Chaos and order in models of black hole pairs

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Chaos in the orbits of black hole pairs has by now been confirmed by several independent groups. While the chaotic behavior of binary black hole orbits is no longer argued, it remains difficult to quantify the importance of chaos to the evolutionary dynamics of a pair of comparable mass black holes. None of our existing approximations are robust enough to offer convincing quantitative conclusions in the most highly nonlinear regime. It is intriguing to note that, in three different approximations to a black hole pair built of a spinning black hole and a *nonspinning* companion, two approximations exhibit chaos and one approximation does not. The fully relativistic scenario of a spinning test mass around a Schwarzschild black hole shows chaos, as does the post-Newtonian Lagrangian approximation. However, the approximately equivalent post-Newtonian Hamiltonian approximation does not show chaos when only one body spins. It is well known in dynamical systems theory that one system can be regular while an approximately related system is chaotic, so there is no formal conflict. However, the physical question remains: Is there chaos for comparable mass binaries when only one object spins? We are unable to answer this question given the poor convergence of the post-Newtonian approximation to the fully relativistic system. A resolution awaits better approximations that can be trusted in the highly nonlinear regime.

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I. INTRODUCTION

Isolated black holes are beautifully simple, as are the orbits of test particles around them. The elegance of the Kerr metric for a rotating black hole is impressive. Impressive too is Carter's [1] characterization of the orbits of a test particle in the Kerr spacetime. Carter [1] found enough constants of motion to prove that the geodesics around a Kerr black hole were perfectly regular; that is to say, integrable. However, general relativity no longer looks simple once we begin to consider two black holes. Two black holes pose a notoriously difficult problem and as yet have defied all attempts at a solution. Even more, the dynamics of two spinning black holes shows chaotic episodes [2–5].

In retrospect, it is no surprise that there is chaos in the orbits of two spinning black holes. The inner orbits around a nonspinning Schwarzschild black hole, although absolutely integrable, already provide prime terrain for the onset of chaos. Most notable is the presence of a hyperbolic fixed point, better known as an unstable circular orbit, with which is associated a homoclinic orbit—an orbit that approaches the circular orbit both in the infinite past and the infinite future. It is well known that under perturbation homoclinic orbits can give rise to a homoclinic tangle, an infinite intersection of the stable and unstable manifolds of a hyperbolic fixed point. The presence of a spinning companion can give rise to a homoclinic tangle and in the increasingly complicated set of bound orbits that results one finds chaos.

Although, in retrospect it is no surprise, the existence of chaos in the orbits of two spinning black holes has met with intense resistance. Still, there it is. Chaos was originally found and confirmed in several different approximations to the two-body problem [2–8]. Whether or not chaos will affect future detections from the gravitational wave observatories is still a cloudy issue but it is fair to say the experiments will not be very adversely affected and in an ideal world we might even have the opportunity one day to witness the onset of chaos.

A fascinating subtlety which deserves a little attention is the following: In three different approximations to the same physical system, two exhibit chaos and one does not. The physical system is one nonspinning black hole with a spinning companion. The three different approximations are (1) the extreme-mass-ratio limit of a Schwarzschild black hole orbited by a spinning companion (2) the post-Newtonian (PN) *Lagrangian* formulation of the two-black hole system with one body spinning, and (3) the PN *Hamiltonian* formulation of the two-black hole system with one body spinning. All three systems are purely conservative, so radiation reaction is effectively turned off.

Saying this another way, there is chaos in the full relativistic system when only one body spins. The chaos is absent in the PN Hamiltonian approximation to this fully relativistic system—at least it is absent up to order 3PN [9–11]. This is a reflection of the PN expansion's poor convergence to the full nonlinearities of general relativity. The chaos appears in an approximation to the approximation, namely, the PN Lagrangian formulation but at an order higher than the approximation can be trusted.

There is actually no explicit conflict between these results. They can all be correct. There can be chaos in the full system that goes away in an approximation to the system. The chaos that went away in an approximation can reappear in a related system. It is well known in dynamical systems theory that a regular, that is to say nonchaotic, system can become chaotic under a small perturbation. In fact there is a famous theorem that helped identify the locus of chaos under small perturbations, the Kolmogorov-Arnol'd-Moser (KAM) theorem [12–14].

A quick sketch of the argument begins with a regular Hamiltonian system H_0 with N coordinates and N conjugate momenta. If there are N constants of motion, then the system is decidedly integrable and not at all chaotic. A canonical transformation to action-angle coordinates, $(\Theta, \mathbf{I})$, can be performed so that each of the new conjugate momenta are set equal to one of the N constants of motion and the Hamilton equations become

$$\dot{\mathbf{I}} = 0, \qquad \dot{\Theta} = \frac{\partial H}{\partial \mathbf{I}} = \omega(\mathbf{I}). \tag{1}$$

The angular frequencies only depend on the constant $\mathbf{I}$ and so are also constant. Therefore the motion in each coordinate direction is cyclical and orbits are confined to an N-dimensional torus. Now the KAM theorem shows that, under a small perturbation, the motion will remain quasiperiodic for most initial data and that the remaining orbits that do not remain quasiperiodic occupy a region of phase space as small as the perturbation is small. Generally speaking, these latter orbits correspond to resonant tori which are destroyed under perturbation and allow for the onset of chaos. The gist is that one system can be regular while an approximately related system is chaotic. So, for instance, the Lagrangian formulation of the PN two-black hole dynamics can show chaos while the approximately equivalent Hamiltonian formulation does not. (It should be emphasized that the Lagrangian and Hamiltonian equations of motion are related by a gauge transformation *and an approximation*.)

The deeper question that emerges is this: If there is chaos in one formulation of the dynamics but not in an approximately equivalent formulation, which one is right? That is, is there chaos for one spinning body or not? The answer is this: The full relativistic system exhibits chaos when only one body spins [5]. That should be the final story. The absence of chaos in the PN Hamiltonian approximation to the case of one body spinning must be a consequence of the slow convergence of the PN expansion to the fully relativistic system. Some of the nonlinearity is omitted in the approximation. So physically, the fully relativistic result is the correct result.

However, it was also shown in Ref. [5] that around a Schwarzschild black hole a spinning test particle will become chaotic only for unphysically large spins. If the heavy black hole has mass m_1 and the light companion has mass m_2, then the spin of the light companion must be $S_2 > 1\mu M = 1m_2^2(m_1/m_2)$, where $M = m_1 + m_2$ and $\mu = (m_1 m_2)/M$. This spin is much larger than maximal given the extreme mass ratio $m_1/m_2 \gg 1$. The physical question we can worry about now is: Will there be chaos for *comparable mass* binaries at a physical value of the spin, $S \leq m^2$, when only one object spins? This has yet to be determined.

For comparable mass systems with physically accessible spins, the 2PN Hamiltonian dynamics says no, there is no chaos if only one object spins, while the 2PN Lagrangian dynamics says yes, there can be chaos even if only one body spins. Neither is definitive. After all, the 2PN Hamiltonian approach says there is no chaos in the extreme mass ratio case when we know that there is chaos in the fully relativistic system. On the other hand, the chaos seen in the 2PN Lagrangian system is of higher order than the approximation can be trusted.

The physical question can only be resolved when the chaos appears or disappears consistently at the same order as the approximation is valid. So I do not claim to resolve it here. Instead, I take a moment in the following section to give a quick demonstration of the difference between the Hamiltonian and Lagrangian approaches and to show that the former is regular and the latter allows chaos when only one body spins.

II. THE PN HAMILTONIAN FORMULATION

The absence of chaos in the Hamiltonian formulation was recently argued in Ref. [9] for the dynamics of two compact objects when only one of the bodies spins. (They find similarly that there can be no chaos when the binaries are of equal mass.) To be clear, the dynamics is conservative, computed to second order in the post-Newtonian expansion, and spin effects are limited to the spin-orbit couplings only. In all other situations, there can be chaos—if both objects spin and are not of equal mass and/or spin-spin couplings are included. When (i) only one body spins or (ii) when the binaries are equal mass, the argument against chaos in the Hamiltonian dynamics is that there are enough (exactly conserved) constants of motion to prove that the system is integrable in these two simplified cases. Even more, the authors were able to find parametric solutions to the Hamiltonian dynamics if only one body spins. In accord with their claim, the technique of fractal basin boundaries used in Refs. [2,3,7] indeed confirms that there is no chaos in the Hamiltonian formulation when only one body spins, as will be shown here. We will confirm that the basin boundaries for this case are smooth and not fractal and therefore are entirely consistent with regular, nonchaotic dynamics.

The Hamiltonian is currently available to 3PN order. However, for comparison with the Lagrangian formulation, we will only write the Hamiltonian explicitly up to 2PN order. The reduced 2PN-Hamiltonian in ADM coordinates is (with $\mathbf{r}$ measured in units of M and $\mathbf{p}$ measured in units of μ)

$$H = H_N + H_{1\mathrm{PN}} + H_{2\mathrm{PN}} \tag{2}$$

with terms

$$H_N = \frac{\mathbf{p}^2}{2} - \frac{1}{r}, \tag{3}$$

$$H_{1PN} = \frac{1}{8}(3\eta - 1)(\mathbf{p}^2)^2 - \frac{1}{2}[(3+\eta)\mathbf{p}^2 + \eta(\mathbf{n}\cdot\mathbf{p})^2]\frac{1}{r} + \frac{1}{2r^2}, \tag{4}$$

$$H_{2PN} = \frac{1}{16}(1 - 5\eta + 5\eta^2)(\mathbf{p}^2)^3 + \frac{1}{8}[(5 - 20\eta - 3\eta^2)(\mathbf{p}^2)^2 - 2\eta^2(\mathbf{n}\cdot\mathbf{p})^2\mathbf{p}^2 - 3\eta^2(\mathbf{n}\cdot\mathbf{p})^4]\frac{1}{r} + \frac{1}{2}[(5+8\eta)\mathbf{p}^2 + 3\eta(\mathbf{n}\cdot\mathbf{p})^2]\frac{1}{r^2} - \frac{1}{4}(1+3\eta)\frac{1}{r^3}. \tag{5}$$

The spin-orbit Hamiltonian is

$$H_{SO} = \frac{\mathbf{L}\cdot\mathbf{S}_{\rm eff}}{r^3} \tag{6}$$

with the reduced angular momentum $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ and

$$\mathbf{S}_{\rm eff} = \left(2 + \frac{3m_2}{2m_1}\right)\mathbf{S}_1 + \left(2 + \frac{3m_1}{2m_2}\right)\mathbf{S}_2. \tag{7}$$

Spin-spin coupling terms are not included. The spins $\mathbf{S}_i$ are measured in units of M^2. To be clear, $\mathbf{S}_i = \mathbf{A}_i(m_i/M)^2$ with physical values of the amplitude $|\vec{A}_i| \leq 1$.

The equations of motion are given by

$$\dot{\mathbf{r}} = \frac{\partial H}{\partial \mathbf{p}}, \qquad \dot{\mathbf{p}} = -\frac{\partial H}{\partial \mathbf{r}} \tag{8}$$

and the evolution equation for the spins and the angular momentum can be found from the Poisson brackets:

$$\dot{\mathbf{S}}_1 = \{\mathbf{S}_1, H\} = \left(2 + \frac{3m_2}{2m_1}\right)\frac{\mathbf{L}\times\mathbf{S}_1}{r^3} \tag{9}$$

$$\dot{\mathbf{S}}_2 = \{\mathbf{S}_2, H\} = \left(2 + \frac{3m_1}{2m_2}\right)\frac{\mathbf{L}\times\mathbf{S}_2}{r^3} \tag{10}$$

$$\dot{\mathbf{L}} = \{\mathbf{L}, H\} = \frac{\mathbf{S}_{\rm eff}\times\mathbf{L}}{r^3}. \tag{11}$$

A parametric solution for the equations of motion including spin-orbit coupling (but not including spin-spin coupling) has been found for one body spinning [9]. Even without a solution, a count of conserved quantities shows that motion must lie on a torus. Therefore this one case is integrable and should show no evidence of chaos. The fractal basin boundary method [15] confirms that there is no chaos when only one body spins. Figure 1 shows the smooth basin between outcomes for such a black hole pair. The pair is evolved using the 2PN Hamiltonian including spin-orbit couplings for 40 000 different initial conditions.

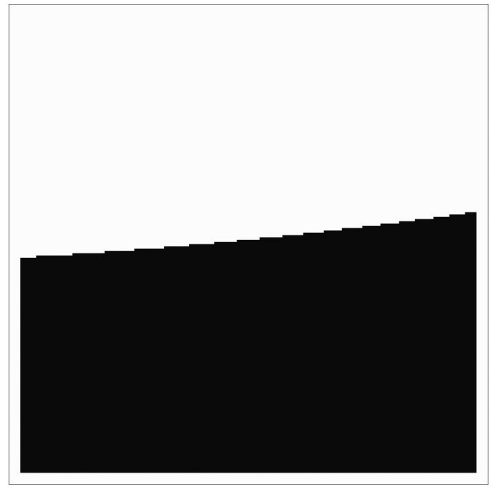

FIG. 1. Basin boundaries in the 2PN-Hamiltonian system with spin-orbit couplings. The pair has mass ratio $m_2/m_1 = 1/3$. The heavier black hole is maximally spinning ($S_1 = m_1^2$) with an initial angle with respect to the $\hat{z}$-axis of 95° while the lighter companion is not spinning ($S_2 = 0$). The initial center of mass separation in ADM coordinates is $r_i/M = 5$. The orbital initial conditions vary along the x-axis from $0.02 \leq p_r \leq 0.05$ and along the y-axis from $0.1195 \leq p_\phi \leq 0.1200$. 200×200 orbits are shown. Initial conditions that are color-coded white correspond to stable orbits and those color-coded black correspond to merging pairs. The basin boundary is clearly smooth and not fractal. Therefore, there is no evidence of chaos.

The orbits differ in initial p_r and initial p_ϕ. If the pair merges, the initial condition is color-coded black. If the pair executes more than 50 windings, the initial condition is color-coded white. The boundary between initial basins—merger and stability—is smooth. There is no evidence of chaos in this slice through phase space nor any of the others surveyed. Chaos manifests as an extreme sensitivity to initial conditions and a mixing of trajectories. A smooth boundary shows no such sensitivity or mixing of orbits; that is, stable orbits remain on one side of the clean boundary and do not mix with unstable orbits, which remain on the other side of the smooth boundary.

The exact location and shape of the boundary can depend on the exit criteria, but the criteria cannot turn a smooth basin into a fractal. For instance, the criteria used are that merger occurs when the coordinate $r \leq 1$ and stability corresponds to more than 50 windings. It is worth noting that the basin boundaries remain smooth when the 3PN-Hamiltonian is used as well.

It should be emphasized that, while the presence of fractal basin boundaries provides an unambiguous signal of chaos, the *absence* of fractals at the basin boundary does not prove the system is integrable. We only show the smooth basins here to demonstrate consistency between the two different approaches of Refs. [2,3,7,9]. As no fractals were found despite many scans of various regions of phase space, the basin boundaries are consistent with the

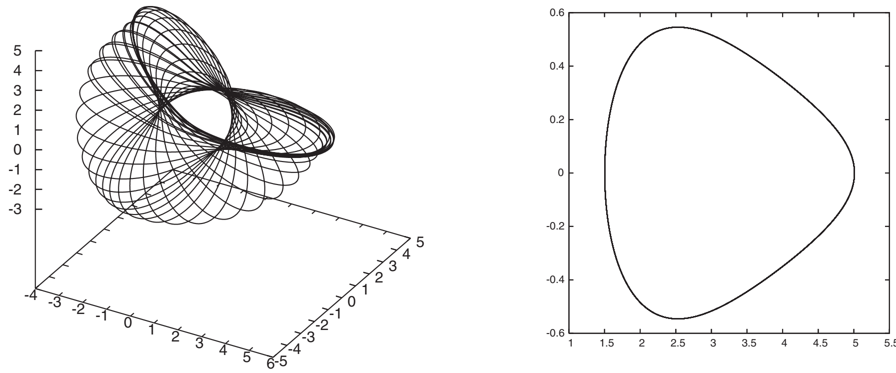

FIG. 2. Left: An orbit taken from the stable basin of Fig. 1. The initial conditions for the orbit are $p_r = 0.03$ and $p_\phi = 0.206\,45$. Right: A projection of the phase space motion onto the (r, p_r) plane. The topology of the projection confirms that the motion clearly lies on a torus and therefore is not chaotic.

system being integrable, although it does not provide a proof of integrability.

An orbit drawn from near the smooth basin boundary of Fig. 1 is shown in Fig. 2. A Poincaré surface of section could be taken to verify that the motion is confined to a torus in phase space. However this is not necessary since even a projection of the full phase space motion onto the (r, p_r) plane lies on a line with the topology of a torus. That projection is also shown in Fig. 2. As must be the case, both the collective behavior of orbits and the individual behavior of orbits is regular and not chaotic.

III. THE PN LAGRANGIAN FORMULATION

There is chaos in the *approximately* equivalent Lagrangian formulation of the dynamics when only one body spins—for a very narrow range of parameters [3]. The chaos appears in some sense at higher than second order in the PN parameters.

The equations of motion in harmonic coordinates [16,17] do not correspond to a conventional Lagrangian [18,19] but can be derived from a generalized Lagrangian that depends on the coordinates, the velocities, as well as a relative acceleration. In the following, the variables $(\mathbf{r}, \mathbf{p})$ will continue to refer to ADM coordinates and their conjugate momenta while harmonic coordinates and their velocities are denoted $(\mathbf{x}, \mathbf{v})$ with $\mathbf{v} = \dot{\mathbf{x}}$.

The generalized Lagrangian can be expressed in terms of the ADM Hamiltonian. In Ref. [20], the map between the ADM coordinates and the harmonic coordinates is given explicitly. Notice that the derivative on $\mathbf{r}$ in the ADM Lagrangian will generate higher derivative acceleration $(\ddot{\mathbf{x}})$ terms when reexpressed in harmonic coordinates:

$$L(\mathbf{x}, \dot{\mathbf{x}}, \ddot{\mathbf{x}}) = \mathbf{p}[\mathbf{x}, \dot{\mathbf{x}}] \cdot \dot{\mathbf{r}}[\mathbf{x}, \dot{\mathbf{x}}, \ddot{\mathbf{x}}] - H(\mathbf{r}[\mathbf{x}, \dot{\mathbf{x}}], \mathbf{p}[\mathbf{x}, \dot{\mathbf{x}}]). \tag{12}$$

Reference [20] presents the expression for the harmonic Lagrangian derived from Eq. (12) explicitly. The Euler-Lagrange equations are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\mathbf{x}}} - \frac{d}{dt}\frac{\partial L}{\partial \ddot{\mathbf{x}}}\right) = \frac{\partial L}{\partial \mathbf{x}}. \tag{13}$$

For our purposes it is important to recognize that in order to derive the Lagrangian equations of motion of Refs. [16,17], wherever the acceleration appears in higher-order terms in the Euler-Lagrange equation, the lower-order equation of motion is substituted in its place. This is an essential point. As a result of this substitution, the Lagrangian formulation and the Hamiltonian formulation are only *approximately* related. Higher-order terms that make them exactly equal are explicitly discarded to render them only approximately equal. According to the KAM theorem, while most tori survive and therefore lead to nonchaotic orbits, there is a complement set related to the size of the perturbation between them for which tori are often destroyed leading to chaos.

Furthermore, it is worth noting that for similar reasons the "constants of motion" are only *approximately* conserved in the Lagrangian system. This is to be contrasted with the Hamiltonian system for which no such substitution is required and the equations of motion that result from Hamilton's equations *exactly* conserve the constants of motion. Therefore there can never be chaos in the ADM-Hamiltonian formulation when only one body spins but *there can be chaos* in the harmonic-Lagrangian equations when only one body spins. The chaos must be at a higher order than the order at which the equations are valid since the constants of motion are only violated at higher orders. This is consistent with the KAM theorem statement that tori will only be destroyed for a set of size related to the size of the perturbation.

Put yet another way, the ADM-Hamiltonian equations of motion and the harmonic-Lagrangian equations of motion are equivalent to 2PN order. However, from a dynamical systems theory perspective, one can take these two sets of equations and ask if they are identical, which would require that they are identical to all orders. Since they are not identical to all orders, they can show different features and they do. The different features must be higher than 2PN order, and they are. So a numerical integration of the two sets of equations of motion will show higher order differences.

The resultant Lagrangian equations of motion in $\mathbf{x}$, $\mathbf{v}$ with spins added are (with $x = \sqrt{\mathbf{x}\cdot\mathbf{x}}$ the harmonic radial coordinate) [16,17]

$$\dot{\mathbf{x}} = \mathbf{v} \tag{14}$$

$$\ddot{\mathbf{x}} = \mathbf{a}_N + \mathbf{a}_{1PN} + \mathbf{a}_{2PN} + \mathbf{a}_{SO} \tag{15}$$

and

$$\begin{aligned}
\mathbf{a}_N &= -\frac{\mathbf{n}}{x^2},\\
\mathbf{a}_{1PN} &= -\frac{\mathbf{n}}{x^2}\left\{(1+3\eta)\mathbf{v}^2 - 2(2+\eta)\frac{1}{x} - \frac{3}{2}\eta\dot{x}^2\right\} - \frac{\mathbf{v}}{x^2}2(2-\eta)\dot{x},\\
\mathbf{a}_{2PN} &= -\frac{\mathbf{n}}{x^2}\left\{\frac{3}{4}(12+29\eta)\frac{1}{x^2} + \eta(3-4\eta)(\mathbf{v}^2)^2 + \frac{15}{8}\eta(1-3\eta)\dot{x}^4\right.\\
&\quad\left. - \frac{3}{2}\eta(3-4\eta)\mathbf{v}^2\dot{x}^2 - \frac{1}{2}\eta(12-4\eta)\frac{\mathbf{v}^2}{x} - (2+25\eta+2\eta^2)\frac{\dot{x}^2}{x}\right\}\\
&\quad - \frac{\mathbf{v}}{x^2}\left\{-\frac{\dot{x}}{2}\left[\eta(15+4\eta)\mathbf{v}^2 - (4+41\eta+8\eta^2)\frac{1}{x} - 3\eta(3+2\eta)\dot{x}^2\right]\right\}.
\end{aligned} \tag{16}$$

The spin-orbit contribution to the acceleration is

$$\begin{aligned}
\mathbf{a}_{SO} = \frac{1}{x^2}\Bigg\{&6\mathbf{n}\left[(\mathbf{n}\times\mathbf{v})\cdot\left(2\mathbf{S} + \frac{\delta M}{M}\Delta\right)\right]\\
&- \left[\mathbf{v}\times\left(7\mathbf{S} + 3\frac{\delta M}{M}\Delta\right)\right]\\
&+ 3\dot{x}\left[\mathbf{n}\times\left(3\mathbf{S} + \frac{\delta M}{M}\Delta\right)\right]\Bigg\},
\end{aligned} \tag{17}$$

with $\Delta = M(\mathbf{S}_2/m_2 - \mathbf{S}_1/m1)$, $\eta = \mu/M$, and $\delta M = m_1 - m_2$. There are also spin-spin terms but they vanish in the event that one of the black holes is spinless and so are omitted here in the comparison. Finally, the spins precess according to

$$\begin{aligned}
\dot{\mathbf{S}}_1 &= \frac{(\mathbf{x}\times\mu\mathbf{v})\times\mathbf{S}_1}{x^3}\left(2 + \frac{3m_2}{2m_1}\right),\\
\dot{\mathbf{S}}_2 &= \frac{(\mathbf{x}\times\mu\mathbf{v})\times\mathbf{S}_2}{x^3}\left(2 + \frac{3m_1}{2m_2}\right).
\end{aligned} \tag{18}$$

Equations (14)–(18) constitute the Lagrangian dynamical system.

Figure 3 shows fractal boundaries at the basins of stability and merger in a slice through phase space. The fractal boundaries prove that there is chaotic scattering in this region of phase space for the Lagrangian approximation [21]. Two basins are shown for black holes with a mass ratio $m_2/m_1 = 1/3$. For the basin boundaries shown on the left of Fig. 3, the heavier black hole is maximally spinning while the lighter companion has no spin. For the basin boundaries on the right of the figure, the lighter black hole is maximally spinning while the heavier black hole has no spin. In both cases, the boundaries show extreme sensitivity to initial conditions and a mixing of orbits.

To be clear, this is the same physical system (although it is not the exact same slice through phase space) as that shown in Fig. 1 for the Hamiltonian approximation. But unlike the Hamiltonian approximation, which always had smooth boundaries, the Lagrangian approximation shows fractal boundaries. It is also essential to note that these

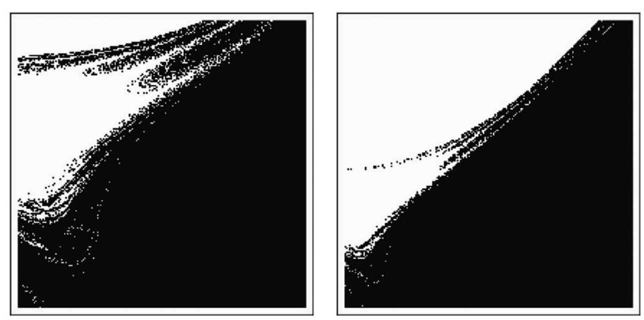

FIG. 3. The pair has mass ratio $m_2/m_1 = 1/3$. The initial center of mass separation in harmonic coordinates is $5M$. Using the notation $\mathbf{x} = (x^{(1)}, x^{(2)}, x^{(3)})$, the orbital initial conditions vary along the x-axis from $0 \le \dot{x}^{(1)} \le 0.035$ and along the y-axis from $0.425 \le \dot{x}^{(2)} \le 0.443125$. 200×200 orbits are shown. Initial conditions that are color-coded white correspond to stable orbits and those color-coded black correspond to merging pairs. The basin boundary is fractal indicating at least a thin region of chaos. Left: A reproduction of Fig. 4 from Ref. [3]. The heavier black hole is maximally spinning ($S_1 = m_1^2$) with an initial angle with respect to the $\hat{z}$-axis of 95° while the lighter companion is not spinning ($S_2 = 0$). Right: The heavier black hole is not spinning ($S_1 = 0$) while the lighter object is maximally spinning ($S_2 = m_2^2$) with an initial angle with respect to the $\hat{z}$-axis of 95°.

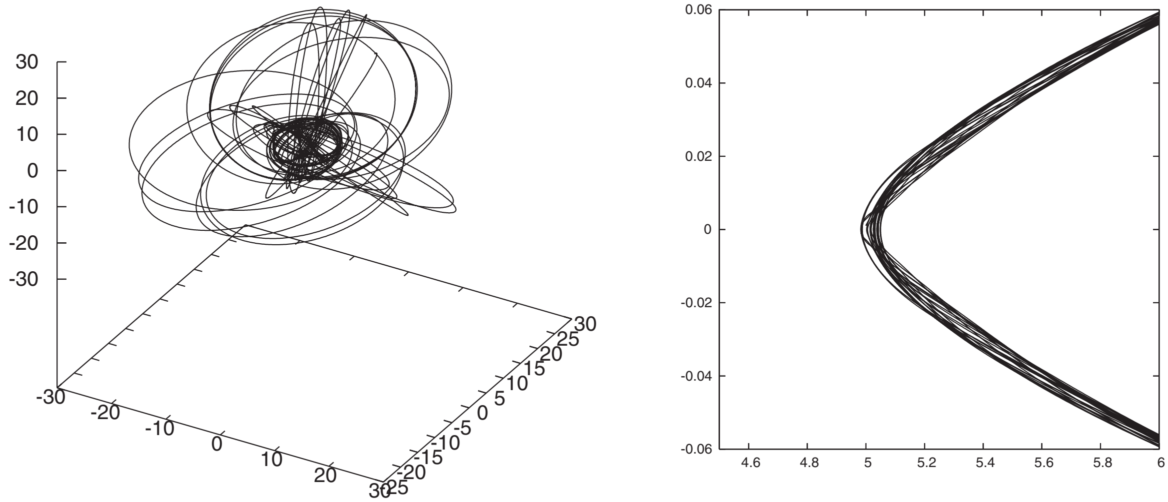

FIG. 4. Left: An orbit taken from the stable basin of Fig. 3. The initial conditions for the orbit are $\dot{x}^{(1)} = 0.001\,05$ and $\dot{x}^{(2)} = 0.430\,74$. Right: A detail of the projection of the motion in phase space onto the radial $(x, \dot{x})$ plane, where again $x = \sqrt{\mathbf{x}\cdot\mathbf{x}}$.

boundaries are extremely thin. For the slice through phase space in Fig. 3, 1PN corrections are roughly $v^2 < 0.2$, 2PN corrections are roughly $v^4 < 0.04$, and so corrections higher than the order at which the approximation is valid appear around $v^6 < 0.008$. The sensitivity to initial conditions seen in the boundaries only appears for differences in initial velocity of $\ll 0.005$, which indicates that 3PN corrections are at least this large. The 2PN approximation is being pushed beyond its limits. Again, this speaks to the consistency of the two approximations at the 2PN level. The conclusion can only be that there is chaos in the 2PN Lagrangian approximation when only one body spins but it is difficult to draw physical conclusions since it only appears at such high decimal places.

An orbit drawn from the stable basin of Fig. 3 is shown in Fig. 4. Also shown is a detail of the projection of the motion onto the radial $(x, \dot{x})$ plane, where again $x = \sqrt{\mathbf{x}\cdot\mathbf{x}}$. The projection does not lie on a line with the topology of a torus, indicating that the motion may not be regular. Now being careful, it might be the case that the spread off of a torus shown in this detail is an artifact of projecting the multidimensional coordinate space down to the radial $(x, \dot{x})$ plane. Consequently, this does not prove that the motion is lifted off of a torus, it only suggests that it has diffused off of a torus. It is therefore not proof of chaos, only a suggestion of chaotic motion. A proper Poincaré surface of section in a properly defined phase space would need to be taken to prove the orbit is chaotic. Instead, however, we confirmed that this orbit does have a positive Lyapunov exponent [6,7], which proves that this orbit is indeed chaotic.

Not all of the orbits drawn from the stable basin will be chaotic. Some will remain on tori in phase space and will have zero Lyapunov exponent. Those that do show chaos seem to lie very close to the already thin fractal basin boundary. This again confirms the expectation that those tori that are destroyed occupy a very thin region of phase space corresponding to perturbations to the Hamiltonian system that are higher than second order.

IV. SUMMARY

In the previous section, it was shown that the same methods applied to the 2PN-Hamiltonian formulation and the 2PN-Lagrangian formulation found no chaos in the former and chaos in the latter. There is no conflict between these results as the Hamiltonian approach and the Lagrangian approach are only approximately related. One of the very underpinnings of the development of chaos theory is the realization that a regular Hamiltonian system can become chaotic under small perturbation. The Lagrangian formulation can be viewed as a small perturbation to the Hamiltonian system and so the emergence of chaos is permitted—if subtle.

It could be argued that the Hamiltonian approach is more appealing analytically since the constants of motion are exactly conserved. This is always an advantage, particularly in numerical studies in which the constants can be tracked and their constancy continually checked. The derivation of the equations of motion is also cleaner and more direct. Regardless, we have not resolved the physical question: Is there chaos in the orbits of comparable mass binaries when only one spins? It is highly likely that when spin-spin terms are included, the Hamiltonian approximation will also show chaos even for the special case of one body spinning—although the effect may continue to appear at orders higher than the approximation can be trusted. All we can be sure of at this point is that we have another reflection of the poor convergence of the PN expansion to the full relativistic system. In our hopes to provide gravitational wave templates for the gravitational wave observatories, we are tempted to push these approximations into regimes where they are not faring well [22]. In an ideal world, we would have an excellent approximation that remained valid at small separations, near the innermost stable circular orbit, and in the most highly nonlinear regime. That we do not have. In the interim, all we can do is treat the approximations as dynamical systems and see what emerges. Quite fascinatingly, what has emerged is two different claims on the chaotic behavior of comparable mass binaries.

Just to be clear, there is inarguably chaos at physically accessible values of the spins when *both objects spin*. This was shown in the initial papers [2,3,7] and was verified in the Hamiltonian formulation in Ref. [4]. The authors of Ref. [4] were able to investigate a wide range of parameters to conclude that chaos did not appear to have a huge impact on gravitational waves in the LIGO bandwidth. Similarly, we have shown that the damping effects of dissipation when the radiation reaction is included [6–8] can squelch chaos. One can imagine that these conclusions will remain more or less intact even at higher order. But we cannot know for certain. One can also imagine that the chaos will worsen as our currently poor approximations to the full relativistic problem improve. My projection is that the chaos will worsen a bit but will not plague the ground-based gravitational wave observatories in a detrimental way. Regardless, chaos will continue to be important primarily at very late stages of inspiral, such as the transition from inspiral to plunge, and cannot be disregarded in our theoretical attempts to understand the dynamics of black hole pairs.

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