Forward scattering amplitudes and the thermal operator representation

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(Received 25 September 2006; published 8 December 2006)

We develop systematically to all orders the forward scattering description for retarded amplitudes in field theories at zero temperature. Subsequently, through the application of the thermal operator, we establish the forward scattering description at finite temperature. We argue that, beyond providing a graphical relation between the zero temperature and the finite temperature amplitudes, this method is calculationally quite useful. As an example, we derive the important features of the one-loop retarded gluon self-energy in the hard thermal loop approximation from the corresponding properties of the zero temperature amplitude.

DOI: 10.1103/PhysRevD.74.125005 PACS numbers: 11.10.Wx

I. INTRODUCTION

In a series of papers [1–4], the idea of a thermal operator representation [5,6] has been developed extensively in both the imaginary time formalism and the real time formalism of the closed time path. In simple terms, the thermal operator representation relates a Feynman graph at finite temperature (with or without a chemical potential) to the corresponding graph at zero temperature. As we have argued earlier, the thermal operator representation offers a powerful method for studying various questions at finite temperature. As an example, we have shown in an earlier paper [7] how the cutting rules at finite temperature (with or without a chemical potential), in the closed time path formalism, can be derived starting from those at zero temperature. This derivation also clarifies the miraculous cancellations that arise in an explicit demonstration of a cutting description for the imaginary part of a thermal amplitude [8,9].

At finite temperature, retarded amplitudes play a significant role in studying various physical phenomena. Plasma oscillations provide a very simple example of this. When a thermal plasma is perturbed weakly, the subsequent response of the plasma to the perturbation is studied using the linear response theory [10–12]. In particular, the damping of the oscillation in the plasma is understood by analyzing the poles of the retarded propagators of the particles moving through the plasma. Of course, at finite temperature very few quantities can be evaluated exactly, but the forms of thermal amplitudes simplify considerably either in the low temperature or the high temperature limit. In many phenomena of physical interest (such as quark-gluon plasma phase transitions, early universe, etc.), it is the high temperature behavior that is relevant. While there are many ways of evaluating the high temperature behavior (also known as the hard thermal loop approximation [13]) of thermal amplitudes, the forward scattering description for the retarded amplitudes provides an efficient calculational tool [14]. This can be seen from the following simple example. Let us consider a scalar field theory with a cubic interaction in six dimensions (which is similar to non-Abelian gauge theories in four dimensions). The thermal correction to the one-loop retarded self-energy can be directly calculated (see Fig. 1) in this theory and, after doing the internal energy integral, leads to

$$\Sigma_{\rm R}^{(1)\beta}(p_0,\vec{p}) = \lambda^2 \int \frac{d^5k}{(2\pi)^5} \frac{n_{\rm B}(E_k)}{4E_kE_{k+p}} \Big[\frac{1}{p_0 - E_k - E_{k+p}} - \frac{1}{p_0 + E_k + E_{k+p}} + \frac{1}{p_0 + E_k - E_{k+p}} - \frac{1}{p_0 - E_k + E_{k+p}} \Big], \tag{1}$$

where $E_k = \sqrt{\vec{k}^2 + m^2}$, $E_{k+p} = \sqrt{(\vec{k}+\vec{p})^2 + m^2}$. Furthermore, $n_{\rm B}(E_k)$ denotes the bosonic distribution function and p_0 is assumed to correspond to $p_0 + i\epsilon$ which is necessary for the retarded self-energy. At very high temperatures where $|\vec{k}| \gg p_\mu, m$, the masses can be neglected, and we see from (1) that the high temperature limit needs to be calculated carefully since there are energy differences in the denominator.

On the other hand, the forward scattering description of the same retarded self-energy involves diagrams where one of the internal propagators in the loop is thermal and on shell while the other corresponds to a zero temperature

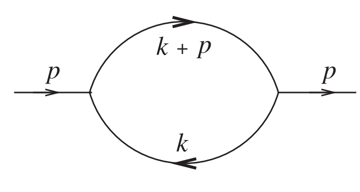


FIG. 1. One-loop retarded self-energy in ϕ^3 theory.

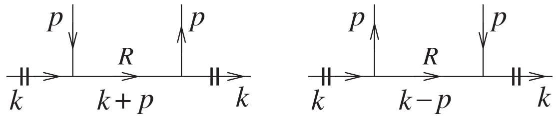


FIG. 2. Two forward scattering amplitudes for the one-loop retarded self-energy.

retarded propagator. They are shown in Fig. 2 and lead immediately to

$$\Sigma_{\rm R}^{(1)\beta}(p_0, \vec{p}) = \lambda^2 \int \frac{d^5k}{(2\pi)^5} \frac{n_{\rm B}(E_k)}{2E_k} \left[\frac{1}{(k+p)^2 - m^2} + \frac{1}{(k-p)^2 - m^2} \right]. \quad (2)$$

Here the "$i\epsilon$" in p_0 (denoting a retarded propagator) as well as $k_0 = E_k$ are to be understood. At high temperature where masses can be neglected, this takes the simple form

$$\Sigma_{\rm R}^{(1)\beta}(p_0, \vec{p}) = -\lambda^2 \int \frac{d^5k}{(2\pi)^5} \frac{n_{\rm B}(|\vec{k}|)}{4|\vec{k}|} \frac{p^2}{(k \cdot p)^2}. \quad (3)$$

The structure resulting directly from the forward scattering description is very simple and interesting (of course, the same structure would also arise in a direct calculation, but limits have to be taken carefully and terms have to be grouped properly before this simple structure is obtained). First of all, we note that the coefficient of the term $\frac{n_{\rm B}(|\vec{k}|)}{|\vec{k}|}$ in the integrand is manifestly Lorentz covariant and is a homogeneous function of degree zero in the external momentum and of degree (-2) in the internal momentum. The manifest Lorentz covariance is broken at high temperature only when the angular integration is carried out. In fact, if we carry out the integration over $|\vec{k}|$, the high temperature limit is obtained to be

$$\Sigma_{\rm R}^{(1)\beta}(p_0, \vec{p}) = -\frac{\lambda^2 \pi^2 T^2}{24} \int \frac{d\Omega}{(2\pi)^5} \left(\frac{p^2}{(p \cdot \hat{k})^2} \right), \quad (4)$$

where we have defined $\hat{k}^\mu = (1, \hat{\vec{k}})$. This Lorentz covariant structure of the integrand in (3) (which results directly in the forward scattering description) is very helpful and has been used to derive in a simple way, in the hard thermal loop approximation, the effective action for QCD as well as the energy-momentum tensor for the quark-gluon plasma [15]. This method is also convenient for the analysis of the high temperature behavior of gauge field theories in a curved space-time [16]. It is also worth noting that the study of the solution of the transport equation at high temperature leads to structures naturally arising in the forward scattering method [17].

In spite of its success, a simple and general derivation of the forward scattering amplitudes to all orders at finite temperature is so far lacking. As we have already argued, the thermal operator representation [1,2,7] provides a powerful method for obtaining results at finite temperature starting from zero temperature, and in this paper we show how the forward scattering amplitudes for retarded thermal n-point functions can be derived through the use of the thermal operator representation. The thermal operator representation is clearly meaningful in studying this question if there exists a forward scattering description at zero temperature. In this paper we derive the forward scattering description for retarded amplitudes in zero temperature field theories. The thermal operator representation then directly leads to the forward scattering description at finite temperature and clarifies the origin of the nice structures observed in the context of the forward scattering amplitudes at high temperature.

The paper is organized as follows. In Sec. II, we develop the forward scattering description for retarded amplitudes of a scalar field theory at zero temperature. In Sec. III, we show how the thermal operator representation leads directly to the forward scattering description for retarded thermal amplitudes. In this section, we also point out various interesting features of retarded amplitudes both at zero and at finite temperature. In Sec. IV we discuss the forward scattering description for the Yang-Mills theory. In particular, we emphasize that various nice properties in the hard thermal loop approximation, such as transversality, manifest Lorentz covariance and gauge invariance of the integrand of the one-loop retarded self-energy, follow simply from the properties of the zero temperature amplitude through the thermal operator representation. We conclude with a brief summary in Sec. V.

II. FORWARD SCATTERING DESCRIPTION AT ZERO TEMPERATURE

The idea of a forward scattering description basically already exists even at zero temperature, although it is not as well developed and is certainly not widely known. Therefore, in this section, we will develop the idea of forward scattering amplitudes at zero temperature systematically for retarded amplitudes so that the thermal operator representation can lead directly to the forward scattering amplitudes at finite temperature. The basic idea behind a forward scattering description at zero temperature [18] is the simple observation that a (time ordered) Feynman propagator (for a massive scalar particle, for simplicity) can be expressed as

$$\frac{i}{k^2 - m^2 + i\epsilon} = \frac{i}{(k_0 + i\epsilon)^2 - E_k^2} + 2\pi\theta(-k_0)\delta(k^2 - m^2), \quad (5)$$

where we have identified $E_k^2 = \vec{k}^2 + m^2$. Namely, the Feynman propagator is the sum of the retarded propagator and a negative energy propagator. As a result, if we have a simple one-loop diagram with n scalar propagators, the amplitude (neglecting vertex factors) can be written as

$$\Gamma_n^{(1)} = \int \frac{d^4k}{(2\pi)^4} \prod_{i=1}^{n} \frac{i}{k_i^2 - m^2 + i\epsilon} = \int \frac{d^4k}{(2\pi)^4} \prod_{i=1}^{n} \left(\frac{i}{(k_{i0} + i\epsilon)^2 - E_{k_i}^2} + 2\pi\theta(-k_{i0})\delta(k_i^2 - m^2) \right), \tag{6}$$

where we have denoted the momentum in the ith propagator as k_i which is a sum of the loop momentum k and some combination of the external momenta whose explicit form is not relevant for our discussion. The form of the integrand in (6) is quite interesting. The first term in the product which will involve only products of retarded propagators would vanish when integrated over energy (which can be seen simply as a consequence of the fact that all the poles lie on the lower half of the complex plane and, therefore, the contour can be closed in the upper half plane to yield zero). The other terms in the expansion of the right-hand side would involve terms with a number of retarded propagators and the rest of the propagators on shell. If we assume that an on-shell propagator can be thought of as a cut-open line representing an on-shell particle coming in and going out, this is very roughly a forward scattering description; namely, a Feynman amplitude can be expressed as a sum of diagrams that involve a number of on-shell particles scattering in the forward direction (their momenta are unchanged in the scattering) and retarded propagators. It is worth noting from the form of (6) that the series of forward scattering diagrams may involve completely disconnected diagrams (which is not the case for retarded amplitudes at finite temperature), but we would like to point out that this is only a consequence of the fact that we are looking at a time ordered Feynman amplitude.

On the other hand, we are interested in retarded amplitudes and, as is well known, these are quite hard to construct at zero temperature within the context of the conventional Feynman propagators. However, if we double the degrees of freedom (the theory with the doubled degrees of freedom can be taken as the zero temperature limit of the theory at finite temperature in the closed time path formalism as described in [7,9]), a diagrammatic representation of retarded amplitudes can be constructed in a straightforward manner. With this in mind, let us look at a scalar field theory with a ϕ^3 interaction with doubled degrees of freedom; we denote the two field degrees of freedom as ϕ_+ and ϕ_-. The propagator for the doubled theory corresponds to a 2×2 matrix,

$$\Delta = \begin{pmatrix} \Delta_{++} & \Delta_{+-} \\ \Delta_{-+} & \Delta_{--} \end{pmatrix}, \tag{7}$$

and, in the momentum space, the components take the forms

$$\begin{aligned} \Delta_{++}(p) &= \lim_{\epsilon\to 0} \frac{i}{p^2 - m^2 + i\epsilon}, \\ \Delta_{+-}(p) &= 2\pi\theta(-p_0)\delta(p^2 - m^2), \\ \Delta_{-+}(p) &= 2\pi\theta(p_0)\delta(p^2 - m^2), \\ \Delta_{--}(p) &= \lim_{\epsilon\to 0} - \frac{i}{p^2 - m^2 - i\epsilon}. \end{aligned} \tag{8}$$

[Unlike in our earlier papers [1,2,7], here we will follow the simple convention of representing quantities at zero temperature without any superscript ($T = 0$). We will reserve the superscript (T) only for quantities at nonzero temperature in order to simplify the notation.)

However, since the forward scattering amplitudes have a physical description in the mixed space (we will discuss this later), we will analyze the problem in this context (the momentum space analysis that is normally done can be obtained from our results through a Fourier transformation). In the mixed space, the components of the propagator can be obtained from a Fourier transformation of (8) and they have the forms (see also [7] for various notations and conventions)

$$\begin{aligned} \Delta_{++}(t, E) &= \frac{1}{2E}[\theta(t)e^{-iEt} + \theta(-t)e^{iEt}], \\ \Delta_{+-}(t, E) &= \frac{1}{2E}e^{iEt}, \\ \Delta_{-+}(t, E) &= \frac{1}{2E}e^{-iEt}, \\ \Delta_{--}(t, E) &= \frac{1}{2E}[\theta(t)e^{iEt} + \theta(-t)e^{-iEt}], \end{aligned} \tag{9}$$

where $E = \sqrt{\vec{p}^2 + m^2}$ describes the on-shell energy of the particle and we are suppressing the "$i\epsilon$" in the exponents for simplicity. Thus, we see that $\Delta_{\pm\mp}$ describe, respectively, the on-shell negative and positive energy propagators. The vertices involving the ϕ_+ fields and the ϕ_- fields differ by a relative negative sign.

The time ordered components of the propagators in (9) satisfy the constraint

$$\Delta_{++} + \Delta_{--} = \Delta_{+-} + \Delta_{-+}. \tag{10}$$

It is now simple to see that the retarded and the advanced propagators of the theory can be identified with

$$\begin{aligned} \Delta_R(t, E) &= \Delta_{++}(t, E) - \Delta_{+-}(t, E) \\ &= \theta(t)\frac{1}{2E}(e^{-iEt} - e^{iEt}), \\ \Delta_A(t, E) &= \Delta_{++}(t, E) - \Delta_{-+}(t, E) \\ &= \theta(-t)\frac{1}{2E}(e^{iEt} - e^{-iEt}), \end{aligned} \tag{11}$$

much like at finite temperature [9]. From these definitions, we note that

$$\Delta_R(-t, E) = \Delta_A(t, E), \tag{12}$$

as we would expect. As a result of these relations, we can decompose the matrix propagator in (7) as

$$\Delta = P + \Delta_{+-} Q, \tag{13}$$

where

$$P(\Delta_{\mathrm{R}}, \Delta_{\mathrm{A}}) = \begin{pmatrix} \Delta_{\mathrm{R}} & 0 \\ \Delta_{\mathrm{R}} - \Delta_{\mathrm{A}} & -\Delta_{\mathrm{A}} \end{pmatrix}, \qquad Q = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}. \tag{14}$$

Let us next note that a retarded n-point amplitude at any loop can be defined as follows. If we assume that the time t_1 corresponding to the first index of the amplitude is the largest among the time coordinates, then we have

$$\Gamma_{n,\mathrm{R}}(t_1, \ldots, t_n) = \sum_{a_i=\pm} \Gamma_{+a_1\ldots a_{n-1}}(t_1, \ldots, t_n), \tag{15}$$

where we have suppressed the energy dependence of the amplitude for simplicity. Here a_i denote the "thermal indices" of the fields which can take the values "$\pm$." For example, the retarded two-point function (self-energy) at one loop would correspond to the sum of the two diagrams in Fig. 3. Let us also note here for future use that, for any n-point amplitude,

$$\sum_{a_i=\pm} \Gamma_{a_1 a_2 a_3 \ldots a_n} = 0, \tag{16}$$

which follows from the largest time equation [7]. We are now ready to derive the forward scattering description for retarded amplitudes to all orders at zero temperature and we do so in two steps.

A. Forward scattering amplitudes at one loop

The forward scattering amplitudes for retarded n-point functions can be derived algebraically at one loop (which is the reason for separating the derivation into two cases). First we note from (15) that the retarded amplitude consists of terms where each vertex other than the largest time is summed over the thermal index "$\pm$." Furthermore, as we have already pointed out, the vertex for the ϕ_- field has a relative negative sign compared to that for the ϕ_+ field. Thus, summing over the thermal index of the graph at one loop can be effected by multiplying the matrix propagator with a 2×2 matrix σ_3 at the vertex where the thermal index is being summed. For example, for the case of the retarded self-energy at one loop (see Fig. 3), we note that (once again we are neglecting factors associated with the vertices as well as the dependence on external energies for simplicity)

$$\begin{aligned} \Sigma_{\mathrm{R}}^{(1)}(t_1 - t_2) &= \int \frac{d^3k}{(2\pi)^3} [\Delta_{++}(t_1 - t_2, E_1)\Delta_{++}(t_2 - t_1, E_2) \\ &\quad - \Delta_{+-}(t_1 - t_2, E_1)\Delta_{-+}(t_2 - t_1, E_2)] \\ &= \int \frac{d^3k}{(2\pi)^3} [\Delta(t_1 - t_2, E_1)\sigma_3 \Delta(t_2 - t_1, E_2)]_{++}. \end{aligned} \tag{17}$$

Note also that the retarded amplitude is obtained by taking the "$++$" component in the matrix product (simply because we start from a "$+$" vertex and end at the same vertex). As a result of this simplification, the retarded n-point amplitude at one loop, shown in Fig. 4, can be written as

$$\Gamma_{n,\mathrm{R}}^{(1)}(t_1, \ldots, t_n) = \int \frac{d^3k}{(2\pi)^3} \bigg[\bigg(\prod_{i=1}^{n-1} \Delta(t_i - t_{i+1}, E_i)\sigma_3 \bigg) \times \Delta(t_n - t_1, E_n) \bigg]_{++}. \tag{18}$$

We can use the decomposition (13) of the propagator in terms of the P, Q matrices which satisfy many interesting relations. We list below some of the relations that are useful for our discussion.

$$\begin{aligned} Q\sigma_3 Q &= 0, \\ Q\sigma_3 P(\Delta_{\mathrm{R}}, \Delta_{\mathrm{A}}) &= \Delta_{\mathrm{A}} Q, \\ P(\Delta_{\mathrm{R}}, \Delta_{\mathrm{A}})\sigma_3 Q &= \Delta_{\mathrm{R}} Q, \\ P(\Delta_{1,\mathrm{R}}, \Delta_{1,\mathrm{A}})\sigma_3 P(\Delta_{2,\mathrm{R}}, \Delta_{2,\mathrm{A}}) &= P(\Delta_{1,\mathrm{R}}\Delta_{2,\mathrm{R}}, \Delta_{1,\mathrm{A}}\Delta_{2,\mathrm{A}}). \end{aligned} \tag{19}$$

Using these relations, the n-point amplitude in (18) can be simplified considerably. First, we note that the expression on the right-hand side can at most be linear in Q and, therefore, can only have at most a single propagator of the type Δ_{+-} which, as we have seen, can describe on-shell particles [see, for example, (8)]. Furthermore, if we assume that an on-shell propagator can be thought of as a cut-open line (representing an on-shell particle), it is clear that the retarded n-point amplitude will involve only connected diagrams (not disconnected as can be the case in a time

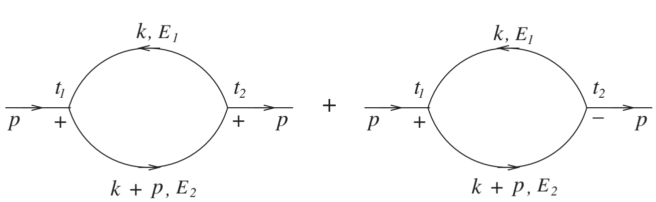


FIG. 3. Sum of two diagrams which gives the retarded self-energy at one loop.

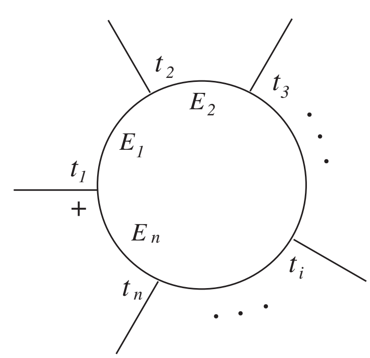


FIG. 4. One-loop diagram for the retarded n-point function. The sum over the thermal indices from t_2 to t_n is to be understood.

ordered Feynman amplitude which we have seen earlier). In fact, an explicit evaluation of (18) leads to

$$\Gamma^{(1)}_{n,\mathrm{R}} = \int \frac{d^3k}{(2\pi)^3}\bigg[\prod_{i=1}^{n}\Delta_{\mathrm{R}}(t_i - t_{i+1}, E_i) + \sum_{m=0}^{n-1}\bigg(\prod_{i=1}^{n-m-1}\Delta_{\mathrm{R}}(t_i - t_{i+1}, E_i)\bigg) \times \Delta_{+-}(t_{n-m} - t_{n-m+1}, E_{n-m}) \times \bigg(\prod_{j=1}^{m}\Delta_{\mathrm{A}}(t_{n-m+j} - t_{n-m+j+1}, E_{n-m+j})\bigg)\bigg], \tag{20}$$

where we are identifying $t_{n+1} = t_1$, $E_{n+1} = E_1$. We are also using the convention that, when $m = 0$ (or $m = n - 1$), the term in the parentheses has the value

$$\bigg(\prod_{i=1}^{0}\Delta(t_i)\bigg) = 1. \tag{21}$$

It is obvious that the first term in the bracket that involves only a product of retarded propagators would vanish when integrated. Furthermore, using (12) we can convert all the advanced propagators into retarded ones and write

$$\Gamma^{(1)}_{n,\mathrm{R}} = \int \frac{d^3k}{(2\pi)^3}\sum_{m=0}^{n-1}\bigg(\prod_{i=1}^{n-m-1}\Delta_{\mathrm{R}}(t_i - t_{i+1}, E_i)\bigg) \times \Delta_{+-}(t_{n-m} - t_{n-m+1}, E_{n-m}) \times \bigg(\prod_{j=1}^{m}\Delta_{\mathrm{R}}(-t_{n-m+j} + t_{n-m+j+1}, E_{n-m+j})\bigg). \tag{22}$$

This gives a forward scattering description for the retarded n-point amplitude at one loop at zero temperature. Each term in the series is a number of retarded propagators with one on-shell propagator (Δ_{+-}) leading to the forward scattering of a single on-shell particle in all possible manners in a connected causal manner. Unlike the Feynman amplitude in (6), the forward scattering description for the retarded n-point function does not involve disconnected diagrams, which is also reflected in the basic definition of the retarded amplitudes in terms of nested commutators that we will discuss in Sec. III. From (22), we can easily derive a recursion relation for the integrands of the one-loop retarded amplitudes of the form

$$\gamma^{(1)}_{n+1,\mathrm{R}} = \prod_{i=1}^{n}\Delta_{\mathrm{R}}(t_i - t_{i+1}, E_i)\Delta_{+-}(t_{n+1} - t_1, E_{n+1}) + \gamma^{(1)}_{n,\mathrm{R}}\Delta_{\mathrm{R}}(-t_{n+1} + t_1, E_{n+1}). \tag{23}$$

B. Forward scattering amplitude at higher loops

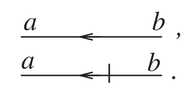
The simple algebraic derivation of the one-loop forward scattering amplitudes for the zero temperature retarded n-point function does not carry over to higher loops in general. This is simply because of the fact that, at higher loops, more than two propagators (internal lines) may be connected to a given vertex. In such a case, the convenient matrix structure for retarded amplitudes that arises in one loop (because only two propagators can be connected to a vertex) is not present. Nonetheless, the forward scattering description for some simple higher loop graphs can be easily derived algebraically as follows. Let us consider the scalar ϕ^{n+2} theory. In this case, the retarded self-energy at n loops (see Fig. 5) can be written as

$$\begin{aligned}\Sigma^{(n)}_{\mathrm{R}} &= \int \frac{d^3k_n}{(2\pi)^3}[\Sigma^{(n-1)}_{++}\Delta_{++}(E_{n+1}) + \Sigma^{(n-1)}_{+-}\Delta_{+-}(E_{n+1})] \\ &= \int \frac{d^3k_n}{(2\pi)^3}[\Sigma^{(n-1)}_{\mathrm{R}}\Delta_{+-}(E_{n+1}) - \Sigma^{(n-1)}_{+-}\Delta_{\mathrm{R}}(E_{n+1})] \\ &= \int \frac{d^3k_n}{(2\pi)^3}\Sigma^{(n-1)}_{\mathrm{R}}\Delta_{+-}(E_{n+1}) \\ &\quad + \int \prod_{i=1}^{n}\frac{d^3k_i}{(2\pi)^3}\Delta_{+-}(E_i)\Delta_{\mathrm{R}}(E_{n+1}).\end{aligned} \tag{24}$$

Here k_i, $i = 1, 2, \cdots, n$ denote the n independent momenta of the loops and, in the intermediate steps, we have used various relations such as (11) and have neglected terms involving products of retarded quantities in the integrand (which will vanish upon integration). The recursion relation in (24) is interesting for two reasons. First, it shows the generic feature in higher loops that any retarded amplitude at n loops can be given a forward scattering description in terms of retarded amplitudes at lower order. Second, the recursion relation (24) can be thought of as a recipe for opening up loops [18] for this particular diagram.

Although the forward scattering description for some simple higher loop diagrams can be derived algebraically, for a general higher loop amplitude, this is best established diagrammatically. For this purpose, let us introduce the graphical representation for the two parts of the matrix propagator in (13) as

P_{ab} =
$\Delta_{+-}Q_{ab}$ =



(25)

There are two important things to note here. First, the "cut" propagator corresponds to the on-shell propagator and all the elements of the matrix Q have the value unity so that the form of the cut propagator is the same, independent of the indices $a, b = \pm$. Second, since the propagator P is directional, we choose the convention of taking the direction of time flow to be towards the "+" vertex in a retarded amplitude (which corresponds to the largest time). This simplifies the derivation and is physically meaningful to give a causal evolution for the amplitudes. The direction of the time flow at other vertices, where the thermal indices

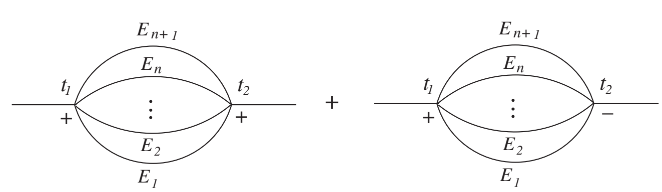


FIG. 5. Sum of diagrams which gives the n-loop retarded self-energy in ϕ^{n+2} theory.

are being summed over, is unimportant. It is clear that a graphical decomposition of the propagator in the manner as described in (25) would allow us to write any diagram as a sum of diagrams each consisting of certain numbers of "P" propagators and the remaining ones cut propagators. Each cut propagator corresponds to an on-shell propagator and, therefore, can be thought of as a cut-open line representing forward scattering of an on-shell particle. This can, therefore, also be thought of as a graphical description of the opening up of loops.

In this process of "opening up of loops," we may run into disconnected diagrams. As we will discuss in more detail in the next section, the retarded amplitudes correspond to vacuum expectation values of products of nested commutators and as such cannot have disconnected diagrams (which would correspond to products of vacuum expectation values). This can, of course, be checked graph by graph at any order as was done at finite temperature in [19]. However, this can also be seen graphically as follows. Suppose, in this process of opening up of a diagram, it separates into two disconnected parts as shown in Fig. 6.

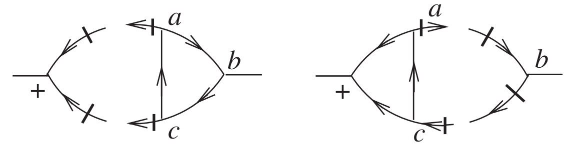


FIG. 6. Two ways of cutting the two-loop self-energy diagram with two cut propagators which render the graph disconnected. The thermal indices a, b, and c are being summed.

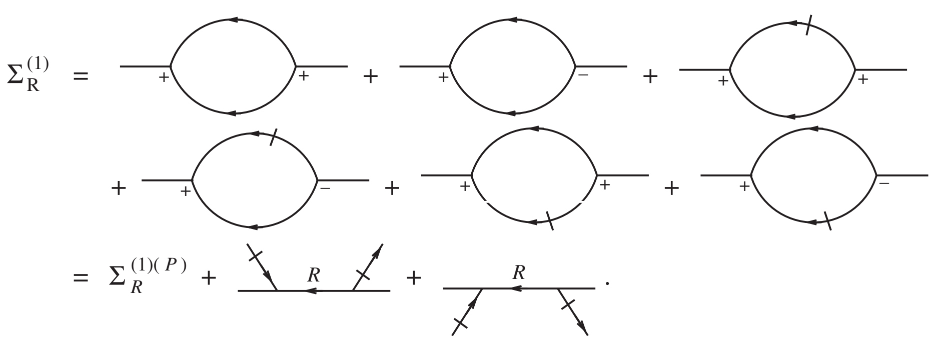
In this case, there are two distinct possibilities. First, one of the disconnected parts contains the "+" vertex corresponding to the largest time and the other is a connected part involving only vertices whose thermal indices are being summed over. In this case, the second part would vanish because of the identity (16). The other possibility is that one of the disconnected parts is a connected diagram involving the "+" vertex and the other simply consists of a vertex whose thermal index is being summed over. Once again, when we sum over the thermal index of this disconnected vertex (with a fixed distribution of the other thermal indices), the diagram would sum to zero (since the "+" and the "−" vertices have a relative negative sign). The crucial ingredient that allows this argument to go through is the special property that a cut propagator is the same for any value of the thermal indices. As a consequence of this nice result, it follows now that, for an arbitrary retarded amplitude at n loops, there can, at the most, be n number of cut propagators in a diagram because more cuts than that would render the graph disconnected. Furthermore, only those propagators in a diagram can be cut propagators (even when their number is less than or equal to n) if they do not render the diagram disconnected. The propagators that are not cut correspond to the "P" propagators which can be seen explicitly from (14) to be lower triangular with

$$P_{++} = \Delta_R, \qquad P_{+-} = 0, \qquad P_R = P_{++} - P_{+-} = \Delta_R. \tag{26}$$

As a result, in a retarded amplitude, the uncut propagators simply correspond to Δ_R propagators (which can be thought of as retarded "P" propagators).

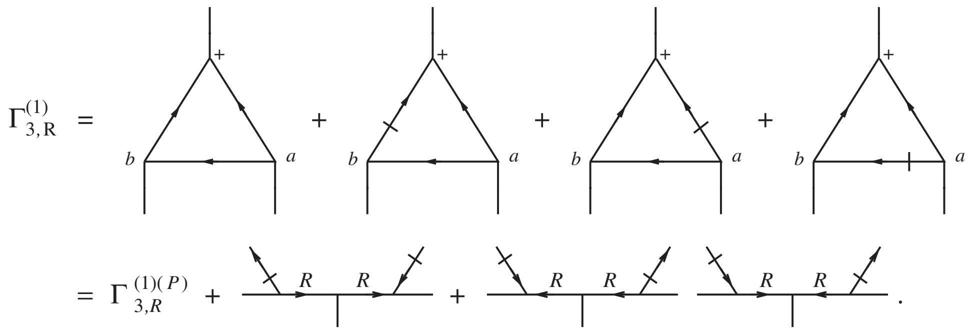
From these interesting properties follows a simple recipe for constructing the forward scattering amplitudes for a retarded graph at n loops. Start with a given graph and write it as sum of all possible diagrams involving "uncut" and cut propagators (open lines) such that none of the diagrams is disconnected and there are at the most n number of cut propagators. The diagram with n number of cut propagators would correspond to a tree-level forward scattering diagram with intermediate retarded propagators. Any diagram with the number of cut propagators less than n would involve vertex diagrams of lower order (loop) as well as intermediate propagators that are retarded. The vertex diagrams of lower order would correspond to retarded diagrams with respect to the "P" propagators and, therefore, would involve only Δ_R propagators. This is the forward scattering description for a retarded diagram at any loop at zero temperature.

The above recipe is already obvious in the examples that we have discussed before. Let us illustrate these as well as some nontrivial examples at higher loops in a graphical manner. First, let us look at the one-loop retarded self-energy in the ϕ^3 theory which can be written as



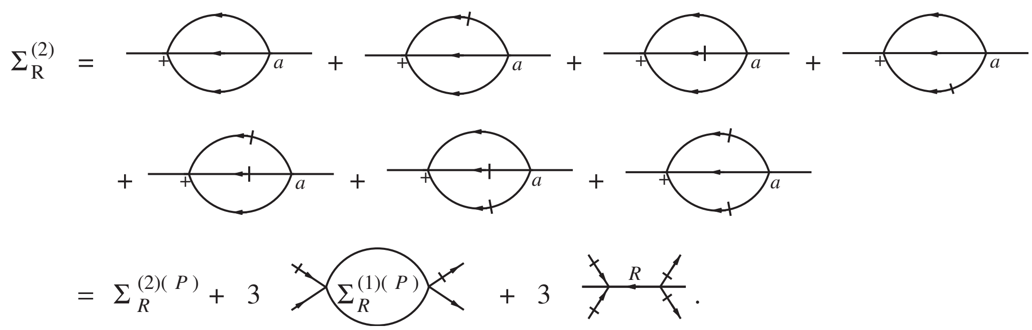
(27)

Similarly, the retarded three-point function at one loop in the ϕ^3 theory takes the form



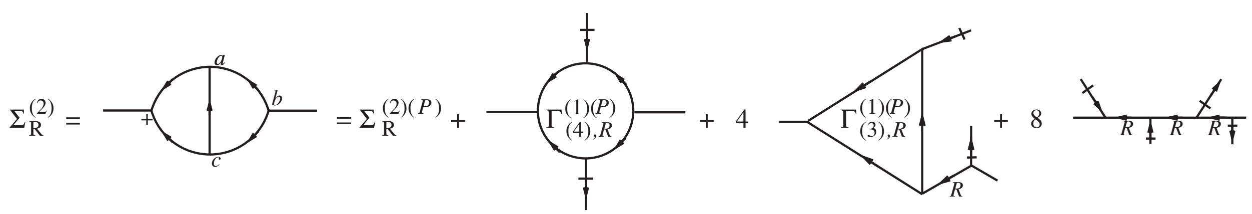
(28)

Let us next look at the retarded self-energy at two loops in the ϕ^4 theory, which takes the form



(29)

Here and in what follows a multiplicative factor in a graph denotes symbolically the number of distinct graphs of the same topology that can be drawn. Let us next look at a nontrivial diagram for the retarded self-energy at two loops in the ϕ^3 theory which takes the form



(30)

All these examples illustrate how the recipe works for an arbitrary retarded amplitude at n loops, and demonstrate the forward scattering description for a retarded amplitude at zero temperature.

III. FORWARD SCATTERING DESCRIPTION AT FINITE TEMPERATURE

Given the forward scattering description for retarded amplitudes at zero temperature, it is now straightforward to derive the forward scattering description at finite temperature through the use of the thermal operator. Let us recall that, in the closed time path formalism, the thermal propagator for a scalar field can be related to the zero temperature one through the thermal operator as [1]

$$\Delta^{(T)}(t, E) = \mathcal{O}^{(T)}(E)\Delta(t, E), \tag{31}$$

where

$$\mathcal{O}^{(T)}(E) = 1 + n_{\rm B}(E)(1 - S(E)). \tag{32}$$

Here $n_{\rm B}(E)$ represents the bosonic distribution function and $S(E)$ is a reflection operator that takes $E \rightarrow -E$. The thermal operator is the same for each component of the propagator (it is a scalar multiplicative operator) and leads to

$$\begin{aligned}\Delta^{(T)}_{+-}(t, E) &= \mathcal{O}^{(T)}(E)\Delta_{+-}(t, E)\\ &= \Delta_{+-}(t, E) + \Delta^{\beta}_{+-}(t, E),\end{aligned} \tag{33}$$

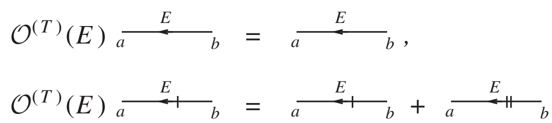
where we have identified the temperature dependent part of the propagator to be (this notation is an attempt to be consistent with the notation in [19], although we have denoted the propagator in those papers by G)

$$\Delta^{\beta}_{+-}(t, E) = \frac{n_{\rm B}(E)}{2E}(e^{-iEt} + e^{iEt}). \tag{34}$$

The other interesting thing to note is that

$$\Delta_{\rm R}^{(T)}(t,E)=\mathcal{O}^{T}(E)\Delta_{\rm R}(t,E)=\Delta_{\rm R}(t,E),$$
$$\Delta_{\rm A}^{(T)}(t,E)=\mathcal{O}^{(T)}(E)\Delta_{\rm A}(t,E)=\Delta_{\rm A}(t,E). \tag{35}$$

Namely, physical propagators such as the retarded and the advanced propagators (at the tree level) are independent of temperature. Graphically, these results can be written as



(36)

where the "double cut" propagator can be identified with $\Delta^{\beta}_{+-}Q$ (completely parallel with the notation of [19]).

We know that any Feynman graph at finite temperature is related to the corresponding zero temperature graph through a thermal operator [1,2] that can be built out of the basic thermal operator in (32). For example, the integrand of a graph (after the internal time integrations are done in the mixed space or the energy integrations are done in the energy-momentum space) with N scalar propagators carrying energy E_i, $i=1,2,\cdots,N$ at finite temperature is related to the integrand of the corresponding graph at zero temperature by the thermal operator

$$\mathcal{O}^{(T)}=\prod_{i=1}^{N}\mathcal{O}^{(T)}(E_i). \tag{37}$$

This is a consequence of the simple fact that at finite temperature only the propagators of the theory are modified (because of the periodicity properties) while the interaction vertices remain unaltered. As a result, the finite temperature forward scattering description for a retarded diagram can be obtained directly from the zero temperature one by simply applying the thermal operator appropriate to the particular diagram. Of course, this also clarifies the origin of the finite temperature forward scattering description; namely, it exists because there is a corresponding description at zero temperature.

It is clear from (36) that, since the thermal operator does not change the "P" propagators, all the uncut lines in a diagram in the forward scattering description will continue to be the zero temperature retarded propagator, $\Delta_{\rm R}$. The thermal operator will only change the cut propagators (the open lines) to a cut plus a "double cut" propagator. (Like the cut propagators, the "double cut" propagators are also on shell with a factor of the distribution function $n_{\rm B}$ and correspondingly can be thought of as representing thermal on-shell incoming and outgoing particles.) Thus, one can organize the graphs in the number of "double cut" propagators. There will be diagrams with no "double cut" propagator, a single "double cut" propagator and so on, and the maximum number of "double cut" propagators will be n for a retarded amplitude at n loops. The diagrams without any "double cut" propagator will, of course, correspond to the zero temperature retarded amplitude. The diagrams with the maximum number of "double cut" propagators (n in the case under study) will represent tree-level forward scattering amplitudes for thermal on-shell particles. The diagrams with the number of "double cut" propagators less than n will all arrange into forward scattering amplitudes for thermal on-shell particles with (zero temperature) retarded vertices of lower order ($n-1$ and lower). Namely, the effect of applying the thermal operator to a retarded amplitude at zero temperature is to change all the "P" retarded vertices in the forward scattering description to genuine retarded vertices at zero temperature and replace all the on-shell forward scattering particles by thermal on-shell forward scattering particles. If we ignore the zero temperature retarded amplitude, the rest of the diagrams yield the temperature dependent forward scattering amplitudes.

The fact that the graphs will arrange as described above can be seen symbolically as follows. Let us identify $\Delta_{\rm R}=P$, $\Delta_{+-}=y$, $\Delta^{\beta}_{+-}=y^{\beta}$. Then, a retarded graph with N propagators at zero temperature can be symbolically represented as $(P+y)^N$. If the graph is at n loops, then we can expand it in terms of the number of on-shell propagators (y) and write it symbolically as

$$\Gamma^{(n)}_{(N),{\rm R}}=(P+y)^N=P^N+a_1P^{N-1}y+\cdots+a_nP^{N-n}y^n. \tag{38}$$

Here the multiplicities a_i, $i=1,2,\ldots,n$ are assumed to denote the number of ways the forward scattering on-shell particles can occur in a graph without disconnecting the diagram (which is why these cannot correspond to the pure binomial coefficients). Furthermore, the terms involving powers of P represent intermediate retarded propagators as well as "P" retarded vertices. As we have seen, under the action of the thermal operator,

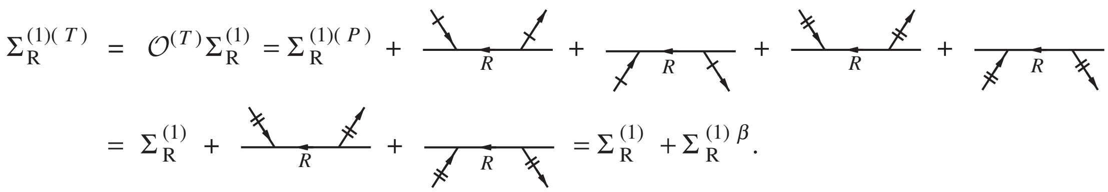
$$\mathcal{O}^{(T)}(P+y)=(P+y+y^{\beta}). \tag{39}$$

As a result, using the thermal operator representation, we obtain

$$\begin{aligned}\Gamma^{(n)(T)}_{(N),{\rm R}}&=\mathcal{O}^{(T)}\Gamma^{(n)}_{(N),{\rm R}}=(P+y+y^{\beta})^N\\&=(P+y)^N+a_1(P+y)^{N-1}y^{\beta}+\cdots+\cdots\\&\quad+a_n(P+y)^{N-n}(y^{\beta})^n\\&=\Gamma^{(n)}_{(N),{\rm R}}+y^{\beta}(\Gamma^{(n-1)}_{(N-1),{\rm R}}+P\Gamma^{(n-1)}_{(N-2),{\rm R}}+\cdots)\\&\quad+(y^{\beta})^2(\Gamma^{(n-2)}_{(N-2),{\rm R}}+\cdots)+\cdots.\end{aligned} \tag{40}$$

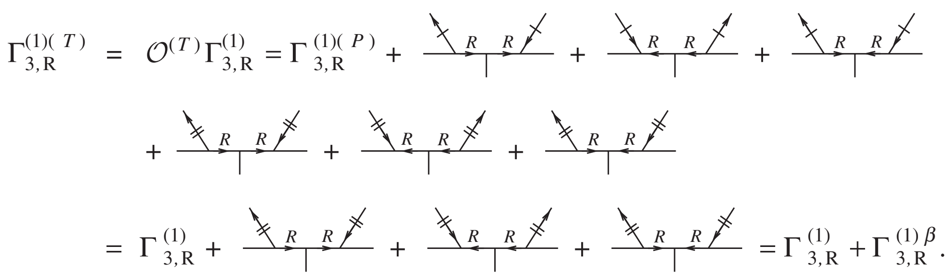
It is useful to remember that the sum of the power of y and y^{β} in any term on the right-hand side of (40) can at the most be n at n loops.

Let us illustrate these relations with some examples. Applying the thermal operator to the forward scattering description of the retarded self-energy in the ϕ^3 theory at one loop [see (27)], we obtain



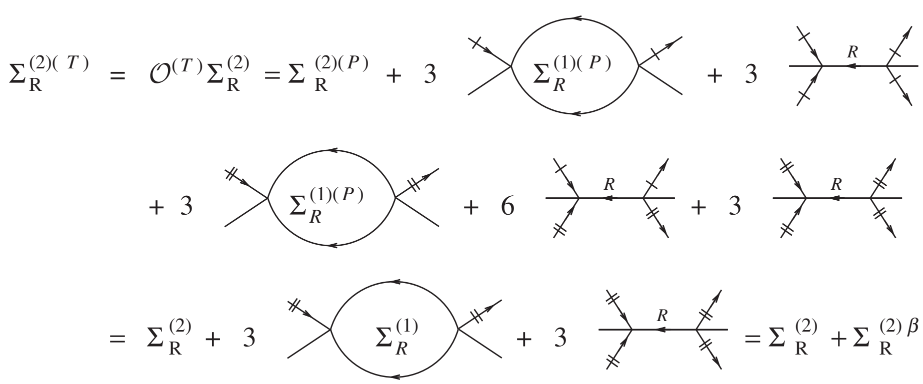
(41)

Similarly, the one-loop retarded three-point function [see (28)] at finite temperature takes the form



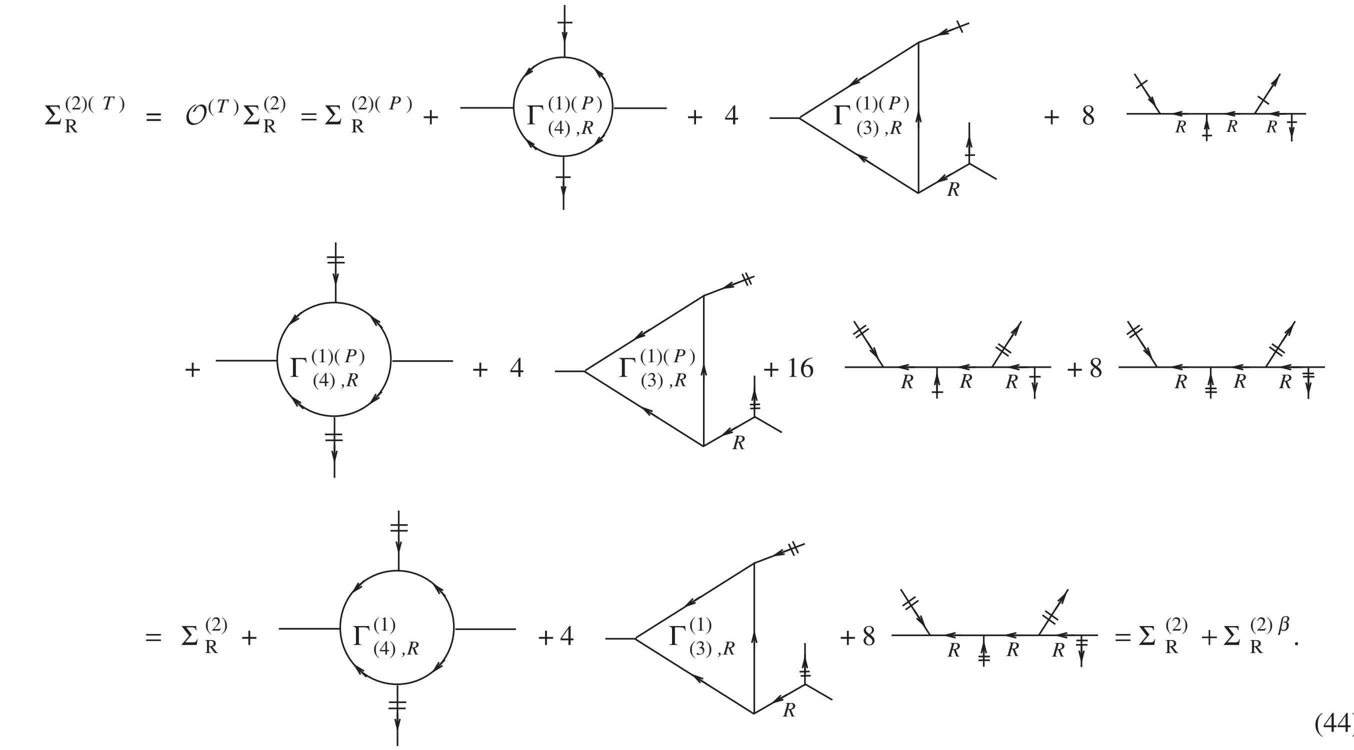
(42)

A slightly more complicated example would correspond to the two-loop retarded self-energy in the ϕ^4 theory [see (29)] at finite temperature which takes the form



(43)

This shows explicitly how the "P" retarded vertices of lower order rearrange themselves into full retarded vertices. Finally, let us consider the nontrivial example of the two-loop retarded self-energy diagram [see (30)] at finite temperature,



(44)

This nontrivial example once again demonstrates how the "P" retarded vertices of lower order rearrange themselves into full retarded vertices. We note that the thermal operators $\mathcal{O}^{(T)}$ for the different amplitudes in (41)–(44) are different and their appropriate forms can be obtained from (37). Furthermore, the temperature dependent forward scattering amplitudes are denoted with a superscript β to coincide with the definition in [19] and in the applicable examples can be checked to agree with the results there.

We would now like to make some observations on the structure of retarded amplitudes in general which are not directly related to the main goal of this paper, but are quite important in understanding their structures. First, we note that the retarded N-point amplitude at any loop is defined algebraically in the coordinate space in terms of the original fields of the theory as the vacuum expectation value of the nested commutators [20],

$$\Gamma_{N,\mathrm{R}}(t_1, t_2, \ldots, t_N) = (-i)^{N-1}\theta(t_1 - t_2)\theta(t_2 - t_3)\ldots \times \theta(t_{N-1} - t_N)\langle 0|[[\ldots[\phi(x_1), \phi(x_2)], \phi(x_3)]\ldots, \phi(x_N)]|0\rangle + \text{permutations}. \tag{45}$$

Here we have assumed that the time coordinate t_1 is the largest among all the coordinates and the "permutations" refer to symmetrizing in all the other coordinates (other than the largest time) and fields. As a result, the retarded amplitude is symmetric in all the coordinates other than the largest time coordinate. For a real scalar field, $\phi(x)$ is a Hermitian operator and, therefore, the factor $(-i)^{N-1}$ shows that the retarded amplitudes are real in coordinate space. Under Hermitian conjugation, the change in the sign of each factor of i is compensated by the change in sign coming from each commutator. We have already argued in Sec. II graphically that a retarded amplitude cannot have disconnected parts which would correspond to products of vacuum expectation values. This can also be seen from the above definition as follows. Let us denote the nested commutator involving the first $N - 1$ fields as

$$A = [[\ldots[\phi(x_1), \phi(x_2)], \phi(x_3)]\ldots, \phi(x_{N-1})]. \tag{46}$$

Then, we can write the vacuum expectation value of the nested commutator in (45) as

$$\langle 0|[A, \phi(x_N)]|0\rangle. \tag{47}$$

Inserting a complete set of intermediate states (say, discrete energy eigenstates or the particle number states), this can be written as

$$\begin{aligned}\langle 0|[A, \phi(x_N)]|0\rangle &= \sum_{n=0}^{\infty}(\langle 0|A|n\rangle\langle n|\phi(x_N)|0\rangle - \langle 0|\phi(x_N)|n\rangle\langle n|A|0\rangle) \\ &= \sum_{n=1}^{\infty}(\langle 0|A|n\rangle\langle n|\phi(x_N)|0\rangle - \langle 0|\phi(x_N)|n\rangle\langle n|A|0\rangle).\end{aligned} \tag{48}$$

Namely, the intermediate vacuum states cancel out in the vacuum expectation value of the commutator and, as a result, the retarded amplitudes contain only connected graphs which we have explicitly seen earlier.

It was noted earlier [7] that the retarded self-energy for a real scalar field in the mixed space is a real quantity. As we have already argued, the retarded amplitudes, by definition, are real in the coordinate space. However, in going to the mixed space, one Fourier transforms the spatial coordinates into spatial momenta, and Fourier transformation does not maintain the reality of a function in general. Let us comment here briefly on when the retarded amplitudes will be real in the mixed space for a scalar theory. The definition of the retarded amplitudes in the mixed space takes the form [see (45)]

$$\begin{aligned}\tilde{\Gamma}_{N,\mathrm{R}}(t_1, t_2, \ldots, t_N) = {}& (-i)^{N-1}\theta(t_1 - t_2)\theta(t_2 - t_3)\ldots \\ & \times \theta(t_{N-1} - t_N)\langle 0|[\ldots[\phi(t_1, \vec{p}_1), \\ & \phi(t_2, \vec{p}_2)], \ldots, \phi(t_N, \vec{p}_N)]|0\rangle \\ & + \text{permutations.}\end{aligned} \tag{49}$$

Under Hermitian conjugation, the change in sign in each factor of "i" is still compensated for by the change in sign coming from each commutator. However, since under Hermitian conjugation

$$\phi(t, \vec{p}) \rightarrow \phi(t, -\vec{p}), \tag{50}$$

the retarded amplitude is not real in general. In fact, let us note explicitly that, under Hermitian conjugation,

$$\begin{aligned}\tilde{\Gamma}^*_{N,\mathrm{R}}(t_1, t_2, \ldots, t_N) = {}& (-i)^{N-1}\theta(t_1 - t_2)\theta(t_2 - t_3)\ldots \\ & \times \theta(t_{N-1} - t_N)\langle 0|[\ldots[\phi(t_1, -\vec{p}_1), \\ & \phi(t_2, -\vec{p}_2)], \ldots, \phi(t_N, -\vec{p}_N)]|0\rangle \\ & + \text{permutations.}\end{aligned} \tag{51}$$

We note that, if the scalar field transforms under parity as

$$\phi(t, \vec{p}) \rightarrow \phi(t, -\vec{p}) = (-1)^{\alpha}\phi(t, \vec{p}), \tag{52}$$

where $\eta = (-1)^{\alpha}$ denotes the intrinsic parity of the scalar field, then we can write (51) as

$$\begin{aligned}\tilde{\Gamma}^*_{N,\mathrm{R}}(t_1, t_2, \ldots, t_N) = {}& (-1)^{N\alpha}(-i)^{N-1}\theta(t_1 - t_2)\theta(t_2 - t_3)\ldots \\ & \times \theta(t_{N-1} - t_N)\langle 0|[\ldots[\phi(t_1, \vec{p}_1), \\ & \phi(t_2, \vec{p}_2)], \ldots, \phi(t_N, \vec{p}_N)]|0\rangle \\ & + \text{permutations} \\ = {}& (-1)^{N\alpha}\tilde{\Gamma}_{N,\mathrm{R}}(t_1, t_2, \ldots, t_N).\end{aligned} \tag{53}$$

For a scalar field of even parity, $\alpha = 0$ and we see that the retarded amplitudes will continue to be real even in the mixed space. However, for a pseudoscalar field, $\alpha = 1$ and we note that only parity conserving retarded amplitudes will be real while the parity violating retarded amplitudes will be purely imaginary in the mixed space. We would like to emphasize that the reality of an amplitude in the coordinate space/mixed space is not contradictory to the existence of dispersion relations in the energy-momentum space since the imaginary parts of the amplitudes in energy-momentum space arise from the imaginary parts of the step functions [$\theta(t)$] in the integral representation.

IV. FORWARD SCATTERING DESCRIPTION FOR YANG-MILLS THEORY

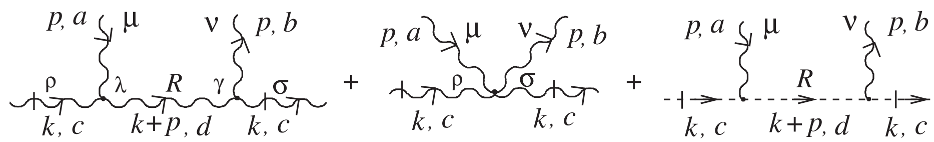
The results of the earlier sections show that the forward scattering description at finite temperature can be obtained from the forward scattering description at zero temperature by the use of the thermal operator. Although we have done this explicitly for scalar field theories, this can be generalized easily to other theories. We would like to emphasize that this correspondence between the finite temperature and zero temperature forward scattering descriptions should not be thought of as only useful in establishing a graphical identification. It is also quite useful as a calculational tool as well as in clarifying various aspects of field theories at high temperature. To give an example of this, we will next derive the retarded gluon self-energy in the Yang-Mills theory [belonging to $SU(N)$] at one loop in the hard thermal loop approximation from the zero temperature result, which will also clarify the structure of this thermal amplitude.

A lot is known about the structure of Yang-Mills theories at high temperature [11,12]. It is known, for example, that in the hard thermal loop approximation the one-loop retarded self-energy in the forward scattering description is independent of the gauge fixing parameter and has a manifestly gauge invariant (transverse) and Lorentz covariant structure (before carrying out the angular integrations). However, the reason for such a structure at finite temperature is not well understood. We will see below that such a structure of the integrand for the retarded self-energy already exists at zero temperature in the appropriate regime and, since the thermal operator is gauge invariant, the thermal amplitude obtained through the application of the thermal operator representation preserves these properties.

From the discussions in Sec. II, we can immediately write down the forward scattering description for the one-loop retarded self-energy for the Yang-Mills field (including the ghost contributions) easily. Our discussion in Sec. II has been completely within the context of the mixed space simply because we wanted to bring out the physical nature of the retarded amplitudes as evolving forward in time in a connected manner. However, the thermal operator representation holds equally well in the energy-momentum space (which can be seen simply by Fourier transforming the external time coordinates) where the thermal operator acts on the integrand after all the energy integrations have been carried out. Since all the calculations of thermal amplitudes in the hard thermal loop approximation have been carried out in the energy-momentum space, in this section we will also work in the energy-momentum space and use the thermal operator representation in this space.

The forward scattering description for the one-loop retarded self-energy for the Yang-Mills field [belonging to $SU(N)$] at zero temperature can be written as

$$\Pi^{ab(1)}_{\mu\nu,\mathrm{R}}(p) =$$



$$+(k \longrightarrow -k)$$

$$= \frac{g^2}{2}\int \frac{d^4k}{(2\pi)^3}\delta^+(k^2)\Big[f^{adc}f^{bcd}V_{\mu\lambda\rho}(p,-k-p,k)\frac{d^{\lambda\gamma}(k+p)}{(k+p)^2}d^{\rho\sigma}(k)V_{\nu\sigma\gamma}(-p,-k,k+p) + W^{abcc}_{\mu\nu\rho\sigma}d^{\rho\sigma}(k) - 2f^{adc}f^{bcd}\frac{k_\mu(k+p)_\nu}{(k+p)^2} + (k\to -k)\Big], \tag{54}$$

where wavy and dotted lines in the graphs denote, respectively, the gluon and the ghost propagators, which, in an arbitrary covariant gauge, have the forms

$$D^{ab}_{\mu\nu}(k) = -\frac{i\delta^{ab}}{k^2}d_{\mu\nu}(k) = -\frac{i\delta^{ab}}{k^2}\left(\eta_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2}\right), \tag{55}$$
$$D^{ab}(k) = \frac{i\delta^{ab}}{k^2}.$$

Here ξ denotes the gauge fixing parameter, and the appropriate "$i\epsilon$" factors in the denominators of the propagators are understood. We also note here that a product such as $\delta^+(k^2)\frac{1}{k^2}$ in (54) has to be understood in a regularized sense as has been described in [21]. Similarly, the gauge and the ghost vertices have the forms (we are suppressing the energy-momentum conserving delta functions and are defining $V^{abc}_{\mu\nu\lambda} = -gf^{abc}V_{\mu\nu\lambda}$ and $V^{abc}_{\mu} = gf^{abc}V_\mu$)

$$V_{\mu\nu\lambda}(p,k,q) = \eta_{\mu\nu}(p-k)_\lambda + \eta_{\nu\lambda}(k-q)_\mu + \eta_{\lambda\mu}(q-p)_\nu,$$
$$V_\mu(p,k,q) = k_\mu. \tag{56}$$

The quartic vertex, $(-ig^2W^{abcd}_{\mu\nu\rho\sigma})$, can be read off from [22]. Furthermore, we have defined

$$\delta^+(k^2) = \theta(k_0)\delta(k^2). \tag{57}$$

Since the ghost propagator as well as the gauge interaction vertex are independent of ξ, for the purpose of understanding the ξ dependence in (54), we can restrict ourselves only to the diagrams involving intermediate gauge field propagators.

The tree-level gauge field vertices (56) satisfy the identity

$$k^\nu V_{\mu\nu\lambda}(p,k,q) = (\eta_{\mu\lambda}p^2 - p_\mu p_\lambda - \eta_{\mu\lambda}q^2 + q_\mu q_\lambda), \tag{58}$$

where we have used the fact that $k = -q - p$ because of energy-momentum conservation. In the region where $k_\mu \gg p_\mu$ (where we are assuming that k, q are the internal momenta and p is the external one), the first two terms in (58) can be neglected for our purpose, and what remains is a structure that is transverse to the other internal momentum. As a result, we see that the terms quadratic in the parameter ξ vanish so that the self-energy in (54) can at most depend linearly on the gauge fixing parameter ξ. An explicit evaluation of the diagrams involving the gauge field propagators shows that the linear terms in ξ cancel out among the two classes of diagrams at the integrand level. Consequently, in the limit $k_\mu \gg p_\mu$, the leading order contribution to the self-energy is independent of the gauge fixing parameter ξ.

At zero temperature, the amplitudes are manifestly Lorentz covariant. Furthermore, the Ward identity for the gluon self-energy requires that it be transverse to the external momentum,

$$p^\mu \Pi^{ab(1)}_{\mu\nu,\mathrm{R}}(p) = 0. \tag{59}$$

Since the leading part of the retarded self-energy is independent of the gauge fixing parameter ξ, in evaluating (54) in this regime, we can choose it to have the value $\xi = 1$ for simplicity (Feynman gauge). In this case, the numerator of the gauge propagator in (55) is simply the metric tensor independent of any momentum. The momentum dependence in the numerator in (54) comes from the vertices and is, therefore, at most quadratic in the momenta. (Note that the three-point vertex depends on the momentum, but the quartic vertex is independent of the momentum.) Since we are interested in the "hard" internal momentum region where $k_\mu \gg p_\mu$, we can expand the numerator in powers of the internal momentum. Similarly, we can also expand the denominator in (54) in this region as (recall that $k^2 = 0$ because of the delta function)

$$\frac{1}{(p+k)^2} = \frac{1}{2p\cdot k} - \frac{p^2}{(2k\cdot p)^2} + \cdots. \tag{60}$$

Using these in (54), one explicitly finds that the leading contribution in this region comes from terms in the integrand (multiplying the delta function) which are of degree zero in both the external and the internal momentum. There is only one such tensor structure available at zero temperature which is consistent with the Ward identity (59), namely,

$$\eta_{\mu\nu} - \frac{p_\mu k_\nu + p_\nu k_\mu}{p\cdot k} + \frac{p^2 k_\mu k_\nu}{(p\cdot k)^2}, \tag{61}$$

and the integrand has to be proportional to this structure. Explicit calculation indeed shows that, in this region, (54) takes the form

$$\Pi^{ab(1)}_{\mu\nu,\mathrm{R}}(p) = -g^2 N\delta^{ab}\int\frac{d^4k}{(2\pi)^3}\delta^+(k^2) \times\left(\eta_{\mu\nu} - \frac{p_\mu k_\nu + p_\nu k_\mu}{p\cdot k} + \frac{p^2 k_\mu k_\nu}{(p\cdot k)^2}\right). \tag{62}$$

This is the leading contribution in the hard internal momentum region and, as we will show shortly, this leads to the hard thermal loop results at high temperature.

There are several things to note from the structure in (62). First, the quantity in the parentheses is manifestly Lorentz covariant of degree zero in both the internal and the external momenta. Furthermore, it is manifestly transverse (gauge invariant). We note that the integral in (62) is quadratically divergent and is usually set to zero in the dimensional regularization. However, if we only carry out the k_0 integration, it does not vanish and, as we will show shortly, it is this leading contribution that leads to the hard thermal loop results at high temperature. (We remark here, parenthetically, that the conventional ultraviolet divergent terms in the self-energy are, in contrast, gauge dependent and are only logarithmically divergent, yielding at high temperature only $\ln T$ contributions [23].) We want to emphasize that the reason for carrying out only the k_0 integration is that the thermal operator acts on the integrand after the energy integrations have been carried out and before evaluating the integrations over the spatial momenta [1]. To apply the thermal operator representation, we need to integrate over the k_0 variable in (62) and this leads to

$$\Pi^{ab(1)}_{\mu\nu,\mathrm{R}}(p) = -g^2 N\delta^{ab}\int\frac{d^3k}{(2\pi)^3}\frac{1}{2E_k} \times\left(\eta_{\mu\nu} - \frac{p_\mu \hat{k}_\nu + p_\nu \hat{k}_\mu}{p\cdot \hat{k}} + \frac{p^2 \hat{k}_\mu \hat{k}_\nu}{(p\cdot \hat{k})^2}\right), \tag{63}$$

where $E_k = |\vec{k}|$ and we have defined $\hat{k}_\mu = (1, -\hat{\vec{k}})$. Applying now the thermal operator to the integrand, we obtain [It is worth clarifying here once again that although some of the diagrams in (54) involve two propagators, only the thermal operator corresponding to the on-shell propagator leads to a nontrivial result. The retarded propagators are unchanged by the application of the thermal operator as discussed in (36).]

$$\begin{aligned}\Pi^{ab(1)(T)}_{\mu\nu,\mathrm{R}} &= -g^2 N\delta^{ab}\int\frac{d^3k}{(2\pi)^3}\mathcal{O}^{(T)}(E_k)\frac{1}{2E_k} \times\left(\eta_{\mu\nu} - \frac{p_\mu \hat{k}_\nu + p_\nu \hat{k}_\mu}{p\cdot \hat{k}} + \frac{p^2 \hat{k}_\mu \hat{k}_\nu}{(p\cdot \hat{k})^2}\right)\\ &= \Pi^{ab(1)}_{\mu\nu,\mathrm{R}}(p) - g^2 N\delta^{ab}\int\frac{d^3k}{(2\pi)^3}\frac{n_\mathrm{B}(E_k)}{E_k} \times\left(\eta_{\mu\nu} - \frac{p_\mu \hat{k}_\nu + p_\nu \hat{k}_\mu}{p\cdot \hat{k}} + \frac{p^2 \hat{k}_\mu \hat{k}_\nu}{(p\cdot \hat{k})^2}\right)\\ &= \Pi^{ab(1)}_{\mu\nu,\mathrm{R}}(p) + \Pi^{ab(1)\beta}_{\mu\nu,\mathrm{R}}.\end{aligned} \tag{64}$$

The temperature independent part $\Pi^{ab(1)}_{\mu\nu,\mathrm{R}}$ can now be set to zero using dimensional regularization for the spatial momentum.

We note that, since $E_k = |\vec{k}|$, the radial momentum integration in the second term in (64) can be done to yield

$$\int_0^\infty dk k n_\mathrm{B}(k) = \frac{\pi^2 T^2}{6}, \tag{65}$$

which leads us to the temperature dependent part of the self-energy

$$\Pi^{ab(1)\beta}_{\mu\nu,\mathrm{R}}(p) = -\frac{g^2 N T^2\delta^{ab}}{48\pi}\int d\Omega\left(\eta_{\mu\nu} - \frac{p_\mu \hat{k}_\nu + p_\nu \hat{k}_\mu}{p\cdot \hat{k}} + \frac{p^2 \hat{k}_\mu \hat{k}_\nu}{(p\cdot \hat{k})^2}\right). \tag{66}$$

This is the well-known result for the temperature dependent part of the retarded self-energy in the hard thermal loop approximation in the forward scattering description. The angular integrations break the manifest Lorentz invariance. (At finite temperature, Lorentz invariance is broken because the rest frame of the heat bath defines a preferred reference frame.) However, the beautiful properties of the integrand follow simply from the properties of the zero temperature amplitude.

This example demonstrates that, in addition to establishing a graphical correspondence, the thermal operator representation can also be conveniently used to calculate the thermal amplitudes in the hard thermal loop approximation from the hard internal momentum amplitudes at zero temperature. Such a calculation also clarifies various important features of the thermal amplitudes at high temperature.

V. CONCLUSION

In this paper, we have systematically derived the forward scattering description for retarded amplitudes to all orders at zero temperature. This graphical derivation then allows us to obtain the forward scattering description for such amplitudes to all orders at finite temperature through the thermal operator representation. Although our derivation has been within the context of a scalar field theory, the derivation can be generalized easily to other theories with or without a chemical potential. Furthermore, although we have used the real time formalism of the closed time path for our discussions for simplicity, the results also hold in the imaginary time formalism (which we do not go into). Besides giving a graphical derivation of the forward scattering description at finite temperature, such a relation can be used as a powerful tool for calculations at high temperature and it clarifies various properties of thermal amplitudes. As an example, we have calculated the one-loop retarded self-energy for gluons in the Yang-Mills theory at finite temperature starting from the forward scattering description at zero temperature. This derivation emphasizes that various nice features of these amplitudes such as gauge invariance, transversality, manifest Lorentz covariance, etc. arise simply because the zero temperature amplitude already possesses such properties. This description of the forward scattering amplitudes at finite temperature provides yet another example of the usefulness of the thermal operator representation.

ACKNOWLEDGMENTS

We would like to thank Gerald Dunne for making us aware of Ref. [18]. This work was supported in part by the US DOE Grant No. DE-FG 02-91ER40685, and by MCT/CNPq as well as by FAPESP, Brazil.

[1] F. T. Brandt, A. Das, O. Espinosa, J. Frenkel, and S. Perez, Phys. Rev. D **72**, 085006 (2005).
[2] F. T. Brandt, A. Das, O. Espinosa, J. Frenkel, and S. Perez, Phys. Rev. D **73**, 065010 (2006).
[3] M. Inui, H. Kohyama, and A. Niegawa, Phys. Rev. D **73**, 047702 (2006).
[4] F. T. Brandt, A. Das, O. Espinosa, J. Frenkel, and S. Perez, Phys. Rev. D **73**, 067702 (2006).
[5] O. Espinosa and E. Stockmeyer, Phys. Rev. D **69**, 065004 (2004).
[6] O. Espinosa, Phys. Rev. D **71**, 065009 (2005).
[7] F. T. Brandt, A. Das, O. Espinosa, J. Frenkel, and S. Perez, Phys. Rev. D **74**, 085006 (2006).
[8] P. F. Bedaque, A. Das, and S. Naik, Mod. Phys. Lett. A **12**, 2481 (1997).
[9] A. Das, *Finite Temperature Field Theory* (World Scientific, New York, 1997).
[10] A. L. Fetter and J. D. Walecka, *Quantum Theory of Many Particle Systems* (McGraw-Hill, New York, 1971).
[11] J. I. Kapusta, *Finite Temperature Field Theory* (Cambridge University Press, Cambridge, England, 1989).
[12] M. L. Bellac, *Thermal Field Theory* (Cambridge University Press, Cambridge, England, 1996).
[13] E. Braaten and R. Pisarski, Nucl. Phys. **B337**, 569 (1990); **B339**, 310 (1990).
[14] F. T. Brandt and J. Frenkel, Phys. Rev. D **47**, 4688 (1993); **56**, 2453 (1997).
[15] J. Frenkel and J. C. Taylor, Nucl. Phys. **B374**, 156 (1992); F. T. Brandt, J. Frenkel, and J. C. Taylor, Nucl. Phys. **B410**, 3 (1993).
[16] J. Frenkel, E. A. Gaffney, and J. C. Taylor, Nucl. Phys. **B439**, 131 (1995).
[17] F. T. Brandt, J. Frenkel, and J. C. Taylor, Nucl. Phys. **B437**, 433 (1995).
[18] R. P. Feynman, in *Magic without Magic: John Archibald Wheeler; A Collection of Essays in Honor of His Sixtieth Birthday*, edited by J. R. Klauder (W. H. Freeman, San Francisco, 1972).
[19] F. T. Brandt, A. Das, J. Frenkel, and A. J. da Silva, Phys. Rev. D **59**, 065004 (1999); F. T. Brandt, A. Das, and J. Frenkel, Phys. Rev. D **60**, 105008 (1999).
[20] S. S. Schweber, *Introduction to Relativistic Quantum Field Theory* (Row, Peterson, Illinois, 1961), p. 751.
[21] P. F. Bedaque and A. Das, Phys. Rev. D **45**, 2906 (1992); **47**, 601 (1993).
[22] F. Gross, *Relativistic Quantum Mechanics and Field Theory* (*John Wiley*, New York, 1993).
[23] F. T. Brandt and J. Frenkel, Phys. Rev. D **60**, 107701 (1999).