the nonanalytical contributions correspond exactly to the long-range corrections of the potential.

We will employ the same conventions as in Refs. [5,6]. The mostly minus Minkowski metric convention $(1, -1,$

*Electronic address: butmu@nbi.dk

$-1, -1)$ will be used and the natural units are ($\hbar = c = 1$) when nothing else is stated.

In Sec. II we will see how to combine QED with general relativity by using the vierbein formalism, and, moreover, introduce the ghost fields. Next, in Sec. III, we will focus first on the distinction between nonanalytical and analytical contributions to the scattering matrix amplitude, whereafter we will define the potential. Finally, in Sec. IV, we will evaluate the Feynman diagrams contributing nonanalytically to the scattering matrix, in order to construct the leading corrections to the nonrelativistic Newtonian and Coulomb potentials. We will end this paper with a discussion in relation to [6]. In the appendixes, the vertex rules together with a table of relevant integrals are presented.

II. THE DIRAC-EINSTEIN SYSTEM AS A COMBINED EFFECTIVE FIELD THEORY

The combined theory of QED in a gravitational field is given by the sum of the QED and Einstein Lagrangian densities,

$$\mathcal{L} = \mathcal{L}_{\text{gravity}} + \mathcal{L}_{\text{QED}}. \tag{1}$$

The interacting field theory for quantum electrodynamics is well known, with the Dirac equation minimally coupled to the electromagnetic field,

$$\begin{aligned}\mathcal{L}_{\text{QED}} &= \mathcal{L}_{\text{Dirac}} + \mathcal{L}_{\text{Maxwell}} \\ &= \bar{\psi}(i\gamma^\mu D_\mu - m)\psi - \tfrac{1}{4}g^{\alpha\mu}g^{\beta\nu}F_{\alpha\nu}F_{\mu\beta},\end{aligned} \tag{2}$$

where m is the mass, e_q [20] is the electron charge with $e_q = |e_q|$, and finally $D_\mu \equiv \partial_\mu - ie_qA_\mu(x)$ is the covariant derivative.

To make the action of the Dirac Lagrangian density invariant under general coordinate transformations, we follow the general procedure, i.e. multiply it with $\sqrt{-g}$, and at the same time introduce a proper covariant derivative,

$$\begin{aligned}\mathcal{L}_{\text{Dirac}} &= \sqrt{-g}\bar{\psi}(i\gamma^\mu D_\mu - m)\psi \\ &= e\bar{\psi}(i\gamma^d e_d{}^\mu D_\mu - m)\psi.\end{aligned} \tag{3}$$

Now $D_\mu = \partial_\mu - ie_qA_\mu + \frac{1}{2}\sigma^{ab}\omega_{\mu ab}$ and we have used $\sqrt{-g} = \det(e^a{}_\mu) \equiv e$, i.e. the determinant of the vierbein is the matrix square root of the metric. Finally, $\gamma^\mu = \gamma^a e_a{}^\mu$.

The full generally covariant Lagrangian density including the fermionic degrees of freedom may collectively be written as

$$\begin{aligned}\mathcal{L} = e\frac{2}{\kappa^2}R + e(\bar{\psi}i\gamma^a e_a{}^\mu D_\mu\psi - \bar{\psi}\psi m) \\ - \frac{1}{4}\sqrt{-g}g^{\alpha\mu}g^{\beta\nu}F_{\alpha\nu}F_{\mu\beta}.\end{aligned} \tag{4}$$

This will account for our full theory. The Lagrangian density is to be expanded in powers of $c_{\mu\nu}$ [where we choose $c_{\mu\nu}$ to be linearly symmetrically equal to $h_{\mu\nu}$ (A9)] in the case of the Dirac field and only $h_{\mu\nu}$ in the case of the Maxwell fields; specifically, we expand the Lagrangian density as follows:

$$\mathcal{L} = \mathcal{L}_{\text{background}} + \mathcal{L}_{\text{linear order}} + \ldots, \tag{5}$$

where the ellipses denote second and higher order terms that will not contribute at the 1-loop level calculations.

The Lagrangian density for the photon field can now be expanded in powers of $h_{\mu\nu}$,

$$\begin{aligned}\mathcal{L}_{\text{Maxwell}} = -\frac{\kappa}{4}h(\partial_\mu A_\nu\partial^\mu A^\nu - \partial_\nu A_\mu\partial^\mu A^\nu) \\ + \frac{\kappa}{2}h^{\rho\sigma}(\partial_\alpha A_\sigma\partial^\alpha A_\rho + \partial_\alpha A_\alpha\partial_\rho A^\alpha \\ - \partial_\alpha A_\sigma\partial_\rho A^\alpha - \partial_\sigma A_\alpha\partial^\alpha A_\rho),\end{aligned} \tag{6}$$

where the trace of $h \equiv h^\alpha{}_\alpha = h^\alpha_\alpha = h$; Eq. (6) is also found in [21] and many other places. Likewise, we obtain for the fermionic part

$$\begin{aligned}\mathcal{L}_{\text{Dirac}} = e_q\bar{\psi}\gamma^\mu A_\mu\psi \\ + \frac{i\kappa e_q}{2}\bar{\psi}\gamma^a(I_a{}^{\mu\beta\alpha} - \delta_a{}^\mu\eta^{\alpha\beta})h_{\alpha\beta}A_\mu\psi \\ + \frac{\kappa}{2}h(\bar{\psi}i\gamma^\mu\partial_\mu\psi - \bar{\psi}m\psi) - \frac{\kappa}{2}\bar{\psi}\gamma^d h^\mu{}_d\partial_\mu\psi \\ + \frac{\kappa}{2}\bar{\psi}\gamma^\mu\sigma^{ab}\partial_b h_{a\mu}\psi,\end{aligned} \tag{7}$$

where the symmetric identity $I_a{}^{\mu\beta\alpha} = \frac{1}{2}\eta_a{}^{\{\beta}\eta^{\alpha\}\mu}$, which can also be found in [21].

All the necessary lowest order interaction vertices of fermions, gravitons, and photons can be found for the theory from the linear order expansions as stated above in Eqs. (6) and (7). A summary of these rules is presented in the appendixes; see also [13,21] for comparison.

In principle, we should also expand

$$\mathcal{L} = e\frac{2}{\kappa^2}R; \tag{8}$$

however, it is known [9] that the metric and vierbein formulations are equivalent for fields with only covariant vector indices. The coupling to the vierbein field only occurs as a symmetric combination of vierbein fields; the symmetric combination is, as we have seen, equal to the metric tensor to linear order. No new aspects of the traditional quantization of the pure gravitational action are introduced. We will therefore use the known vertices and propagators for the bosonic and gravitational fields.

We have excluded the antisymmetric fields in all our expressions. This is due to the fact that the antisymmetric fields have propagators that go as $\sim\kappa^2$. This can be seen when we fix the gauges in our quantization scheme. Our theory (i.e. fermions including gravitational effects) has two types of invariances. One is the general coordinate transformation, under which the fermions behave as scalars (since they are defined with respect to the local Lorentz frame). The other is the local Lorentz transformation, under which the fermions transform as spinors. If the Einstein action is included, then the coordinate gauge can be fixed by choosing the harmonic (de Donder) gauge,

$$\mathcal{L}^C = -\tfrac{1}{2}\sqrt{-\bar{g}}(h_{\mu\nu}{}^{,\nu} - \tfrac{1}{2}h_\nu{}^\nu{}_{,\mu})^2, \tag{9}$$

whereas the local Lorentz invariance is broken by choosing the sum of the squares of the antisymmetric vierbein components,

$$\mathcal{L}^L = -\tfrac{1}{2}e\kappa^{-2}a_{\mu\nu}^2. \tag{10}$$

Gauge fixing of both these fields will result in an introduction of two sets of ghost fields. We do not need to be concerned about the ghost introduced due to the antisymmetric field. In a vierbein description of pure gravity, the ghosts are never external. Furthermore, neither the antisymmetric vierbein fields nor its ghosts propagate (they cancel each other [9]); thus we will not need to calculate vertices for the external ghosts fields. This is very reassuring since the pure gravity theory in vierbein formulation can be covariantly quantized and is equivalent to the quantized metric approach. That is, we could, in principle, describe the theory without introducing these variables. But if we do not have pure gravity and include fermions, the antisymmetric fields become coupled to the vertices. We need only consider the *symmetric* fields of the interactions. This is due to the fact that we will only be interested in the long-range corrections to the background field, and the antisymmetric fields do not produce nonanalytic terms to the order at which we are working, due to the proportionality factor of their propagator $\sim\kappa^2$. In fact, a diagram consisting of at least an antisymmetric field and a graviton vertex will at least go as $\sim\kappa^3$ which is an order higher than $\sim\kappa^2$. However, in a full treatment of gravitational interaction between fermionic matter, these fields will have important contributions. They will most likely contribute to higher order calculations.

III. THE SCATTERING MATRIX AND THE POTENTIAL

The nonanalytical parts that occur in the calculations originate from the propagation of massless particles (photons and gravitons). These cannot originate from massive particles like the fermions, since it is not possible to expand them in a Taylor series. From

$$\frac{1}{q^2 - m^2} = -\frac{1}{m^2}\left(1 + \frac{q^2}{m^2} + \ldots\right) \tag{11}$$

we see that no $\sim\frac{1}{q^2}$-type terms are generated in the above expansion of the massive propagator. The nonanalytical effects are terms in the S matrix that will go as $\sim\ln(-q^2)$ or $\sim\frac{1}{\sqrt{-q^2}}$. These terms will generate corrections to the long-ranged forces we are interested in. The analytic parts of the S matrix will only generate corrections to local effects and will not contribute in the low energy regime, hence they will be disregarded.

Defining the potential

The S matrix is defined as the scattering matrix between incoming and outgoing particles. The invariant matrix element $i\mathcal{M}$ originating from the diagrams is

$$\langle k_1 k_2 \ldots | iT | k_A k_B \rangle = (2\pi)^4 \delta^4(k_A + k_B - \Sigma k_{\text{final}})(i\mathcal{M}); \tag{12}$$

here we have two incoming particles. If we Fourier transform the earlier mentioned nonanalytic terms to real space, we easily see how the nonanalytic terms contribute to the long-ranged corrections,

$$\int \frac{d^3q}{(2\pi)^3} e^{i\vec{q}\cdot\vec{r}} \frac{1}{|\mathbf{q}^2|} = \frac{1}{4\pi r}, \tag{13}$$

$$\int \frac{d^3q}{(2\pi)^3} e^{i\vec{q}\cdot\vec{r}} \frac{1}{|\mathbf{q}|} = \frac{1}{2\pi^2 r^2}, \tag{14}$$

$$\int \frac{d^3q}{(2\pi)^3} e^{i\vec{q}\cdot\vec{r}} \ln(\mathbf{q}^2) = -\frac{1}{2\pi r^3}. \tag{15}$$

Obviously these terms do indeed contribute to the long-range corrections. When we calculate the tree diagrams, we explicitly see that the nonanalytic contribution of the type (13) will correspond to the Coulomb and Newtonian part of the potentials and the higher power of $\frac{1}{r}$ will generate the leading order and classical corrections to the Coulomb and Newtonian potentials. Explicitly, the invariant matrix element will look like

$$\mathcal{M} = \bigg(A + Bq^2 + (\alpha_1\kappa^2 + \alpha_2 e^2)\frac{1}{q^2} + \beta_1\kappa^2 e^2 q^2 \ln(-q^2) + \beta_2\kappa^2 e^2 q^2 \frac{m}{\sqrt{-q^2}}\ldots\bigg), \tag{16}$$

where the high energy regime of the effective field theory is represented by the terms $A, B, \ldots$ corresponding to the analytical, local, and short-ranged interactions; hence these terms will be the dominating terms at the high energy range. The low energy range will be dominated by $\alpha_1, \alpha_2, \ldots$ and $\beta_1, \beta_2, \ldots$ terms corresponding to the leading nonanalytic, nonlocal, long-range contributions to the amplitude. Many diagrams will yield pure analytic contributions to the S matrix; such diagrams will not be necessary in our calculations, and we will only consider the nonanalytic contributions from the 1-loop diagrams. The diagrams which will yield nonanalytic contributions to the S-matrix amplitude are those containing two or more massless propagating particles.

Relating the Born approximation to the scattering amplitude in nonrelativistic quantum mechanics, we get in terms of iT

$$\langle k_1 k_2 \ldots | iT | k_A k_B \rangle = -i\tilde{V}(\mathbf{q})(2\pi)\delta(E - E'), \quad (17)$$

where $\mathbf{q} = \mathbf{p}' - \mathbf{p}$ and $\tilde{V}(\mathbf{q})$ is the nonrelativistic potential transformed in momentum space. We should be careful when comparing with (10); in $(i\mathcal{M})$ factors of $(2m_1 \times 2m_2)$ arise due to relativistic normalization conventions, thus we divide with these to obtain the nonrelativistic limit. Equating the two we deduce

$$-i\tilde{V}(\mathbf{q})(2\pi)\delta(E - E') \sim (2\pi)^4 \delta^4(k_A + k_B - \Sigma k_{\text{final}})(i\mathcal{M}) \quad (18)$$

or rather

$$\tilde{V}(\mathbf{q}) = -\frac{1}{2m_1}\frac{1}{2m_2}\int \frac{d^3k}{(2\pi)^3}(2\pi)^3 \times \delta^3(k_A + k_B - \Sigma k_{\text{final}})(\mathcal{M}). \quad (19)$$

Momentum integration yields the nonrelativistic potential

$$\tilde{V}(\mathbf{q}) = -\frac{1}{2m_1}\frac{1}{2m_2}\mathcal{M} \quad (20)$$

or, in coordinate space,

$$V(\mathrm{x}) = -\frac{1}{2m_1}\frac{1}{2m_2}\int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x}}\mathcal{M}. \quad (21)$$

In our calculations, $\mathcal{M}$ will only contain the nonanalytic contributions of the amplitude of the scattering process to 1-loop order, and we will not compute the full amplitude of the S matrix; only the long-range corrections will be of interest to us. In order to obtain their contribution to the potential, only a subclass of scattering matrix diagrams will be required. If we wanted to find the full total nonrelativistic potential, we would merely have to include the remaining 1-loop diagrams. This type of calculation has e.g. been performed in [22] (who also have used the same definition of the potential as us), where the full amplitude is considered. Their choice of potential included all 1-loop diagrams; hence they obtained a gauge invariant definition of the potential. This choice of the potential makes good physical sense since it is gauge invariant, but other choices are also possible. The most convenient choice could depend on the physical situation at hand or how the total energy is defined. The gauge invariant choice is also equivalent to the suggestion in [23], where it is suggested that one should use the full set of diagrams constituting the scattering matrix, from which one can decide the nonrelativistic potential from the total sum of the 1-loop diagrams. However, it is worthwhile to note that we consider all the nonanalytic corrections to 1-loop order; thus, if we had the full amplitude to 1-loop order, we would still need to extract the nonanalytical parts. We will continue using this definition of the potential.

IV. RESULTS FOR THE FEYNMAN DIAGRAMS

A. Diagrams contributing to the nonanalytic parts of the scattering matrix potential

In this section we shall extract the nonanalytical parts of a limited set of 1-loop diagrams needed for the 1-loop scattering matrix in the combined quantum theory of QED and general relativity (however, it is a practically complete set of diagrams in terms of nonanalytical contributions to the scattering matrix). We will explicitly see that the nonanalytic contributions indeed correspond to the long-range corrections of the potential. This will become obvious when the amplitudes are Fourier transformed to produce the scattering potential, whence all the analytic pieces are disregarded. The resulting nonanalytic piece of the scattering amplitude will then be used to construct the leading corrections to the nonrelativistic gravitational potential.

B. Classical physics

Here we will look at the tree diagrams. The fermion-fermion scattering process at tree level should, of course, reproduce the results of classical physics both for gravitational interactions and for electromagnetic interactions.

Tree diagrams

Given in Fig. 1, we have depicted the scattering process, where the (incoming/outgoing) momenta for the first particle are (k/k') with the (mass/charge) being (m_1/e_1), and similarly for the second particle with (p, p') being the (incoming/outgoing) momenta and (m_2/e_2) being the (mass/charge). This is assigned for all the other diagrams as well. The formal expression for the diagram depicted in Fig. 1(a), the scattering process involving a photon exchange, is

$$i\mathcal{M}_{1(a)} = \bar{u}(p')[\tau^{\alpha}]u(p)\bar{u}(k')[\tau^{\beta}]u(k)\left[-\frac{i\eta_{\alpha\beta}}{q^2}\right] \quad (22)$$

and the expression for Fig. 1(b), a graviton exchange, is

$$i\mathcal{M}_{1(b)} = \bar{u}(p')[\tau^{\mu\nu}]u(p)\bar{u}(k')[\tau^{\alpha\beta}]u(k)\left[\frac{i\mathcal{P}_{\mu\nu\alpha\beta}}{q^2}\right] \quad (23)$$

yielding the well-known classical results, namely, the

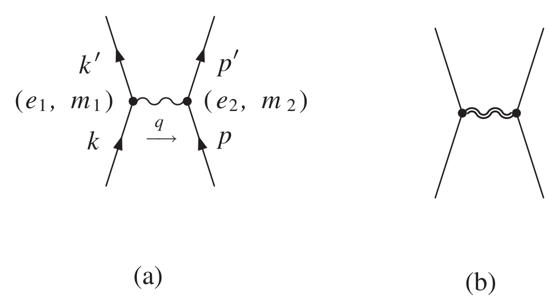


FIG. 1. The set of tree diagrams.

Coulomb,

$$V_{1(a)}(r) = \frac{e_1 e_2}{4\pi r}, \tag{24}$$

and Newtonian,

$$V_{1(b)}(r) = -\frac{G m_1 m_2}{r}, \tag{25}$$

potentials. It is worthwhile to note that already at this stage the level of difficulty is not obvious. There is virtually no problem in working out the Coulomb term for the interaction; technically and mathematically it is straightforward. However, in comparison, the Newtonian term is much more sophisticated to work out. This is mainly due to the many γ matrices involved and the complicated vertex rules in matter theories coupled to gravity. This difference will play a much bigger role when more complicated diagrams are involved. Indeed, the next set of diagrams is perhaps the most challenging of them all, the box and crossed box diagrams.

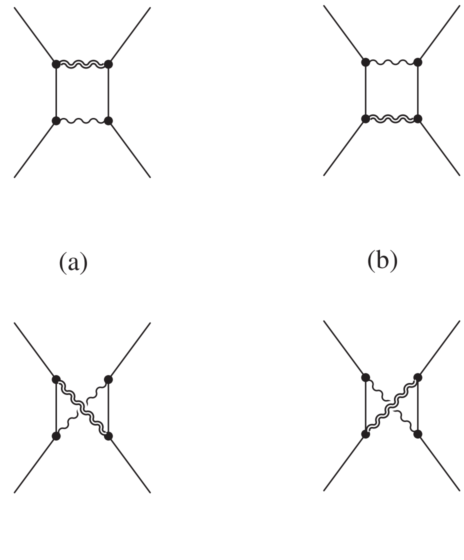
C. The 1PI diagrams

We will calculate all the relevant 1PI diagrams necessary to find the long-range corrections to the potentials. We will start with the box and crossed box diagrams and continue with the set of triangular diagrams. Lastly, we will work out the circular loop diagram.

1. The box and crossed box diagrams

There are in all four distinct diagrams: two box and two crossed box diagrams; these are depicted in Fig. 2. We will not treat all of these diagrams here. We will only look at one of these; the rest can be treated in the same manner. Explicitly, Fig. 2(a) is defined by

$$\begin{aligned} i\mathcal{M}_{2(a)} = \int \frac{d^4\ell}{(2\pi)^4}\left[-\frac{i\eta_{\delta\gamma}}{\ell^2}\right]\left[\frac{i\mathcal{P}_{\mu\nu\rho\sigma}}{(\ell+q)^2}\right]\bar{u}(p') \\ \times [\tau^{\rho\sigma}(p-\ell, p')D_F(p-\ell)\tau^{\delta}(p, p-\ell)] \\ \times u(p)\bar{u}(k')[\tau^{\mu\nu}(k+\ell, k')D_F(k+\ell) \\ \times \tau^{\gamma}(k, k+\ell)]u(k). \end{aligned} \tag{26}$$



(a) (b) (c) (d)

FIG. 2. The set of box diagrams contributing to the potential.

The methods and techniques to work with these diagrams are identical to those shown in [6]; we will repeat them briefly here.

The only difference lies in the fact that these diagrams require four different integrals that were not worked out previously. We have worked them out, and the coefficients can be obtained by contacting us; they are too tedious to be written down in the appendixes. Other than these integrals, these diagrams simply had to be worked out even though they involved enormous amounts of calculations. The level of difficulty is much higher than in the previous case, due to the reasons mentioned earlier. All the box diagrams have been calculated by using symbolic manipulation on a computer. These have partly been checked by hand. On the mass shell we will have the following types of identities:

$$\begin{aligned} \ell \cdot q &= \tfrac{1}{2}((\ell+q)^2 - q^2 - \ell^2), \\ \ell \cdot k &= \tfrac{1}{2}((\ell+k)^2 - m_1^2 - \ell^2), \\ \ell \cdot p &= -\tfrac{1}{2}((\ell-p)^2 - m_2^2 - \ell^2), \end{aligned} \tag{27}$$

so

$$\begin{aligned} q_\mu K^{\mu\nu} &= \int \frac{d^4\ell}{(2\pi)^4} \frac{(\ell\cdot q)\ell^\nu}{\ell^2(\ell+q)^2[(\ell+k)^2 - m_1^2][(\ell-p)^2 - m_2^2]} \\ &\to -\frac{q^2}{2}\int \frac{d^4\ell}{(2\pi)^4}\frac{\ell^\nu}{\ell^2(\ell+q)^2[(\ell+k)^2 - m_1^2][(\ell-p)^2 - m_2^2]} = -\frac{q^2}{2}K^\nu \end{aligned} \tag{28}$$

since the terms with $(\ell+q)^2$ and ℓ^2 simply do not contribute with nonanalytical results.

A more drastic reduction of the integrals takes place when the integral is contracted with the source momenta instead of the exchange momenta,

$$k_\mu K^{\mu\nu} = \int \frac{d^4\ell}{(2\pi)^4} \frac{(\ell\cdot k)\ell^\nu}{\ell^2(\ell+q)^2[(\ell+k)^2-m_1^2][(\ell-p)^2-m_2^2]} \to \frac{1}{2}\int \frac{d^4\ell}{(2\pi)^4} \frac{\ell^\nu}{\ell^2(\ell+q)^2[(\ell-p)^2-m_2^2]} = \frac{1}{2}I_p^\nu \quad (29)$$

or, in a similar manner,

$$p_\mu K^{\mu\nu} = \int \frac{d^4\ell}{(2\pi)^4} \frac{(\ell\cdot p)\ell^\nu}{\ell^2(\ell+q)^2[(\ell+k)^2-m_1^2][(\ell-p)^2-m_2^2]} \to -\frac{1}{2}\int \frac{d^4\ell}{(2\pi)^4} \frac{\ell^\nu}{\ell^2(\ell+q)^2[(\ell+k)^2-m_1^2]} = -\frac{1}{2}I_k^\nu, \quad (30)$$

where the subscripts k and p on the I's are written to indicate that the propagators left in the integrals are either from the particle with momentum k or p. Thus a loop momentum contracted with a source momentum simplifies our integrals considerably. The kinematics are (on the mass shell)

$$k\cdot q = p'\cdot q = \frac{q^2}{2}, \qquad k'\cdot q = p\cdot q = -\frac{q^2}{2}, \quad (31)$$

$$k\cdot k' = m_1^2 - \frac{q^2}{2}, \qquad p\cdot p' = m_2^2 - \frac{q^2}{2}. \quad (32)$$

The potential contribution from these diagrams are found to be

$$V(r)_{2(a)+2(b)+2(c)+2(d)} = \frac{3}{4}\frac{e_1e_2(m_1+m_2)G}{\pi r^2} - \frac{118}{48}\frac{Ge_1e_2}{\pi^2 r^3}. \quad (33)$$

These diagrams yield both a classical contribution—$\sim\frac{1}{r^2}$—and a quantum correction—$\sim\frac{1}{r^3}$.

2. The triangular diagrams

Diagrammatically, the triangular diagrams are given in Fig. 3; these are the only triangular diagrams that contribute with nonanalytic terms. We will again only consider one instance of these diagrams, namely, Fig. 3(a). Formally, it is written down as follows:

$$\begin{aligned} i\mathcal{M}_{3(a)} = \int \frac{d^4\ell}{(2\pi)^4}\bar{u}(p')[\tau^{\rho\sigma(\delta)}]u(p)\bar{u}(k') \\ \times[\tau^{\mu\nu}(-\ell-k,k')D_F(-\ell-k)\tau^\gamma(k,-\ell-k)] \\ \times u(k)\Big[-\frac{i\eta_{\delta\gamma}}{\ell^2}\Big]\Big[\frac{i\mathcal{P}_{\mu\nu\rho\sigma}}{(\ell+q)^2}\Big]. \end{aligned} \quad (34)$$

Upon applying contractions and insertion of the relevant integrals, whereafter Fourier transformations are performed, we end up with the potential contribution

$$V_{3(a)+3(b)+3(c)+3(d)}(r) = -\frac{9Ge_1e_2}{4\pi^2 r^3}. \quad (35)$$

3. The circular diagram

The circular diagram is depicted in Fig. 4. Formally, it can be written as

$$\begin{aligned} i\mathcal{M}_{4(a)} = \int \frac{d^4\ell}{(2\pi)^4}\bar{u}(p')[\tau^{\rho\sigma(\delta)}]u(p)\Big[-\frac{i\eta_{\delta\gamma}}{\ell^2}\Big] \\ \times\Big[\frac{i\mathcal{P}_{\mu\nu\rho\sigma}}{(\ell+q)^2}\Big]\bar{u}(k')[\tau^{\mu\nu(\gamma)}]u(k). \end{aligned} \quad (36)$$

After doing all the contractions and rearranging the γ matrices, one arrives at the result that the contribution to the potential from the circular loop diagram is precisely equal to nill,

$$V_{4(a)} = 0. \quad (37)$$

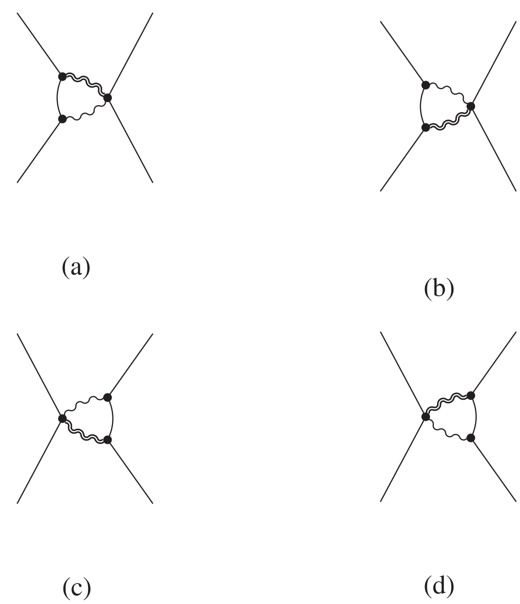


FIG. 3. The set of triangular diagrams contributing nonanalytically to the potential.

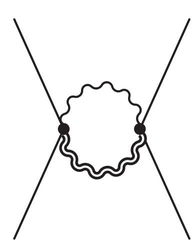


FIG. 4. The circular diagram with nonanalytic contributions.

D. The vertex correction diagrams

There are several sets of 1PR diagrams. All these are presented in this section.

1. The 1PR diagram

The first set of these 1PR diagrams is given in Fig. 5. These diagrams are the only ones corresponding to the gravitational vertex corrections. Again, we will only consider one instance of these diagrams. The matrix element originating from Fig. 5(a) is

$$\begin{aligned} i\mathcal{M}_{5(a)} = \int \frac{d^4\ell}{(2\pi)^4} &\bar{u}(p')[\tau^{\beta} D_F(p-\ell)\tau^{\alpha}\tau^{\rho\sigma(\gamma\delta)}(\ell, \ell+q)] \\ &\times u(p)\left[\frac{-i\eta_{\alpha\gamma}}{\ell^2}\right]\left[\frac{-i\eta_{\beta\delta}}{(\ell+q)^2}\right]\left[\frac{i\mathcal{P}_{\mu\nu\rho\sigma}}{q^2}\right] \\ &\times \bar{u}(k')[\tau^{\mu\nu}(k,k')]u(k) \end{aligned} \tag{38}$$

which yields the following potential when all the diagrams are summed,

$$V_{5(a)+5(b)}(r) = G\left(\frac{m_2 e_1^2 + m_1 e_2^2}{8\pi r^2} - \frac{e_1^2 \frac{m_2}{m_1} + e_2^2 \frac{m_1}{m_2}}{3\pi^2 r^3}\right). \tag{39}$$

This checks with [5] where it has also been calculated.

Of the next set of 1PR diagrams, depicted in Fig. 6, we will also only consider the first one. This is the first of the set of diagrams corresponding to the photonic vertex corrections. Formally, Fig. 6(a) is given by

$$\begin{aligned} i\mathcal{M}_{6(a)} = \int \frac{d^4\ell}{(2\pi)^4} &\bar{u}(p')[\tau^{\beta} D_F(p-\ell)\tau^{\mu\nu}(p, p-\ell)] \\ &\times u(p)\left[\frac{-i\eta_{\alpha\beta}}{(\ell+q)^2}\right]\left[\frac{i\mathcal{P}_{\rho\sigma\mu\nu}}{\ell^2}\right][\tau^{\rho\sigma(\delta\alpha)}(q, \ell+q)] \\ &\times \left[\frac{-i\eta_{\gamma\delta}}{q^2}\right]\bar{u}(k')[\tau^{\gamma}]u(k), \end{aligned} \tag{40}$$

giving the potential

$$V_{6(a)+6(b)+6(c)+6(d)}(r) = -\frac{3Ge_1e_2}{4\pi^2 r^3}. \tag{41}$$

The second set of diagrams corresponding to the photonic vertex corrections is given in Fig. 7. Formally,

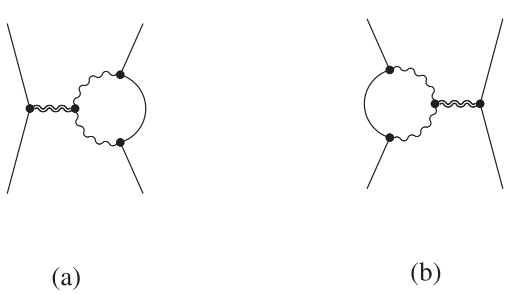


FIG. 5. The first set of 1PR diagrams contributing nonanalytically to the potential.

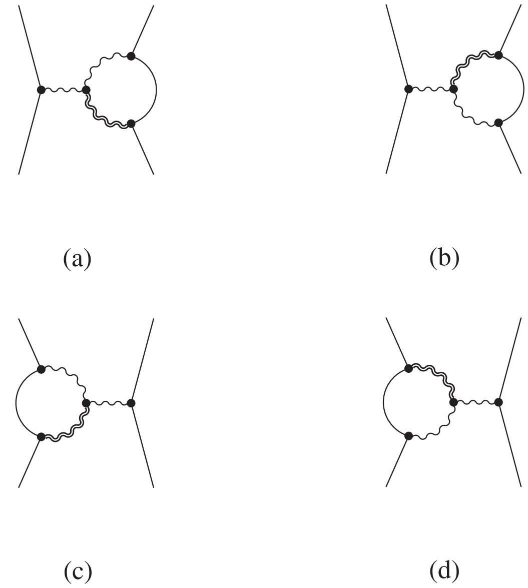


FIG. 6. The second set of 1PR diagrams contributing to the potential.

Fig. 7(a) is given by

$$\begin{aligned} i\mathcal{M}_{7(a)} = \int \frac{d^4\ell}{(2\pi)^4} &\tau^{\mu\nu(\delta\alpha)}(q, q+\ell)\left[\frac{i\mathcal{P}_{\mu\nu\sigma\rho}}{\ell^2}\right] \\ &\times \left[\frac{-i\eta_{\alpha\beta}}{(\ell+q)^2}\right]\bar{u}(p')[\tau^{\sigma\rho(\beta)}(p, p', e_2)]u(p) \\ &\times \left[\frac{-i\eta_{\gamma\delta}}{q^2}\right]\bar{u}(k')[\tau^{\gamma}]u(k), \end{aligned} \tag{42}$$

yielding the potential

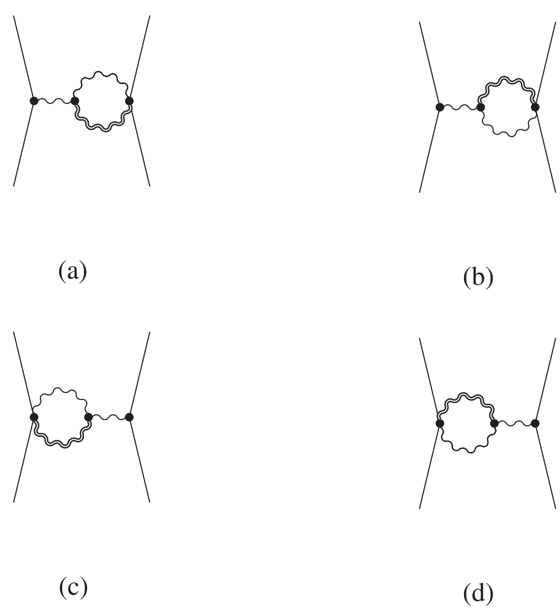


FIG. 7. The third set of 1PR diagrams contributing to the potential.

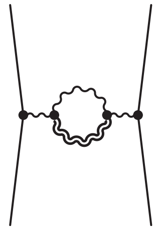


FIG. 8. The vacuum polarization diagram contribution to the nonrelativistic potential.

$$V_{7(a)+7(b)+7(c)+7(d)}(r) = \frac{3Ge_1e_2}{2\pi^2 r^3}. \tag{43}$$

2. The vacuum polarization diagram

The diagram is depicted in Fig. 8. The formal expression for this diagram is

$$\begin{aligned} i\mathcal{M}_{8(a)} = & \int \frac{d^4\ell}{(2\pi)^4}\bar{u}(p')[\tau^\gamma]u(p)\left[\frac{-i\eta_{\gamma\delta}}{q^2}\right]\tau^{\sigma\rho(\delta\alpha)}(q,-\ell) \\ & \times \left[\frac{-i\eta_{\alpha\beta}}{\ell^2}\right]\left[\frac{i\mathcal{P}_{\mu\nu\rho\sigma}}{(\ell+q)^2}\right]\tau^{\mu\nu(\beta\epsilon)}(-\ell,q)\left[\frac{-i\eta_{\phi\epsilon}}{q^2}\right] \\ & \times \bar{u}(k')[\tau^\phi]u(k). \end{aligned} \tag{44}$$

This is the only instance in these calculations that γ matrices are not explicitly involved. Simple index contractions are done and one obtains the potential contribution after going to the nonrelativistic limit

$$V_{8(a)}(r) = \frac{Ge_1e_2}{6\pi^2 r^3} \tag{45}$$

which is identical to the bosonic version.

V. THE RESULTS FOR THE POTENTIAL

When adding up all the separate contributions, we end up with

$$\begin{aligned} V(r) = & -\frac{Gm_1m_2}{r} + \frac{\tilde{\alpha}\tilde{e}_1\tilde{e}_2}{r} + \frac{1}{2}(m_2\tilde{e}_1^2 + m_1\tilde{e}_2^2)\frac{G\tilde{\alpha}}{c^2r^2} \\ & + 3\frac{\tilde{e}_1\tilde{e}_2(m_1+m_2)\tilde{\alpha}G}{c^2r^2} - \frac{4}{3}\frac{G\tilde{\alpha}\hbar}{\pi c^3 r^3}\left(e_1^2\frac{m_2}{m_1} + e_2^2\frac{m_1}{m_2}\right) \\ & - 15\frac{1}{6}\frac{G\tilde{\alpha}\hbar\tilde{e}_1\tilde{e}_2}{\pi r^3}, \end{aligned} \tag{46}$$

where we have included the appropriate physical factors of $\hbar$, c and we have further rescaled everything in terms of $\tilde{\alpha} = \frac{\hbar c}{137}$; lastly, $(\tilde{e}_1, \tilde{e}_2)$ are the normalized charges in units of elementary charge. The result is divided into three separate parts: the first two terms represent the Newtonian and Coulomb potentials; the next two terms represent the classical post-Newtonian corrections to the potential, which also can be found by pure classical treatment of general relativity with the inclusion of charged matter sources [24]. It is interesting to see that loop calculations also reproduce classical results, and not only quantum corrections. Finally, the last two terms are the leading 1-loop quantum corrections. We have, to a greater extent, reproduced the results of [6], except for the last quantum correction where we get the factor $(-15\frac{1}{6})$ instead of (-8) as in [6]. Other than the box diagrams, the remaining diagrams have been done by hand, and all the diagrams done in this paper have been checked by symbolic manipulation on a computer.

VI. DISCUSSION

We have examined the leading order quantum corrections to gravitational coupling of a spin-$\frac{1}{2}$ massive charged particle. Explicitly, we have extracted the nonanalytic terms from the diagrams, which exactly manifested themselves as corrections to the long-range forces; this was realized when we Fourier transformed these terms into coordinate space. These terms originated from the propagation of the massless particles, here the photons and gravitons. We have obtained similar results for most of the contributions to the corrections of the potentials, when compared with [6]. Only one of the leading quantum corrections *differs* from the bosonic calculation.

One could, in a similar manner, do a QED-pure gravity scattering calculation as in [8]. However, difficult problems would appear. First of all, one would have to find many new vertex rules involving the antisymmetric fields. Moreover, we would have to derive second order Lagrangian densities in terms of both the symmetric and antisymmetric fields. Indeed, in this direction there lies a considerably interesting project ahead.

ACKNOWLEDGMENTS

I would like to thank Poul Henrik Damgaard for his supervision and guidance and for many suggestions. I would also like to thank N. E. J. Bjerrum-Bohr for invaluable discussions, and also for supervision and guidance during the initial stage of this work.

APPENDIX A: THE METRIC FIELD AND THE VIERBEIN FIELDS

In this section we will briefly give the most essential ingredients needed to expand the Lagrangians and derive the vertex rules presented in this paper. The following relations between the vierbein fields exist:

$$e^a{}_\mu e_b{}^\mu = \delta^a_b, \tag{A1a}$$

$$e_a{}^\mu e^a{}_\nu = \delta^\mu_\nu, \tag{A1b}$$

and the metric in terms of the vierbein is

$$g_{\mu\nu} = e^a{}_\mu e^b{}_\nu \eta_{ab}. \tag{A2}$$

The generators of the Lorentz group are $\sigma_{ab} = \frac{1}{4}[\gamma_a, \gamma_b]$

for spinors belonging to a given representation $S(\Lambda) = e^{1/2\lambda^{ab}\sigma_{ab}}$; the covariant derivative for spinor fields is

$$D_\mu \psi \equiv (\partial_\mu + \tfrac{1}{2}\sigma_{ab}\omega_\mu^{ab})\psi, \tag{A3}$$

where the spin connection is given by

$$\omega_\mu^{ab} = \tfrac{1}{2}(e^{[a\nu}\partial_{[\mu}e^{b]}{}_{\nu]} + e^{a\rho}e^{b\sigma}\partial_{[\sigma}e_{c\rho]}e^c{}_\mu). \tag{A4}$$

Note that Latin letters commute with Latin letters and Greek letters commute with Greek letters.

The metric and the vierbein fields are expanded into two separate contributions, a classical background field and a quantum field,

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \kappa h_{\mu\nu}, \tag{A5}$$

$$e^a{}_\mu = \bar{e}^a{}_\mu + \kappa c^a{}_\mu, \tag{A6}$$

where $\kappa^2 = 32\pi G$ and the background fields are denoted as $\bar{g}_{\mu\nu}$ and $\bar{e}^a{}_\mu$. The quantum part—the graviton field—is denoted by $h_{\mu\nu}$ and $c^a{}_\mu$, the sum of these being the full metric and vierbein, respectively. The following inverses and other relations are deduced:

$$e^{a\mu} = \bar{e}^{a\mu} - \kappa c^{\mu a} + \ldots \tag{A7}$$

for the vierbeins, and

$$\begin{aligned} g_{\mu\nu} &= \bar{g}_{\mu\nu} + \kappa h_{\mu\nu} = \bar{g}_{\mu\nu} + \kappa(c_{\mu\nu} + c_{\nu\mu}) + \kappa^2 c_{a\mu}c^a{}_\nu, \\ g^{\mu\nu} &= \bar{g}^{\mu\nu} - \kappa h^{\nu\mu} + \ldots = \bar{g}^{\mu\nu} - \kappa(c^{\mu\nu} + c^{\nu\mu}) + \ldots \end{aligned} \tag{A8}$$

for the metric. From these relations we see that the metric and vierbein quantum fields are related according to

$$h_{\mu\nu} = c_{\mu\nu} + c_{\nu\mu} + \kappa c_{a\mu}c^a{}_\nu = s_{\mu\nu} + \kappa c_{a\mu}c^a{}_\nu, \tag{A9}$$

showing us that the quantized metric field is equal to the quantized symmetric vierbein field to first order in the quantum fields, i.e. $h_{\mu\nu} = c_{\mu\nu} + c_{\nu\mu} \equiv s_{\mu\nu}$. The determinants of the vierbein and metric fields are expanded into

$$e = \det[e^a{}_\mu] \approx \tilde{e}(1 + \kappa c^\alpha{}_\alpha + \ldots) \tag{A10}$$

with $\tilde{e} \equiv \det \bar{e}^a{}_\alpha$, as well as the square root of the metric tensor $\sqrt{-g} = \sqrt{-\det(g_{\mu\nu})}$,

$$\sqrt{-g} \approx \sqrt{-\bar{g}}\left(1 + \frac{\kappa}{2}h_\alpha^\alpha + \ldots\right) \tag{A11}$$

with $\bar{g} \equiv \det \bar{g}_{\mu\alpha}$. Finally, when expanding the spin connection we get

$$\omega_{\mu ab}^{\text{background}} = 0, \tag{A12a}$$

$$\omega_{\mu ab}^{\text{first order}} = \tfrac{1}{2}\partial_\mu a_{ba} + \tfrac{1}{2}\partial_b s_{a\mu} - \tfrac{1}{2}\partial_a s_{b\mu}, \tag{A12b}$$

where we have defined a new field, an antisymmetric field $a_{\mu\nu} = c_{\mu\nu} - c_{\nu\mu}$.

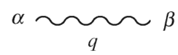
APPENDIX B: FEYNMAN RULES

1. Propagators

The relevant propagators are presented in this section.

a. Photon propagator

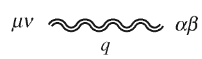
The photon propagator is no stranger in QFT. In Feynman gauge the propagator becomes



$$= \frac{-i\eta_{\alpha\beta}}{q^2 + i\epsilon}$$

b. Graviton propagator

The graviton propagator is perhaps a stranger. However, it has been worked out in several places. In the harmonic gauge we get the following for the graviton propagator:



$$= \frac{i\mathcal{P}_{\mu\nu\alpha\beta}}{q^2 + i\epsilon}$$

with the projection operator

$$\mathcal{P}_{\mu\nu\alpha\beta} = \frac{\frac{1}{2}(\eta_{\alpha\{\mu}\eta_{\nu\}\beta} - \eta_{\mu\nu}\eta_{\alpha\beta})}{q^2 + i\epsilon}.$$

c. Fermion propagator

The fermion propagator can be found in many places in literature; it is very well known,

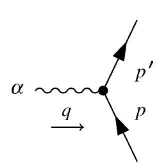
$$= \frac{i}{(\not{k} - m)} = \frac{i(\not{k} + m)}{k^2 - m^2}$$

2. Vertices

The vertices are presented here. They are all derived in [25]. For all vertices, the rules of momentum conservation has been applied.

a. 1-photon-2-fermion vertex

The 1-photon-2-fermion vertex can also be looked up in the literature; it is worked out to be



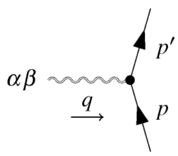
$$= \tau^\alpha(p, p')$$

with

$$\tau^\alpha(p, p') = ie_q\gamma^\alpha.$$

b. 1-graviton-2-fermion vertex

The 1-graviton-2-fermion vertex is found to be



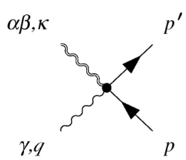
$$= \quad \tau^{\alpha\beta}(p, p')$$

where

$$\tau^{\alpha\beta}(p, p') = \frac{i\kappa}{2}\left[\eta^{\alpha\beta}\left(\frac{1}{2}(\not{p} + \not{p}') - m\right) - \frac{1}{4}\gamma^{\{\alpha}(p + p')^{\beta\}}\right].$$

c. 1-photon-1-graviton-2-fermion vertex

The 1-photon-1-graviton-2-fermion vertex is not known from the literature; however, it is found to be



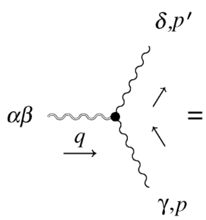
$$= \quad \tau^{\alpha\beta(\gamma)}(p, p')$$

where

$$\tau^{\alpha\beta(\gamma)}(p, p') = \frac{i\kappa e_q}{4}\gamma_a(2\eta^{\gamma a}\eta^{\alpha\beta} - \eta^{\gamma\{\alpha}\eta^{\beta\}a}).$$

d. 1-graviton-2-photon vertex

We have derived the following for the 1-graviton-2-photon vertex:



$$= \quad \tau^{\alpha\beta(\gamma\delta)}(p, p')$$

where

$$\begin{aligned}\tau^{\alpha\beta(\gamma\delta)}(p, p') = i\kappa[&\mathcal{P}^{\alpha\beta(\gamma\delta)}(p \cdot p') \\ &+ \tfrac{1}{2}(\eta^{\alpha\beta}p^{\delta}p'^{\gamma} + \eta^{\gamma\delta}p^{\{\beta}p'^{\alpha\}} \\ &- p^{\delta}p'^{\{\alpha}\eta^{\beta\}\gamma} - p'^{\gamma}p^{\{\alpha}\eta^{\beta\}\delta})].\end{aligned}$$

$\mathcal{P}^{\alpha\beta(\gamma\delta)}$ is defined as above.

APPENDIX C: TABLE OF RELEVANT INTEGRALS

The following integrals are needed; note that in these integrals only the lowest order nonanalytical terms are presented:

$$J = \int \frac{d^4\ell}{(2\pi)^4}\frac{1}{\ell^2(\ell + q)^2} = \frac{i}{32\pi^2}[-2L] + \ldots, \tag{C1}$$

$$J_\mu = \int \frac{d^4\ell}{(2\pi)^4}\frac{\ell_\mu}{\ell^2(\ell + q)^2} = \frac{i}{32\pi^2}[q_\mu L] + \ldots, \tag{C2}$$

$$\begin{aligned}J_{\mu\nu} &= \int \frac{d^4\ell}{(2\pi)^4}\frac{\ell_\mu\ell_\nu}{\ell^2(\ell + q)^2} \\ &= \frac{i}{32\pi^2}\left[q_\mu q_\nu\left(-\frac{2}{3}L\right) - q^2\eta_{\mu\nu}\left(-\frac{1}{6}L\right)\right] + \ldots,\end{aligned} \tag{C3}$$

together with

$$\begin{aligned}I &= \int \frac{d^4\ell}{(2\pi)^4}\frac{1}{\ell^2(\ell + q)^2((\ell + k)^2 - m^2)} \\ &= \frac{i}{32\pi^2 m^2}[-L - S] + \ldots,\end{aligned} \tag{C4}$$

$$\begin{aligned}I_\mu &= \int \frac{d^4\ell}{(2\pi)^4}\frac{\ell_\mu}{\ell^2(\ell + q)^2((\ell + k)^2 - m^2)} \\ &= \frac{i}{32\pi^2 m^2}\left[k_\mu\left(\left(-1 - \frac{1}{2}\frac{q^2}{m^2}\right)L - \frac{1}{4}\frac{q^2}{m^2}S\right)\right. \\ &\quad \left. + q_\mu\left(L + \frac{1}{2}S\right)\right] + \ldots,\end{aligned} \tag{C5}$$

$$\begin{aligned}I_{\mu\nu} &= \int \frac{d^4\ell}{(2\pi)^4}\frac{\ell_\mu\ell_\nu}{\ell^2(\ell + q)^2((\ell + k)^2 - m^2)} \\ &= \frac{i}{32\pi^2 m^2}\left[q_\mu q_\nu\left(-L - \frac{3}{8}S\right)\right. \\ &\quad + k_\mu k_\nu\left(-\frac{1}{2}\frac{q^2}{m^2}L - \frac{1}{8}\frac{q^2}{m^2}S\right) + (q_\mu k_\nu + q_\nu k_\mu) \\ &\quad \times\left(\left(\frac{1}{2} + \frac{1}{2}\frac{q^2}{m^2}\right)L + \frac{3}{16}\frac{q^2}{m^2}S\right) \\ &\quad \left. + q^2\eta_{\mu\nu}\left(\frac{1}{4}L + \frac{1}{8}S\right)\right] + \ldots,\end{aligned} \tag{C6}$$

$$\begin{aligned} I_{\mu\nu\alpha} &= \int \frac{d^4\ell}{(2\pi)^4} \frac{\ell_\mu \ell_\nu \ell_\alpha}{\ell^2(\ell+q)^2((\ell+k)^2-m^2)} \\ &= \frac{i}{32\pi^2 m^2}\bigg[q_\mu q_\nu q_\alpha\Big(L+\frac{5}{16}S\Big) + k_\mu k_\nu k_\alpha\Big(-\frac{1}{6}\frac{q^2}{m^2}L\Big) + (q_\mu k_\nu k_\alpha + q_\nu k_\mu k_\alpha + q_\alpha k_\mu k_\nu)\Big(\frac{1}{3}\frac{q^2}{m^2}L + \frac{1}{16}\frac{q^2}{m^2}S\Big) \\ &\quad + (q_\mu q_\nu k_\alpha + q_\mu q_\alpha k_\nu + q_\nu q_\alpha k_\mu)\Big(\Big(-\frac{1}{3}-\frac{1}{2}\frac{q^2}{m^2}\Big)L - \frac{5}{32}\frac{q^2}{m^2}S\Big) + (\eta_{\mu\nu}k_\alpha + \eta_{\mu\alpha}k_\nu + \eta_{\nu\alpha}k_\mu)\Big(\frac{1}{12}q^2 L\Big) \\ &\quad + (\eta_{\mu\nu}q_\alpha + \eta_{\mu\alpha}q_\nu + \eta_{\nu\alpha}q_\mu)\Big(-\frac{1}{6}q^2 L - \frac{1}{16}q^2 S\Big)\bigg] + \ldots, \end{aligned} \tag{C7}$$

where $S = \frac{\pi^2 m}{\sqrt{-q^2}}$ and $L = \ln(-q^2)$. The ellipses denote higher order nonanalytical contributions as well as the neglected analytical terms. Please note that there seems to be a typo in [6]; in $I_{\mu\nu\alpha}$ the factor after $(k_\mu k_\nu k_\alpha)$ is lacking an L. Other than this typo, all the integrals have been checked and are agreed upon.

The following integrals are needed to do the box diagrams.

$$\begin{aligned} K &= \int \frac{d^4\ell}{(2\pi)^4} \frac{1}{\ell^2(\ell+q)^2((\ell+k)^2-m_1^2)((\ell-p)^2-m_2^2)} \\ &= \frac{i}{16\pi^2 m_1 m_2 q^2}\bigg[\Big(1-\frac{w}{3m_1m_2}\Big)L\bigg] + \ldots, \end{aligned} \tag{C8}$$

$$K' = \int \frac{d^4\ell}{(2\pi)^4} \frac{1}{\ell^2(\ell+q)^2((\ell+k)^2-m_1^2)((\ell+p')^2-m_2^2)} = \frac{i}{16\pi^2 m_1 m_2 q^2}\bigg[\Big(-1+\frac{W}{3m_1m_2}\Big)L\bigg] + \ldots, \tag{C9}$$

$$K^\mu = \int \frac{d^4\ell}{(2\pi)^4} \frac{\ell^\mu}{\ell^2(\ell+q)^2[(\ell+k)^2-m_1^2][(\ell-p)^2-m_2^2]} = \alpha q^\mu + \beta k^\mu + \gamma p^\mu, \tag{C10}$$

$$K'^\mu = \int \frac{d^4\ell}{(2\pi)^4} \frac{\ell^\mu}{\ell^2(\ell+q)^2[(\ell+k)^2-m_1^2][(\ell+p')^2-m_2^2]} = \alpha' q^\mu + \beta' k^\mu + \gamma' p'^\mu, \tag{C11}$$

$$\begin{aligned} K^{\mu\nu} &= \int \frac{d^4\ell}{(2\pi)^4} \frac{\ell^\mu \ell^\nu}{\ell^2(\ell+q)^2[(\ell+k)^2-m_1^2][(\ell-p)^2-m_2^2]} \\ &= [q_\mu q_\nu a + k_\mu k_\nu b + p_\mu p_\nu c] + [(q_\mu k_\nu + q_\nu k_\mu)d + (q_\mu p_\nu + q_\nu p_\mu)e] + (p_\mu k_\nu + p_\nu k_\mu)f + \eta_{\mu\nu}q^2 g, \end{aligned} \tag{C12}$$

$$\begin{aligned} K'^{\mu\nu} &= \int \frac{d^4\ell}{(2\pi)^4} \frac{\ell^\mu \ell^\nu}{\ell^2(\ell+q)^2[(\ell+k)^2-m_1^2][(\ell+p')^2-m_2^2]} \\ &= [q_\mu q_\nu a' + k_\mu k_\nu b' + p'_\mu p'_\nu c'] + (q_\mu k_\nu + q_\nu k_\mu)d' + (q_\mu p'_\nu + q_\nu p'_\mu)e' + (p'_\mu k_\nu + p'_\nu k_\mu)f' + \eta_{\mu\nu}q^2 g'. \end{aligned} \tag{C13}$$

Here we have defined $w = (k\cdot p) - m_1 m_2$ and $W = (k\cdot p') - m_1 m_2$. From these, we can deduce an important relation that becomes vital during the calculations, namely, $W - w = k\cdot(p'-p) = (k\cdot q) = \frac{-q^2}{2}$. The w and W are displayed here only for the K and K' integrals; for the rest of the integrals we have used $W = (k\cdot p) - m_1 m_2 - \frac{q^2}{2}$ (see [25] for derivations). The coefficients to these are long and tedious to write down properly; however, if required, they can be obtained by contacting us.

For the above integrals the following constraints for the nonanalytical terms can be verified directly on the mass shell. We will use ${}_kI \sim \frac{1}{\ell^2(\ell+q)^2[(\ell+k)^2-m_1^2]}$ for k, ${}_pI \sim \frac{1}{\ell^2(\ell+q)^2[(\ell-p)^2-m_2^2]}$ for p, ${}_{p'}I \sim \frac{1}{\ell^2(\ell+q)^2[(\ell+p')^2-m_2^2]}$ for p', and no particular choice is needed for contractions with q:

$$K_{\mu\nu}\eta^{\mu\nu} = K'_{\mu\nu}\eta^{\mu\nu} = I_{\mu\nu\alpha}\eta^{\mu\nu} = I_{\mu\nu}\eta^{\mu\nu} = J_{\mu\nu}\eta^{\mu\nu} = 0, \qquad K_{\mu\nu}q^{\mu} = -\frac{q^2}{2}K_{\nu}, \qquad K_{\mu}q^{\mu} = -\frac{q^2}{2}K,$$

$$K'_{\mu\nu}q^{\mu} = -\frac{q^2}{2}K'_{\nu}, \qquad K'_{\nu}q^{\nu} = -\frac{q^2}{2}K', \qquad K_{\mu\nu}p^{\mu} = -\frac{1}{2}{}_kI_{\nu}, \qquad K_{\mu}p^{\mu} = -\frac{1}{2}{}_kI, \qquad K_{\mu\nu}k^{\mu} = \frac{1}{2}{}_pI_{\nu},$$

$$K_{\nu}k^{\nu} = \frac{1}{2}{}_pI, \qquad K'_{\mu\nu}p'^{\mu} = \frac{1}{2}{}_kI_{\nu}, \qquad K'_{\nu}p'^{\nu} = \frac{1}{2}{}_kI, \qquad K'_{\mu\nu}k^{\mu} = \frac{1}{2}{}_{p'}I_{\nu}, \qquad K'_{\nu}k^{\nu} = \frac{1}{2}{}_{p'}I, \qquad I_{\mu\nu\alpha}q^{\alpha} = -\frac{q^2}{2}I_{\mu\nu},$$

$$I_{\mu\nu}q^{\nu} = -\frac{q^2}{2}I_{\mu}, \qquad I_{\mu}q^{\mu} = -\frac{q^2}{2}I, \qquad J_{\mu\nu}q^{\nu} = -\frac{q^2}{2}J_{\mu}, \qquad J_{\mu}q^{\mu} = -\frac{q^2}{2}J, \qquad {}_kI_{\mu\nu\alpha}k^{\alpha} = \frac{1}{2}J_{\mu\nu},$$

$${}_kI_{\mu\nu}k^{\nu} = \frac{1}{2}J_{\mu}, \qquad {}_kI_{\mu}k^{\mu} = \frac{1}{2}J, \qquad {}_pI_{\mu\nu\alpha}p^{\alpha} = -\frac{1}{2}J_{\mu\nu}, \qquad {}_pI_{\mu\nu}p^{\nu} = -\frac{1}{2}J_{\mu}, \qquad {}_pI_{\mu}p^{\mu} = -\frac{1}{2}J.$$

There seems to be a typo in [6]; the metric in $I_{\mu\nu\alpha}\eta^{\mu\nu}$ is written as $I_{\mu\nu\alpha}\eta^{\alpha\beta}$.

[1] J. F. Donoghue, gr-qc/9512024; Phys. Rev. D **50**, 3874 (1994); Phys. Rev. Lett. **72**, 2996 (1994).
[2] S. Weinberg, Physica A (Amsterdam) **96**, 327 (1979).
[3] B. S. De Witt, Phys. Rev. **160**, 1113 (1967); **162**, 1195 (1967); **162**, 1239 (1967).
[4] L. F. Abbott, Nucl. Phys. **B185**, 189 (1981).
[5] J. F. Donoghue, B. R. Holstein, B. Garbrecht, and T. Konstandin, Phys. Lett. B **529**, 132 (2002).
[6] N. E. J. Bjerrum-Bohr, Phys. Rev. D **66**, 084023 (2002).
[7] N. E. J. Bjerrum-Bohr, J. F. Donoghue, and B. R. Holstein, Phys. Rev. D **68**, 084005 (2003); **71**, 069904(E) (2005).
[8] N. E. J. Bjerrum-Bohr, J. F. Donoghue, and B. R. Holstein, Phys. Rev. D **67**, 084033 (2003); **71**, 069903(E) (2005).
[9] S. Deser and P. van Nieuwenhuizen, Phys. Rev. Lett. **32**, 245 (1974); Phys. Rev. D **10**, 401 (1974); **10**, 411 (1974).
[10] F. Berends and R. Gastmans, Phys. Lett. **55B**, 311 (1975).
[11] M. J. G. Veltman, in *Proceedings of Les Houches, Session XXVIII, 1975*, edited by R. Balian and J. Zinn-Justin (North Holland Publishing Company, Amsterdam, 1976).
[12] G. t' Hooft and M. J. G. Veltman, Ann. Poincare Phys. Theor. A **20**, 69 (1974).
[13] José F. Nieves and Palash B. Pal Phys. Rev. D **58**, 096005 (1998).
[14] José F. Nieves and Palash B. Pal Phys. Rev. D **64**, 085004 (2001).
[15] H. Weyl, Z. Phys. **56**, 330 (1929).
[16] E. Cartan, Lecons sur la Théorie des Spineurs (Paris, 1938) [The Theory of Spinors (Hermann, Paris, 1966)].
[17] R. Utiyama, Phys. Rev. **101**, 1597 (1956).
[18] N. D. Birrell and P. C. W. Davies, *Quantum Fields in Curved Space* (Cambridge University Press, Cambridge, England, 1982).
[19] Kimball A. Milton, Phys. Rev. D **15**, 538 (1977).
[20] The q is attached to distinguish the charges from the vierbeins.
[21] S. Y. Choi, J. S. Shim, and H. S. Song, Phys. Rev. D **51**, 2751 (1995).
[22] H. W. Hamber and S. Liu, Phys. Lett. B **357**, 51 (1995).
[23] K. A. Kazakov, Phys. Rev. D **63**, 044004 (2001).
[24] B. M. Barker and R. F. O'Connell, J. Math. Phys. (N.Y.) **18**, 1818 (1977); **19**, 1231(E) (1978).
[25] M. S. Butt, M. Sc. thesis, University of Copenhagen, 2005.