

General computations without fixing the gauge

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Within the framework of a manifestly gauge invariant exact renormalization group for $SU(N)$ Yang-Mills, we derive a simple expression for the expectation value of an arbitrary gauge invariant operator. We illustrate the use of this formula by computing the $O(g^2)$ correction to the rectangular, Euclidean Wilson loop with sides $T \gg L$. The standard result is trivially obtained, directly in the continuum, for the first time without fixing the gauge. We comment on possible future applications of the formalism.

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I. INTRODUCTION

The necessity of gauge fixing in order to compute in Yang-Mills theories is, in common lore, practically taken for granted and, for perturbative calculations, generally considered obligatory. This point of view is lent considerable weight both by Feynman's unitarity argument for the existence of Faddeev-Popov ghosts [1] and by the elegance and power of the resulting BRST symmetries [2].

However, there is no reason in principle why gauge invariant quantities, as opposed to Green's functions in gauge fixed formulations, cannot be computed in a manifestly gauge invariant manner. Indeed, in a nonperturbative context, this is routinely exploited on the lattice, where calculations can be performed without gauge fixing. Excitingly, in a series of works [3–20], a formalism has been developed which allows manifestly gauge invariant computations to be performed directly in the continuum.

The benefits of this scheme are numerous. The gauge field is protected from field strength renormalization and the Ward identities take a particularly simple form since the Wilsonian effective action is built only from gauge invariant combinations of the covariant derivative, even at the quantum level [5]. In the nonperturbative domain, the difficult technical issue of Gribov copies [21] is entirely avoided. Furthermore, it should be possible to make statements about phenomena such as confinement in a completely gauge independent manner, and it is surely this which gives a manifestly gauge invariant scheme much of its appeal.

The framework developed in [3–20] is based on the exact renormalization group (ERG), the continuum version of Wilson's RG [22–24]. The essential physical idea behind this approach is that of integrating out degrees of freedom between the bare scale of the quantum theory and some effective scale, Λ . The effects of these modes are encoded in the Wilsonian effective action, S_Λ , which describes the physics of the theory in terms of parameters relevant to the effective scale.

The possibility of constructing manifestly gauge invariant ERGs arises, fundamentally, from the huge freedom inherent in the approach [25]. For any given quantum field theory, there are an infinite number of ERGs corresponding to the infinite number of different ways in which the high energy degrees of freedom can be averaged over (the continuum version of blocking on the lattice) [8,16,25]. In Yang-Mills theory, an infinite subset of these schemes allows the computation of the gauge invariant Wilsonian effective action, without fixing the gauge.¹

Central to the ERG methodology is the ERG equation, which determines how the Wilsonian effective action changes under infinitesimal changes of the scale. Part of the reason for the considerable amount of work put into adapting the ERG for Yang-Mills (see [26] for a summary of the various approaches) is that the ERG equation, by relating physics at different scales, provides access to the low energy dynamics of the theory. Indeed, more generally, the ERG has proven itself to be a flexible and powerful tool for studying both perturbative and nonperturbative problems in a range of field theories (see [27–35] for reviews). A particular advantage conferred by the ERG is that renormalization is built in: solutions to the flow equation (in pretty much any approximation scheme), from which physics can be extracted, are naturally phrased directly in terms of renormalized parameters. It is thus clear that a manifestly gauge invariant formalism, based on the ERG, has considerable potential. Furthermore, an interesting link between this formalism and the AdS/CFT correspondence has recently been made [36].

The majority of the work into the scheme employed in this paper has focused on constructing and testing the formalism, culminating in the successful reproduction of the $SU(N)$ Yang-Mills two-loop β -function [15,17]. Subsequent to this, the powerful diagrammatic techniques developed to facilitate this calculation have been refined and applied in the context of β -function coefficients at arbitrary loop order [18–20]. These substantial works have

*Electronic address: orosten@stp.dias.ie¹In practice we further specialize to those ERGs which allow convenient renormalization to any loop order [15–17].

paved the way for more general application of the formalism; in this paper, we describe how to compute the expectation values of gauge invariant operators, without fixing the gauge, and illustrate the formalism with a very simple computation of the $O(g^2)$ correction to the rectangular, Euclidean Wilson loop with sides $T \gg L$. [There have been attempts to compute the perturbative corrections to this Wilson loop in (gauge fixed) ERG studies, in the past. In particular, using the axial gauge flow equation proposed by [37], it was found in [38] that, while the $O(g^2)$ result could be correctly reproduced, the formalism failed at $O(g^4)$. However, the flow equation of [37] is not a flow equation in the Wilsonian sense: the implementation of the cutoff, Λ , is not sufficient to regularize the theory, and dimensional regularization has to be employed as well. This is the reason for the negative result of [38]; as recognized in [39], properly Wilsonian axial gauge flow equations can be constructed, which work perfectly well. In the formalism used in this paper, the above issues never arise, since the implementation of the (gauge invariant) cutoff is sufficient to regularize the theory, as proven in [9].]

The outline of this paper is as follows. In Sec. II we review the setup of our manifestly gauge invariant ERG. Section III is devoted to the methodology for computing the expectation of gauge invariant operators. The basic idea, for which little prior knowledge is required, is detailed in the short Sec. III A. In the remainder of Sec. III, the machinery for performing calculations in perturbation theory is developed. This section concludes with a fantastically compact diagrammatic expression for the perturba-

tive corrections to the expectation value of any gauge invariant operator. In Sec. IV, we specialize to the computation of the expectation values of (renormalized) Wilson loops. After covering some general features in Sec. IVA, in Sec. IVB we compute the $O(g^2)$ correction to the Euclidean, rectangular Wilson loop with sides $T \gg L$ and recover the standard result. We conclude in Sec. V.

II. REVIEW

A. Elements of $SU(N|N)$ gauge theory

Throughout this paper, we work in Euclidean dimension, D . We regularize $SU(N)$ Yang-Mills, carried by the physical gauge field A_μ^1 , by embedding it in spontaneously broken $SU(N|N)$ Yang-Mills, which is itself regularized by covariant higher derivatives [9]. The massive gauge fields arising from the spontaneous symmetry breaking play the role of gauge invariant Pauli-Villars (PV) fields, furnishing the necessary extra regularization to supplement the covariant higher derivatives. In order to unambiguously define contributions which are finite only by virtue of the PV regularization, a preregulator must be used in $D = 4$ [9]. We will use dimensional regularization, emphasizing that this makes sense nonperturbatively, since it is not being used to renormalize the theory, but rather as a prescription for discarding surface terms in loop integrals [9].

The supergauge invariant Wilsonian effective action has an expansion in terms of supertraces and products of supertraces [11]:

$$\begin{aligned}
 S = & \sum_{n=1}^{\infty} \frac{1}{s_n} \int d^D x_1 \cdots d^D x_n S_{a_1 \cdots a_n}^{X_1 \cdots X_n}(x_1, \dots, x_n) \text{str} X_1^{a_1}(x_1) \cdots X_n^{a_n}(x_n) \\
 & + \frac{1}{2!} \sum_{m,n=0}^{\infty} \frac{1}{s_n s_m} \int d^D x_1 \cdots d^D x_n d^D y_1 \cdots d^D y_m S_{a_1 \cdots a_n, b_1 \cdots b_m}^{X_1 \cdots X_n, Y_1 \cdots Y_m}(x_1, \dots, x_n; y_1 \cdots y_m) \text{str} X_1^{a_1}(x_1) \cdots X_n^{a_n}(x_n) \\
 & \times \text{str} Y_1^{b_1}(y_1) \cdots Y_m^{b_m}(y_m) + \cdots,
 \end{aligned} \tag{2.1}$$

where the $X_i^{a_i}$ and $Y_j^{b_j}$ are embeddings of broken phase fields into supermatrices. We take only one cyclic ordering for the lists $X_1 \cdots X_n$, $Y_1 \cdots Y_m$ in the sums over n , m . If any term is invariant under some nontrivial cyclic permutations of its arguments, then s_n (s_m) is the order of the cyclic subgroup, otherwise $s_n = 1$ ($s_m = 1$).

The momentum space vertices are written

$$\begin{aligned}
 & S_{a_1 \cdots a_n}^{X_1 \cdots X_n}(p_1, \dots, p_n) (2\pi)^D \delta\left(\sum_{i=1}^n p_i\right) \\
 & = \int d^D x_1 \cdots d^D x_n e^{-i \sum_i x_i \cdot p_i} S_{a_1 \cdots a_n}^{X_1 \cdots X_n}(x_1, \dots, x_n),
 \end{aligned}$$

where all momenta are taken to point into the vertex. We will employ the shorthand

$$S_{a_1 a_2}^{X_1 X_2}(p) \equiv S_{a_1 a_2}^{X_1 X_2}(p, -p).$$

In addition to the coupling, g , of the physical gauge field, there is a second dimensionless coupling, g_2 , associated with one of the unphysical regulator fields, A_μ^2 [11,15–17]. For convenience, we work not with g_2 directly but with

$$\alpha \equiv g_2^2/g^2. \tag{2.2}$$

The coupling g (similarly α) is defined through its renormalization condition:

$$S[A^1] = \frac{1}{2g^2} \text{tr} \int d^D x (F_{\mu\nu}^1)^2 + \cdots, \tag{2.3}$$

where the ellipses stand for higher dimension operators and

the ignored vacuum energy. Equation (2.3) constrains the classical two-point vertex of the physical field, $S_{0\mu\nu}^{A^1 A^1}(p) \equiv S_{0\mu\nu}^{11}(p)$, as follows:

$$\begin{aligned} S_{0\mu\nu}^{11}(p) &= 2(p^2 \delta_{\mu\nu} - p_\mu p_\nu) + O(p^4) \\ &\equiv 2\Box_{\mu\nu}(p) + O(p^4). \end{aligned} \quad (2.4)$$

B. Diagrammatics

In this section, we introduce and describe the diagrammatics necessary for this paper. For a comprehensive description of the diagrammatics, see [15,16].

1. Diagrammatics for the action

The *vertex coefficient functions* belonging to the action (2.1) have a simple diagrammatic representation:

$$\left[\textcircled{S} \right]^{\{f\}} \equiv \textcircled{S} \text{ with } \{f\} \text{ fields attached} \quad (2.5)$$

represents all vertex coefficient functions corresponding to all cyclically independent orderings of the set of broken phase fields, $\{f\}$, distributed over all possible supertrace structures. For example,

$$\left[\textcircled{S} \right]^{A^1 A^1} \quad (2.6)$$

represents the coefficient functions $S^{A^1 A^1}$ which, from (2.1), is associated with the (super)trace structure $\text{tr} A^1 A^1$. This diagram would also represent the coefficient function S^{A^1, A^1} , were it not for the fact that this does not exist, on account of $\text{tr} A^1 = 0$.

2. Diagrammatics for the exact flow equation

The diagrammatic representation of the flow equation is shown in Fig. 1 [15,16].

The left-hand side just depicts the flow of all cyclically independent Wilsonian effective action vertex coefficient functions. The objects on the right-hand side of Fig. 1 have two different types of component. The lobes represent vertices of action functionals, where $\Sigma_g \equiv g^2 S - 2\hat{S}$, \hat{S} being the seed action [10–12,14–17]: a functional which respects the same symmetries as the Wilsonian effective action, S , and has the same structure. Physically, the seed

$$-\Lambda \partial_\Lambda \left[\textcircled{S} \right]^{\{f\}} = \frac{1}{2} \left[\begin{array}{c} \textcircled{\Sigma_g} \\ \bullet \\ \textcircled{S} \end{array} - \textcircled{\Sigma_g} \right]^{\{f\}}$$

FIG. 1. The diagrammatic form of the flow equation.

action can be thought of as (partially) parametrizing a general Kadanoff blocking in the continuum [16,25].

The object attaching to the various lobes, \bullet , is the sum over vertices of the covariantized ERG kernels [5,11] and, like the action vertices, can be decorated by fields belonging to $\{f\}$. The fields of the action vertex (vertices) to which the vertices of the kernels attach act as labels for the ERG kernels. We loosely refer to both individual and summed over vertices of the kernels simply as a kernel.

The rule for decorating the diagrams on the right-hand side is simple: the set of fields, $\{f\}$, are distributed in all independent ways between the component objects of each diagram.

Following [4–6,11,12,15,16], it is technically convenient to use the freedom inherent in \hat{S} by choosing the two-point, classical seed action vertices equal to the corresponding Wilsonian effective action vertices. The effect of this is that the kernels, integrated with respect to $\ln \Lambda$ (at constant α), turn out to be the inverses of the classical, two-point vertices in the transverse space. For example, in the A^1 -sector we find that

$$S_{0\mu\alpha}^{11}(p) \Delta_{\alpha\nu}^{11}(p) = \delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}, \quad (2.7)$$

where Δ^{11} in the integrated A^1 sector kernel. It is apparent that Δ^{11} is the inverse of the corresponding classical, two-point vertex up to a remainder term which, since it is forced to be there as a consequence of the manifest gauge invariance, we call a “gauge remainder.” In recognition of the similarities of the integrated kernels to propagators, in both form and diagrammatic role, we refer to them as effective propagators [11]. However, we emphasize that at no point is gauge fixing required in their definition and that our diagrams do not correspond, in any way, to conventional Feynman diagrams. Equation (2.7) can be diagrammatically generalized to hold in all sectors:

$$M \textcircled{0} \text{---} \langle \text{---} \rangle = M \text{---} \langle \text{---} \rangle - M \blacktriangleright \langle \text{---} \rangle = M \text{---} \langle \text{---} \rangle - M \triangleright \langle \text{---} \rangle. \quad (2.8)$$

We have attached the effective propagator, denoted by a solid line, to an arbitrary structure since it only ever appears as an internal line. The field labeled by M can be any of the broken phase fields. The object $\blacktriangleright \equiv \triangleright$ is a

gauge remainder. The individual components of $\triangleright \blacktriangleright$ will often be loosely referred to as gauge remainders; where it is necessary to unambiguously refer to the composite structure, we will use the terminology “full gauge remain-

der.” Equation (2.8) is referred to as the effective propagator relation. From (2.4) and (2.7), it follows that

$$\Delta_{\rho\sigma}^{11}(p) = \frac{\delta_{\rho\sigma}}{2p^2}(1 + \mathcal{O}(p^2/\Lambda^2)), \quad (2.9)$$

which we will use later.

Embedded within the diagrammatic rules is a prescription for evaluating the group theory factors. Suppose that we wish to focus on the flow of a particular vertex coefficient function which, necessarily, has a unique supertrace structure.

On the right-hand side of the flow equation, we must focus on the components of each diagram with precisely the same supertrace structure as the left-hand side, noting that the kernel, like the vertices, has multisupertrace contributions (for more details see [15,16]). In this more explicit diagrammatic picture, the kernel is to be considered a double sided object. Thus, while the dumbbell like term of Fig. 1 has at least one associated supertrace, the next diagram has at least two, on account of the loop (this is strictly true only in the case that kernel attaches to fields on the same supertrace). If a closed circuit formed by a kernel is devoid of fields then it contributes a factor of $\pm N$, depending on the flavors of the fields to which the kernel forming the loop attaches. This is most easily appreciated by defining the projectors

$$\sigma_{\pm} \equiv \frac{1}{2}(1 \pm \sigma)$$

and noting that $\text{str}\sigma_{\pm} = \pm N$. In the counterclockwise sense, either a σ_+ or σ_- , as appropriate, can always be inserted for free after any of the broken phase fields.

The above prescription for evaluating the group theory factors receives $1/N$ corrections in the A^1 and A^2 sectors. If a kernel attaches to an A^1 or A^2 , it comprises a direct attachment and an indirect attachment. In the former case, one supertrace associated with some vertex coefficient function is “broken open” by an end of a kernel: the fields on this supertrace and the single supertrace component of the kernel are on the same circuit. In the latter case, the kernel does not break anything open and so the two sides of the kernel pinch together at the end associated with the indirect attachment. This is illustrated in Fig. 2; for more detail, see [15,16].

We can thus consider the diagram on the left-hand side as having been unpackaged, to give the terms on the right-hand side. The dotted lines in the diagrams with indirect attachments serve to remind us where the loose end of the kernel attaches in the parent diagram.

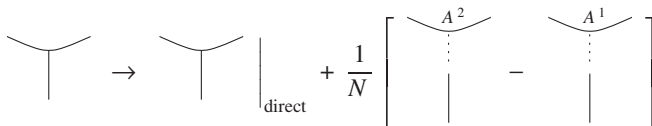


FIG. 2. The $1/N$ corrections to the group theory factors.

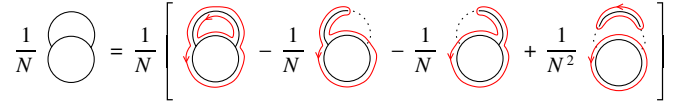


FIG. 3 (color online). An example showing how to evaluate the group theory factor of a diagram in which the kernel is taken to be in the A^1 sector.

As an example, which will be of use later, consider the group theory factor of the diagram on the left-hand side of Fig. 3, where we suppose that the kernel forming the loop is in the A^1 sector.

On the right-hand side, we have unpackaged the parent diagram and explicitly indicated, in red, how many circuits each diagram has. To evaluate the corresponding group theory factors, we simply take each circuit to contribute $\text{str}\sigma_+$ (σ_+ because we are taking the kernel to be in the A^1 sector). Therefore, the overall group theory factor is

$$\frac{1}{N} \left(N^2 - 2 \frac{1}{N} N + \frac{1}{N^2} N^2 \right) = \frac{1}{N} (N^2 - 1) = 2C_2(F),$$

where $C_2(F)$ is the quadratic Casimir operator for the fundamental representation of $SU(N)$.

III. METHODOLOGY

A. Basics

In this section we describe the strategy for computing the renormalized expectation value of the gauge invariant operator, \mathcal{O} . Denoting the set of (dynamical) broken phase fields by Φ , we aim to compute

$$\langle \mathcal{O} \rangle_R = \frac{1}{Z_0} \int_{\Lambda_0} \mathcal{D}\Phi \mathcal{O}_{\Lambda_0}[A^1] e^{-S_{\Lambda_0}[\Phi]}, \quad (3.1)$$

where the subscript R stands for renormalized and $Z_0 = \int_{\Lambda_0} \mathcal{D}\Phi e^{-S_{\Lambda_0}[\Phi]}$. Notice that we have explicitly tagged the functional integral, action and \mathcal{O} with Λ_0 . This is to remind us that the expression is defined at the bare scale, Λ_0 . At this scale, the gauge invariant operator is taken to be a functional of just the physical gauge field, A^1 . The limit $\Lambda_0 \rightarrow \infty$, which essentially corresponds to the continuum limit [28], is taken at the end of a calculation.

Introducing the source, J , we rewrite (3.1) in the usual way:

$$\langle \mathcal{O} \rangle_R = - \frac{\partial}{\partial J} \ln Z_J \Big|_{J=0}, \quad (3.2)$$

where

$$Z_J = \int_{\Lambda_0} \mathcal{D}\Phi e^{-S_{\Lambda_0}[\Phi] - J \mathcal{O}_{\Lambda_0}[A^1]}.$$

The key step now is to integrate out modes between the bare scale and the effective scale, Λ , to yield

Notice that we have replaced the upper limits of the sums over n with the finite n' . By taking n' to infinity, (3.21) becomes exact. However, the form given above is suitable for the order-by-order computation of corrections to $\langle \mathcal{O} \rangle_R$.

IV. WILSON LOOPS

In this section, we will illustrate (3.21) by using it to compute perturbative corrections to Wilson loops. Before taking the explicit case of the rectangular Wilson loop with sides $T \gg L$, we discuss some general features of Wilson loop calculations, within our framework.

A. General considerations

For some closed path, Γ , the path ordered phase factor, *a.k.a.* the Wilson loop, is defined to be

$$\phi(\Gamma) = \frac{1}{N} \text{tr} P \exp \left[i \oint_{\Gamma} dx_{\mu} A_{\mu}^1(x) \right].$$

It is well known [40–42] that the expectation value of this object,

$$W(\Gamma) = \langle \phi(\Gamma) \rangle, \quad (4.1)$$

is divergent even after renormalization of the coupling and, in the case of a gauge fixed formulation, field strength renormalization. In our manifestly gauge invariant formulation, where the gauge field does not suffer from field strength renormalization, (4.1) is defined such that the renormalization of the coupling has been done.

The remaining divergences have two sources. For smooth, simple loops, there is a divergence $e^{-\kappa \Lambda_0 l(\Gamma)}$, where κ is a dimensionless parameter and $l(\Gamma)$ is the length of Γ . The linearly divergent $K \equiv \kappa \Lambda_0$ can be interpreted as a mass divergence. The other divergences come from any (finite) number of cusps and intersections, parameterized by the angles θ^i and ϑ^i , respectively. The renormalized expectation value of the Wilson loop with cusps but no intersections is defined to be [42]

$$W_R(\Gamma_{\theta^i}) = Z(\theta^i) e^{-\bar{m}_0 \Lambda_0 l(\Gamma)} W(\Gamma_{\theta^i}),$$

where we have used powers of Λ_0 to replace the bare mass, m_0 , with a dimensionless parameter, \bar{m}_0 . The renormalized mass, m , is

$$m = K + m_0$$

and the multiplicative renormalization constant factorizes:

$$Z(\theta^i) = Z(\theta^1) Z(\theta^2) \cdots.$$

[In the case that Γ includes intersections, $W_R(\Gamma)$ no longer renormalizes by itself, and must be considered together with expectation values of a family of other loop functions.]

$Z(\theta)$ and \bar{m} have the following expansions:

$$Z(\theta, g) = 1 + \sum_{i=1}^{\infty} g^{2i} Z_i(\theta) \quad \bar{m}(g) = \sum_{i=1}^{\infty} g^{2i} \bar{m}_i.$$

With these points in mind, we identify the boundary value of our gauge invariant operator with

$$\mathcal{O}_{\Lambda_0} = \frac{1}{N} \text{tr} P \exp \left[i \oint_{\Gamma} dx_{\mu} A_{\mu}^1(x) \right] e^{-\bar{m}_0 \Lambda_0 l(\Gamma)} Z(\theta^i). \quad (4.2)$$

We can use the fact that \mathcal{O}_{Λ_0} does not possess an $\mathcal{O}(1/g^2)$ component to simplify the following analysis. To this end, consider the classical flow of \mathcal{O} :

$$\left[\begin{array}{c} \bullet \\ \square \\ 0 \end{array} \right]^{\{f\}} = \left[\begin{array}{c} \square \\ 0 \\ \bullet \\ \bigcirc \\ \Pi_0 \end{array} \right]^{\{f\}}.$$

Now, for the right-hand side not to vanish, the Π_0 vertex must be decorated by at least two fields belonging to $\{f\}$. This is because both seed action and Wilsonian effective action one-point vertices vanish at tree level [11,16] and $\Pi_{0RS}^{XX}(k) = 0$ due to our choice to set the seed action, two-point, classical vertices equal to their Wilsonian effective action counterparts.

From this it follows that

$$\left[\begin{array}{c} \bullet \\ \square \\ 0 \end{array} \right] = 0, \quad \left[\begin{array}{c} \bullet \\ \square \\ 0 \end{array} \right]^X = 0,$$

where X is any field. Integrating up and using the fact that all classical vertices vanish at the boundary [see (3.9) and (4.2)], we find that

$$\left[\begin{array}{c} \square \\ 0 \end{array} \right] = 0, \quad \left[\begin{array}{c} \square \\ 0 \end{array} \right]^X = 0.$$

But, these relationships, together with the vanishing of $\Pi_{0RS}^{XX}(k)$ and the boundary condition imply that

$$\left[\begin{array}{c} \square \\ 0 \end{array} \right]^{XX} = 0.$$

Iterating this argument, it is clear that, in fact,

$$\left[\begin{array}{c} \square \\ 0 \end{array} \right]^{\{f\}} = 0.$$

Given that the \mathcal{O} vertex of $\bar{\mathcal{Q}}$ must, therefore, have an argument of at least one, this allows us to reduce the maximum value of j by unity [20].

In a similar fashion, we can demonstrate that

$$\left[\begin{array}{c} \square \\ 1 \end{array} \right]^X = 0.$$

With these points in mind, let us apply (3.21) for $n' = 1$. Using the boundary condition, we obtain the expected (trivial) result that

$$\left[\boxed{1} \right]_{\Lambda=0} = 1. \quad (4.3)$$

[Note that (4.3) is in fact exact, not requiring supplementation at $O(g^2)$. This follows because, in the weak coupling expansion, the vertex $\boxed{1}$ multiplies g^0 .] At the next order we find

$$\begin{aligned} [g^2 \boxed{2}]_{\Lambda=0} &= \left[g^2 \left(\boxed{2} - 2\Upsilon_{1,0} \text{ (loop)} \right) \right]_{\Lambda=\Lambda_0} \\ &+ O(g^4). \end{aligned} \quad (4.4)$$

Equation (4.4) gives the first nontrivial correction to the renormalized Wilson loop parametrized by a contour with an arbitrary (finite) number of cusps (generalization to include intersections is straightforward, as indicated earlier). To evaluate (4.4) we feed in the boundary condition (4.2). The first term on the right-hand side possesses precisely those contributions necessary to cancel the diver-

gences in the second term. With these divergences canceled, we can safely take the continuum limit, $\Lambda_0 \rightarrow \infty$.

B. The rectangular Wilson loop with sides $T \gg L$

To illustrate the application of (4.4) in a way which will allow us to compare directly with known results, we must compute a quantity which is independent of the renormalization prescription. To this end, we focus on the rectangular Wilson loop, $\bar{\Gamma}$, with sides T and L , in the limit where $T \gg L$. The leading order contribution in this limit is universal, being directly related to the lowest order Coulomb potential of the physical $SU(N)$ Yang-Mills theory.

At the boundary, the expression for the first term on the right-hand side of (4.4) follows, directly, from (4.2) upon expanding the exponentials and identifying the $O(g^2)$, field-independent contribution. For the second term we must work a little harder, since we need to relate the two-point vertex to the boundary condition. To do this, we expand the exponential of (4.2) and focus on the coefficient of $\text{tr} A_\mu^1 A_\nu^1$ at $O(g^0)$:

$$-\frac{1}{2N} \oint_{\bar{\Gamma}} dx_\mu \oint_{\bar{\Gamma}} dy_\nu = -\frac{1}{2N} \int d^D x \int dt \frac{dx_\mu(t)}{dt} \delta(x - x(t)) \int d^D y \int ds \frac{dy_\nu(s)}{ds} \delta(y - y(s)).$$

The recasting on the right-hand side allows us to directly compare this expression with the field expansion of \mathcal{O} , given by the analogue of (2.1) with S replaced by \mathcal{O} . Therefore,

$$\left. \begin{array}{c} x \quad \mu \\ | \\ \boxed{1} \\ | \\ y \quad \nu \end{array} \right|_{\Lambda_0} = -\frac{1}{N} \int dt \frac{dx_\mu(t)}{dt} \delta(x - x(t)) \int ds \frac{dy_\nu(s)}{ds} \delta(y - y(s)).$$

The other components of the second diagram on the right-hand side of (4.4) are the effective propagator, $\Delta_{\mu\nu}^{11}(x, y)$,⁴ the group theory factor (which can be evaluated according to Fig. 3) and an integral over the undetermined coordinates, x and y .

Using (3.17), Eq. (4.4) becomes

$$\lim_{T/L \rightarrow \infty} W_R^{(2)}(\bar{\Gamma}) = \lim_{T/L \rightarrow \infty} \lim_{\Lambda_0 \rightarrow \infty} \left[g^2 \left(\boxed{2} - \frac{N^2 - 1}{2N} \oint_{\bar{\Gamma}} dx_\mu \oint_{\bar{\Gamma}} dx_\nu \Delta_{\mu\nu}^{11}(x, y, \Lambda_0) \right) \right], \quad (4.5)$$

where we have changed notation slightly to make the path dependence of the left-hand side explicit. Since we are taking the $T/L \rightarrow \infty$ limit, we do not need to be too precise about our renormalization prescription: the associated finite terms are subleading and so we have

$$\begin{aligned} \lim_{T/L \rightarrow \infty} W_R^{(2)}(\bar{\Gamma}) &= \lim_{T/L \rightarrow \infty} \left[-g^2 C_2(F) \right. \\ &\quad \left. \times \oint_{\bar{\Gamma}} dx_\mu \oint_{\bar{\Gamma}} dx_\nu \Delta_{\mu\nu}^{11}(x, y) \Big|_{\text{finite}} \right]. \end{aligned} \quad (4.6)$$

Writing

$$\Delta_{\mu\nu}^{11}(x, y) = \int \frac{d^D p}{(2\pi)^D} e^{ip \cdot (x-y)} \Delta_{\mu\nu}^{11}(p^2/\Lambda^2)$$

⁴The fields must be in the A^1 sector at the bare scale, since this is the only sector in which the vertex to which the effective propagator attaches has support.

and recalling (2.9), it is clear that our expression corresponds to the usual one. However, we emphasize once again that, despite obvious similarities, the object $\Delta_{\mu\nu}^{11}(p^2/\Lambda^2)$ is not a (regularized) Feynman propagator and that at no stage have we fixed the gauge. Notice that we can immediately take $D \rightarrow 4$, since preregularization plays no role here. Indeed, this highlights the fact that we only ever use dimensional regularization as a prescription for removing finite surface terms present as a consequence of the Pauli-Villars regularization provided by the $SU(N|N)$ scheme [9]. All necessary UV regularization in (4.5) and (4.6) is provided by the cutoff functions buried in the effective propagator.

Explicitly evaluating the contour integrals we find that

$$\lim_{T/L \rightarrow \infty} W_R^{(2)}(\bar{\Gamma}) = g^2 \frac{C_2(F)T}{4\pi L},$$

recovering the standard result.

V. CONCLUSION

We have described how to compute the expectation values of renormalized gauge invariant operators in a manifestly gauge invariant way, within the framework of the exact renormalization group. The methodology has been illustrated with a computation of the $O(g^2)$ correction to the rectangular Wilson loop with sides $T \gg L$.

The key elements of the methodology are as follows. Given our regularized $SU(N)$ gauge theory defined at the bare scale, Λ_0 , we add a source term $J\mathcal{O}_{\Lambda_0}$ for the gauge invariant operator, \mathcal{O}_{Λ_0} . As we integrate out modes, so the source term evolves. Although this generates a Taylor series in J , the only term which contributes to $\langle \mathcal{O} \rangle_R$ is the one linear in J which, after specializing to the small J limit, we denote by \mathcal{O}_Λ . Figure 4 gives the flow of this component.

We then derived Eq. (3.7), which states that the expectation value of our gauge invariant operator is simply given by $\mathcal{O}_{\Lambda=0}$. Thus, in conjunction, Figs. 1 and 4 and Eq. (3.7) allow us to compute the expectation value of an arbitrary gauge invariant operator (in some approximation scheme).

The rest of the paper was devoted to exploring the formalism in the perturbative domain. It was here that

the considerable effort invested in [15,18–20] to understand the structure of perturbative β -function coefficients really paid off. The associated developments allowed us to directly obtain (3.21), which gives an extremely compact diagrammatic expression for the perturbative corrections to $\langle \mathcal{O} \rangle_R$. We note that this expression makes use of the diagrammatic function, \bar{Q} , given by (3.15) and (3.16). This function depends only on Wilsonian effective action vertices, effective propagators and (components of) gauge remainders. There is no explicit dependence on either the seed action or the covariantization of the ERG kernels.

Whilst the perturbative treatment is useful both to gain experience with the techniques and also to demonstrate that practical calculations can be straightforwardly (and correctly) performed, the real challenge is to apply the formalism nonperturbatively. Of course, the key results shown in Figs. 1 and 4 and Eq. (3.7) are defined nonperturbatively. The main difficulty is deciding how best to approximate the flow equation where there is no obviously small parameter in which to expand (for speculations on whether it might be possible to perform a strong coupling expansion in the inverse of the *renormalized* coupling see [43]). However, some inspiration for this may be provided by the perturbative treatment. We know that, for operators which correspond to physical observables, the expression for $\mathcal{O}_{\Lambda=0}$ must be universal. Obviously, such an expression is independent of the details of the seed action or the covariantization of the ERG kernels. Thus, it is natural to speculate whether, nonperturbatively, $\mathcal{O}_{\Lambda=0}$ can be written in terms of a generalization of \bar{Q} ; indeed, this generalization has now been found [44]. Nevertheless, this generalized diagrammatic function possesses an infinite number of vertices and so much work remains to be done to extract useful information. However, this surely represents a desirable, direct starting point for attacking nonperturbative problems within the ERG formalism.

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