Renormalized masses of heavy Kaluza-Klein states

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Several ways of computing the radiative corrections to the heavy boson masses in Kaluza-Klein theory are discussed. It is argued that only an intrinsically higher dimensional approach embodies all the desired physical properties. This contradicts earlier results in the literature.

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I. INTRODUCTION

The vacuum polarization of a gauge theory is a valuable source of information. It conveys information on the running of the corresponding gauge coupling and, in principle, it can be used to compute would-be radiative corrections to the mass of the gauge bosons. These corrections vanish in the usual four-dimensional theory owing to unbroken gauge invariance; i.e., the Ward-Takahashi identity. However, this last statement cannot be directly applied to a theory defined in dimensions greater than four because of its peculiarities, in particular, the presence of an infinite tower of Kaluza-Klein states from the four-dimensional point of view.

This has motivated a vast number of studies on these issues, including the possibility of a power law behavior of the couplings [1,2] and finiteness of the radiative Higgs mass in Gauge-Higgs unification models [3,4]. In this models the Higgs is identified with the extra components of a gauge field in higher dimensions. We want to focus our attention on the calculation of the radiative mass of the extra-dimensional gauge boson with trivial holonomy, i.e., we will not consider noncontractible Wilson loops.

The physical intuition behind these ideas is that higher dimensional gauge invariance somewhat protects the Higgs from getting radiative contributions to its mass. And for this to be true, it is plain that at very short distances, physics must be really higher dimensional.

There are essentially three different ways to compute these corrections. We shall comment on them in turn, and argue eventually that if we want to formally implement the aforementioned ideas, a full higher dimensional computation is mandatory.

The correction has been often computed diagramatically once the mode expansion and the integral over the compact manifold had been performed, which means that in some sense this computation is purely four-dimensional because the Feynman rules applied correspond to a theory with an infinite number of Kaluza-Klein (KK) modes and their corresponding interactions (see for example [5,6]). The result of this kind of calculations is a one-loop finite mass for the Higgs field proportional to the compactification scale.

The main purpose of this paper is to repeat the calculation from a different point of view based on the discussion done in [7], wherein a systematic method for computing directly in higher dimensions is introduced, and, as we have already said, it is claimed that this is crucial because it is the only one that actually implements the physical intuition that at very short distances all dimensions should become visible, and at any rate, it does not give the same results as a purely four-dimensional calculation.

Let us indeed analyze the divergences of a quantum field theory using 't Hooft's ideas [8] applied to higher dimensions (which we shall always take to be either $n = 5$, or $n = 6$ for the sake of the argument)

We shall expand all fields[1] around an arbitrary background, $\bar{\phi}$,

$$\Phi_i = \bar{\phi}_i + \phi_i \tag{1}$$

(where the subindex stands for spacetime as well as internal degrees of freedom). The expansion of the action up to quadratic order in the fields is

$$iS = i\int d^n x\bigg[S(\bar{\phi}_k) + \bigg(\frac{\delta S}{\delta \Phi_l}(\bar{\phi}) + J_l\bigg)\phi_l + J_l\bar{\phi}_l + \phi_i\bigg(-\frac{1}{2}\Box\delta_{ij} - N^{\mu}_{ij}(\bar{\phi})\partial_\mu - \frac{1}{2}M_{ij}(\bar{\phi})\bigg)\phi_j\bigg]. \tag{2}$$

The term linear in the fluctuations is absent whenever the background is a solution of the equations of motion, which we will assume from here on. The partition function is given by the Gaussian integral

$$Z(\bar{\phi}, J) \equiv \int \mathcal{D}\phi e^{iS} = e^{i[S(\bar{\phi}) + \int d^n x J_l \bar{\phi}_l]}\det{}^{-1/2}[\mathcal{M}_{ij}], \tag{3}$$

where $\mathcal{M}_{ij}(\bar{\phi}) \equiv -\frac{1}{2}\Box\delta_{ij} - N^{\mu}_{ij}(\bar{\phi})\partial_\mu - \frac{1}{2}M_{ij}(\bar{\phi})$. It follows that

[1] Although we are doing this for bosonic fields only, it can be easily generalized to fermions with only minor modifications, along the lines of 't Hooft's original paper.

$$W(\bar{\phi}, J) \equiv -i \log Z(\bar{\phi}, J) = S(\bar{\phi}) + \int d^n x J_l \bar{\phi}_l + \frac{i}{2} \log \det[\mathcal{M}_{ij}(\bar{\phi})]. \tag{4}$$

The piece involving the determinant reads

$$\frac{i}{2} \log \det\left[\frac{i}{2}\Box\right] + \frac{i}{2} \log \det[\delta_{ij} + \Box^{-1}(2N^{\mu}_{ij}\partial_\mu + M_{ij})] = \frac{i}{2} \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n} \mathrm{tr}[\Box^{-1}(2N^{\mu}_{ij}\partial_\mu + M_{ij})]^n. \tag{5}$$

The last equality indicates the way of computing the determinant using Feynman diagrams. In four dimensions, the counterterm has got dimension four. The most general counterterm of mass dimension four is (taking into account that $[M] = 2$ and $[N] = 1$)

$$\Delta L = \frac{1}{8\pi^2 \epsilon} \mathrm{tr}[a_0 M^2 + a_1(\partial_\mu N_\nu)^2 + a_2(\partial_\mu N^\mu)^2 + a_3 M N_\alpha N^\alpha + a_4 N_\mu N_\nu \partial^\mu N^\nu + a_5(N_\alpha N^\alpha)^2 + a_6(N_\mu N_\nu)^2]. \tag{6}$$

In order to compute the coefficients, it is enough to compute a few selected diagrams. Before doing that, and also before studying the appropriate extension to five and six dimensions, it is convenient to recall a hidden symmetry again uncovered by 't Hooft.

Let us rewrite the Lagrangian in a compact notation as

$$L = \tfrac{1}{2}(\partial_\mu \phi + N_\mu \phi)^2 - \tfrac{1}{2}\phi X \phi \tag{7}$$

with

$$X \equiv M - N_\alpha N^\alpha. \tag{8}$$

There is now a manifest $O(N)$ invariance

$$\delta\phi = \Lambda(x)\phi(x), \qquad \delta N_\mu = -\partial_\mu \Lambda + [\Lambda, N_\mu], \qquad \delta X = [\Lambda, X]. \tag{9}$$

The one-loop counterterm must respect this symmetry. The most general Lorentz invariant, dimension four operator with this property reads

$$\Delta L_{n=4} = \frac{1}{8\pi^2 \epsilon} \mathrm{tr}[aX^2 + bF_{\mu\nu}F^{\mu\nu}], \tag{10}$$

where $F_{\mu\nu} \equiv \partial_\mu N_\nu - \partial_\nu N_\mu + [N_\mu, N_\nu]$.

Explicit computation yields

$$a = \tfrac{1}{4} \qquad b = \tfrac{1}{24}. \tag{11}$$

The whole of the preceding reasoning goes through to six dimensions, and actually to any *even* dimension. The most general counterterm (before using the background equations of motion) is given by

$$\Delta L_{n=6} = \frac{1}{8\pi^2 \epsilon} \mathrm{tr}[aX^3 + bD_\alpha F_{\beta\gamma} D^\alpha F^{\beta\gamma} + cXF_{\mu\nu}F^{\mu\nu} + dD_\alpha X D^\alpha X + eF_{\mu\nu}F^{\nu\rho}F^{\mu}_{\rho} + fD^2 D^2 X + gD_\alpha F^{\alpha\beta} D^\gamma F_{\gamma\beta}]. \tag{12}$$

Again, computation of a few diagrams fully determine the numerical coefficients. More efficient techniques based upon the heat kernel can also be applied (cf. [7]).

This same reasoning gives no candidate counterterms in five dimensions (nor in any *odd* dimension). What happens is the following. When using the proper time (explained, for example, in Collins' book [9]) representation of a propagator

$$\frac{1}{(p^2 - m^2)^a} = \int_0^\infty d\tau \tau^{a-1} e^{-\tau(p^2 - m^2)} \tag{13}$$

after using Feynman parameters in an arbitrary dimension, one ends up with integrals of the type

$$I \equiv \int_0^1 dx \int_0^\infty d\tau \tau \left(\frac{\pi}{-\tau}\right)^{n/2} e^{-\tau k^2 x(1-x)}. \tag{14}$$

The integral over proper time is then done, yielding a Gamma function

$$\Gamma(2 - n/2) \tag{15}$$

which has poles only for even values of the dimension. This integral over proper time involves an analytic continuation. The divergence can be isolated by imposing a cutoff in the lower limit of the integral (this is *not* a cutoff in momentum space, and is compatible with whatever symmetries the theory enjoys), and expanding the integrand in powers of τ

$$I \equiv \int_0^1 dx \int_{\Lambda^{-2}}^\infty d\tau \tau \left(\frac{\pi}{-\tau}\right)^{n/2} (1 - \tau k^2 x(1-x) + O(\tau^2)). \tag{16}$$

Using this procedure, we get in four dimensions a logarithmic divergence; just another language to express the divergence previously studied; symbolically,

$$\frac{1}{\epsilon} \sim \log\frac{\Lambda}{\mu}, \tag{17}$$

μ being an infrared cutoff. The difference is that this gives in five dimensions a nontrivial result, namely, a linear divergence. We can then write

$$\Delta L_{n=5} = \Lambda_{n=5} \mathrm{tr}[aX^2 + bF_{\mu\nu}F^{\mu\nu}]. \tag{18}$$

In six dimensions this gives both a quadratic and a logarithmic divergence. The general structure of the six-dimensional counterterm would then be

$$\Delta L_{n=6} = \Lambda_{n=6}^2 \operatorname{tr}[aX^2 + bF_{\mu\nu}F^{\mu\nu}] + \log\frac{\Lambda_{n=6}}{\mu_{n=6}} \operatorname{tr}[cX^3 + dD_\alpha F_{\beta\gamma}D^\alpha F^{\beta\gamma} + eXF_{\mu\nu}F^{\mu\nu} + fD_\alpha XD^\alpha X + gF_{\mu\nu}F^{\nu\rho}F_\rho^\mu + hD^2D^2X + iD_\alpha F^{\alpha\beta}D^\gamma F_{\gamma\beta}]. \quad (19)$$

We can now try to make precise the physical intuition that tells us that a five-dimensional theory in $\mathbb{R}^4 \times S^1$ would become four-dimensional in the limit in which the Kaluza-Klein scale M (the inverse of the radius of the circle) gets much bigger than any other scale. In this limit we can approach any five-dimensional integral by

$$\int d^5 x f(x^\mu, x_4) \sim \frac{1}{M}\int d^4 x \bar{f}(x^\mu) \quad (20)$$

(where x^μ are the usual four-dimensional coordinates). It is then possible (and necessary for mathematical consistency of the physical intuition) to choose the five and four-dimensional cutoffs in such a way that

$$\frac{\Lambda_{n=5}}{M} = \log\frac{\Lambda_{n=4}}{\mu_{n=4}}. \quad (21)$$

We can then contemplate a chain of reductions from six dimensions to five dimensions (at a scale M_6) and from five to four (at a scale M_5)[2]

$$\frac{\Lambda_{n=6}^2}{M_6} = \Lambda_{n=5} \qquad \frac{\Lambda_{n=5}}{M_5} = \log\frac{\Lambda_{n=4}}{\mu_{n=4}}. \quad (22)$$

The six-dimensional logarithmic divergence appears then in four dimensions as a log log divergence.

The preceding ideas lead to a general procedure to renormalize theories in a way consistent with dimensional reduction [7]. Let us examine its consequences in a couple of examples, namely, a version of $\mathrm{QED}_{n=5}$ and $\mathrm{QED}_{n=6}$.

II. SIX-DIMENSIONAL VACUUM POLARIZATION

As we show in the appendix, there are inherent ambiguities in a four-dimensional calculation and we will try to avoid them by computing directly in the higher dimensional space.

Suppose we have QED_6, quantum electrodynamics on a six-dimensional manifold. The theory is of course nonrenormalizable because the coupling constant has mass dimension $[e] = -1$. Nevertheless it is possible to identify and study all divergences appearing at one-loop order [or $O(e^2)$] [7].

The one-loop divergences are given, for a flat manifold and in terms of the gauge background field $\bar{A}_M$, and the fermionic backgrounds $\bar{\eta}$, η

$$\Delta L_{n=6} = \Lambda_{n=6}^2 \int \frac{d^6x}{(4\pi)^3}\left(\frac{4}{3}e^2\bar{F}_{MN}^2 + 4e^2\bar{\eta}\,\bar{\not{D}}\,\eta + 12me^2\bar{\eta}\eta\right) + \log\frac{\Lambda_{n=6}}{\mu_{n=6}}\int\frac{d^6x}{(4\pi)^3}\left(-\frac{4}{3}e^2m^2\bar{F}_{MN}\bar{F}^{MN} - 2e^2m^2\bar{\eta}\gamma^M\bar{D}_M\eta - 6e^2m^3\bar{\eta}\eta - -\frac{11}{45}e^2\bar{D}_R\bar{F}_{MN}\bar{D}^R\bar{F}^{MN} + \frac{23}{9}e^2\bar{D}_M\bar{F}^{MN}\bar{D}^R\bar{F}_{RN} + \frac{19}{15}e^2m\bar{\eta}\bar{D}_M\bar{D}^M\eta\right) + O(e^3). \quad (23)$$

In addition to the counterterms corresponding to operators that were already present in the original Lagrangian higher order operators have been generated radiatively. The appearance of these terms was discussed in [2,11]. If we want to absorb their divergences we must include them in the bare Lagrangian

$$\mathcal{L} = \mathcal{L}_0 + \mu D_M F^{MN}D^R F_{RN} + \lambda D_R F_{MN}D^R F^{MN} + \ldots, \quad (24)$$

where $[\mu] = [\lambda] = -2$. We have written explicitly only the extra terms that are quadratic in the gauge field and therefore the ones that modify the extra-dimensional vacuum polarization. Once we perform the mode expansion the same operators will yield the mass of the tower coming from the gauge field. If we define

$$A_M^0 = Z_3^{1/2}A_M, \quad (25)$$

we get

$$Z_3 = 1 + \frac{e^2}{12\pi^3}\Lambda_{(d=6)}^2 - \frac{e^2m^2}{12\pi^3}\log\frac{\Lambda_{(d=6)}^2}{\mu_{(d=6)}^2}. \quad (26)$$

It is easy to see then that the pole in F_{MN}^2 is absorbed in the wave function renormalization of the gauge field so from an extra-dimensional point of view there is no renormalization of the mass of the gauge boson.[3] This is expected in some sense due to gauge invariance. In four dimensions it is well known that even if we include a mass term for the photon in the bare Lagrangian its mass does not renormal-

[2]A somewhat similar analysis is done in the heat kernel language starting from supergravity in 11 dimensions by Fradkin and Tseytlin in [10].

[3]This can be seen by introducing a mass term $m_B^2 A_M A^M$ in the bare Lagrangian. Repeating the computation of [7] it is found that the only effect at one loop is the appearance of extra fermionic terms like $m_B^2 e^2(\bar{\psi}\not{D}\psi + m_f\bar{\psi}\psi)$ so m_B remains unrenormalized

ize. Nevertheless, gauge invariance is not enough to ensure a massless photon as we know from the Schwinger model in two dimensions. The lesson to learn from this is that the number of dimensions is crucial. Since in Gauge-Higgs unification the Higgs boson is identified with the extra-dimensional components of the gauge field once the mode expansion has been performed its mass does not renormalize either.

Concerning higher order terms, its divergences can be absorbed in arbitrary dimensionful couplings like μ and λ in (24) if we define

$$\mu_0 = Z_\mu \mu, \qquad \lambda_0 = Z_\lambda \lambda. \tag{27}$$

The conclusion is the very same as for F^2_{MN}: once we have renormalized the theory in six dimensions the mass coming from the mode expansions does not renormalize because the divergences are absorbed in Z_3 and Z_μ, Z_λ. Of course, to all orders of perturbation theory we would need an infinite number of arbitrary couplings to fit with experiments and this is precisely the benchmark for a nonrenormalizable field theory.

It is interesting to study the effects of this extra operators at tree level. First of all they induce corrections to the mass of the gauge bosons once the compactification has been performed. For example in six dimensions compactification of (24) yields terms like

$$(\mu + 2\lambda)|N|^4 A_\mu^{-n} A_n^\mu \tag{28}$$

and similar ones (i.e. of order $(2\lambda + \mu)M^4$) for the scalar field. Observe that at one loop we find a renormalization of the dimensionful couplings μ and λ that induces a running for the masses through (28) which is suppressed by M^{-2} (with respect to the usual mass).

Concerning the propagator suppose now that we include higher order terms in the form

$$F^2_{MN} + \frac{c_1}{\Lambda^2} F_{MN} \partial^2 F^{MN} + \frac{c_2}{\Lambda^2} F_{MN} \partial^M \partial_R F^{RN}, \tag{29}$$

where Λ is a parameter (naturally of the order of the compactification mass) in order to make c_1 and c_2 dimensionless. Then the propagator of the gauge field is

$$A_M D^{-1}_{MN} A_N = A_M \left(1 - \frac{2c_1 + c_2}{\Lambda^2} p^2\right)(p^2 \delta_{MN} - p_M p_N) A_N \tag{30}$$

It has the usual pole in $p = 0$, but also depending on the sign of the couplings c_1 and c_2 it can have another one

$$p^2 \sim \frac{M^2}{2c_1 + c_2}. \tag{31}$$

It may be possible to use arguments [12] concerning superluminal fluctuations around nontrivial backgrounds to fix the sign of the couplings and avoid this second pole. In any case possible poles coming from this higher order terms can be absorbed in dimensional coupling constants introduced in the bare Lagrangian in the form (24). Therefore, in some sense, the mass of the gauge field is protected from renormalization. In this respect, it is interesting to point out Smilga's conjecture [13] that there might exist consistent higher derivative theories, in particular, in six dimensions (although the zero mode instability is always present).

III. FIVE-DIMENSIONAL VACUUM POLARIZATION

Let us turn our attention to five-dimensional QED, on $\mathbb{R}^4 \times S^1$

$$S = \int d^5x \left(\frac{1}{4} F^2_{MN} + \bar{\psi}^i (\not{D}_{ij} + m_f \delta_{ij}) \psi^j \right). \tag{32}$$

We have doubled the fermion content of the theory and defined new matrices $(i, j = 1, 2)$

$$\gamma^M_{ij} \equiv \gamma^M \otimes \sigma^2_{ij} \tag{33}$$

that satisfy a modified Clifford algebra

$$\{\gamma^M, \gamma^N\}_{ij} = 2\delta^{MN} \otimes \delta_{ij} \tag{34}$$

for computational reasons.

Standard computations lead to the counterterm

$$\Delta L_{n=5} = \Lambda_{n=5} \int \frac{d^5 x}{(4\pi)^{5/2}} \left(\frac{4}{3} e^2 \bar{F}^2_{MN} + 3e^2 \bar{\eta}^i \bar{\not{D}}_{ij} \eta^j + 10 e^2 m_f \bar{\eta}^i \eta^i \right) \tag{35}$$

There are some differences with respect to the six-dimensional theory of the previous section. To be specific, there is no logarithmic divergence, neither are higher dimensional operators generated as counterterms (they start to appear at two-loop order).[4]

This result disagrees with the one obtained from a four-dimensional analysis even if one takes into account the whole KK tower [which will be reviewed in the Appendix, cf. the formula (A11)] unless we find some consistent definition of the infinite sums in such a way that each one vanishes.

In particular, as in six dimensions, gauge invariance forbids a mass term for the gauge boson. Therefore the masses of the modes once we make the expansion are protected because we can absorb the divergence in Z_3 as we argued in the last section. Explicitly

$$Z_3 = 1 + \frac{16 e^2}{3(4\pi)^{5/2}} \Lambda_{n=5}. \tag{36}$$

This is not the case from the four-dimensional point of view, as we see from (A11). A detailed description of the

[4] Dimensional regularization analysis yields a finite answer from a five-dimensional point of view. This is a well known result common to odd dimensional spacetimes at one-loop order, as we already mentioned in the introduction.

renormalization from the point of view of four dimensions is given in the appendix, where we point out the ambiguities and inconsistencies of that choice.

IV. CONCLUSIONS

We have analyzed radiative corrections in extra-dimensional theories not involving gravity. It has been shown that a four dimensional calculation is at least ambiguous when one considers the theory at one loop. There are two different ways of computing diagrams according to the place where the mode sum over all the KK tower is performed. When the sum is done after the momentum integral (which corresponds to the calculation of the appendix with the whole tower) usual four-dimensional divergences are found along with extra divergences coming from infinite sums. Also we find many problems with the divergence of the mass of the zero mode scalar because it is massless at tree level.

If one adopts, as we advocate, the higher dimensional point of view with the purpose of renormalizing the theory then the possible counterterms are dictated by gauge and Lorentz invariance in the extra-dimensional manifold. This fixes the form of the possible mass terms for the four-dimensional gauge boson as well as the Higgs in Gauge-Higgs unification. Therefore, it is easy to convince oneself that every divergence may be absorbed in the wave function renormalization of the gauge background $\bar{A}_M$ and the renormalization of the couplings of higher dimension operators such as μ and λ in (24).

This approach is, in our opinion, the only one that embodies the physical intuition, which we believe correct, that at very short distances all dimensions should appear at the same foot, and physics should be higher dimensional.

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APPENDIX: FOUR-DIMENSIONAL VACUUM POLARIZATION CORRESPONDING TO THE KK TOWER

In this appendix we will review the diagrammatical four-dimensional calculation in order to illustrate its inherent difficulties. We will follow closely the computation done in [5] but performing the sum over the extra-dimensional momentum at the end. Consider the vacuum polarization function of QED$_5$. If one of the dimensions corresponds to a circle S^1 with radius R then the momentum in that dimensions is quantized in units of $R^{-1} \equiv M$ and the integral has to be replaced by a sum. Taking into account the Feynman rules the vacuum polarization has the form ($p^2 = p_\mu p^\mu$)

$$\begin{aligned} i\Pi_{\mu\nu}(p^2, p_5^2) &= -e^2 \sum_{k_5} \int \frac{d^4k}{(2\pi)^4} \mathrm{tr}\Big(\gamma_\mu \frac{1}{\not{k} + i\gamma^5 k_5} \gamma_\nu \frac{1}{(\not{k} - \not{p}) + i\gamma^5(k_5 - p_5)}\Big) \\ &= -4e^2 \sum_{k_5} \int \frac{d^4k}{(2\pi)^4} \frac{k_\mu(k_\nu - p_\nu) + k_\nu(k_\mu - p_\mu) - g_{\mu\nu}k(k-p) + g_{\mu\nu}k_5(k_5 - p_5)}{(k^2 - k_5^2)((k-p)^2 - (k_5 - p_5)^2)}. \end{aligned} \tag{A1}$$

Introducing a Feynman parameter and doing the usual shift in the four-momentum $k'_\mu = k_\mu - \alpha p_\mu$ as well as a shift in the compact dimension $k'_5 = k_5 - \alpha p_5$ we get

$$i\Pi_{\mu\nu} = -4e^2 \sum_{k_5} \int_0^1 d\alpha \int \frac{d^4k}{(2\pi)^4} \times \frac{N_{\mu\nu}}{(k^2 - k_5'^2 + \alpha(1-\alpha)(p^2 - p_5^2))^2}, \tag{A2}$$

where the numerator is

$$N_{\mu\nu} = 2k_\mu k_\nu + g_{\mu\nu}(-k^2 + \alpha(1-\alpha)(p^2 - p_5^2) + (2\alpha - 1)p_5 k'_5 + k_5'^2) - 2\alpha(1-\alpha)p_\mu p_\nu \tag{A3}$$

And we have neglected terms linear in k_μ which vanish because of the angular integral. Let us then split the vacuum polarization into two pieces.

$$\Pi_{\mu\nu} \equiv g_{\mu\nu}\Pi_1 - p_\mu p_\nu \Pi_2 \tag{A4}$$

After Wick rotation to Euclidean space

$$\begin{aligned} \Pi_1 &= -4e^2 \sum_{k_5} \int_0^1 d\alpha \int \frac{d^4k}{(2\pi)^4} \times \frac{\frac{k^2}{2} + \alpha(\alpha - 1)(p^2 + p_5^2) + (2\alpha - 1)p_5 k'_5 + k_5'^2}{(k^2 + k_5'^2 + \alpha(1-\alpha)(p^2 + p_5^2))^2} \\ \Pi_2 &= 8e^2 \sum_{k_5} \int_0^1 d\alpha (1-\alpha)\alpha \int \frac{d^4k}{(2\pi)^4} \times \frac{1}{(k^2 + k_5'^2 + \alpha(1-\alpha)(p^2 + p_5^2))^2}. \end{aligned} \tag{A5}$$

Using a proper time parametrization the first piece can be put into the form

$$\Pi_1 = -4e^2 \sum_{k_5} \int_0^1 d\alpha \int_0^\infty d\tau \tau \int \frac{d^4k}{(2\pi)^4} \left(\frac{k^2}{2} + \alpha(\alpha - 1)(p^2 + p_5^2) + (2\alpha - 1) p_5 k'_5 + k'^2_5 \right) \times e^{-\tau(k^2 + k'^2_5 + \alpha(\alpha-1)(p^2+p_5^2))}. \tag{A6}$$

The integral in momentum space is obviously quadratically divergent, but it can be computed in dimensional regularization:

$$\Pi_1 = -\frac{4e^2\pi^{n/2}}{(2\pi)^n} \sum_{k_5} \int_0^1 d\alpha \int_0^\infty d\tau \tau \left(\frac{n}{4\tau^{(n/2)+1}} + \frac{\alpha(\alpha - 1)(p^2 + p_5^2) + (2\alpha - 1)p_5 k'_5 + k'^2_5}{\tau^{n/2}} \right) \times e^{-\tau(k'^2_5 + \alpha(\alpha-1)(p^2+p_5^2))}. \tag{A7}$$

It is now easy to perform the integral in proper time and particularize to $n = 4 + \epsilon$ dimensions to get

$$\begin{aligned} \Pi_1 &= -\frac{4e^2\pi^{n/2}}{(2\pi)^n} \Gamma\left(2 - \frac{n}{2}\right) \sum_{k_5} \int_0^1 d\alpha \left(\frac{n}{4(1 - \frac{n}{2})} (k'^2_5 + \alpha(\alpha - 1)(p^2 + p_5^2))^{(n/2)-1} \right. \\ &\quad \left. + (\alpha(\alpha - 1)(p^2 + p_5^2) + (2\alpha - 1)p_5 k'_5 + k'^2_5)(k'^2_5 + \alpha(1 - \alpha)(p^2 + p_5^2))^{(n/2)-2} \right) \\ &= \frac{e^2}{12\pi^2} \Gamma\left(-\frac{\epsilon}{2}\right) \left(p^2 + p_5^2 + \frac{1}{2} p_5^2 \right) \sum_{k_5} 1. \end{aligned} \tag{A8}$$

Similar manipulations with Π_2 yield

$$\Pi_{\mu\nu}(p^2, p_5^2) = \frac{e^2}{12\pi^2} \Gamma\left(-\frac{\epsilon}{2}\right) \left(\left(p^2 + p_5^2 + \frac{1}{2} p_5^2 \right) g_{\mu\nu} - p_\mu p_\nu \right) \sum_{k_5} 1. \tag{A9}$$

Note that the vacuum polarization of the four-dimensional photon $A_\mu^{(0)}$ (which means $p_5 = 0$) verifies the Ward-Takahashi identity

$$p^\mu \Pi_{\mu\nu}(p^2, p_5 = 0) = 0. \tag{A10}$$

From a four-dimensional point of view this result is not surprising at all. For fixed k_5 it corresponds to the contribution to the pole of a single fermionic loop. If we now consider an infinite number of fermions coupled with the same strength to the gauge bosons we have an additional divergence coming from the sum over the whole tower. The heat kernel computation done with the tower of modes in four dimensions seems to support this conclusion. The corresponding counterterm is

$$\begin{aligned} \Delta L_{n=4} &= \int \frac{d^4x}{(4\pi)^2} \sum_l \left(\frac{4}{3} e^2 \sum_n \bar{F}^n_{\mu\nu} \bar{F}^{\mu\nu}_{-n} - 4e^2 \sum_n \bar{A}_5^{-n} \Box \bar{A}_5^n - 16ie \frac{l}{R} \left(\frac{l^2}{R^2} + m_f^2 \right) \bar{A}_5^0 \right. \\ &\quad + 4e^2 \sum_n \left(2m_f^2 + \frac{2n^2 + (n + l)^2}{R^2} \right) \bar{A}_5^{n-l} \bar{A}_5^{l-n} + 4ie^3 \sum_{n,m} \frac{2m + l + n}{R} \bar{A}_5^{m-l} \bar{A}_5^{l-n} \bar{A}_5^{n-m} \\ &\quad - 4e^4 \sum_{n,m,s} \bar{A}_5^{m-l} \bar{A}_5^{l-s} \bar{A}_5^{s-n} \bar{A}_5^{n-m} + 8ie^2 \sum_n \frac{n}{R} \partial_\mu \bar{A}_5^n \bar{A}^\mu_{-n} + 4e^2 \sum_n \frac{n^2}{R^2} \bar{A}_n^\mu \bar{A}_\mu^{-n} + 6e^2 \sum_{n\neq 0} \bar{\eta}^i_{l-n} \partial\!\!\!/_{ij} \eta^j_{l-n} \\ &\quad - 6e^3 \sum_{n,m\neq 0} \bar{\eta}^i_{l-m} A\!\!\!/_{ij}^{l-n} \eta^j_{n-m} - 12e^3 \sum_{n,m\neq 0} \bar{\eta}^i_{l-m} \gamma^5_{ij} \bar{A}_5^{l-n} \eta^j_{n-m} + 12i \sum_{n\neq 0} \frac{l}{R} \bar{\eta}^i_{l-n} \gamma^5_{ij} \eta^j_{l-n} \\ &\quad + 20m_f \sum_{n\neq 0} \bar{\eta}^i_{l-n} \eta^i_{l-n} + 3e^2 \bar{\eta}^i_l \partial\!\!\!/_{ij} \eta^j_l - 3e^3 \sum_n \bar{\eta}^i_n A\!\!\!/_{ij}^{n-l} \eta^j_l - 6e^3 \sum_n \bar{\eta}^i_n \gamma^5_{ij} \bar{A}_5^{n-l} \eta^j_l + 6i \frac{l}{R} \bar{\eta}^i_l \gamma^5_{ij} \eta^j_l + 10m_f \bar{\eta}^i_l \eta^i_l \\ &\quad \left. + \left(6\frac{l^4}{R^4} - 8m_f^2 \frac{l^2}{R^2} - 4m_f^4 \right) \right) \log \frac{\Lambda_{n=4}}{\mu}. \end{aligned} \tag{A11}$$

From (A11), if we define the renormalized field and mass ($m_n^2 = \frac{n^2}{R^2}$)

$$A^n_{\mu(0)} = Z_3^{1/2} A^n_\mu, \qquad m^2_{n(0)} = Z_m m_n^2. \tag{A12}$$

Then we get

$$Z_3 = 1 + \frac{e^2}{3\pi^2\epsilon}\sum_l 1 \qquad Z_m = 1 + \frac{e^2}{6\pi^2\epsilon}\sum_l 1 \tag{A13}$$

There is an obvious divergence due to the infinite sum, which could be regularized, for example, by using a zeta function. Our results are, however, independent of the particular definition chosen. With the renormalization group functions

$$\beta_e \equiv \frac{\partial e}{\partial \log\mu} = \frac{e^3}{12\pi^2}\sum_l 1,$$
$$\beta_{m_n} \equiv \frac{\partial m_n}{\partial \log\mu} = -\frac{e^2 m_n}{12\pi^2}\sum_l 1. \tag{A14}$$

Notice that the beta function of the fine structure constant embodies an infinite number of identical fermion contributions. The behavior of the couplings is

$$e^2 = \frac{e_0^2}{1 - \frac{e_0^2}{6\pi^2}\sum_l 1 \log\frac{\mu}{\mu_0}},$$
$$m_n = m_n^0\left(1 - \frac{e_0^2}{6\pi^2}\sum_l 1 \log\frac{\mu}{\mu_0}\right)^{1/2}. \tag{A15}$$

The case of the scalar A_5^n, whose zero mode would play the role of the Higgs, is much more complicated. In any case one thing is clear: the correction is not finite even for the zero mode in the chiral theory $m_f = 0$. In fact, since A_5^0 is massless at tree level we cannot absorb the divergence at one loop. For consistency of the theory one must include a mass term in the original Lagrangian

$$\mathcal{L}_m = \tfrac{1}{2} m_B^2 A_M A^M \supset \tfrac{1}{2} m_B^2 A_5^0 A_5^0. \tag{A16}$$

But this is clearly non-gauge-invariant (except precisely for the zero mode). Another possibility is to include a mass term only for the zero mode in the compactified Lagrangian but it would make the theory lose all the advantages of Gauge-Higgs unification coming from extra-dimensional gauge invariance and the problems associated with the mass of a scalar would reappear.

Nevertheless this interpretation is in sharp contrast with the (also four-dimensional) one in [5] where a totally finite result was obtained.[5] In particular the correction to the mass of the nonzero KK modes is found to be

$$\delta m^2 = -\frac{e^2\zeta(3)}{4\pi^4}M^2. \tag{A17}$$

In the approximation $p^2 = p_5^2$. The reason of the difference is of course the point where the sum over the extra-dimensional momentum is performed.[6]

Suppose we are trying to do a purely five-dimensional calculation of the diagram. Before the compactification of the theory, let us say to $\mathbb{R}^4 \times S^1$, we have a full $O(1,4)$ invariance. In that case the momentum integral has trivially the property

$$\int \frac{d^5k}{(2\pi)^5} f(k^2) = \int \frac{d^4k}{(2\pi)^4}\int \frac{dk_5}{2\pi} f(k^2)$$
$$= \int \frac{dk_5}{2\pi}\int \frac{d^4k}{(2\pi)^4} f(k^2), \tag{A18}$$

which means that it is strictly equivalent to perform the integral first over the extra dimension and then the four dimensional one or vice-versa. If we now compactify the theory the full five-dimensional Lorentz invariance is spontaneously broken to $O(1,3)\times O(2)$. An essential ambiguity[7] appears then if we insist in interpreting the diagrams as five dimensional because clearly

$$\sum_{k_5}\int d^4k f(k_\mu, k_5) \neq \int d^4k \sum_{k_5} f(k_\mu, k_5). \tag{A19}$$

When the integral (or the sum) is divergent. Those two alternatives are then the two different four-dimensional calculations we were referring to above.

This observation is not new and a lot of effort has been put into studying its possible consequences, also when the expressions are not formally divergent. In [17] a brane Gaussian distribution along the extra dimension was used to regularize the theory while KK modes were not truncated. The integral can be performed and after the infinite sum the result is claimed to be finite. Similar conclusions were reached in [18] using Pauli-Villars and an adapted version of dimensional regularization. Both regulators are

[5]Some authors [14,15] have found quadratically divergent corrections with similar calculations, which suggests that this kind of computation is not very well established. There is also a quadratically divergent Fayet-Iliopoulos term in supersymmetric theories on orbifolds [16].

[6]In [5] a Poisson resummation is done before the proper time integral.

[7]The ambiguity is related to considering k_5 as a component of the five-momentum, but usually it is treated as a mass for the higher KK modes. Then, it is natural to do the summation after the evaluation of a single diagram because in that case k_5 simply labels fermions with different masses.

supposed to preserve the symmetries. The most explicit study of the validity of (A19) is that of [19] were a method to dimensionally regularize KK sums using Mellin transform and analytic extension of special functions is proposed. With this procedure it is believed that the ambiguity is resolved. Works with a similar philosophy can be found in [20] where the tower is summed using a pole function and in [21] were the sum is regularized using a ζ-function.

In any case, we believe that none of these works is fully satisfactory.

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