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Evolution of universes in quadratic theories of gravity

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We use a dynamical systems approach to investigate Bianchi type I and II universes in quadratic theories of gravity. Because of the complicated nature of the equations of motion we focus on the stability of exact solutions and find that there exists an isotropic Friedmann-Robertson-Walker (FRW) universe acting as a past attractor. This may indicate that there is an isotropization mechanism at early times for these kind of theories. We also discuss the Kasner universes, elucidate the associated center manifold structure, and show that there exists a set of nonzero measure which has the Kasner solutions as a past attractor. Regarding the late-time behavior, the stability shows a dependence of the parameters of the theory. We give the conditions under which the de Sitter solution is stable and also show that for certain values of the parameters there is a possible late-time behavior with phantomlike behavior. New types of anisotropic inflationary behavior are found which do not have counterparts in general relativity.

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I. INTRODUCTION

In this paper we are going to study the dynamical evolution of a class of anisotropic universes in quadratic theories of gravity. These extensions of general relativity (GR) provide guidance as to the possible effects of quantum corrections to the Einstein equations. They permit us to investigate the possible effects on singularity formation, inflation, and the expansion dynamics of the early universe. Past studies of these extensions have focused on the isotropic Friedmann metrics, where it is sufficient to consider only the effects of an R^2 term in the gravitational Lagrangian to the field equations [1]. However, the R^2 contribution has fairly predictable cosmological consequences because the resulting quadratic vacuum theory is conformally equivalent to GR with a scalar field moving in a potential with a single asymmetric minimum [2,3]. This type of solution has been well studied in connection with inflation [4] and in the situation of pure power-law Lagrangians of the form R^n [5–8]. The addition of the $R_{\mu\nu}R^{\mu\nu}$ Ricci term to the Lagrangian in the case of an anisotropic universe creates a much richer diversity of cosmological behaviors that are harder to summarize in terms of simple modifications of the general-relativistic situation. The effective stresses that are contributed to the field equations by the quadratic Ricci terms in the Lagrangian can mimic a wide range of fluids which violate the strong and weak energy conditions. This allows completely different behavior to occur than is found in GR or its quadratic extension with pure R^2 contributions. In particular, we have shown elsewhere that the cosmic no-hair theorems no longer hold: vacuum universes with positive cosmological constant do not necessarily approach de Sitter, but can inflate anisotropically [9]. Moreover, the addition of quadratic Ricci terms to the Lagrangian can create cosmological models which have no counterpart in the GR limit of the theory.

In what follows, we are going to widen this investigation by considering the global dynamics of the Bianchi type I and II universes in the presence of quadratic Ricci and scalar curvature contributions to the Lagrangian. Whilst these anisotropic universes contain some mathematical simplifications, they include spatially homogeneous expanding universes with isotropic (type I) and anisotropic (type II) spatial 3-curvatures as well as expansion shear anisotropy. To this end, we consider the gravitational action

$$S_G = \frac{1}{2\kappa}\int_M d^4x\sqrt{|g|}(R + \alpha R^2 + \beta R_{\mu\nu}R^{\mu\nu} - 2\Lambda). \quad (1)$$

Variation of this action leads to the following generalized Einstein equations (see, e.g., [10]):

$$G_{\mu\nu} + \Phi_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}, \quad (2)$$

where $T_{\mu\nu}$ is the energy-momentum tensor of the ordinary matter sources, which in this paper we will assume to be zero, for simplicity,

$$G_{\mu\nu} \equiv R_{\mu\nu} - \tfrac{1}{2}Rg_{\mu\nu}, \quad (3)$$

$$\Phi_{\mu\nu} \equiv 2\alpha R(R_{\mu\nu} - \tfrac{1}{4}Rg_{\mu\nu}) + (2\alpha + \beta)(g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu)R + \beta\Box(R_{\mu\nu} - \tfrac{1}{2}Rg_{\mu\nu}) + 2\beta(R_{\mu\sigma\nu\rho} - \tfrac{1}{4}g_{\mu\nu}R_{\sigma\rho})R^{\sigma\rho}, \quad (4)$$

with $\Box \equiv \nabla^\mu\nabla_\mu$ and $g_{\mu\nu}$ is the metric tensor and Λ the cosmological constant. The effective stress tensor $\Phi_{\mu\nu}$ incorporates the deviation from regular Einstein gravity introduced by the quadratic terms in the action, and we see that $\alpha = \beta = 0$ implies $\Phi_{\mu\nu} = 0$, although the converse need not be true.

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Similarly, by rescaling $(\alpha, \beta) \mapsto \kappa(\alpha, \beta)$ we can consider the limit $\kappa \rightarrow \infty$. This limit, whenever it exists, corresponds to the case where the action only contains quadratic terms, hence Eq. (2) reduces to (for vacuum, $T_{\mu\nu} = 0$)

$$\Phi_{\mu\nu} = 0. \tag{5}$$

It is important to consider the vacuum solutions to the pure quadratic theories; that is, solutions to Eq. (6). First, we note that any Einstein metric with $R_{\mu\nu} = \lambda g_{\mu\nu}$, necessarily obeys $\Phi_{\mu\nu} = 0$. In particular, this implies that *any GR vacuum solution* ($G_{\mu\nu} = 0$) *will also be a vacuum solution to the quadratic theory* ($G_{\mu\nu} + \Phi_{\mu\nu} = 0$). This means that there will be chaotic dynamics in the quadratic theory because it is present in a number of general-relativistic vacuum Bianchi models (types $\mathrm{VI}^*_{-1/9}$, VIII, and IX) near an initial curvature singularity; however it might not be stable in solutions to the quadratic theories in the way that it appears to be for these solutions in GR. Second, there are some specific solutions to $\Phi_{\mu\nu} = 0$, which are not solutions to $G_{\mu\nu} = 0$, and so have no counterparts in the $(\alpha, \beta) \rightarrow (0, 0)$ general-relativity limit of the quadratic theory. This feature plays an important role in the evolution of universes that arise in these theories. For example, one such intrinsically quadratic solution is a Friedmann-Robertson-Walker (FRW) universe which has expansion dynamics similar to those of a radiation-dominated universe in GR. This means that if we take a FRW universe with spatial curvature parameter k, there is a simple solution to $\Phi_{\mu\nu} = 0$ given by the metric that solves the field equations of GR ($G_{\mu\nu} = \kappa T_{\mu\nu}$) in the presence of blackbody radiation:

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + (t - kt^2)\left[\frac{\mathrm{d}r^2}{1 - kr^2} + r^2(\mathrm{d}\phi^2 + \sin^2\phi \mathrm{d}\theta^2)\right]. \tag{6}$$

Note that despite the fact that $T_{\mu\nu} = 0$ this quadratic universe behaves as if it is radiation-dominated. This radiationlike universe seems to play a special role in the theory as it will be found to act as a past attractor for the anisotropic universes we study. Other interesting conclusions can also be deduced from this simple solution. We see, for example, that unlike for the GR case, in the quadratic theory there is a closed ($k = 1$) FRW *vacuum* solution which recollapses after a finite time ($t = 1$) and a flat ($k = 0$) vacuum solution.

There are some further geometrical observations regarding solutions to these quadratic theories that will be useful. In 4D spacetimes we can use the Weyl curvature invariant and the Euler density E defined by,

$$\begin{aligned} C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} &= R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 2R_{\mu\nu}R^{\mu\nu} + \tfrac{1}{3}R^2, \\ E &= R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2, \end{aligned} \tag{7}$$

to replace the quadratic Ricci invariant in the action, since

$$\alpha R^2 + \beta R_{\mu\nu}R^{\mu\nu} = \tfrac{1}{3}(3\alpha + \beta)R^2 + \tfrac{\beta}{2}(C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} - E).$$

Since integration over the Euler density is a topological invariant, the variation of E will not contribute to the equations of motion. From this we see that there are two special cases

(a) $(3\alpha + \beta) = 0$: the conformally invariant Bach-Weyl gravity theory [11].

(b) $\beta = 0$: a special case of $f(R)$ gravity.

Since the equations of motion change their structure in these two special cases, a separate analysis is required for each. So, in what follows we will assume that

$$3\alpha + \beta \neq 0, \qquad \beta \neq 0.$$

On the other hand, our formulation of the equations of motion is well defined in the $\kappa \rightarrow \infty$ limit (as explained above).

In our analysis, we will adopt the dynamical systems approach by introducing expansion-normalized variables. This approach has proven to be extremely successful in the analysis of spatially homogeneous Bianchi universes in GR [12,13]. However, by considering quadratic theories of gravity there are some extra complications added to the formalism, as the universe can bounce or recollapse at expansion minima and maxima, which leads to infinities in the expansion-normalized variables. Here, we will discuss these infinities and explain how a sequence of expanding and contracting phases can be considered. On the other hand, initial singularities and future asymptotes are all finite in these variables, and the dynamical systems approach is ideal for studying such regimes.

In a recent paper by Leach *et al* [14], a dynamical systems approach to the local rotational symmetric Bianchi type I universes was considered within a class of $f(R)$ theories. In particular, they were interested in the shear dynamics of such universes, and, interestingly, they found that there is a possibility for an isotropic singularity. The introduction of a $R_{\mu\nu}R^{\mu\nu}$ term introduces extra shear degrees of freedom and their higher time derivatives, which considerately complicate the equations of motion. However, in spite of the fact that the case we study is quite separate, and we consider more general Bianchi models, we also find the possibility for an isotropic singularity, as did Cotsakis *et al.* [15] in their study of Bianchi type IX universes in quadratic theories. Indeed, we will argue that this isotropic singularity is past stable for *all Bianchi models*.

II. EQUATIONS OF MOTION

Our starting point is the generalized vacuum field equations:

$$G_{\mu\nu} + \Phi_{\mu\nu} + \Lambda g_{\mu\nu} = 0.$$

We note that the GR-limit can be obtained by letting $(\alpha, \beta) \to (0, 0)$. Now consider the spatially homogeneous Bianchi metrics. We can always write their metric line elements as

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \delta_{ab}\boldsymbol{\omega}^a\boldsymbol{\omega}^b,$$

where $\boldsymbol{\omega}^a$ is a triad of one-forms obeying

$$(\mathrm{d}\boldsymbol{\omega}^a)_\perp = -\tfrac{1}{2}C^a{}_{bc}\boldsymbol{\omega}^b \wedge \boldsymbol{\omega}^c,$$

the $C^a{}_{bc}$ depend only on time and at any moment of constant time are the structure constants of the Bianchi group-type under consideration, and $(-)_\perp$ means projection onto the spatially homogeneous hypersurfaces. The structure constants can be split into a vector part a_b and a symmetric tensor n^{ab} in the standard way by

$$C^a{}_{bc} = \varepsilon_{bcd}n^{da} - \delta^a{}_b a_c + \delta^a{}_c a_b.$$

Using the Jacobi identity, a_b is in the kernel of n^{ab}

$$n^{ab}a_b = 0.$$

The different structures of n^{ab} and a_b define the various Bianchi types (see, e.g., [12]).

Defining the timelike hypersurface-orthogonal vector $\mathbf{u} = \partial/\partial t$, we can define the Hubble scalar H and the shear σ_{ab} as follows:

$$H \equiv \tfrac{1}{3}u^\mu_{;\mu}, \qquad \sigma_{ab} = u_{(a;b)} - H\delta_{ab}.$$

We will also restrict attention to cosmological models where the shear is diagonal, so we can write

$$\sigma_{ab} = \mathrm{diag}(-2\sigma_+, \sigma_+ + \sqrt{3}\sigma_-, \sigma_+ - \sqrt{3}\sigma_-).$$

We are interested in the Bianchi type I and type II models for which $a_b = 0$ and we can write

$$n_{ab} = 0 \quad (\text{type I}), \qquad n_{ab} = \mathrm{diag}(n_{11}, 0, 0) \quad (\text{type II}).$$

We define the dimensionless expansion-normalized variables by scaling out appropriate powers of H

$$B = \frac{1}{(3\alpha+\beta)H^2}, \qquad \chi = \frac{\beta}{3\alpha+\beta}, \qquad Q = \frac{\dot{H}}{H^2},$$
$$Q_2 = \frac{\ddot{H}}{H^3}, \qquad \Omega_\Lambda = \frac{\Lambda}{3H^2}, \qquad N = \frac{n_{11}}{\sqrt{3}H}, \tag{8}$$
$$\Sigma_\pm = \frac{\sigma_\pm}{H}, \qquad \Sigma_{\pm1} = \frac{\dot{\sigma}_\pm}{H^2}, \qquad \Sigma_{\pm2} = \frac{\ddot{\sigma}_\pm}{H^3}.$$

Note the presence of time derivatives of the variables Q_2 and $\Sigma_{\pm2}$; this reflects the 4th-order time derivatives in the field equations of the quadratic theory[1]; χ is a constant. We also introduce the dynamical time variable τ by

$$\frac{\mathrm{d}\tau}{\mathrm{d}t} = H,$$

and we assume that the cosmological constant is positive: $\Omega_\Lambda > 0$.

The equations of motion and the constraint (''Friedmann-like'') equation are:

$$B' = -2QB, \tag{9}$$

$$\Omega'_\Lambda = -2Q\Omega_\Lambda, \tag{10}$$

$$N' = -(Q + 1 + 4\Sigma_+)N, \tag{11}$$

$$Q' = -2Q^2 + Q_2, \tag{12}$$

$$\begin{aligned} Q_2' = {} & -3(Q+2)Q_2 - \tfrac{9}{2}(Q+2)Q - \tfrac{3}{4}B(1 + \Sigma^2 - \Omega_\Lambda \\ & + \tfrac{2}{3}Q - \tfrac{1}{3}N^2) - \tfrac{3}{2}(1+2\chi)\Sigma^4 - \tfrac{1}{4}(8+\chi)\Sigma_1^2 \\ & - (4-\chi)(\Sigma\cdot\Sigma_1) - \tfrac{1}{4}(4-\chi)(3\Sigma^2 + 2\Sigma\cdot\Sigma_2 \\ & + 2Q\Sigma^2) - (1+2Q)N^2 + N^2[\tfrac{1}{2}(1+8\chi)N^2 \\ & + 5(13+3\chi)\Sigma_+^2 + 8(2\Sigma_+ - \Sigma_{+1}) + (1-\chi)\Sigma_-^2], \end{aligned} \tag{13}$$

$$\Sigma'_\pm = -Q\Sigma_\pm + \Sigma_{\pm1}, \tag{14}$$

$$\Sigma'_{\pm1} = -2Q\Sigma_{\pm1} + \Sigma_{\pm2}, \tag{15}$$

$$\begin{aligned} \Sigma'_{+2} = {} & -3(Q+2)\Sigma_{+2} + \frac{\Sigma_{+1}}{\chi}[B - (11\chi - 8) + 4Q(1-\chi) + 4\Sigma^2(1+2\chi)] \\ & + \frac{\Sigma_+}{\chi}[3B + (4-\chi)(6 + Q_2 + 7Q) + 4(1+2\chi)(3\Sigma^2 + 2\Sigma\cdot\Sigma_1)] - \frac{4}{\chi}N^2[B + 8 + 4Q - 4(1+8\chi)N^2] \\ & - \frac{4}{\chi}N^2[(1+15\chi)(\Sigma_+ + \Sigma_{+1} - 4\Sigma_+^2) + 4(1-\chi)\Sigma_-^2], \end{aligned} \tag{16}$$

[1]The variable Q_2 can be related to the statefinders q and j (see e.g. [16]), by $Q_2 = j + 3q + 2$.

$$\Sigma'_{-2} = -3(Q+2)\Sigma_{-2} + \frac{\Sigma_{-1}}{\chi}[B - (11\chi - 8) + 4Q(1-\chi) + 4\Sigma^2(1+2\chi)] + \frac{\Sigma_-}{\chi}[3B + (4-\chi)(6 + Q_2 + 7Q) + 4(1+2\chi)(3\Sigma^2 + 2\Sigma\cdot\Sigma_1)] - \frac{4(1-\chi)}{\chi}N^2(\Sigma_- + \Sigma_{-1} - 8\Sigma_-\Sigma_+). \quad (17)$$

These equations are subject to the constraint:

$$0 = B(1 - \Omega_\Lambda - \Sigma^2 - N^2) + 12Q - 2Q^2 + 4Q_2 - (4-\chi)(3+2Q)\Sigma^2 - 6(1+2\chi)\Sigma^4 - \chi(\Sigma_1^2 - 2\Sigma\cdot\Sigma_2) + 4(2+\chi)(\Sigma\cdot\Sigma_1) + 4N^2[\tfrac{1}{2}(1+8\chi)N^2 + 1 + (1+15\chi)\Sigma_+^2 + 8\Sigma_+ + (1-\chi)\Sigma_-^2], \quad (18)$$

where, we have introduced the short-hand notation $\Sigma_n \equiv (\Sigma_{+n}, \Sigma_{-n})$ and $(\Sigma_n \cdot \Sigma_m) \equiv \Sigma_{+n}\Sigma_{+m} + \Sigma_{-n}\Sigma_{-m}$.

There are two points to note. First, the parameter Q is related to the usual deceleration parameter q via

$$q = -(1+Q).$$

Second, the variable B measures how greatly the quadratic part of the Lagrangian dominates over the general-relativistic Einstein-Hilbert term $R - 2\Lambda$. In particular, the larger the value of B, the "closer" we are to GR. The $B = 0$ case corresponds to a purely quadratic Lagrangian theory whose equations of motion reduce to $\Phi_{\mu\nu} = 0$.

The equations of motion define a dynamical flow in a large phase volume, and the behavior can be analyzed qualitatively by standard techniques from the theory of ordinary differential equations. Of particular interest are the exact solutions which define the critical points of the system and their stability at small and large times.

III. SOLUTIONS AND THEIR STABILITY

A. The de Sitter solution: dS

The de Sitter solution is characterized by the critical points where

$$Q = Q_2 = \Sigma_\pm = \Sigma_{\pm1} = \Sigma_{\pm2} = N = 0, \qquad \Omega_\Lambda = 1, \qquad B \neq 0.$$

Its stability is determined by the eigenvalues[2]:

$$0, \quad -1, \quad -\frac{3}{2}\left(1 \pm \sqrt{1 - \frac{2B}{9}}\right), \quad -3\ [\times 2], \quad -\frac{3}{2}\left(1 \pm \sqrt{1 + \frac{4[B + 2(4-\chi)]}{9\chi}}\right)[\times 2].$$

The zero eigenvalue corresponds to the variable B and appears because these solutions are a one-parameter family. The last three eigenvalues arise from the shear equations and come therefore in pairs. We also see that the limit $\chi \to 0$ is ill defined for these eigenvalues and these arise therefore only for theories with a nonzero $R_{\mu\nu}R^{\mu\nu}$-term.[3]

We note that

$$B > 0 \Rightarrow (3\alpha + \beta) > 0,$$
$$\frac{B + 2(4-\chi)}{\chi} < 0 \Rightarrow \frac{1 + 2\Lambda(4\alpha + \beta)}{\beta} < 0.$$

In particular, this means that whenever $\alpha = 0$, $\beta > 0$ the de Sitter solution has two unstable modes. These modes stem from the shear equations and are therefore not present for the FRW models studied earlier. On the other hand, if $\beta < 0$ and $(3\alpha + \beta) > 0$ then the de Sitter solution is stable.

A related set of equilibrium points is

$$Q = Q_2 = \Sigma_\pm = \Sigma_{\pm1} = \Sigma_{\pm2} = N = B = 0, \qquad \Omega_\Lambda = \text{constant} \neq 0.$$

The eigenvalues for this line bifurcation is the same as for the de Sitter solution restricted to $B = 0$. Consequently, there exist two zero eigenvalues for this set of solutions. However, it can be shown that these solutions are always unstable.

B. The Kasner circle: $\mathcal{K}$

The Kasner circle of equilibrium points is given by

$$Q = -3, \qquad (\Sigma_+, \Sigma_-) = (\cos\phi, \sin\phi), \qquad (\Sigma_{+1}, \Sigma_{-1}) = -3(\cos\phi, \sin\phi),$$
$$(\Sigma_{+2}, \Sigma_{-2}) = 18(\cos\phi, \sin\phi), \qquad \Omega_\Lambda = B = N = 0. \quad (19)$$

The exact Kasner solutions are completed by integrating the B equation so that $B = B_0 e^{6\tau}$; however, here we will

[2] The stability of de Sitter for these models was also studied independently by A. Toporensky and collaborators [17].

[3] As can be seen from the equations of motion, Eqs. (27) and (30) are ill defined in the $\chi \to 0$ limit. The case $\chi = 0$ induces a relation between Σ_1 and the other variables which makes it possible to eliminate Σ_2 and Σ_1 from the equations of motion. Therefore the case $\chi = 0$ has 4 fewer variables, which correspond to the last two pairs of eigenvalues given above.

mostly be interested in the limit $\tau \to -\infty$ for which case the equilibrium points suffice.

Stability is determined by the eigenvalues:

$$0\,[\times 2],\quad 2(1-2\cos\phi),\quad 6\,[\times 7].$$

We note that there are two zero eigenvalues. One of the zero eigenvalues just corresponds to the fact that this is a line of nonisolated equilibria. The second eigenvalue corresponds to a one-dimensional center manifold and in order to determine the stability of the Kasner circle an analysis of this center manifold is needed. Note also that the eigenvalue $2(1-2\cos\phi)$ arises from the curvature variable N. This is the same mode as in the GR case, where it causes vacuum type II transitions between points on the Kasner circle (see e.g. [12]).

The center manifold

Consider the invariant subspace given by (Q, Σ, F, G) defined as follows:

$$(\Sigma_+, \Sigma_-) = \Sigma(\cos\phi, \sin\phi),$$
$$(\Sigma_{+1}, \Sigma_{-1}) = F\Sigma(\cos\phi, \sin\phi),$$
$$(\Sigma_{+2}, \Sigma_{-2}) = G\Sigma(\cos\phi, \sin\phi),$$
$$N = B = \Omega_\Lambda = 0,$$
$$\phi = \text{constant},$$

and Q_2 is given by the constraint equation.

We consider the following perturbation:

$$(Q, \Sigma, F, G) = (-3 + x, 1 + y, -3 + z, 18 + w).$$

Close to the Kasner circle, the center manifold can be parametrized by the linear combination defined by the variable

$$X = \left(\frac{14+\chi}{72}\right)x - \left(\frac{\chi-4}{6}\right)y + \left(\frac{5\chi-2}{72}\right)z + \left(\frac{2+\chi}{72}\right)w,$$

and the equations of motion on the center manifold turn into

$$X' = X^2 + \mathcal{O}(X^3).$$

This implies that close to $X = 0$, X is always increasing. In essence, the Kasner circle acts as a past attractor for some orbits, while for others it is unstable to the past.

The center manifold has a special importance for solutions asymptoting to the equilibrium point. Because of the zero eigenvalue the decay of the solutions as they asymptote to the equilibrium point will be power law (compared to exponential) in the dynamical time τ. Hence, at sufficiently early (or late) time, the decay rates will be dominated by the power-law behavior. In the phase space this can be illustrated as follows. The solutions will approach the center manifold exponentially rapidly, and then move along the center manifold towards the equilibrium point in a power-law manner (see Fig. 1). The center manifold will therefore dominate the behavior at sufficiently early (or late) times.

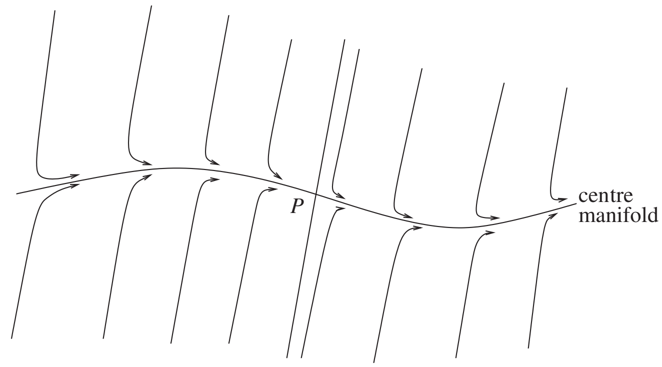

FIG. 1. The dynamical effect of a center manifold: the solutions approach a typical center manifold and move along the center manifold towards, or away from, the equilibrium point P.

A comment on the results of Ref. [15] is in order. They also consider the stability of the Kasner solutions in quadratic theories of gravity, however, they conclude that the Kasner solutions are unstable. Technically, this is correct and they base their result on the presence of a logarithmic term in t. However, as we have seen here, there exist typical orbits having the Kasner solutions as a past attractor (in Ref. [15] this amounts to choosing a negative k in their Eq. (30) which would cause the singularity to appear for $t = k > 0$ and thus $\ln t$ would be finite there). The cause of this problem is the center manifold and it is our opinion that the dynamical system approach unravels the true nature of the (in)stability of the Kasner solutions more clearly. We therefore conclude that: in the space of solutions, there exists a set of nonzero measure which has the Kasner solutions as a past attractor.

C. Quasi-FRW solution: $\mathcal{I}$

This solution is given by

$$\Sigma_\pm = \Sigma_{\pm 1} = \Sigma_{\pm 2} = N = B = \Omega_\Lambda = 0,$$
$$Q = -2,\qquad Q_2 = 8.$$

Its stability is determined by the eigenvalues

$$1\,[\times 3],\quad 2\,[\times 2],\quad 3\,[\times 3],\quad 4\,[\times 2].$$

The presence of all positive eigenvalues (i.e. unstable to the future) here implies that the solution is a past attractor.

The metric corresponding to this equilibrium point is

$$ds^2 = -dt^2 + t(dx^2 + dy^2 + dz^2),$$

and has $\Phi_{\mu\nu} = 0$, as can be checked explicitly—it is the $k = 0$ subcase of the radiationlike solution of Eq. (6) and evolves like a flat radiation-dominated GR universe despite being a solution of the pure quadratic theory.

D. Kasner-like solutions: $\tilde{\mathcal{K}}(I)$

There is also a set of Kasner-like solutions when $\chi < -1/2$:

$$\begin{aligned} Q &= -1, \\ (\Sigma_+, \Sigma_-) &= \Sigma(\cos\phi, \sin\phi), \\ (\Sigma_{+1}, \Sigma_{-1}) &= -\Sigma(\cos\phi, \sin\phi), \\ (\Sigma_{+2}, \Sigma_{-2}) &= 2\Sigma(\cos\phi, \sin\phi), \\ \Sigma^2 &= \frac{1+\sqrt{-2\chi}}{-(1+2\chi)}, \\ \Omega_\Lambda &= B = N = 0. \end{aligned} \tag{21}$$

These equilibrium points have both negative and positive eigenvalues and are therefore unstable.

The metric corresponding to this exact solution of $\Phi_{\mu\nu} = 0$ is

$$\begin{aligned} ds^2 &= -dt^2 + t^2[t^{-4\sigma_+}dx^2 + t^{2(\sigma_+ + \sqrt{3}\sigma_-)}dy^2 \\ &\quad + t^{2(\sigma_+ - \sqrt{3}\sigma_-)}dz^2], \\ (\sigma_+^2 + \sigma_-^2) &= \frac{1+\sqrt{-2\chi}}{-(1+2\chi)}. \end{aligned} \tag{22}$$

E. Superinflating FRW universe: E

This solution is given by

$$Q = Q_2 = \Sigma_\pm = \Sigma_{\pm 1} = \Sigma_{\pm 2} = N = \Omega_\Lambda = B = 0.$$

Its stability is determined by the eigenvalues:

$$0\ [\times 2], \quad -1, \quad -3\ [\times 3],$$
$$-\frac{3}{2}\left(1 \pm \sqrt{1 + \frac{8(4-\chi)}{9\chi}}\right)\ [\times 2].$$

The exact solution corresponds to an inflating Einstein metric $G_{\mu\nu} = \lambda g_{\mu\nu}$ with λ arbitrary; hence it is necessarily also a solution to $\Phi_{\mu\nu} = 0$.

However, interestingly, this solution has a nontrivial center manifold which causes a peculiar phenomenon. On the 2-dimensional center manifold we can use $B = X$ and $\Omega_\Lambda = Y$ as coordinates. To lowest order in its neighborhood we get

$$X' \approx \tfrac{1}{6}X^2, \qquad Y' \approx \tfrac{1}{6}XY.$$

So if $(4-\chi)/\chi < 0$, $B < 0$, this solution is stable, but is otherwise unstable.

We can show that, by approximating the solution close to the equilibrium point, and assuming that the solution is stable, the evolution of the scale factor is given by $a(t) \propto e^{H_0 t^2/2}$ where t is the cosmological time. The Hubble scalar diverges linearly $H = H_0 t$ and there are divergent curvature modes with $R \sim t^2$ as $t \to \infty$. Hence, generic solutions with $B < 0$ approaching E will have deceleration parameter $q = -[1 + 1/(H_0 t^2)] < -1$ with $q \to -1$ at late times. These models are therefore "superinflationary" (or marginally "phantomlike).

F. Anisotropically inflating type I universes: $\mathcal{A}(I)$

Unusually, for certain values of χ and B, there are also exact solutions that describe anisotropic inflationary solutions of Bianchi type I:

$$(\Sigma_+, \Sigma_-) = \Sigma(\cos\phi, \sin\phi), \qquad \Sigma^2 = -\frac{2(4-\chi)+B}{4(2\chi+1)},$$
$$Q = \Sigma_{\pm 1} = \Sigma_{\pm 2} = N = 0.$$

There are two classes of such solutions, depending on the values of B and Ω_Λ:

(i)

$$B = \text{constant}, \qquad \Omega_\Lambda = \frac{18\chi - B}{8(2\chi+1)},$$

(ii)

$$B = 0, \qquad \Omega_\Lambda = \text{constant}.$$

As long as χ and B take values for which $\Sigma^2 > 0$, these solutions exist. Their eigenvalues can be jointly expressed as:

$$0\ [\times 2], \quad -3\ [\times 3], \quad -\frac{3}{2}\left(1 \pm \sqrt{1 - \frac{2B}{9}}\right),$$
$$-\frac{3}{2}\left(1 \pm \sqrt{1 + 8\Sigma^2}\right), \quad -(1 + 4\Sigma_+).$$

Two of the zero eigenvalues appear because these are 2-parameter families of equilibrium points. However, since $\Sigma^2 > 0$, there will always be one unstable mode, and hence, these solutions are saddle points.

The metrics corresponding to the case where $B \neq 0$ can be written

$$\begin{aligned} ds^2 &= -dt^2 + e^{2bt}[e^{-4\sigma_+ t}dx^2 + e^{2(\sigma_+ + \sqrt{3}\sigma_-)t}dy^2 \\ &\quad + e^{2(\sigma_+ - \sqrt{3}\sigma_-)t}dz^2], \\ b^2 &= \frac{1 + 8\Lambda(\alpha+\beta)}{9\beta}, \\ (\sigma_+^2 + \sigma_-^2) &= -\frac{1 + 2\Lambda(4\alpha+\beta)}{18\beta}. \end{aligned} \tag{23}$$

This set of solutions is defined so long as $b^2 > 0$ and $(\sigma_+^2 + \sigma_-^2) > 0$. As we can see, these solutions inflate anisotropically. Note that they are solutions of the theory in which $\Lambda > 0$ but are not de Sitter spacetimes. Therefore, they show explicitly that the usual cosmic no-hair theorem of GR [18–22] does not hold in these theories. In effect, the nonlinear ($\Phi_{\mu\nu} \neq 0$) terms contribute an effective stress tensor to the vacuum equations which violates the strong-energy condition needed for the cosmic no-hair theorem to hold in GR [23]. The essential features of this novel solution arise because of the contribution of the quadratic Ricci terms. If we put $\alpha = 0$ then there is no essential change in the character of the solution but note that then the solution does not have a GR limit when we take $\beta \to 0$: it is an intrinsically quadratic solution of the higher-order theory.

G. Anisotropically inflating type II universe: $\mathcal{A}(\boldsymbol{II})$

The phenomenon of anisotropic inflation is not confined to the Bianchi type I models. An anisotropically inflating solution of Bianchi type II is described by the critical point

$$\Sigma_+ = -\tfrac{1}{4}, \qquad \Sigma_- = \Sigma_{\pm1} = \Sigma_{\pm2} = Q = Q_2 = 0,$$
$$\Omega_\Lambda = \tfrac{33}{32} - \tfrac{1}{2}N^2, \qquad B = 4(1 + 8\chi)N^2 - \tfrac{3}{4}(11 - 2\chi). \tag{24}$$

This is the type II solution given in [9].

Its stability is determined by the eigenvalues

$$0, \quad -\frac{3}{2}\left(1 \pm \sqrt{1 - \frac{2B}{9}}\right), \quad -\frac{3}{2}\left(1 \pm \frac{1}{6}\sqrt{a \pm \sqrt{b}}\right), \quad -3,$$
$$-\frac{3}{2}(1 \pm \sqrt{1 + 16N^2}),$$

where

$$a = 15(3 - 16N^2), \qquad b = 9(9 - 6240N^2 - 24\,320N^4).$$

The zero eigenvalue corresponds to the fact that this solution can be parametrized by B. Because of the presence of a positive eigenvalue, this solution is unstable to the future.

There is a related set of equilibrium points for which

$$\Sigma_+ = -\frac{1}{4}, \qquad \Sigma_- = \Sigma_{\pm1} = \Sigma_{\pm2} = Q = Q_2 = B = 0,$$
$$N^2 = \frac{3}{16}\left(\frac{11 - 2\chi}{1 + 8\chi}\right), \qquad \Omega_\Lambda = \text{constant}. \tag{25}$$

These equilibrium points are defined for all χ, as long as $N^2 > 0$.

H. Other solutions

We must stress that this list of equilibrium points of type II is not exhaustive. There are further critical points of the full dynamical system, which we do not consider here.

IV. BEHAVIORS AT INFINITY

A. FRW case

Consider the behavior at infinity for the FRW case; i.e., with $\Sigma_\pm = \Sigma_{\pm1} = \Sigma_{\pm2} = N = 0$. We note that for large values of Q there are solutions which, to lowest order, give (noting that, after performing a translation of time, we can assume that Q diverges at $\tau = 0$ without loss of generality):

$$Q = \frac{1}{2\tau}, \qquad Q_2 = \frac{Q_{2,0}}{|\tau|^{3/2}}, \qquad B = \frac{B_0}{|\tau|}, \qquad \Omega_\Lambda = \frac{\Omega_{\Lambda0}}{|\tau|}, \tag{26}$$

where $Q_{2,0}$, B_0, $\Omega_{\Lambda,0}$ are constants fulfilling the condition

$$B_0\Omega_{\Lambda,0} = -\tfrac{1}{2}.$$

Hence, for these solutions at infinity to exist, the constants B_0 and $\Omega_{\Lambda,0}$ need to have opposite sign; so, since $\Omega_\Lambda > 0$, we must have $B_0 < 0$.

Therefore, these behaviors at infinity come in two classes, according to whether $\tau < 0$, which implies $Q < 0$ and Q diverges into the future, or $\tau > 0$, which implies $Q > 0$ and Q diverges to the past. These two choices for the sign of Q correspond to *recollapsing* and *bouncing* cosmological solutions, respectively.

Consider the case $\tau < 0$; by direct integration, we get for the Hubble scalar and the proper time t:

$$H \approx H_0|\tau|^{1/2}, \qquad |\tau|^{1/2} \approx \frac{H_0}{2}(t_0 - t),$$

and hence, $H = (H_0^2/2)(t_0 - t)$ and the universe reaches a maximum size at t_0 and contracts thereafter. The divergence of the variables for these cases is just a feature of the expansion-normalized variables.

Let us now consider the FRW case in more detail. Assuming a FRW metric, the equations of motion can be written (without introducing expansion-normalized variables, but assuming $(3\alpha + \beta) < 0$):

$$6H^2\dot{H} + 2H\ddot{H} - \dot{H}^2 - \eta H^2 + \omega^2 = 0, \tag{27}$$

where

$$\eta = -\frac{1}{2(3\alpha + \beta)}, \qquad \omega^2 = -\frac{\Lambda}{6(3\alpha + \beta)},$$

and overdots denote d/dt. We note that ω can have either sign and $\eta > 0$.

We are interested in the case where there is a point where $H = 0$ (i.e., the evolution has a turning point). Let us assume that this occurs for $t = 0$. Then we can find solutions which, close to $t = 0$, can be expanded as

$$H = \omega t + \frac{H_2}{2}t^2 - \frac{\omega}{6}(6\omega - \eta)t^3 - \frac{H_2}{24}(15\omega - \eta)t^4 + \mathcal{O}(t^5) \tag{28}$$

with H_2 constant. As Eq. (27) is ill posed through $H = 0$, we have assumed that $H(t)$ is analytic at $t = 0$.

We note that in the case $\omega < 0$, H goes from being positive to negative, hence the universe goes from an expanding to a collapsing phase. If $\omega > 0$, the universe goes from a collapsing to an expanding phase, and so the universe experiences a bounce at $t = 0$. We also note that this turning point is not symmetric with respect to $t = 0$ as long as $H_2 \neq 0$. The constant H_2 is proportional to the constant $Q_{2,0}$ from above. Moreover, by comparing the approximate solution Eq. (28), we can identify the behavior at infinity, as given by Eq. (26), as turning points of the expansion of the universe. Note that for the special case $\omega = \eta = 0$ the universe expands, reaches $H = 0$, and then continues to expand.

B. General case

In the general Bianchi type I and II cases we can also find approximate solutions at infinity, with

$$Q = \frac{1}{2\tau}, \qquad Q_2 = \frac{Q_{2,0}}{|\tau|^{3/2}}, \qquad B = \frac{B_0}{|\tau|},$$
$$\Omega_\Lambda = \frac{\Omega_{\Lambda 0}}{|\tau|}, \qquad \Sigma_\pm = \frac{\Sigma_{\pm,0}}{|\tau|^{1/2}}, \qquad \Sigma_{\pm 1} = \frac{\Sigma_{\pm 1,0}}{|\tau|}, \quad (29)$$
$$\Sigma_{\pm 2} = \frac{\Sigma_{\pm 2,0}}{|\tau|^{3/2}}, \qquad N = \frac{N_0}{|\tau|^{1/2}}.$$

The constants B_0, $\Omega_{\Lambda 0}$, $\Sigma_{\pm,0}$, $\Sigma_{\pm 1,0}$, $\Sigma_{\pm 2,0}$, and N_0 must fulfill a complicated constraint arising from the $1/\tau^2$ term of the constraint equation.

From this behavior, we can again show that these infinities correspond to turning points of the expansion of the universe between states of expansion and contraction.

C. Transitions at infinity

Since the infinities described above correspond to turning points in the evolution, we can regularly pass through these infinities by switching to, for example, cosmological time. By choosing these infinities to occur at $\tau = 0$, we can pass through $\tau = 0$ ($\tau < 0 \mapsto \tau > 0$ will correspond to a transition from $H > 0$ to $H < 0$ to the future, while the reverse will be a bounce in the past). In terms of the approximations, Eq. (29) we get the following transitions for the expansion-normalized variables and the dynamical time at infinity:

$$(\tau, Q, Q_2, B, \Omega_\Lambda, \Sigma_\pm, \Sigma_{\pm 1}, \Sigma_{\pm 2}, N)$$
$$\mapsto (-\tau, Q, -Q_2, B, \Omega_\Lambda, -\Sigma_\pm, \Sigma_{\pm 1}, -\Sigma_{\pm 2}, -N).$$

These transitions can now be used to continue these solutions through the formal infinity arising in state space. We should emphasize that for some invariant subspaces there might be different transitions to the one given above.

V. BEHAVIOR OF FRW UNIVERSES

Let us consider the FRW case in further detail. It is defined by $\Sigma_\pm = \Sigma_{\pm 1} = \Sigma_{\pm 2} = N = 0$ and we have $\Omega_\Lambda > 0$. It is also useful to define the following quantity:

$$K = \frac{\Omega_\Lambda}{B}, \qquad K' = 0.$$

So K is a constant and we will subsequently replace all occurrences of Ω_Λ with KB. Note also that $B < 0$ implies $K < 0$, and $B > 0$ implies $K > 0$. Moreover, we will use the constraint equation to solve for Q_2. The equations of motion are then reduced to a two-dimensional system:

$$B' = -2QB, \quad (30)$$

$$Q' = -\frac{3}{2}Q^2 - 3Q - \frac{1}{4}B + \frac{K}{4}B^2, \quad (31)$$

which can now be analyzed by using standard techniques.

It is also useful to map the two-dimensional state space onto the compact two-dimensional unit disk D^2. This can be done by introducing polar coordinates $(B, Q) = R(\cos\phi, \sin\phi)$, followed by, for example, the transformation: $R \mapsto R/\sqrt{1 + R^2}$. The behavior of the state space at infinity is now mapped onto the unit circle $S^1 = \partial D^2$. On the unit circle there are two points of special importance, labeled by the value of the angular variable, ϕ:

$$\mathcal{B}_\infty\colon\ \tan\phi = -\sqrt{\frac{|K|}{2}}, \qquad \phi \in \left(\frac{\pi}{2}, \pi\right),$$
$$\mathcal{R}_\infty\colon\ \tan\phi = +\sqrt{\frac{|K|}{2}}, \qquad \phi \in \left(\pi, \frac{3\pi}{2}\right). \quad (32)$$

The points $\mathcal{B}_\infty$ and $\mathcal{R}_\infty$ correspond to the possible turning points of the expansion (bounce and recollapse, respectively).

There are two other points on the unit circle that correspond to $H = 0$, namely $\phi = \pi/2$ and $\phi = 3\pi/2$, which will be denoted $\mathcal{T}^+_\infty$ and $\mathcal{T}^-_\infty$, respectively. Both of these points are in the invariant subspace $B = 0$, but are both unstable for models with $B \neq 0$. These two points correspond to $\omega = \eta = 0$ in Eq. (28).

For $B = 0$, we can find the exact solution

$$Q(\tau) = \frac{-2}{1 + Ce^{3\tau}},$$

for which the metrics can be written

$$ds^2 = -\frac{k^2 d\tau^2}{(1 + \epsilon e^{-3\tau})^{4/3}} + e^{2\tau}(dx^2 + dy^2 + dz^2),$$
$$\epsilon = -1, 1.$$

These metrics are solutions to $\Phi_{\mu\nu} = 0$ and describe universes evolving as follows (see Fig. 2):

(1) $\epsilon = 1$: From $\mathcal{I}$ to E.

(2) $\epsilon = -1$: For $\tau < 0$, from $\mathcal{I}$ to $\mathcal{T}^-_\infty$, while for $\tau > 0$, from $\mathcal{T}^+_\infty$ to E. Note that, after requiring analyticity, we have the transition $\mathcal{T}^-_\infty \to \mathcal{T}^+_\infty$ at $\tau = 0$.

For $B = 0$ on the approach to E we have $H = H_0$ at late times unlike the general case ($B \neq 0$) for which the evolution gives $H = H_0 t$ at late times.

The behavior of general FRW universes ($B \neq 0$) can be split into 5 different possibilities. The evolution can be as follows (see Fig. 2):

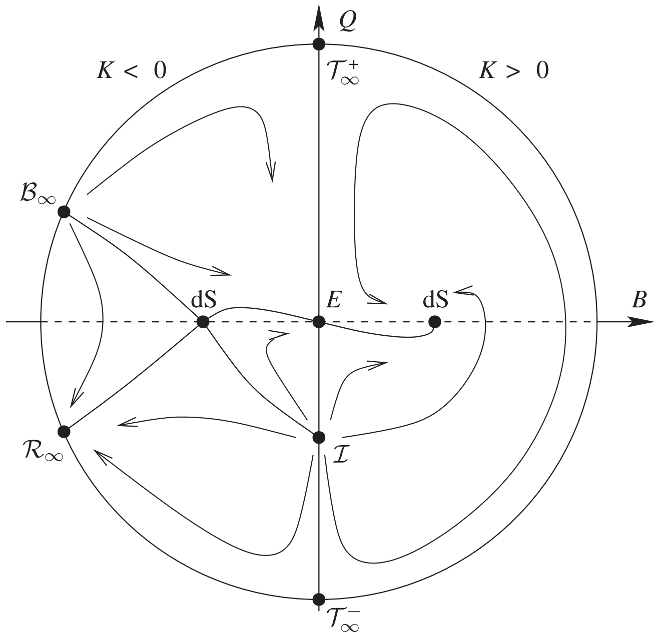

FIG. 2. A sketch of the dynamical behavior of vacuum FRW universes. For illustration, the dynamical system is mapped onto the unit disc as explained in the text.

(1) $B > 0$: $\mathcal{I} \to$ dS.
(2) $B < 0$: $\mathcal{I} \to E$.
(3) $B < 0$: $\mathcal{I} \to \mathcal{R}_\infty$.
(4) $B < 0$: $\mathcal{B}_\infty \to E$.
(5) $B < 0$: $\mathcal{B}_\infty \to \mathcal{R}_\infty$.

The last possibility suggests that it might even be possible for a universe to experience a sequence of expanding and collapsing phases (recall that a collapsing phase can be described by the same equations by running the dynamical time τ backwards).

VI. PAST BEHAVIOR OF BIANCHI-TYPE UNIVERSES

In the analysis above we have seen that there are (at least) three possibilities for past behavior of Bianchi models. Let us discuss each of these in turn. Again, we should emphasize that the list of equilibrium points for the Bianchi type II case is not exhaustive, and therefore behaviors other than these might occur in the general type II solution.

A. Isotropic singularity

We have already seen that the radiationlike FRW universe denoted $\mathcal{I}$, is an attractor for the models studied here. This creates the possibility for an isotropic past singularity in these theories. Hence, the quadratic theories of gravity studied here seem to have an isotropization mechanism into the past provided by the quadratic curvature contributions. Let us give an argument that this isotropic singularity is indeed an attractor *for all Bianchi models*.

1. Curvature modes

As explained earlier, the various Bianchi models can be described in terms of the structure coefficients via the symmetric tensor n_{ab} and the vector a_b. The evolution of these curvature variables are determined by the Jacobi identity; hence, the evolution of n_{ab} and a_b are therefore theory-independent. Introducing expansion-normalizsed variables in the standard manner, we obtain the equations of motion close to the isotropic singularity:

$$N_i' = -(Q+1)N_i, \qquad A' = -(Q+1)A.$$

Since $Q = -2$ for the attractor $\mathcal{I}$, we have

$$N_i \approx N_{i,0}e^{\tau}, \qquad A \approx A_0 e^{\tau},$$

and hence each quantity decays to the past $\tau \to -\infty$.

2. Shear modes

We do expect extra shear modes to appear; however, if we introduce the corresponding expansion-normalized variables for these modes, the evolution equations are, to linear order, the same as for $\Sigma_\pm$. Consequently, this would give eigenvalues $\{1, 2, 3\}$ for each of the additional shear modes. Therefore, we expect the shear to decay as

$$\Sigma_{ij} = C_{ij}e^{\tau} + \mathcal{O}(e^{2\tau}),$$

at early times as $\tau \to -\infty$. This is consistent with the situation found for the Bianchi type IX model by Cotsakis *et al.* [15].

3. Perfect-fluid matter

With regards to the influence of matter on the evolution, let us assume the presence of a comoving perfect fluid with a barotropic equation of state:

$$p = (\gamma - 1)\rho.$$

Introducing an expansion-normalized energy density, $\Omega \equiv \rho/(3H^2)$, the conservation of energy yields the equation of motion:

$$\Omega' = -[2(Q+1) + (3\gamma - 2)]\Omega.$$

Hence, we have for the evolution of the fluid density with volume expansion, $\Omega \propto e^{(4-3\gamma)\tau}$, and there is a change of behavior of Ω at $\gamma = 4/3$. However, in the evolution equations Ω only appears as part of the product $B\Omega \propto e^{(8-3\gamma)\tau}$. This means that for all regular matter with $\gamma \leq 2$, the matter term is subdominant and the isotropic singularity is stable to the past. In GR the situation is very different, and stability is only possible when $\gamma = 2$ [24].

These conclusions have been drawn for the case of irrotational matter only. The situation with rotation or noncomoving 4-velocities, where $\mathbf{u}$ is not hypersurface orthogonal remains to be investigated. We know from experience with Bianchi-type universes in general relativity that the introduction of all the possible noncomoving velocity components can produce a situation that is delicately sensitive to the equation of state because of the role played by the pressure and density in the conservation of angular momenta around the three orthogonal expansion axes, see Refs. [13,25]. The evolution of fluids possessing anisotropic pressures is also interesting and is likely to play a very important role in the late-time evolution of these universes, notably in the case where the anisotropic fluid has vanishing trace and is accompanied by an isotropic blackbody radiation fluid. However, the way in which the quadratic Ricci terms have been found to mimic the effect of an isotropic radiation stress implies that the presence of a pure trace-free stress (for example a pure magnetic field) may evolve in an influential way in these theories, just as was found in the case of general relativity [26,27].

B. Kasner circle

The Kasner circle is an attractor for some orbits of type I and II and in these models this behavior is therefore typical. The type II models cause transitions between different points on the Kasner circle and since this is an exact solution for ordinary GR we expect chaos as we go to the more general vacuum type VIII, IX, and $\mathrm{VI}^*_{-1/9}$ models. However, even though chaotic solutions exist, we do not know whether this chaotic set is an attractor for typical orbits.

C. Bounce

We have seen that there are bouncing solutions for the type I and II universes. However, even though this behavior seems to be typical for these models it is unclear whether they are typical for more general Bianchi models. The bouncing solutions correspond to solutions coming from infinity of the state space and are therefore particularly challenging to analyze. Apart from for the FRW case, rigorous results regarding the generality of these bouncing solutions are not known. The stability of the FRW behavior in the type IX case [15] indicates that the closed FRW bounce behavior will occur there but there may be other behaviors that are stable far from isotropy which have not yet been identified. The possibility for bounce is closely related to the possibility of recollapsing solutions in these theories. However, the analysis of this kind of behavior using expansion-normalized variables is plagued by the fact that the state variables diverge. It does seem that recollapsing solutions are typical but a more rigorous analysis is lacking at the present time.

VII. FUTURE BEHAVIOR

The future behavior of these cosmologies shows a dependence of the parameters of the theory. Unlike in general relativity, there does not exist a simple no-hair theorem when $\Lambda > 0$ for the vacuum case, or where matter obeys the strong-energy condition. As discussed in our earlier paper, Ref. [9], this situation arises because the contribution of the $\Phi_{\mu\nu}$ stresses to the field equations (2) can mimic the form of a fluid with negative density and/or pressures and so the energy conditions assumed to hold in the general-relativistic no-hair theorems are effectively violated by the nonlinear curvature terms. We showed that there exist vacuum solutions with $\Lambda > 0$ which do not approach de Sitter at late times: they inflate anisotropically. A similar effect can be seen in the study of Kaloper into the effects of the Chern-Simons terms on the behavior of a class of Bianchi-type universes [28]. However, conditions can be identified for which de Sitter is an attractor at late times.

A. De Sitter

For de Sitter to be an attractor we need $B > 0$ and $[1 + 2\Lambda(4\alpha + \beta)]/\beta < 0$. However, even in this case, numerical simulations indicate that this behavior is typical but not generic. But in the cases where $B < 0$, another late-time behavior is possible:

B. Superinflating FRW

For $B < 0$ and $\chi < 0$ or $\chi > 4$ the equilibrium point E is a future attractor. Because of the presence of a nontrivial center manifold, the universe generically approaches this point with a power-law evolution. As the solutions approach E, the universe eventually has $H(t) = H_0 t$ and $q = -[1 + 1/(H_0 t^2)]$.

Lately, there has been an interest for so-called "phantom cosmologies" with $q < -1$ which can end in a singularity within finite time (a so-called "big rip"). The evolution described above has eventually $q < -1$ but approaches $q = -1$ sufficiently fast to avoid the big rip [29]. The deceleration parameter approaches -1 from below, while the Hubble scalar diverges linearly in cosmological time. Other types of finite-time singularity are possible in which H and ρ remain finite but there is an infinity of the pressure and the acceleration [30–33]. These "sudden" singularities occur without any violation of the strong-energy condition. It appears likely that they will occur in these quadratic theories also. The situation in the presence of Lagrangians that involved only R was discussed in Ref. [32].

VIII. DISCUSSION

In this paper we have provided a study of a range of anisotropic universes in a quadratic theory of gravity involving all allowed quadratic curvature invariants to appear in the Lagrangian. We have seen that there is a possibility for a stable (or even generic) isotropic singularity in quadratic theories of gravity. There is a isotropic equilibrium point, representing a FRW universe with scale factor $a(t) \propto \sqrt{t}$ that is a solution of the *vacuum* field equations of the quadratic theory, which is stable into the past. Here, for simplicity, we have only considered Bianchi type I and type II models, but these incorporate both shear and 3-curvature anisotropies, and an argument was given that this is also a past attractor for more general Bianchi models. Therefore, it appears that including quadratic terms provides us with a new mechanism for constraining the initial singularity to be isotropic. This is reminiscent of the situation proposed by Weyl curvature hypothesis, which envisages some measure of the Weyl curvature playing the role of a gravitational entropy, so it must initially be small (or zero, up to quantum corrections) in order to provide a subsequent gravitational arrow of time [34–36]. However, it must not be the case that the quadratic stresses drive the expansion towards isotropy on approach to any future singularity if the universe is closed and recollapses.

This study raises many questions for further investigation of anisotropic and inhomogeneous cosmologies that are more general than those considered here, which we will report elsewhere. We have also found that the presence of quadratic curvature terms leads to a range of new possibilities in situations which would be expected to yield simple de Sitter inflation in GR. The presence of a stress with $p = -\rho$ need not produce future evolution towards the isotropic de Sitter metric locally. There is a possibility for inflationary behavior which is anisotropic because of the effective stresses being contributed to the field equations by the quadratic terms.

In ordinary GR, the general Bianchi types have been shown to possess a chaotic behavior as we approach the initial singularity. This chaotic behavior consists of a sequence of Kasner regimes connected via vacuum type II transitions and possibly frame rotations. On a dynamical time scale, the transitions are fairly short compared to the Kasner epochs [37–39]. Therefore, since we noticed that the Kasner circle does indeed act as a past attractor for some type I orbits, we could expect chaos to be present for typical orbits in the more general Bianchi models. Certainly, the chaotic vacuum type IX Mixmaster universe is a particular exact solution of the quadratic theory and its chaotic behavior has been discussed in the presence of the quadratic terms ; it remains to be seen whether another form of chaotic evolution might arise, and be stable, far from the isotropic closed FRW radiationlike model.

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[1] J. D. Barrow and A. C. Ottewill, J. Phys. A **16**, 2757 (1983).
[2] J. D. Barrow and S. Cotsakis, Phys. Lett. B **214**, 515 (1988).
[3] J. D. Barrow, Nucl. Phys. **B296**, 697 (1988).
[4] A. A. Starobinsky, Phys. Lett. B **91**, 99 (1980); A. Guth, Phys. Rev. D **23**, 347 (1981); A. D. Linde, Phys. Lett. B **129**, 177 (1983); V. Müller, H.-J. Schmidt, and A. A. Starobinskii, Phys. Lett. B **202**, 198 (1988); A. Berkin and K. I. Maeda, Phys. Rev. D **44**, 1691 (1991); S. Gottlöber, V. Müller, and A. A. Starobinskii, Phys. Rev. D **43**, 2510 (1991); A. A. Starobinskii and H.-J. Schmidt, Classical Quantum Gravity **4**, 695 (1987); H.-J. Schmidt, Classical Quantum Gravity **5**, 233 (1988); Gen. Relativ. Gravit. **25**, 87 (1993).
[5] H.-J. Schmidt, gr-qc/0407095; V. Müller and H.-J. Schmidt, Gen. Relativ. Gravit. **17**, 769 (1985).
[6] T. Clifton and J. D. Barrow, Classical Quantum Gravity **23**, 2951 (2006).
[7] J. D. Barrow and T. Clifton, Classical Quantum Gravity **22**, L1 (2005).
[8] T. Clifton and J. D. Barrow, Phys. Rev. D **72**, 103005 (2005).
[9] J. D. Barrow and S. Hervik, Phys. Rev. D **73**, 023007 (2006)
[10] S. Deser and B. Tekin, Phys. Rev. D **67**, 084009 (2003).
[11] R. Bach, Math. Z. **9**, 110 (1921).
[12] J. Wainwright and G. F. R. Ellis, *Dynamical Systems in Cosmology* (Cambridge University Press, Cambridge, England, 1997); A. A. Coley, *Dynamical Systems and Cosmology* (Kluwer, Dordrecht, 2003).
[13] A. A. Coley and S. Hervik, Classical Quantum Gravity **21**, 4193 (2004); S. Hervik, R. J. van den Hoogen, and A. A. Coley, Classical Quantum Gravity **22**, 607 (2005); S. Hervik, R. J. van den Hoogen, W. C. Lim, and A. A. Coley, Classical Quantum Gravity **23**, 845 (2006); S. Hervik and W. C. Lim, Classical Quantum Gravity **23**, 3017 (2006).
[14] J. A. Leach, S. Carloni, and P. K. S. Dunsby, Classical Quantum Gravity **23**, 4915 (2006).
[15] S. Cotsakis, J. Demaret, Y. De Rop, and L. Querella, Phys. Rev. D **48**, 4595 (1993); J. Demaret and L. Querella Classical Quantum Gravity **12**, 3085 (1995).
[16] M. Dąbrowski, Ann. Phys. (N.Y.) **15**, 352 (2006).
[17] A. V. Toporensky, in Proceedings of MG11, Berlin, Germany, 2006 (unpublished); A. V. Toporensky and P. V. Tretyakov, gr-qc/0611068.
[18] J. D. Barrow, in *The Very Early Universe*, edited by G. Gibbons, S. W. Hawking, and S. T. C Siklos (Cambridge University Press, Cambridge, England, 1983), p. 267.
[19] W. Boucher and G. W. Gibbons, in *The Very Early Universe*, edited by G. Gibbons, S. W. Hawking, and S. T. C Siklos (Cambridge University Press, Cambridge, England, 1983), p. 273.

[20] A. A. Starobinskii, Sov. Phys. JETP **37**, 66 (1983).
[21] L. G. Jensen and J. Stein-Schabes, Phys. Rev. D **35**, 1146 (1987).
[22] R. Wald, Phys. Rev. D **28**, 2118 (1983).
[23] J. D. Barrow, Phys. Lett. B **187**, 12 (1987).
[24] J. D. Barrow, Nature (London) **272**, 211 (1978).
[25] J. D. Barrow, Mon. Not. R. Astron. Soc. **178**, 625 (1977); C. B. Collins, Commun. Math. Phys. **39**, 131 (1974); G. F. R. Ellis and C. B. Collins, Phys. Rep. **56**, 65 (1979); I. S. Shikin, Sov. Phys. JETP **41**, 794 (1976).
[26] J. D. Barrow, Phys. Rev. D **55**, 7451 (1997).
[27] J. D. Barrow, P. Ferreira, and J. Silk, Phys. Rev. Lett. **78**, 3610 (1997).
[28] N. Kaloper, Phys. Rev. D **44**, 2380 (1991).
[29] A. A. Starobinsky, Gravitation Cosmol. **6**, 157 (2000); R. R. Caldwell, Phys. Lett. B **545**, 23 (2002); A. Shulz and M. J. White, Phys. Rev. D **64**, 043514 (2001); J. Hao and X. Li, Phys. Rev. D **67**, 107303 (2003); G. W. Gibbons, hep-th/0302199; S. Nojiri and S. D. Odintsov, Phys. Lett. B **562**, 147 (2003) **571**, 1 (2003); P. Singh, M. Sami, and N. Dadhich, Phys. Rev. D **68**, 023522 (2003); J. Hao and X. Li, Phys. Rev. D **68**, 043501 (2003); M. Dąbrowski, T. Stachowiak, and M. Sydlowski, Phys. Rev. D **68**, 103519 (2003); P. Elizalde and J. Quiroga, Mod. Phys. Lett. A **19**, 29 (2004); P. F. González-Díaz, Phys. Lett. B **586**, 1 (2004); A. Feinstein and S. Jhingan, Mod. Phys. Lett. A **19**, 457 (2004); L. P. Chimento and R. Lazkoz, Phys. Rev. Lett. **91**, 211301 (2003); E. Elizalde, S. Nojiri, and S. D. Odintsov, Phys. Rev. D **70**, 043539 (2004); L. P. Chimento and R. Lazkoz, Mod. Phys. Lett. A **69**, 123512 (2004).
[30] J. D. Barrow, Classical Quantum Gravity **21**, L79 (2004).
[31] J. D. Barrow, Classical Quantum Gravity **21**, 5619 (2004).
[32] J. D. Barrow and C. G. Tsagas, Classical Quantum Gravity **22**, 1563 (2005).
[33] S. Cotsakis and I. Klaoudatou, J. Geom. Phys. **55**, 306 (2005).
[34] R. Penrose, in *Theoretical Principles in Astrophysics and Relativity*, edited by N. R. Lebovitz, W. M. Reid, and P. O. U. Vandervoort (University of Chicago Press, Chicago, 1978); R. Penrose, *The Road to Reality* (Jonathan Cape, London, 2005).
[35] Ø. Grøn and S. Hervik, Int. J. Theor. Phys. Group Theory Nonlinear Opt. **10**, 29 (2003).
[36] J. D. Barrow and S. Hervik, Classical Quantum Gravity **19**, 5173 (2002).
[37] V. Belinskii, E. M. Lifshitz, and I. Khalatnikov, Adv. Phys. **19**, 525 (1970)
[38] D. Chernoff and J. D. Barrow, Phys. Rev. Lett. **50**, 134 (1983).
[39] J. D. Barrow, in *Classical General Relativity*, edited by W. Bonnor, J. Islam, and M. A. H. MacCallum (Cambridge University Press, Cambridge, England, 1984), p. 25.