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Holography of gravitational action functionals

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Einstein-Hilbert (EH) action can be separated into a bulk and a surface term, with a specific (''holographic'') relationship between the two, so that either can be used to extract information about the other. The surface term can also be interpreted as the entropy of the horizon in a wide class of spacetimes. Since EH action is likely to just the first term in the derivative expansion of an effective theory, it is interesting to ask whether these features continue to hold for more general gravitational actions. We provide a comprehensive analysis of Lagrangians of the form $\sqrt{-g}L = \sqrt{-g}Q_a{}^{bcd}R^a{}_{bcd}$, in which $Q_a{}^{bcd}$ is a tensor with the symmetries of the curvature tensor, made from metric and curvature tensor and satisfies the condition $\nabla_c Q_a{}^{bcd} = 0$, and show that they share these features. The Lanczos-Lovelock Lagrangians are a subset of these in which $Q_a{}^{bcd}$ is a homogeneous function of the curvature tensor. They are all holographic, in a specific sense of the term, and—in all these cases—the surface term can be interpreted as the horizon entropy. The thermodynamics route to gravity, in which the field equations are interpreted as $TdS = dE + pdV$, seems to have a greater degree of validity than the field equations of Einstein gravity itself. The results suggest that the holographic feature of EH action could also serve as a new symmetry principle in constraining the semiclassical corrections to Einstein gravity. The implications are discussed.

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I. INTRODUCTION

Holography, in different guises, has been an attractive and influential concept in high energy physics [1]. This term is used in different contexts to indicate different features, with the common thread being the existence of a relation between the dynamics in $(D-1)$-dimensional space (the ''surface,'' $\partial\mathcal{V}$) and the dynamics in the D-dimensional space (the ''bulk,'' $\mathcal{V}$). In some of the string theory contexts, ''holography'' refers to a relation between two *different* theories, one living on the surface and the other on the bulk. In many other contexts (like in the discussion of holographic bounds on entropy), this term is used to relate the degrees of freedom of the *same* theory in the surface and bulk. In this paper, we adopt the latter usage and will investigate the existence of relations between the surface and bulk terms of classical gravitational action functionals. We will describe some specific features present in a generic, generally covariant action of gravity, which could be termed as holography at the classical level. Our key motivation is the known fact [2] that the Einstein-Hilbert action can be split into a bulk action and a surface term with a specific kind of relationship between the two. (The term holography was previously used in this context.) In the splitting we destroy the manifest general covariance of the action; however, the relationship between the bulk and the surface term and the fact that they add up to produce a generally covariant action specifies each of them uniquely. Our aim will be to generalize this feature to a wider class of gravitational action functionals. We will review in the next section how this splitting can be achieved and motivate the relationship between the split parts from the variational principle. We will further show how this splitting could be generalized for the case of Lanczos-Lovelock actions [3]. We obtain a specific functional relationship between the bulk action and the surface term, which would be a natural generalization of the Einstein-Hilbert case. This procedure would also specify the surface term uniquely. Since this is quite in the spirit of holography, we shall use this term to describe such actions with the clear understanding that the term is used in a specific sense.

In the case of the Einstein-Hilbert action with the Lagrangian $L_{\rm EH}[\partial^2 g, \partial g, g]$, the separation into bulk and surface terms $L_{\rm EH}\sqrt{-g} = L_{\rm bulk}[\partial g, g] + L_{\rm sur}[\partial^2 g, \partial g, g]$ is quite obvious because there exists a $L_{\rm bulk}$ (the usual Γ^2 Lagrangian) which is *independent* of second derivatives of the metric. But when we consider more general Lagrangians, involving higher powers of curvature, say, it is not possible to affect such a simple separation. In fact, no $L_{\rm bulk}$ which is independent of second derivatives of the metric, will exist for such Lagrangians. What is remarkable, however, is that there is indeed a natural way of extending the results [4] obtained for the Einstein-Hilbert Lagrangian to all Lagrangians of the form $\sqrt{-g}L = \sqrt{-g}Q_a{}^{bcd}R^a{}_{bcd}$ in which $Q_a{}^{bcd}$ is a tensor with the symmetries of the curvature tensor, made from metric and curvature tensor and satisfies the condition $\nabla_c Q_a{}^{bcd} = 0$.

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The Lanczos-Lovelock Lagrangians are a subset of these in which $Q_a{}^{bcd}$ is a homogeneous function of the curvature tensor. They are all holographic, in the sense of the term defined above, and will be the focus of our attention in this paper.

The motivation for this analysis is threefold: First, the Lanczos-Lovelock Lagrangian has an interesting geometrical structure and has been extensively investigated in the literature [5]. It would be nice to add to this study new features and new perspectives on previous results. As it sometimes happens, the study of a general structure sheds light on the peculiar features of a special case and here, the study of holographic properties of Lanczos-Lovelock Lagrangians helps to understand the holography of the Einstein-Hilbert action.

Second, there is a point of view, shared by many, that the Einstein-Hilbert action is just the first term in the derivative expansion in a low energy effective theory. At the present, we have no general prescription which allows us to restrict the form of higher order quantum corrections to gravity. The only known low energy symmetry (under diffeomorphism) allows for a wide choice of correction terms. There is some evidence from string theory that not all these choices are realized in nature. Any extra symmetry of the Einstein-Hilbert action (like the action being holographic) will allow us to restrict higher order correction terms and is worth investigating.

Third, and probably most interesting, motivation comes from the relation between gravity and thermodynamics. The surface term in the Einstein-Hilbert action has a clear thermodynamic interpretation and will lead to the entropy of the horizon in a wide class of spacetimes (for a recent review, see Ref. [6]). The notion of the horizon entropy can be generalized to an arbitrary, generally covariant Lagrangian [7] and is often referred to as Wald's entropy. In the case of the Einstein-Hilbert action, one could demonstrate that the surface term evaluated at the horizon is indeed the entropy associated with a very wide class of spacetimes. The surface term we obtain by splitting the Lanczos-Lovelock actions also yields the Wald's entropy when evaluated at the horizon of spherically symmetric spacetimes. This result is highly nontrivial in the sense that, there is no *a priori* reason that our construction of the surface term should be related to Wald's entropy. However, the fact that it is indeed the Wald entropy, gives credence to our approach and supports the point of view that signatures of holographic description of gravity should be present at the classical level itself. In addition to being an interesting result by itself, this also allows one to interpret the equations describing the gravity, including the higher derivative corrections, in thermodynamic terms [4]. This approach is in the spirit of what could be called the Sakharov paradigm [8], in which Einstein's equations are considered similar to those describing the equations of elasticity in solid state physics. (For some of the previous attempts in the same spirit, see Ref. .) It was known that [10], in the case of spherically symmetric spacetimes with horizons, standard Einstein's equations can be explicitly expressed as $TdS = dE + pdV$; recently, this result has been extended to all Lanczos-Lovelock Lagrangians [11]. This is remarkable because it makes the thermodynamic paradigm more fundamental than a specific set of field equations. Einstein's equations will be modified by quantum corrections but some thermodynamic relation like $TdS = dE + pdV$ might remain valid to all orders [4,12]. From a practical point of view this may not seem dramatic since S, E, etc. will pick up quantum corrections but it certainly has deep conceptual significance.

The plan of the paper is as follows: Since we will be dealing with actions which involve second and higher derivatives of dynamical variables, we shall briefly discuss some issues related to such actions, in a simple setting, in Sec. II. Then we will proceed to derive the holographic relationship in a wide class of actions [Sec. III] after briefly reviewing the Einstein-Hilbert case. In Sec. IV we will show that the surface term obtained in our approach correctly gives the entropy of the horizons, thereby strengthening the thermodynamic interpretation. Section V summarizes the conclusions.

II. WARMUP: TOY MODEL WITH HIGHER DERIVATIVE ACTION

The Einstein-Hilbert action and Lanczos-Lovelock actions which we will discuss in the paper contain second derivatives of dynamical variables but their equations of motion do not have higher order terms. The purpose of this section is to demystify this aspect in a simple context and show how one can construct a large family of Lagrangians involving second derivatives of dynamical variables but with the resulting equations still remaining the second order in time.

Consider a dynamical variable $q(t)$ in point mechanics described by a Lagrangian $L_q(q, \dot{q})$. Varying the action obtained from integrating this Lagrangian in the interval (t_1, t_2) and keeping q fixed at the endpoints, gives the Euler-Lagrange equations for the system $(\partial L_q/\partial q) = dp/dt$, where we have defined a function $p(q, \dot{q}) \equiv (\partial L_q/\partial \dot{q})$. (The subscript q on L_q is an indicator of the variable that is kept fixed at the end points.) The Lagrangian contains only up to first derivatives of the dynamical variable and the equations of motion are—in general—second degree in the time derivative.

When the Lagrangian L_q depends on $\ddot{q}$ as well, the theoretical formulation becomes more complicated. For example, if the equations of motion become higher order, then more initial conditions are required to pose a well-defined initial value problem and the corresponding definition of path integral in quantum theory, using the Lagrangian, is nontrivial [13]. Interestingly enough, there exists a wide class of Lagrangians $L(\ddot{q}, \dot{q}, q)$ which depend on $\ddot{q}$ but still lead to equations of motion which are only second order in time. We will now motivate and analyze this class which will lead to the holographic actions in field theory.

To do this, let us consider the following question: We want to modify the Llagrangian L_q such that the same equations of motion are obtained when—instead of fixing q at the end points—we fix some other (given) function $C(q, \dot{q})$ at the end points. This is easily achieved by modifying the Lagrangian by adding a term $-df(q, \dot{q})/dt$ which depends on $\dot{q}$ as well. (The minus sign is just for future convenience.) The new Lagrangian is:

$$L_C(q, \dot{q}, \ddot{q}) = L_q(q, \dot{q}) - \frac{df(q, \dot{q})}{dt}. \tag{1}$$

We want this Lagrangian L_C to lead to the same equations of motion as L_q, when some given function $C(q, \dot{q})$ is held fixed at the end points. We assume L_q and C are given and we need to find f. The standard variation gives

$$\delta A_C = \int_{t_1}^{t_2} dt\left[\left(\frac{\partial L}{\partial q}\right) - \frac{dp}{dt}\right]\delta q - \int_{t_1}^{t_2} dt\frac{d}{dt}[\delta f - p\delta q]. \tag{2}$$

We will now invert the relation $C = C(q, \dot{q})$ to determine $\dot{q} = \dot{q}(q, C)$ and express $p(q, \dot{q})$ in terms of (q, C) obtaining the function $p = p(q, C)$. In the boundary term in Eq. (2) we treat f as a function of q and C, so that the variation of the action can be expressed as:

$$\begin{aligned}\delta A_C &= \int_{t_1}^{t_2} dt\left[\left(\frac{\partial L}{\partial q}\right) - \frac{dp}{dt}\right]\delta q \\ &\quad + \left[p(q, C) - \left(\frac{\partial f}{\partial q}\right)_C\right]\delta q\,\Big|_{t_1}^{t_2} - \left(\frac{\partial f}{\partial C}\right)_q \delta C\,\Big|_{t_1}^{t_2} \\ &= \int_{t_1}^{t_2} dt\left[\left(\frac{\partial L}{\partial q}\right) - \frac{dp}{dt}\right]\delta q \\ &\quad + \left[p(q, C) - \left(\frac{\partial f}{\partial q}\right)_C\right]\delta q\,\Big|_{t_1}^{t_2}\end{aligned} \tag{3}$$

since $\delta C = 0$ at the end points by assumption. To obtain the same Euler-Lagrange equations, the second term should vanish for any δq. This fixes the form of f to be:

$$f(q, C) = \int p(q, C)dq + F(C), \tag{4}$$

where the integration is with constant C and F is an arbitrary function.

Thus, given a Lagrangian $L_q(q, \dot{q})$ which leads to certain equations of motion when q is held fixed, one can construct a *family* of Lagrangians $L_C(q, \dot{q}, \ddot{q})$ which will lead to the same equations of motion when an arbitrary function $C(q, \dot{q})$ is held fixed at the end points. This family is remarkable in the sense that L_C will be a function of not only q, $\dot{q}$, but will also involve $\ddot{q}$. In spite of the existence of the $\ddot{q}$ in the Lagrangian, the equations of motion are still of second order in q because of the special structure of the Lagrangian. (The results obtained above have an interpretation in terms of canonical transformations, etc. which we purposely avoid since we want to stay within the Lagrangian framework.) So, even though a *general* Lagrangian which depends on $\ddot{q}$ will lead to equations of higher order, there is a host of Lagrangians with a *special* structure which will not.

The analysis extends directly to a multicomponent field $q_A(x^a)$ in a spacetime with coordinates x^a where A collectively denotes the tensor indices. Suppose a Lagrangian $L_q(q_A, \partial q_A)$ gives the field equations when the action is varied keeping q_A fixed at the boundary $\partial\mathcal{V}$ of a spacetime region $\mathcal{V}$. We now want to add to the action a four divergence $\partial_a V^a$ such that the same equations are obtained when the action is varied keeping some given functions $U_A^a(q_A, \partial q_A)n_a$ fixed at the boundary where n_a is the normal to $\partial\mathcal{V}$. As before, we will invert the relation $U_A^a = U_A^a(\partial_a q_A, q_A)$ to determine $\partial_a q_A = \partial_a q_A(q_A, U_A^i)$ and using this will express $\pi_A^i = [\partial L/\partial(\partial_i q^A)] = \pi_A^i(q_A, \partial_a q_A)$ in terms of q_A, U_A^i getting $\pi_A^i = \pi_A^i(q_A, U_A^j)$. [We are assuming that there are no constraints in the theory and such inversions are possible, for the purpose of illustration.] Then the Lagrangian we are looking for is

$$L_U(\partial^2 q_A, \partial q_A, q_A) = L_q(q_A, \partial q_A) - \partial_a V^a(q_A, \partial q_A) \tag{5}$$

with

$$V^j(q_A, U_A^b) = \int \pi_A^j(q_A, U_A^b)dq^A + F^j(U_A^b). \tag{6}$$

So the same classical field theory can be obtained from a family of Lagrangians *involving second derivatives* of dynamical variables, provided some arbitrary function of the dynamical variables and their normal derivatives are held fixed at the end points.

When one considers action merely as a tool to obtain the field equations, the above procedure is acceptable with any C or U_A^a. But once the dynamical variables in the theory have been identified, there are two natural boundary conditions which one may impose on the system. The first one holds q fixed at the end points and the second one keeps the canonical momenta p fixed at the end points. In the second case, $C = p$ and Eq. (4) gives $f = qp = q(\partial L/\partial\dot{q})$ (ignoring the integration constant); the corresponding Lagrangian is:

$$L_p = L_q - \frac{d}{dt}\left(q\frac{\partial L_q}{\partial\dot{q}}\right). \tag{7}$$

The L_p will lead to the same equations of motion when $p(q, \dot{q}) = (\partial L/\partial\dot{q})$ is held fixed at the end points as can be directly verified by explicit variation. In the Hamiltonian language this is summarized by

$$dA = L_q dt - d(qp) = -H dt + p dq - d(qp) = -H dt - q dp. \tag{8}$$

Since H treats q, p symmetrically, this is just a transformation from q to p. This result also has a more natural interpretation in quantum theory: It is easy to show that a path integral defined with L_p will lead to the transition amplitude in momentum space $G(p_2, t_2; p_1, t_1)$, just as a path integral with L_q leads to the transition amplitude in coordinate space $K(q_2, t_2; q_1, t_1)$. But our interest is in the existence of higher derivatives of dynamical variables in the Lagrangian which is not transparent in the Hamiltonian language and we will continue to use the Lagrangian picture.

In the case of field theory, $U^a_A = \pi^a_A$ is independent of q_A and the integral in Eq. (6) gives $V^j = \pi^j_A q^A$. So the new Lagrangian is:

$$L_p(\partial^2 q_A, \partial q_A, q_A) = L_q(q_A, \partial q_A) - \partial_i \left[q_A \left(\frac{\partial L_q}{\partial(\partial_i q^A)} \right) \right] \equiv L_{\rm bulk} + L_{\rm sur}. \tag{9}$$

It is obvious from this structure (which we shall call, for brevity, the "$d(qp)$" structure) of the surface term that the surface and bulk terms in the above action are closely related and the knowledge of the surface term will put strong constraints on $L_{\rm bulk} = L_q$. The Einstein-Hilbert action has precisely this form (except for a dimension dependent proportionality constant which becomes unity in $D = 4$!, see Eq. (16) below). This is the key to the holography in the action functionals which we will explore later on.

III. ACTIONS WITH HOLOGRAPHY

We will now describe a class of action functionals which allow a decomposition in terms of surface and bulk terms and exhibit a holographic relationship between the two. We will begin by rapidly summarizing some of the features of the Einstein-Hilbert action in Sec. III A and then generalize them for a wider class in Sec. III B.

A. Some features of the Einstein-Hilbert action

In this subsection we will gather together and summarize several results related to the Einstein-Hilbert action, which we will need later. We will not bother to give detailed proofs of these results since we will be providing such proofs—in a more general context—in the coming sections. Somewhat longer proofs are presented in the appendix so as not to distract the main discussion.

We begin with the form of the Einstein-Hilbert action for gravity in D-dimensions, given by

$$A_{\rm EH} = \int_{\mathcal{V}} d^D x \sqrt{-g} L_{\rm EH} = \int_{\mathcal{V}} d^D x \sqrt{-g} R. \tag{10}$$

Using the standard text book expressions for the scalar curvature, one can write the Lagrangian in several equivalent forms, all which will be of interest to us later. The simplest one is:

$$L_{\rm EH} \equiv Q_a{}^{bcd} R^a{}_{bcd}; \qquad Q_a{}^{bcd} = \tfrac{1}{2}(\delta^c_a g^{bd} - \delta^d_a g^{bc}). \tag{11}$$

The tensor $Q_a{}^{bcd}$ is the only fourth rank tensor that can be constructed from the metric (alone) that has all the symmetries of the curvature tensor. In addition it has zero divergence on all indices, $\nabla_a Q^{abcd} = 0$, etc. Since the curvature tensor $R^a{}_{bcd}$ can be expressed entirely in terms of Γ^i_{kl} and $\partial_j \Gamma^i_{kl}$ without requiring g^{ab}, there is a nice separation between dependence on the metric (through $Q_a{}^{bcd}$ alone) and dependence on connection and its derivative through $R^a{}_{bcd}$. This separation is useful in certain variational calculations when we treat them as independent. With g^{ab}, Γ^i_{kl}, $R^a{}_{bcd}$ (instead of g_{ab} and its first and second derivatives) treated as independent variables, the vacuum Einstein's equations take a very simple form:

$$\left(\frac{\partial \sqrt{-g} L_{\rm EH}}{\partial g^{ab}} \right) = R^a{}_{bcd} \left(\frac{\partial \sqrt{-g} Q_a{}^{bcd}}{\partial g^{ab}} \right) = 0. \tag{12}$$

That is, Einstein's equations arise through ordinary partial differentiation of the Lagrangian density with respect to g^{ab}, keeping Γ^i_{kl} and $\partial_j \Gamma^i_{kl}$ as constant.

If we raise one index of the curvature tensor, the Einstein-Hilbert Lagrangian can be written in another interesting form as

$$L_{\rm EH} \equiv \delta^{cd}_{ab} R^{ab}_{cd}; \qquad \delta^{cd}_{ab} = \tfrac{1}{2}(\delta^c_a \delta^d_b - \delta^d_a \delta^c_b), \tag{13}$$

where δ^{cd}_{ab} is the alternating ("determinant") tensor. The importance of this form lies in the fact that it allows the generalization [3] to a Lagrangian containing a product of, say, m curvature tensors, which—as we shall see—will share many properties of the Einstein-Hilbert action.

We will now turn to more nontrivial aspects of the Einstein-Hilbert action which provide the key motivation to this work. Since $L_{\rm EH}$ is linear in second derivatives of the metric, it is clear that $\sqrt{-g} L_{\rm EH}$ can be written as a sum $L_{\rm bulk} + L_{\rm sur}$ where $L_{\rm bulk}$ is quadratic in the first derivatives of the metric and $L_{\rm sur}$ is a total derivative which leads to a surface term in the action. From Eq. (11) it is easy to obtain this separation as

$$\sqrt{-g} L_{\rm EH} = 2\partial_c [\sqrt{-g} Q_a{}^{bcd} \Gamma^a_{bd}] + 2\sqrt{-g} Q_a{}^{bcd} \Gamma^a_{dk} \Gamma^k_{bc} \equiv L_{\rm sur} + L_{\rm bulk} \tag{14}$$

with

$$L_{\rm bulk} = 2\sqrt{-g} Q_a{}^{bcd} \Gamma^a_{dk} \Gamma^k_{bc}; \qquad L_{\rm sur} = 2\partial_c [\sqrt{-g} Q_a{}^{bcd} \Gamma^a_{bd}] \equiv \partial_c [\sqrt{-g} V^c], \tag{15}$$

where the last equality defines the D-component object V^c, which—of course—is not a vector. (The proof is given in the appendix.) Even in this form, the metric dependence is confined to $Q_a{}^{bcd}$. As is well known, one can obtain Einstein's equations varying only $L_{\rm bulk}$ keeping g_{ab} fixed at the boundary.

The first nontrivial result regarding the Einstein-Hilbert action is a simple relation [6] between $L_{\rm bulk}$ and $L_{\rm sur}$ allowing $L_{\rm sur}$ to be determined completely by $L_{\rm bulk}$. (As discussed in Sec. I, we call such a relation holographic.) Using g_{ab} and $\partial_c g_{ab}$ as the independent variables in $L_{\rm bulk}$ one can prove that:

$$L_{\rm sur} = -\frac{1}{[(D/2)-1]}\partial_i\left(g_{ab}\frac{\partial L_{\rm bulk}}{\partial(\partial_i g_{ab})}\right). \tag{16}$$

The "$d(qp)$" structure of $L_{\rm sur}$ suggests that $L_{\rm EH}$ is obtained from $L_{\rm bulk}$ by a transformation from coordinate space to momentum space, as has been noticed before in literature [6]. One of our aims will be to obtain a suitable generalization of this result to a wider class of Lagrangians. We will prove a more general result, viz. Eq. (41) below, of which Eq. (16) is a special case.

We note, in passing, that there are other ways of stating the holographic relation. For example, we can write a relation of the form:

$$L_{\rm sur} = -\partial_p\left(\delta^q_r\frac{\partial L_{\rm bulk}}{\partial \Gamma^q_{pr}}\right). \tag{17}$$

The proof is straightforward but there is one caveat. In evaluating the partial derivative on the right-hand side we hold $Q_a{}^{bcd}$ fixed and treat all components of Γ^k_{bc} as independent variables with no symmetry requirements; that is, we take $(\partial\Gamma^a_{bc}/\partial\Gamma^i_{jk}) = \delta^a_i\delta^j_b\delta^k_c$ to obtain:

$$\begin{aligned}\delta^q_r\frac{\partial L_{\rm bulk}}{\partial\Gamma^q_{pr}} &= \delta^q_r[2\sqrt{-g}(Q_a{}^{prd}\Gamma^a_{dq} + Q_q{}^{uvp}\Gamma^r_{uv})]\\ &= 2\sqrt{-g}(Q_a{}^{prd}\Gamma^a_{dr} + Q_r{}^{uvp}\Gamma^r_{uv}).\end{aligned} \tag{18}$$

Obviously, the order of lower indices in Γ^a_{bc} in Eq. (15) is important in arriving at this result. After the derivative is computed we will impose the condition that the Γ^k_{bc} are related to the metric by the standard relation. Then, the symmetry of Γ^a_{dr} in d, r makes the first term vanish (since $Q_a{}^{prd} = -Q_a{}^{pdr}$) and the result becomes

$$\delta^q_r\frac{\partial L_{\rm bulk}}{\partial\Gamma^q_{pr}} = 2\sqrt{-g}(Q_r{}^{uvp}\Gamma^r_{uv}) = -2\sqrt{-g}(Q_r{}^{upv}\Gamma^r_{uv}). \tag{19}$$

A comparison with the definition of $L_{\rm sur}$ in Eq. (15) leads to Eq. (17).

In the same manner we can also prove the following results [4] which determine the bulk and total Lagrangians in terms of the surface term (which is probably truer to the spirit of the term holography):

$$L = \frac{1}{2}R^a{}_{bcd}\left(\frac{\partial V^c}{\partial\Gamma^a_{bd}}\right); \qquad L_{\rm bulk} = \sqrt{-g}\left(\frac{\partial V^c}{\partial\Gamma^a_{bd}}\right)\Gamma^a_{dk}\Gamma^k_{bc}. \tag{20}$$

Thus the knowledge of the functional form of $L_{\rm sur}$ or—equivalently—that of V^c allows us to determine $L_{\rm bulk}$ and even L. The first relation also shows that $(\partial V^c/\partial\Gamma^a_{bd})$ is generally covariant in spite of the appearance.

The fact that one needs to first treat Γ^a_{bc} as independent and then impose the metric compatibility makes the above results less attractive than the one stated in Eq. (16). On the other hand, Eq. (17) and Eq. (20) do not use the explicit form of $Q_a{}^{bcd}$. In the case of the Einstein-Hilbert action, $Q_a{}^{bcd}$ is independent of curvature and depends only on metric but in the next section we will consider $Q_a{}^{bcd}$ that is made from metric, curvature tensor and possibly covariant derivatives of the curvature tensor—all of which can be held fixed while varying Γ^a_{bc}, if we choose the metric, the curvature tensor, its covariant derivatives, and also Γ^a_{bc} as independent variables. All such Lagrangians of the form in Eq. (14) will satisfy Eqs. (17) and (20). In contrast, Eq. (16) uses the specific form of $Q_a{}^{bcd}$ given in Eq. (11) (A straightforward proof of Eq. (16), starting from Eq. (17) and changing variables is given in the appendix, thereby connecting up the two and demonstrating where Eq. (11) is used.)

Before we conclude this section, we would like to comment on a few other issues related to the surface term. The separation of the Einstein-Hilbert action into surface and bulk terms in Eq. (14) is a standard text book result. While neither term is generally covariant, the *variation* of either term with respect to the metric is generally covariant (when the metric is held fixed at the boundary) so that, for example, $L_{\rm bulk}$ can lead to the standard field equations when the metric is held fixed at the boundary. It is, of course, possible to add other surface terms to the Einstein-Hilbert action so that the same field equations are obtained under variation of the metric, when the metric is held fixed at the boundary. The most popular one is the Gibbons-Hawking term $A_{\rm GH}$ which is the integral over the trace of the extrinsic curvature of the boundary [14]. The $A_{\rm sur}$ in Eq. (14) is not equal to $A_{\rm GH}$ in general but matches under a particular coordinate choice (See the appendix of [6] for a discussion; of course, the variations of the two terms always match). In the interpretation of Eq. (14) as having the "$d(qp)$ structure," we have treated all components of g_{ab} at the same footing in the spirit of Lagrangian formulation. It is well known from the Hamiltonian structure of the theory that g_{00} and $g_{0\alpha}$ are constraint variables in the sense that their time derivatives do not occur in the Lagrangian. If we integrate the $L_{\rm sur}$ over a volume bounded by two spacelike surfaces $\Sigma_{1,2}$, we will pick up $[g_{\mu\nu}\partial L_{\rm bulk}/\partial(\partial_0 g_{\mu\nu}) \equiv g_{\mu\nu}\pi^{\mu\nu}]$ involving only the correct dynamical variables $g_{\mu\nu}$ and their canonical momenta $\pi^{\mu\nu} = K^{\mu\nu} - g^{\mu\nu}K$ on these surfaces. If we further choose the gauge $g_{0\mu} = 0$, then we obtain the integral over $g_{\mu\nu}\pi^{\mu\nu} = -2K$, in the standard $D = 4$ case, which is the same as $A_{\rm GH}$. So the claim that Eq. (14) has the "$d(qp)$ structure" is quite appropriate even from this perspective.

The $A_{\rm GH}$ has a formal general covariance (which our term lacks) but this is obtained at the cost of foliation dependence. The relation between foliation dependence and general covariance is worth emphasizing: One would have considered a component of a tensor, say, T_{00} as not generally covariant. But a quantity $\rho = T_{ab}u^a u^b$ is a generally covariant scalar which will reduce to T_{00} in a local frame in which $u^a = (1, 0, 0, 0)$. It is appropriate to say that ρ is generally covariant but foliation dependent. In fact, any term which is not generally covariant can be recast in a generally covariant form by introducing a foliation dependence. The $A_{\rm GH}$ uses the normal vector n^i of the boundary in a similar manner. But since our boundary term will reduce to $A_{\rm GH}$ under a particular coordinate choice, all the results which we quote will similarly be applicable, under this coordinate choice, to $A_{\rm GH}$ as well. The situation becomes more complicated in the case of general Lagrangians and we will comment on this later.

B. Actions with holography: Generalization

The Einstein-Hilbert action is usually introduced by using the fact that it is the only generally covariant scalar that can be built from the metric and its derivatives and is linear in the second derivatives. This guarantees that the variational principle could be made to work, albeit with some unusual boundary conditions. It is, therefore, quite surprising that the action possesses several *other* peculiar properties, in particular, the holographic relations between the surface and bulk terms.

On the other hand, it is quite possible that the Einstein-Hilbert action describes the low energy limit of an effective theory and $L_{\rm EH}$ is just a first term in a series of terms which will involve other scalars (like R^2, $R_{ab}R^{ab}$) that can be constructed from the metric and curvature. Several possible choices exist for such low energy effective action all of which are consistent with the diffeomorphism invariance of the low energy theory. Any extra symmetry, like the holographic relation, could then serve as a powerful guiding principle in constraining or determining the higher order corrections to the action principle. This leads us to ask: What is the most general action for gravity which satisfies holographic conditions? We will now address this question.

Since the relations in Eqs. (17) and (20) are linear in the Lagrangian, it follows that if two Lagrangians individually satisfy these relations, so will their sum with arbitrary constant coefficients. This allows us to investigate the individual terms in a sum of terms separately and also allows us to ignore relative coupling constants between them. Further, since our derivation of Eq. (17) (or Eq. (20)) did not use the explicit form of $Q_a{}^{bcd}$ we already know that any Lagrangian of the form

$$\sqrt{-g}L = 2\partial_c[\sqrt{-g}Q_a{}^{bcd}\Gamma^a_{bd}] + 2\sqrt{-g}Q_a{}^{bcd}\Gamma^a_{dk}\Gamma^k_{bc} \equiv L_{\rm sur} + L_{\rm bulk} \tag{21}$$

will satisfy the relations in Eqs. (17) and (20) provided: (i) Q^{abcd} has all the symmetries of curvature tensor and (ii) one can keep Q^{abcd} constant while differentiating with respect to Γ^a_{bc} treating all the components as independent.

Let us now consider a general Lagrangian of the form in Eq. (21) with $Q^{abcd} = Q^{abcd}(g^{ab}, R^a{}_{bcd}, \nabla_j R^a{}_{bcd}\ldots)$ depending on the metric, curvature tensor, and its covariant derivatives. We will follow the standard principle that, when varying an action, the dynamical variable q_A and each of the higher derivatives ∂q_A, $\partial^2 q_A\ldots$, etc. are to be treated as independent. If a Lagrangian L depends on the metric g^{ab}, curvature $R^a{}_{bcd}$, and its derivatives, the dynamical variable and its derivatives are the set $[g^{ab}, \partial_c g^{ab}, \partial_d\partial_c g^{ab}, \ldots]$ and we treat them as independent. Instead of treating $[g^{ab}, \partial_c g^{ab}, \partial_d\partial_c g^{ab}, \ldots]$ as the independent variables, it is convenient to use $[g^{ab}, \Gamma^i_{kl}, R^a{}_{bcd}, \ldots]$ as the independent variables and we trade off the second (and higher) derivatives of the metric $[\partial_d\partial_c g^{ab}, \ldots]$, in favor of the curvature tensor and its derivatives [15]. The curvature tensor $R^a{}_{bcd}$ can be expressed entirely in terms of Γ^i_{kl} and $\partial_j\Gamma^i_{kl}$ and is *independent* of g^{ab}. Then, we can indeed keep $R^a{}_{bcd}$ and its derivatives (as well as the metric itself) constant while differentiating with respect to $\partial_i g_{kl}$. Therefore, the Lagrangian in Eq. (21) with Q^{abcd} being a tensor with the symmetries of curvature tensor, constructed from metric, curvature, and covariant derivatives of the curvature will satisfy Eqs. (17) and (20).

But, in general, the expression in Eq. (21) will *not* be a generally covariant scalar since it is expressed in terms of Γ^a_{bc}, etc. We need to ascertain the condition on Q^{abcd} such that the Lagrangian is generally covariant. This turns out to be surprisingly easy and insightful. By straightforward algebra, one can prove (see the appendix) the following identity:

$$\sqrt{-g}L = 2\partial_c[\sqrt{-g}Q_a{}^{bcd}\Gamma^a_{bd}] + 2\sqrt{-g}Q_a{}^{bcd}\Gamma^a_{dj}\Gamma^j_{bc} = \sqrt{-g}Q_a{}^{bcd}R^a{}_{bcd} + 2\sqrt{-g}\Gamma^a_{bd}\nabla_c Q_a{}^{bcd}. \tag{22}$$

Obviously, general covariance only requires the condition $\nabla_c Q_a{}^{bcd} = 0$. Because of the symmetries of the $Q_a{}^{bcd}$ its divergence on any of the indices vanishes. Thus, we shall hereafter consider Lagrangians of the form:

$$\sqrt{-g}L = \sqrt{-g}Q_a{}^{bcd}R^a{}_{bcd}; \qquad \nabla_c Q_a{}^{bcd} = 0. \tag{23}$$

We have already proved that all such generally covariant Lagrangians are holographic; i.e., they allow a separation into bulk and surface terms which are related by Eqs. (17) and (20).

The simplicity of this result suggests that there could be a more geometric way of interpreting it. This is indeed true [4]. We know that the one can write the curvature tensor in terms of the two form by $\mathcal{R}^a{}_b = (1/2!)R^a{}_{bcd}w^c \wedge w^d$ where w^a are the basis one forms. Similarly one can introduce a two form for Q_{abcd} with $\mathcal{Q}^a{}_b = (1/2!)Q^a{}_{bcd}w^c \wedge w^d$. Further, using $\mathcal{R}^a{}_b = d\Gamma^a{}_b + \Gamma^a{}_c \wedge \Gamma^c{}_b$ where $\Gamma^a{}_b$ are the curvature forms, we can write the Lagrangian in Eq. (23) as

$$\begin{aligned} L &= *\mathcal{Q}^a{}_b \wedge \mathcal{R}^b{}_a = *\mathcal{Q}^a{}_b \wedge (d\Gamma^b{}_a + \Gamma^b{}_c \wedge \Gamma^c{}_a) \\ &= d(*\mathcal{Q}^a{}_b \wedge \Gamma^b{}_a) + *\mathcal{Q}^a{}_b \wedge \Gamma^b{}_c \wedge \Gamma^c{}_a \end{aligned} \tag{24}$$

provided the $\mathcal{Q}^a{}_b$ satisfies the condition: $d(*\mathcal{Q}^a{}_b) = 0$ corresponding to $\nabla_c Q_a{}^{bcd} = 0$. The separation between bulk and surface terms, just as in the case of Eq. (21), is obvious.

While discussing the corresponding situation in the case of the Einstein-Hilbert action we commented on the, alternative, Gibbons-Hawking surface term $A_{\rm GH}$. The situation becomes more complicated when we move to more general Lagrangians. The analogue of $A_{\rm GH}$ for more general Lagrangians is difficult to come by and—as far as the authors know—there is no algorithmic procedure for finding them. The expressions given in literature for even the Gauss-Bonnet case [16] are fairly complicated and their physical meanings are unclear. But our $L_{\rm sur}$ is well defined for a wide class of Lagrangians and possesses some of the attractive properties, which is encouraging. The lack of manifest general covariance is not of much concern since this issue exists even for the Einstein-Hilbert action. (In specific cases, like in asymptotically flat spacetimes possessing a horizon, the surface term actually turns out to be generally covariant and gives the horizon entropy; see Sec. IV.)

The structure of the theory is thus specified by a single divergence-free fourth rank tensor $Q_a{}^{bcd}$ having the symmetries of the curvature tensor. If we think of gravity as low energy effective theory, the semiclassical, action for gravity can now be determined from the derivative expansion of $Q_a{}^{bcd}$ in powers of the number of derivatives:

$$\begin{aligned} Q_a{}^{bcd}(g, R) &= \overset{(0)}{Q}_a{}^{bcd}(g) + \alpha \overset{(1)}{Q}_a{}^{bcd}(g, R) \\ &\quad + \beta \overset{(2)}{Q}_a{}^{bcd}(g, R^2, \nabla R) + \cdots, \end{aligned} \tag{25}$$

where α, β, $\cdots$ are coupling constants. At the lowest order, $Q_a{}^{bcd}$ has to be built from just the metric and the next order correction will have $Q_a{}^{bcd}$ depending on $R^a{}_{bcd}$ linearly as well as on the metric, etc.

Interestingly enough, the condition $\nabla_c Q_a{}^{bcd} = 0$ encompasses all the gravitational theories (in D dimensions) in which the field equations are no higher than second degree, though we did not demand that explicitly [4]. To see this, let us consider the possible fourth rank tensors Q^{abcd} which (i) have the symmetries of curvature tensor; (ii) are divergence-free; (iii) are made from g^{ab} and $R^a{}_{bcd}$. If we do not use the curvature tensor, then we have just one choice made from the metric given in Eq. (11) and will lead to the Einstein-Hilbert action. Next, if we allow for $Q_a{}^{bcd}$ to depend linearly on curvature, then we have the following additional choice of tensor with required symmetries:

$$Q^{abcd} = R^{abcd} - G^{ac}g^{bd} + G^{bc}g^{ad} + R^{ad}g^{bc} - R^{bd}g^{ac}. \tag{26}$$

In four dimensions, this tensor is essentially the double-dual of R_{abcd} and in any dimension can be obtained from R_{abcd} using the alternating tensor [17] we get

$$\begin{aligned} L &= \tfrac{1}{2}(g_{ia}g^{bj}g^{ck}g^{dl} - 4g_{ia}g^{bd}g^{ck}g^{jl} \\ &\quad + \delta^c_a \delta^k_i g^{bd}g^{jl})R^i_{jkl}R^a{}_{bcd} \\ &= \tfrac{1}{2}[R^{abcd}R_{abcd} - 4R^{ab}R_{ab} + R^2]. \end{aligned} \tag{27}$$

This is just the Gauss-Bonnet action which is a pure divergence in four dimensions but not in higher dimensions. The unified procedure for deriving the Einstein-Hilbert action and Gauss-Bonnet action [essentially from the holographic condition and $\nabla_c Q_a{}^{bcd} = 0$] shows that they are more closely related to each other than previously suspected. The fact that several string theoretical models get Gauss-Bonnet-type terms as corrections, after appropriate field redefinitions [18], is noteworthy in this regard.

Further, both the Einstein-Hilbert Lagrangian and Gauss-Bonnet Lagrangian can be written in a condensed notation using alternating tensors as:

$$L_{\rm EH} = \delta^{13}_{24}R^{24}_{13}; \qquad L_{\rm GB} = \delta^{1357}_{2468}R^{24}_{13}R^{68}_{57}, \tag{28}$$

where the numeral n actually stands for an index a_n, etc. The obvious generalization leads to the Lanczos-Lovelock Llagrangian [3]:

$$L_m = \delta^{1357\ldots 2k-1}_{2468\ldots 2k}R^{24}_{13}R^{68}_{57}\ldots R^{2k-22k}_{2k-32k-1}; \qquad k = 2m, \tag{29}$$

where $k = 2m$ is an even number. The L_m is clearly a homogeneous function of the degree m in the curvature tensor R^{ab}_{cd} so that it can also be expressed in the form:

$$L = \frac{1}{m}\left(\frac{\partial L}{\partial R^a{}_{bcd}}\right)R^a{}_{bcd} \equiv \frac{1}{m}P_a{}^{bcd}R^a{}_{bcd}, \tag{30}$$

where we have defined $P_a{}^{bcd} \equiv (\partial L/\partial R^a{}_{bcd})$ so that $P^{abcd} = mQ^{abcd}$. It can be directly verified that for these Lagrangians (see the appendix):

$$\nabla_c P^{ijcd} = 0. \tag{31}$$

Because of the symmetries, P^{abcd} is divergence-free in *all* indices. These Lagrangians, therefore, belong to the class described by Eq. (23) and—more importantly for our purpose—they allow a separation into bulk and surface terms as given by Eq. (21) with the two parts satisfying Eqs. (17) and (20). [It may be noted that in proving

Eqs. (17) and (20), we treated Γ^a_{bc} as independent variables in the spirit of Palatini variation in Einstein's theory. This idea generalizes directly to Lovelock Lagrangians. If we treat Γ^a_{bc} as independent of g^{ab} and vary it, keeping g^{ab} fixed, then it is easy to show, (using the results of the appendix) that $\delta L/\delta\Gamma^a_{bc} = 0$ if $\nabla_a P^{abcd} = 0$. So this condition allows one to vary Γ^a_{bc}s independently of g^{ab} as in Palatini formulation of general relativity. Hence it follows that one may indeed treat the connection as an independent variable in the case of these actions as well and the derivation of Eqs. (17) and (20) holds for all these cases.] The $m = 1$ and $m = 2$ give the Einstein-Hilbert and Gauss-Bonnet Lagrangians. We shall now prove a host of relations for this class of Lagrangians.

The first result is that, the equations of motion for these Lagrangians take a particularly simple form. To see this, let us consider a general action of the form

$$A = \int_{\mathcal{V}} d^D x\sqrt{-g}L[g^{ab}, R^a{}_{bcd}] \tag{32}$$

in which we have ignored higher derivatives of $R^a{}_{bcd}$ for simplicity. The variation of the action can be easily computed to give the result (see the appendix for details)

$$\delta A = \delta \int_{\mathcal{V}} d^D x\sqrt{-g}L = \int_{\mathcal{V}} d^D x\sqrt{-g}E_{ab}\delta g^{ab} + \int_{\mathcal{V}} d^D x\sqrt{-g}\nabla_j \delta v^j, \tag{33}$$

where

$$\sqrt{-g}E_{ab} \equiv \left(\frac{\partial\sqrt{-g}L}{\partial g^{ab}} - 2\sqrt{-g}\nabla^m\nabla^n P_{amnb}\right); \quad P_a{}^{bcd} \equiv (\partial L/\partial R^a{}_{bcd}) \tag{34}$$

and

$$\delta v^j \equiv [2P^{ibjd}(\nabla_b \delta g_{di}) - 2\delta g_{di}(\nabla_c P^{ijcd})]. \tag{35}$$

This result is completely general. We now see that the equations of motion simplify significantly for a subclass of Lagrangians which satisfy Eq. (31) and are given by

$$\frac{\partial\sqrt{-g}L}{\partial g^{ab}} = 0. \tag{36}$$

That is, just setting the *ordinary derivative* of Lagrangian density with respect to g^{ab} to zero will give the equations of motion, as in the case Einstein-Hilbert action.

It also follows that, for the mth Lanczos-Lovelock Lagrangian, L_m [given by Eq. (29)], the trace of the equations of motion is proportional to the Lagrangian:

$$g^{ab}E_{ab} = g^{ab}\frac{\partial\sqrt{-g}L_m}{\partial g^{ab}} = -[(D/2) - m]\sqrt{-g}L_m; \quad g_{ab}E^{ab} = g_{ab}\frac{\partial\sqrt{-g}L_m}{\partial g_{ab}} = [(D/2) - m]\sqrt{-g}L_m. \tag{37}$$

This *off-shell* relation is easy to prove from the fact that we need to introduce m factors of g^{ab} in Eq. (29) to proceed from $R^a{}_{bcd}$ to R^{ab}_{cd} and that $\sqrt{-g}$ is a homogeneous function of g^{ab} of degree $-D/2$. Further, we can prove that (see the appendix for the proof) for *any* Lagrangian:

$$g_{ab}\frac{\delta L}{\delta(\partial_i g_{ab})} = -2\sqrt{-g}\left(\frac{\partial L}{\partial R_{nbid}}\right)\Gamma_{nbd} = -2\sqrt{-g}P^{nbid}\Gamma_{nbd}, \tag{38}$$

where the *Euler derivative* is defined as

$$\frac{\delta K[\phi, \partial_i\phi, \ldots]}{\delta\phi} = \frac{\partial K[\phi, \partial_i\phi, \ldots]}{\partial\phi} - \partial_a\left[\frac{\partial K[\phi, \partial_i\phi, \ldots]}{\partial(\partial_a\phi)}\right] + \cdots \tag{39}$$

In the case of Lanczos-Lovelock Lagrangians, $P^{nbid} = mQ^{nbid}$ so that we get the relation:

$$\partial_i\left[g_{ab}\frac{\delta L}{\delta(\partial_i g_{ab})}\right] = -mL_{\text{sur}}. \tag{40}$$

This shows that m times the surface term is indeed of the "$d(qp)$" structure provided the momentum is defined using the *total* Lagrangian L and Euler derivative. We also know that all Lagrangians of the form in Eq. (21) satisfy Eqs. (17) and (20) as well, with a specific prescription for evaluation of the derivative. Thus we have established three different holographic relations for Lanczos-Lovelock Lagrangians.

Since the Einstein-Hilbert Lagrangian corresponds to the Lanczos-Lovelock Lagrangian with $m = 1$, Eq. (40) is valid for L_{EH} as well. But the relation in Eq. (40) should be distinguished from Eq. (16) which shows that a similar relation also holds with *bulk* Lagrangian L_{bulk} rather than with *total* Lagrangian L_{EH}. We shall now take up the generalization of the relation Eq. (16) (between L_{bulk} and L_{sur}) for the Lanczos-Lovelock case when $Q_a{}^{bcd}$ depends on the metric as well as the curvature. (The result has a direct generalization even for some cases in which $Q_a{}^{bcd}$ depends on the derivatives of the curvature tensor as well [19]; however, to keep the argument transparent, we will discuss the simpler case, which—in any case—is more relevant to us.) We will prove that:

$$[(D/2) - m]L_{\text{sur}} = -\partial_i\left[g_{ab}\frac{\delta L_{\text{bulk}}}{\delta(\partial_i g_{ab})} + \partial_j g_{ab}\frac{\partial L_{\text{bulk}}}{\partial(\partial_i\partial_j g_{ab})}\right]. \tag{41}$$

Before we give the proof, we will make a couple of comments on the result. First, in the case of the Einstein-Hilbert Lagrangian, the L_{bulk} does not involve the second derivatives of the metric. Therefore, the second term in the right-hand side of Eq. (41) is absent and—in the first term—we can replace the Euler derivative by an ordinary partial derivative. This leads to Eq. (16) as it should. Second, the terms in the right-hand side, for the general case, can be thought of as one form of generalization of "$d(qp)$" for theories with higher derivatives.

The proof of Eq. (41) is based on a simple homology argument and combinatorics, generalizing a corresponding proof for Eq. (16) in the Einstein-Hilbert case (given in the appendix of Ref. [6]). Consider any Lagrangian $L(g_{ab}, \partial_i g_{ab}, \partial_i\partial_j g_{ab})$ and let $E^{ab}[L]$ denote the Euler-Lagrange function resulting from L:

$$E^{ab}[L] \equiv \frac{\partial L}{\partial g_{ab}} - \partial_i\left[\frac{\partial L}{\partial(\partial_i g_{ab})}\right] + \partial_i\partial_j\left[\frac{\partial L}{\partial(\partial_i\partial_j g_{ab})}\right]. \tag{42}$$

Forming the contraction $g_{ab}E^{ab}$ and manipulating the terms, we can rewrite this equation as:

$$\begin{aligned} g_{ab}E^{ab}[L] = {}& g_{ab}\frac{\partial L}{\partial g_{ab}} + (\partial_i g_{ab})\frac{\partial L}{\partial(\partial_i g_{ab})} \\ &+ (\partial_i\partial_j g_{ab})\frac{\partial L}{\partial(\partial_i\partial_j g_{ab})} \\ &- \partial_i\left[g_{ab}\frac{\delta L}{\delta(\partial_i g_{ab})} + \partial_j g_{ab}\frac{\partial L}{\partial(\partial_i\partial_j g_{ab})}\right]. \end{aligned} \tag{43}$$

We will now apply this relation to the bulk Lagrangian $L_{\rm bulk}^{(m)} = 2\sqrt{-g}Q_a{}^{bcd}\Gamma^a_{dj}\Gamma^j_{bc}$ corresponding to the mth order Lanczos-Lovelock Lagrangian. (Hereafter, we will simplify notation by just calling it $L_{\rm bulk}$; it is understood that we are dealing with the mth order Lanczos-Lovelock Lagrangian throughout.) Since both L_m and $L_{\rm bulk}$ lead to the same equations of motion, $E^{ab}[L_m] = E^{ab}[L_{\rm bulk}]$. Hence, using Eq. (37), we find the left-hand side of Eq. (43) to be $[(D/2) - m]\sqrt{-g}L_m$. We will next show that the first three terms in the right-hand side add up to give $[(D/2) - m]L_{\rm bulk}$. Bringing this term to the left-hand side and using $L_{\rm sur} = \sqrt{-g}L - L_{\rm bulk}$ will then lead to Eq. (41).

To prove this, let us write $L_{\rm bulk}/\sqrt{-g}$ entirely in terms of g^{ab}, $\partial_i g_{ab}$, and $\partial_j\partial_i g_{ab}$ by multiplying it out completely. In any given term, let us assume there are n_0 factors of g^{ab}, n_1 factors of $\partial_i g_{ab}$, and k factors of $\partial_i\partial_j g_{ab}$. Then homogeneity implies that for this particular term (labeled by k, which is the number of $\partial_i\partial_j g_{ab}$, that occur in it), the first three terms in the right-hand side of Eq. (43) are given by

$$\begin{aligned} g_{ab}\frac{\partial L^{(k)}_{\rm bulk}}{\partial g_{ab}} &= [(D/2) - n_0]L^{(k)}_{\rm bulk}; \\ (\partial_i g_{ab})\frac{\partial L^{(k)}_{\rm bulk}}{\partial(\partial_i g_{ab})} &= n_1 L^{(k)}_{\rm bulk}; \\ (\partial_i\partial_j g_{ab})\frac{\partial L^{(k)}_{\rm bulk}}{\partial(\partial_i\partial_j g_{ab})} &= k L^{(k)}_{\rm bulk}. \end{aligned} \tag{44}$$

(In the first relation $D/2$ comes from the $\sqrt{-g}$ factor and the sign flip on n_0 is because of switching over from g^{ab} to g_{ab}.) Since all the indices—the 2 upper indices from each g^{ab}, 3 lower indices from each $\partial_i g_{ab}$, 4 lower indices from each $\partial_j\partial_i g_{ab}$—are to be contracted out, we must have $2n_0 = 3n_1 + 4k$ which fixes n_0 in terms of n_1 and k. We next note that $Q_a{}^{bcd}$ is made of $(m-1)$ factors of curvature tensor and each curvature tensor has the structure $R \simeq [\partial^2 g + (\partial g)^2]$. If we multiply out $(m-1)$ curvature tensors, a generic term in the product will have k factors of $\partial^2 g$ and $(m-1-k)$ factors of $(\partial g)^2$. In addition, the two Γ's in $L_{\rm bulk} \simeq Q\Gamma\Gamma$ will contribute two more factors of (∂g). So, for this generic term, $n_1 = 2(m-1-k) + 2 = 2(m-k)$. Using our relation $2n_0 = 3n_1 + 4k$, we find $n_0 = 3m - k$. Substituting into Eq. (44) we get

$$\begin{aligned} g_{ab}\frac{\partial L^{(k)}_{\rm bulk}}{\partial g_{ab}} &= [(D/2) - 3m + k]L^{(k)}_{\rm bulk}; \\ (\partial_i g_{ab})\frac{\partial L^{(k)}_{\rm bulk}}{\partial(\partial_i g_{ab})} &= 2(m-k)L^{(k)}_{\rm bulk}; \\ (\partial_i\partial_j g_{ab})\frac{\partial L^{(k)}_{\rm bulk}}{\partial(\partial_i\partial_j g_{ab})} &= k L^{(k)}_{\rm bulk}. \end{aligned} \tag{45}$$

Though each of these terms depends on k, the sum of the three terms is independent of k leading to the same contribution from each term. So we get:

$$\begin{aligned} g_{ab}\frac{\partial L_{\rm bulk}}{\partial g_{ab}} + (\partial_i g_{ab})\frac{\partial L_{\rm bulk}}{\partial(\partial_i g_{ab})} + (\partial_i\partial_j g_{ab})\frac{\partial L_{\rm bulk}}{\partial(\partial_i\partial_j g_{ab})} \\ = [(D/2) - m]L_{\rm bulk}. \end{aligned} \tag{46}$$

Substituting this in Eq. (43), transferring these terms to the left-hand side and using $L\sqrt{-g} - L_{\rm bulk} = L_{\rm sur}$, we get the result in Eq. (41).

The result in Eq. (41) is the appropriate generalization of Eq. (16) in the case of the Einstein-Hilbert action and has a similar (generalized) "$d(qp)$" structure. We shall now turn to the task of connecting up the surface term to horizon entropy so as to provide a thermodynamic interpretation.

IV. THE SURFACE TERM AND THE ENTROPY OF THE HORIZON

Surface terms in actions sometimes assume special significance in a theory and this is particularly true for the Einstein-Hilbert action. In this case, one can relate the surface term to the entropy of the horizons, if the solution possesses bifurcation horizon. This is well known in the case of the black hole horizons. More generally, if the metric near the horizon can be approximated as a Rindler metric, then one can obtain the general result that the entropy per unit transverse area is $1/4$. To see this, we only need to evaluate the surface contribution

$$S_{\rm sur} = 2\int d^Dx\partial_c[\sqrt{-g}Q^{abcd}\Gamma_{abd}] = 2\int d^Dx\partial_c[\sqrt{-g}Q^{abcd}\partial_b g_{ad}] \qquad (47)$$

for a metric in the Rindler approximation:

$$ds^2 = -\kappa^2x^2dt^2 + dx^2 + dx_\perp^2, \qquad (48)$$

where $x_\perp^m$ demotes $(D-2)$ transverse coordinates. For the static metric, the time integration in Eq. (47) is trivial and involves multiplication by the range of integration. Since the surface gravity of the horizon (located at $x=0$) is κ, the natural range for time integration is $(0,\beta)$ where $\beta = 2\pi/\kappa$. (This is most easily seen in the Euclidean sector in which there is a natural periodicity.) Further, it is easy to verify that only the ${Q_{0x}}^{0x}$ term contributes. Then, a simple calculation shows that

$$S_{\rm sur} = 8\pi\int_H d^{D-2}x_\perp({Q_{0x}}^{0x}). \qquad (49)$$

In evaluating this contribution, the x integral in Eq. (47) will range from some $x=a$ to $x=b$ and the result will depend on the behavior of the integrand at both limits. What we have evaluated in Eq. (49) is the contribution of the integral from *one* surface, which is taken to be the location of the horizon. Our Rindler approximation is valid only near the horizon and one cannot say anything about the other contribution without knowing the detailed form of the metric. For example, if the second limit is at infinity, one needs to know whether the metric is asymptotically flat, etc. We need not bother about these issues by evaluating the result on the horizon alone, indicated by the subscript H in Eq. (49). In the case of the Einstein-Hilbert action $Q^{abcd} = (1/32\pi)[g^{ac}g^{bd} - g^{ad}g^{bc}]$ so that

$${Q_{0x}}^{0x} = -\frac{1}{32\pi}; \qquad S_{\rm sur} = -\frac{1}{4}\mathcal{A}_\perp \qquad (50)$$

as expected. (The minus sign arises because of the Minkowski signature we are working with.)

In the context of Einstein's theory, the thermodynamics of black holes, say, can be derived in many different ways, some of which uses boundary terms very crucially [like the Gibbons-Hawking Euclidean approach] while some do not. When one proceeds to study generalized theories of gravity [like the ones considered here], technical complexity prevents one from adopting certain approaches which works in the case of Einstein gravity. In view of this, results which arise directly from the nature of action principles are particularly valuable. We will now show that the above result, relating the boundary term in the action to the entropy of horizons, continue to hold for Lanczos-Lovelock Lagrangians with our definition of $L_{\rm sur}$, thereby providing a thermodynamic underpinning for our holographic separation of Lanczos-Lovelock Lagrangians. [Of course, in general, the surface term will not have any entropic interpretation just as an arbitrary solution to Einstein's theory—say, representing a spherical neutron star—does not have a temperature or entropy associated with it. As we know, such a thermodynamic connection emerges only for particular solutions with horizons. In that context, we will show that the surface term is related to the entropy for a wide class of spacetimes with horizons.]

In the next subsection, we shall provide a proof by comparing contribution of the surface term on the horizon with the Noether charge for these spacetimes. In Sec. IV B we will give a more direct and explicit calculation in the case static, spherically symmetric, solution.

A. The surface term and the Noether charge

To do this we need an expression for the entropy of the horizon in a general context when the Lagrangian depends on ${R^a}_{bcd}$ in a nontrivial manner. Such a formula has been provided by Wald in Ref. [7] and can be expressed as a integral over P^{abcd} on the horizon, evaluated on shell. It can also been shown [7] that this definition is equivalent to interpreting entropy as the Noether charge associated with diffeomorphism invariance. We shall briefly summarize this approach and use this definition.

To define the Noether charge associated with the diffeomorphism invariance, let us consider the variation $x^a \to x^a + \xi^a$ under which the metric changes by $\delta g_{ab} = -(\nabla_a\xi_b + \nabla_b\xi_a)$. The change in the action, when evaluated on shell, is contributed only by the surface term so that we have the relation

$$\delta_\xi A|_{\rm on\ shell} = -\int d^Dx\sqrt{-g}\nabla_a(L\xi^a) = \int d^Dx\sqrt{-g}\nabla_a(\delta_\xi V^a). \qquad (51)$$

[The subscript ξ on $\delta_\xi\ldots$ is a reminder that we are considering the changes due to a particular kind of variation, viz. when the metric changes by $\delta g_{ab} = -(\nabla_a\xi_b + \nabla_b\xi_a)$.] This leads to the conservation law $\nabla_a J^a = 0$ with $J^a = L\xi^a + (\delta_\xi V^a) \equiv \nabla_b J^{ab}$ with the last equality defining the antisymmetric tensor J^{ab}. For a Lagrangian of the type $L = L(g^{ab}, R^a_{bcd})$ direct computation using Eq. (33) shows that J^{ab} is given by (also see [20])

$$J^{ab} = -2P^{abcd}\nabla_c\xi_d + 4\xi_d(\nabla_c P^{abcd}) \qquad (52)$$

with $P_{abcd} \equiv (\partial L/\partial R^{abcd})$. We shall confine ourselves to Lanczos-Lovelock-type Lagrangians for which

$$L = \frac{1}{m}R^{abcd}\left(\frac{\partial L}{\partial R^{abcd}}\right) \equiv R^{abcd}Q_{abcd} \qquad (53)$$

with $\nabla_a P^{abcd} = \nabla_a Q^{abcd} = 0$ so that $J^{ab} = -2P^{abcd}\nabla_c\xi_d$.

We want to evaluate the Noether charge corresponding to the current J^a for a static metric with a bifurcation horizon and a killing vector field $\xi^a = (1, \mathbf{0})$. The location of the horizon is given by the vanishing of the norm $\xi^a\xi_a = g_{00}$, of this killing vector. Using these facts as well as the relations $\nabla_c \xi^d = \Gamma^d_{c0}$, etc., we find that $J^{ab} = 2P_d{}^{0ab}\Gamma^d_{c0}$. Therefore the Noether charge is given by

$$\begin{aligned}\mathcal{N} &= \int_t d^{D-1}x\sqrt{-g}J^0 \\ &= \int_t d^{D-1}x\sqrt{-g}\frac{1}{\sqrt{-g}}\partial_b(\sqrt{-g}J^{b0}) \\ &= \int_{t,r_H} d^{D-2}x\sqrt{-g}J^{r0}\end{aligned} \tag{54}$$

in which we have ignored the contributions arising from $b =$ transverse directions. This is justifiable when transverse directions are compact or in the case of Rindler approximation when nothing changes along the transverse direction. In the radial direction, we have again confined to the contribution at $r = r_H$ which is taken to be the location of the horizon. Using $Q^{r0} = 2P^{dcr0}\Gamma_{dc0} = -2P^{dcr0}\partial_d g_{c0}$ we get

$$\begin{aligned}\mathcal{N} &= -2\int_{t,r_H} d^{D-2}x\sqrt{-g}P^{dcr0}\partial_d g_{c0} \\ &= 2m\int_{t,r_H} d^{D-2}x\sqrt{-g}Q^{cdr0}\partial_d g_{c0}.\end{aligned} \tag{55}$$

Note that the dimension of $\mathcal{N}$ is L^{D-3} which is the area of transverse dimensions divided by a length. Entropy, which has the dimensions of transverse area, is given by the product of $\mathcal{N}$ and the interval in time integration. If the surface gravity of the horizon is κ, the time integration can be limited to the range $(0, \beta)$ where $\beta = 2\pi/\kappa$. The entropy, computed from the Noether charge approach is thus given by

$$S_{\rm Noether} = \beta\mathcal{N} = 2\beta m\int_{t,r_H} d^{D-2}x\sqrt{-g}Q^{cdr0}\partial_d g_{c0}. \tag{56}$$

We will now show that this is the same result one obtains by evaluating our surface term on the horizon except for a proportionality constant. In the stationary case, the contribution of the surface term on the horizon is given by

$$\begin{aligned}S_{\rm sur} &= 2\int d^Dx\partial_c[\sqrt{-g}Q^{abcd}\partial_b g_{ad}] \\ &= 2\int dt\int_{r_H} d^{D-2}x\sqrt{-g}Q^{abrd}\partial_b g_{ad}.\end{aligned} \tag{57}$$

Once again, taking the integration along t to be in the range $(0, \beta)$ and ignoring transverse directions, we get

$$S_{\rm sur} = 2\beta\int_{r_H} d^{D-2}x\sqrt{-g}Q^{abr0}\partial_b g_{a0}. \tag{58}$$

Comparing with Eq. (55), we find that

$$S_{\rm Noether} = mS_{\rm sur}. \tag{59}$$

The overall proportionality factor has a simple physical meaning. Equation (40) tells us that the quantity $mL_{\rm sur}$, rather than $L_{\rm sur}$, which has the "$d(qp)$" structure and it is this particular combination which plays the role of entropy, as to be expected [21].

B. The direct calculation of horizon entropy from the surface term

In this section, we shall provide a brief outline of an explicit computation of the contribution of the surface term on a horizon and show that it is equal to the standard expression for entropy, computed previously in the literature for Lanczos-Lovelock Lagrangians. To this end, we will consider a metric in D-dimensional spacetime in, static, isotropic coordinates with the form

$$ds^2 = -b(r)dt^2 + b^{-1}(r)dr^2 + r^2g_{mn}(x)dx^m dx^n. \tag{60}$$

In general, a static, spherically symmetric metric can have different functions describing g_{00} and g_{rr}. For our purpose we have assumed $g_{00}g_{rr} = -1$ since many solutions relevant to us fall in this category and it simplifies the calculations.

We will now evaluate the surface term for the off-shell metric discussed above in the Euclidean spacetime. Let the integrand (the $L_{\rm sur}$) of the surface term be $\partial_c P^c$. On integration over the radial direction, this will have two contributions: one from the horizon, $P^r(r_+)$ where the horizon is at $r = r_+$ and the other from the surface at infinity $P^r(\infty)$. We will again concentrate on the contribution from the horizon. Let Σ be the surface $r = r_+ + \epsilon$. Then we need to compute:

$$\begin{aligned}&I_\epsilon \equiv \int d\Sigma n_r P^r = \sqrt{b'(r_+)\epsilon}\int_0^\beta dt r_+^{D-2}\int d^{D-2}\Omega n_r P^r; \\ &\beta = \frac{4\pi}{b'(r_+)}.\end{aligned} \tag{61}$$

We have used the measure $d\Sigma$ appropriate for our metric, restricted the range of integration of t to $(0, \beta)$ as explained earlier, and used the fact that the normal to Σ has the nonvanishing component $n_r = -1/\sqrt{b'(r_+)\epsilon}$. The horizon contribution arises from the limit of $\epsilon \to 0$. The $\sqrt{\epsilon}$ term in the measure cancels with the $\sqrt{\epsilon}$ term in the normal. Further, it can be verified that P^r is regular at the horizon. So the contribution to the surface term from the horizon is

$$I_+ = \frac{4\pi}{b'(r_+)}r_+^{D-2}P^r(r_+)\Sigma_{D-2}, \tag{62}$$

where Σ_{D-2} is the (dimensionless) volume of S^{D-2}. Evaluating $P^r(r) = 2Q^{abrd}\partial_b g_{ad}$ explicitly for our metric, we find that

$$I_+ = 8\frac{\pi}{b'(r_+)}\Sigma_{D-2}[Q^{rtrt}\partial_r g_{tt} + Q^{rmrm}\partial_r(r^2g_{mm})]. \tag{63}$$

This result is general in the sense that we have not assumed anything about Q^{abcd} so far.

We will now specialize to the Lanczos-Lovelock-type Lagrangian, for which explicit evaluation shows that nonzero components near the horizon give:

$$Q^{trrt} = \frac{1}{16\pi}\frac{{}^{D-2}L_{m-1}}{2r^{2(m-1)}} + \mathcal{O}(b); \qquad Q^{mrmr} = \mathcal{O}(b), \qquad Q^{tmtm} = \mathcal{O}(b), \tag{64}$$

where ${}^{d-2}L_{m-1}$ is the Lanczos-Lovelock Lagrangian of degree $(m-1)$ evaluated on the horizon. (The vanishing of some of the components can be argued from symmetry considerations. Terms with an odd number of t's will vanish in a static situation, because of time reversal symmetry. Similarly rotational invariance will forbid terms which have an odd number of transverse coordinates. Rotational invariance also implies that $Q^{rmrn} = 0$ if $m \neq n$.). Using Eq. (64) in Eq. (63) we get the final result to be:

$$I_+ = -\frac{r_+^{D-2m}}{4}\Sigma_{D-2}({}^{D-2}L_{m-1}). \tag{65}$$

This is just $(1/m)$ times the Wald's entropy (with a minus sign due to the choice of Minkowski signature) and has been computed in the literature before (see e.g. [22]). In fact whenever Q^{rmrm} vanishes at the horizon, the contribution of the horizon to the surface term is $(1/m)$ times the Wald's entropy.

Finally we would like to make a comment on the general covariance of the result. It is easy to show that, if one changes coordinates from x^a to y^a the results will differ by a term that is proportional to:

$$\sqrt{b'(r_+)\epsilon}\int_0^\beta dt r^{D-2}\int d^{D-2}\Omega n_r\left(2Q^{abrd}(x)g_{ea}(x) \times\left(\frac{\partial^2 y^e}{\partial x^c \partial x^d}\right)\left(\frac{\partial x^c}{\partial y^b}\right)\right). \tag{66}$$

This term has to be evaluated at the horizon as far as the entropy computation is concerned. On Euclidean continuation, the horizon maps to the origin. For the subset of coordinate transformations (a) which are regular at the origin and (b) in which the transformed coordinates also are like polar coordinates near the origin, this extra contribution vanishes at the horizon. This is because $\partial^2 y^e/\partial x^c \partial x^d$ vanishes at the origin, since the only allowed transformation at the origin is a spacetime independent scaling of r and t.

Holographic relations in Lagrangians

$\sqrt{-g}L = \sqrt{-g}Q_a{}^{bcd}R^a{}_{bcd} = 2\partial_c[\sqrt{-g}Q_a{}^{bcd}\Gamma^a_{bd}] + 2\sqrt{-g}Q_a{}^{bcd}\Gamma^a_{dk}\Gamma^k_{bc} \equiv L_{\rm sur} + L_{\rm bulk}$

(1)	$\nabla_c Q_a{}^{bcd} = 0$	$L_{\rm sur} = -\partial_p\left(\delta^q_r \frac{\partial L_{\rm bulk}}{\partial \Gamma^q_{pr}}\right)$
		$L = \frac{1}{2}R^a{}_{bcd}\left(\frac{\partial V^c}{\partial \Gamma^a_{bd}}\right)$; $L_{\rm bulk} = \sqrt{-g}\left(\frac{\partial V^c}{\partial \Gamma^a_{bd}}\right)\Gamma^a_{dk}\Gamma^k_{bc}$
(2)	$Q_a{}^{bcd} = \frac{1}{m}\frac{\partial L}{\partial R^a{}_{bcd}}$	$L_{\rm sur} = -\partial_p\left(\delta^q_r \frac{\partial L_{\rm bulk}}{\partial \Gamma^q_{pr}}\right)$
		$L = \frac{1}{2}R^a{}_{bcd}\left(\frac{\partial V^c}{\partial \Gamma^a_{bd}}\right)$; $L_{\rm bulk} = \sqrt{-g}\left(\frac{\partial V^c}{\partial \Gamma^a_{bd}}\right)\Gamma^a_{dk}\Gamma^k_{bc}$
		$[(D/2)-m]L_{\rm sur} = -\partial_i\left[g_{ab}\frac{\delta L_{\rm bulk}}{\delta(\partial_i g_{ab})} + \partial_j g_{ab}\frac{\partial L_{\rm bulk}}{\partial(\partial_i\partial_j g_{ab})}\right]$
		$mL_{\rm sur} = -\partial_i\left[g_{ab}\frac{\delta L}{\delta(\partial_i g_{ab})}\right]$
(3)	$Q_a{}^{bcd} = \frac{1}{2}(\delta^c_a g^{bd} - \delta^d_a g^{bc})$	$L_{\rm sur} = -\partial_p\left(\delta^q_r \frac{\partial L_{\rm bulk}}{\partial \Gamma^q_{pr}}\right)$
		$L = \frac{1}{2}R^a{}_{bcd}\left(\frac{\partial V^c}{\partial \Gamma^a_{bd}}\right)$; $L_{\rm bulk} = \sqrt{-g}\left(\frac{\partial V^c}{\partial \Gamma^a_{bd}}\right)\Gamma^a_{dk}\Gamma^k_{bc}$
		$L_{\rm sur} = -\frac{1}{[(D/2)-1]}\partial_i\left(g_{ab}\frac{\partial L_{\rm bulk}}{\partial(\partial_i g_{ab})}\right)$
		$L_{\rm sur} = -\partial_i\left[g_{ab}\frac{\delta L}{\delta(\partial_i g_{ab})}\right]$

V. CONCLUSIONS

Our key conclusions are summarized in the table, listed from the most general results to the special case as we proceed down. The title line defines the Lagrangian we consider which, under the condition in (1), is generally covariant and has a specific separation into surface and bulk terms. The most general results are in the first row, which does not assume any structure about $Q_a{}^{bcd}$ other than that $\nabla_c Q_a{}^{bcd} = 0$. These relations in the table show that one can determine $L_{\rm sur}$ and $L_{\rm bulk}$ in terms of each other provided we treat Γ^a_{bc} as independent during the differentiation, etc., as explained in Sec. III A. The next row deals with Lagrangians which are of Lanczos-Lovelock type [which satisfy *both* conditions (1) and (2)]. In addition to the results in the previous row, we obtain two more results expressing $L_{\rm sur}$ in terms of L or $L_{\rm bulk}$. The "$d(qp)$ structure" is obvious in this case. The last row discusses the well-known Einstein-Hilbert Lagrangian which has been our reference point. In this case, the results for the Lanczos-Lovelock Lagrangian with ($m = 1$), of course, continues to be valid; but, in addition, we can simplify one of the relations further.

As we discussed before, the surface term (even in the most general case) has a "$d(qp)$ structure." In the Lagrangian picture we have adopted throughout the paper, we treat all the g_{ab}s at the same footing. However, we know that in any generally covariant theory the choice of coordinates puts D conditions on the g_{ab} which could be conveniently taken to be on g_{00} and $g_{0\alpha}$. Though the Hamiltonian structure for an arbitrary generally covariant Lagrangian is complicated (and—as far as we know—not fully worked out at the same level as, say, the Arnowitt-Deser-Misner (ADM) description in general relativity), the contribution of the surface term on $t =$ constant surfaces will only depend on $g_{ab}[\partial L/\partial(\partial_0 g_{ab})]$. If one can impose a gauge condition that $g_{00} = 1$ and $g_{0\alpha} = 0$, then this will give the standard canonical momenta corresponding to the dynamical variables $g_{\alpha\beta}$ in the Hamiltonian language. This is however a rather formal statement in the absence of a fully developed Hamiltonian formulation for the Lanczos-Lovelock Lagrangian.

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APPENDIX A: PROOFS FOR DIFFERENT RELATIONS

In this appendix we outline the proofs of different equations stated in the text for the sake of completeness.

1. Proof of Eqs. (14) and (22)

Consider a Lagrangian of the form $L = \sqrt{-g} Q_a{}^{bcd} R^a{}_{bcd}$ in which $Q_a{}^{bcd}$ has the algebraic symmetries of a curvature tensor. Expressing $R^a{}_{bcd}$ in terms of Γ^i_{jk} and using the antisymmetry of $Q_a{}^{bcd}$ in c and d, we can write

$$\begin{aligned} L &= \sqrt{-g} Q_a{}^{bcd} R^a{}_{bcd} = 2\sqrt{-g} Q_a{}^{bcd}(\partial_c \Gamma^a_{db} + \Gamma^a_{ck}\Gamma^k_{db}) \\ &= 2\sqrt{-g} Q_a{}^{bcd}\Gamma^a_{ck}\Gamma^k_{db} + 2\partial_c[\sqrt{-g} Q_a{}^{bcd}\Gamma^a_{db}] \\ &\quad - 2\Gamma^a_{db}\partial_c[\sqrt{-g} Q_a{}^{bcd}] \\ &= 2\sqrt{-g} Q_a{}^{bcd}\Gamma^a_{ck}\Gamma^k_{db} + 2\partial_c[\sqrt{-g} Q_a{}^{bcd}\Gamma^a_{db}] \\ &\quad - 2\sqrt{-g}\Gamma^a_{db}\partial_c Q_a{}^{bcd} - 2\sqrt{-g}\Gamma^a_{db}\Gamma^j_{cj} Q_a{}^{bcd}. \end{aligned} \tag{A1}$$

We now express $\partial_c Q_a{}^{bcd}$ in terms of $\nabla_c Q_a{}^{bcd}$ to obtain

$$\begin{aligned} \Gamma^a_{db}\partial_c Q_a{}^{bcd} &= \Gamma^a_{db}\nabla_c Q_a{}^{bcd} - \Gamma^a_{db}\Gamma^b_{kc} Q_a{}^{kcd} \\ &\quad + \Gamma^a_{db}\Gamma^k_{ac} Q_k{}^{bcd} - \Gamma^a_{db}\Gamma^c_{kc} Q_a{}^{bkd}. \end{aligned} \tag{A2}$$

Substituting Eq. (A2) into Eq. (A1) we notice that two pairs of the terms cancel out leaving the result

$$\begin{aligned} \sqrt{-g} Q_a{}^{bcd} R^a{}_{bcd} &= 2\partial_c[\sqrt{-g} Q_a{}^{bcd}\Gamma^a_{bd}] \\ &\quad + 2\sqrt{-g} Q_a{}^{bcd}\Gamma^a_{dj}\Gamma^j_{bc} \\ &\quad - 2\sqrt{-g}\Gamma^a_{bd}\nabla_c Q_a{}^{bcd}. \end{aligned} \tag{A3}$$

This is essentially our result in Eq. (22). Since $\nabla_c Q_a{}^{bcd} = 0$ for the Einstein-Hilbert action, we get Eq. (14).

2. Connecting up Eqs. (16) and (17)

In the text we proved that, for any Lagrangian of the form $L = \sqrt{-g} Q_a{}^{bcd} R^a{}_{bcd}$ with $\nabla_c Q_a{}^{bcd} = 0$ there is a natural separation of the Lagrangian into bulk and surface terms, related by:

$$L_{\rm sur} = -\partial_m\left(\delta^p_n \frac{\partial L_{\rm bulk}}{\partial \Gamma^p_{mn}}\right). \tag{A4}$$

The key caveat in this relation is that one needs to treat all components of Γ^p_{mn} as independent while differentiating and use a particular ordering of indices in the original expression. The purpose of this subsection is to recast this relation in terms of the derivatives of the metric and show the rather special nature of the Einstein-Hilbert Lagrangian. To convert this into a relation involving the partial derivatives of the metric we use the result:

$$\frac{\partial \Gamma^p_{mn}}{\partial(\partial_a g_{bc})} = \frac{1}{2}(-g^{pa}\delta^b_m\delta^c_n + g^{pb}\delta^a_m\delta^c_n + g^{pb}\delta^c_m\delta^a_n) \tag{A5}$$

from which we have the operator identity:

$$g_{bc}\frac{\partial}{\partial(\partial_a g_{bc})} = \frac{1}{2}(-g^{pa}g_{mn} + \delta^p_n\delta^a_m + \delta^p_n\delta^a_m)\frac{\partial}{\partial \Gamma^p_{mn}}. \tag{A6}$$

So that:

$$g_{bc}\frac{\partial L_{\rm bulk}}{\partial(\partial_a g_{bc})} = \frac{1}{2}(-g^{pa}g_{mn} + \delta^p_n\delta^a_m + \delta^p_n\delta^a_m)\frac{\partial L_{\rm bulk}}{\partial \Gamma^p_{mn}}. \tag{A7}$$

Using Eq. (18) to determine $(\partial L_{\rm bulk}/\partial \Gamma^p_{mn})$ and manipulating the terms we get, after some algebra:

$$\begin{aligned} g_{bc}\frac{\partial L_{\rm bulk}}{\partial(\partial_a g_{bc})} &= -\sqrt{-g}(Q_p{}^{bad}\Gamma^p_{bd} \\ &\quad + Q_p{}^{bpd} g^{la}(\Gamma_{lbd} - \Gamma_{bdl})) \\ &= \sqrt{-g}(\partial_n g_{ad})(Q^{anid} + g^{in} Q_p{}^{apd} \\ &\quad - g^{ia} Q_p{}^{npd}). \end{aligned} \tag{A8}$$

Note that, in the last two terms the indices of Q^{abcd} are contracted among themselves; hence, in the general case, it is not possible to proceed further and relate this result to $L_{\rm sur}$ directly. The Einstein-Hilbert Lagrangian is special in the sense that, for $Q_p{}^{bad} = (1/2)(\delta^a_p g^{bd} - \delta^d_p g^{ba})$, we have $Q_p{}^{bpd} = (1/2)(D-1)g^{bd}$ and the last two terms can be combined with the first to give Eq. (16).

3. Proof of Eq. (31)

For the mth Lanczos-Lovelock Lagrangian $L = R_{a_1b_1}{}^{c_1d_1}\ldots R_{a_mb_m}{}^{c_md_m}\delta^{a_1b_1\ldots a_mb_m}_{c_1d_1\ldots c_md_m}$ we have the result:

$$P^{ab}{}_{cd} = \frac{\partial L}{\partial R_{ab}{}^{cd}} = mR_{a_2b_2}{}^{c_2d_2}\ldots R_{a_mb_m}{}^{c_md_m}\delta^{aba_2b_2\ldots a_mb_m}_{cdc_1d_1\ldots c_md_m}. \tag{A9}$$

Therefore,

$$\nabla_a P^{ab}{}_{cd} = m\nabla_a R_{a_2b_2}{}^{c_2d_2}\ldots R_{a_mb_m}{}^{c_md_m}\delta^{aba_2b_2\ldots a_mb_m}_{cdc_1d_1\ldots c_md_m}. \tag{A10}$$

We note that the derivatives of the curvature tensor appearing in the expression are rendered completely antisymmetric in all the lower indices due to the contraction with the alternating tensor. But Bianchi identity states that $\nabla_{[a}R_{a_2b_2]}{}^{c_2d_2} = 0$ and thus we get $\nabla_a P^{ab}{}_{cd} = 0$.

4. Proof of Eq. (33)

Consider the variation of the quantity $L\sqrt{-g}$ where L is a generally covariant scalar made from g^{ab} and $R^a{}_{bcd}$. We can express its variation in the form

$$\begin{aligned}\delta(L\sqrt{-g}) &= \left(\frac{\partial L\sqrt{-g}}{\partial g^{ab}}\right)\delta g^{ab} + \left(\frac{\partial L\sqrt{-g}}{\partial R^a{}_{bcd}}\right)\delta R^a{}_{bcd}\\ &= \left(\frac{\partial L\sqrt{-g}}{\partial g^{ab}}\right)\delta g^{ab} + \sqrt{-g}P_a{}^{bcd}\delta R^a{}_{bcd}.\end{aligned} \tag{A11}$$

The term $P_a{}^{bcd}\delta R^a{}_{bcd}$ is generally covariant and hence can be evaluated in the local inertial frame using

$$\begin{aligned}\delta R^a{}_{bcd} &= \nabla_c(\delta\Gamma^a_{db}) - \nabla_d(\delta\Gamma^a_{cb})\\ &= \tfrac{1}{2}\nabla_c[g^{ai}(-\nabla_i\delta g_{db} + \nabla_d\delta g_{bi} + \nabla_b\delta g_{di})]\\ &\quad - \{\text{term with } c \leftrightarrow d\}.\end{aligned} \tag{A12}$$

When this expression is multiplied by $P_a{}^{bcd}$ the middle term $g^{ai}\nabla_d\delta g_{bi}$ does not contribute because of the antisymmetry of P^{ibcd} in i and b. The other two terms contribute equally and we get a similar contribution from the term with c and d interchanged. Hence we get

$$P_a{}^{bcd}\delta R^a{}_{bcd} = 2P^{ibcd}\nabla_c\nabla_d(\delta g_{di}). \tag{A13}$$

Manipulating the covariant derivative, this can be reexpressed in the form

$$\begin{aligned}P_a{}^{bcd}\delta R^a{}_{bcd} &= 2\nabla_c[P^{ibcd}\nabla_b\delta g_{di}] - 2\nabla_b[\delta g_{di}\nabla_c P^{ibcd}]\\ &\quad + 2\delta g_{di}\nabla_b\nabla_c P^{ibcd}\end{aligned} \tag{A14}$$

Combining this with the first term in Eq. (A11) and rearranging the expression, we get

$$\begin{aligned}\delta L\sqrt{-g} &= \left(\frac{\partial L\sqrt{-g}}{\partial g^{ab}} - 2\sqrt{-g}\nabla^m\nabla^n P_{amnb}\right)\delta g_{ab}\\ &\quad + \sqrt{-g}\nabla_j[2P^{ibjd}(\nabla_b\delta g_{di}) - 2\delta g_{di}\nabla_c P^{ijcd}]\end{aligned} \tag{A15}$$

which is the same as Eq. (33).

5. Proof of Eqs. (38) and (40)

To prove Eq. (40) we shall prove that

$$\begin{aligned} g_{np}\frac{\partial L\sqrt{-g}}{\partial(\partial_m g_{np})} &= 2\sqrt{-g}[P_a{}^{bac}\Gamma^m_{bc} - 2P^{nbmd}\Gamma_{ndb}],\\ g_{np}\partial_s\frac{\partial L\sqrt{-g}}{\partial(\partial_s\partial_m g_{np})} &= 2\sqrt{-g}[P_a{}^{bac}\Gamma^m_{bc} - P^{nbmd}\Gamma_{ndb}].\end{aligned} \tag{A16}$$

The Euler derivative on the left-hand side of Eq. (40) is the difference between the two quantities evaluated above. On subtraction, two terms on the right-hand side cancel out and what remains leads to Eq. (38) for a *general* Lagrangian. For Lanczos-Lovelock Lagrangians we are led to Eq. (40) when we use $P^{abcd} = mQ^{abcd}$.

Equation (A16) can be proved by direct computation but a somewhat quicker route is the following: We begin by noting that $(\partial_m g_{np})$ and $(\partial_s\partial_m g_{np})$ occurs in L only through $R^a{}_{bcd}$. So if we keep $\delta g_{ab} = 0$ but vary $\partial_m g_{np}$ and $\partial_s\partial_m g_{np}$ in L and get an expression of the form:

$$\delta L = A^{mnp}\delta(\partial_m g_{np}) + B^{smnp}\delta(\partial_s\partial_m g_{np}) \tag{A17}$$

we can read of the terms we need in Eq. (A16). To do this we start with Eq. (A13) which gives, when $\delta g_{ab} = 0$:

$$\begin{aligned}\delta L &= \left(\frac{\partial L}{\partial R^a{}_{bcd}}\right)\delta R^a{}_{bcd} = P_a{}^{bcd}\delta R^a{}_{bcd}\\ &= 2P^{ibcd}\nabla_c\nabla_d(\delta g_{di}).\end{aligned} \tag{A18}$$

We now expand out $\nabla_c\nabla_d(\delta g_{di})$, using $\delta g_{ab} = 0$ repeatedly to get:

$$\begin{aligned}\delta L &= 2P^{ibcd}[\delta(\partial_b\partial_c g_{di}) - \Gamma^k_{db}\delta(\partial_c g_{ki}) - \Gamma^k_{ic}\delta(\partial_b g_{dk})\\ &\quad - \Gamma^k_{bc}\delta(\partial_k g_{di})].\end{aligned} \tag{A19}$$

We have also used the fact that when $\delta g_{ab} = 0$, $\delta\partial_c g_{ab} \neq 0$, we can write $\nabla_j\delta g_{ab} = \partial_j\delta g_{ab} = \delta\partial_j g_{ab}$, etc. We can now read off $\partial L\sqrt{-g}/\partial(\partial_m g_{np})$ and $\partial L\sqrt{-g}/\partial(\partial_s\partial_m g_{np})$ from Eq. (A19) since $\sqrt{-g}$ goes for a ride. Contracting with g_{np} and using the symmetries immediately gives the first of the equations in Eq. (A16) as well as the result

$$g_{np}\partial_s \frac{\partial L\sqrt{-g}}{\partial(\partial_s\partial_m g_{np})} = 2g_{np}\partial_s(\sqrt{-g}P^{psmn}). \qquad \text{(A20)}$$

Finally we use the fact that, when $\nabla_c P^{abcd} = 0$, we have the relation:

$$\partial_c[\sqrt{-g}P^{abcd}] = -\sqrt{-g}[\Gamma^a_{kc}P^{kbcd} + \Gamma^b_{kc}P^{akcd}]. \qquad \text{(A21)}$$

Using this to simplify the right-hand side of Eq. (A20) leads to Eq. (A16).

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