

Gravitationally collapsing shells in (2 + 1) dimensions

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We study gravitationally collapsing models of pressureless dust, fluids with pressure, and the generalized Chaplygin gas (GCG) shell in (2 + 1)-dimensional spacetimes. Various collapse scenarios are investigated under a variety of the background configurations such as anti-de Sitter (AdS) black hole, de Sitter (dS) space, flat and AdS space with a conical deficit. As with the case of a disk of dust, we find that the collapse of a dust shell coincides with the Oppenheimer-Snyder type collapse to a black hole provided the initial density is sufficiently large. We also find—for all types of shell—that collapse to a naked singularity is possible under a broad variety of initial conditions. For shells with pressure this singularity can occur for a finite radius of the shell. We also find that GCG shells exhibit diverse collapse scenarios, which can be easily demonstrated by an effective potential analysis.

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I. INTRODUCTION

Over the last few decades, general relativity in (2 + 1) dimensions has fascinated both field theorists and relativists because of its fertility as a test-bed for ideas about quantum gravity. One particular feature of interest is manifest when a negative cosmological constant is present. Despite the fact that the spacetime geometry of this solution is an anti-de Sitter (AdS) spacetime, possessing negative constant curvature, a black hole can be present under a suitable choice of topological identifications [1]. This solution has drawn much attention since its inception from a wide variety of perspectives [2].

Shortly after the black hole solution was obtained, it was shown that it can be formed from a disk of pressureless dust undergoing gravitational collapse [3] (the three-dimensional analogue of Oppenheimer-Snyder type collapse), generalizing earlier results that suggested matter could collapse to form conical singularities [4]. Further study on this subject has been carried out from several viewpoints, including the formation of a black hole from colliding point particles [5] and the more recent demonstration of critical phenomena in the context of collapse [6]. These results are consistent with other results in four dimensions as well as results in two dimensions [7].

Recently, a cosmological model of a (generalized) Chaplygin gas (GCG) was introduced as a possible explanation of the present acceleration of the universe, the existence of dark energy, and the unification of dark energy and dark matter [8–10]. Historically its original motivation was to account for the lifting force on a plane wing in aerodynamics [11]. Afterwards, the same equation of state was rediscovered in the context of aerodynamics [12,13]. A more interesting feature of this gas was recently renewed in an intriguing connection with string theory, insofar as its equation of state can be obtained from the Nambu-Goto

action for d -branes moving in a $(d + 2)$ -dimensional spacetime in the light-cone frame [14]. In addition, it has been shown that the Chaplygin gas is, to date, the only fluid that admits a supersymmetric generalization [15]; the relevant symmetry group was described in Ref. [16]. Moreover, further theoretical developments of the GCG were given in terms of cosmology and astrophysics [17]. Inspired by the fact that the Chaplygin gas has a negative pressure, violating the energy conditions (in particular the null energy condition (NEC)), traversable wormhole solutions were found in four dimensions [18].

It is natural to ask whether or not a black hole can be formed from gravitational collapse of this gas in a finite collapse time. Much of the work on black hole formation deals with pressureless dust collapse; collapse of this kind of exotic fluid to black holes so far has not received much treatment. Recent work [19] involved investigation of spherically symmetric clouds of a collapsing modified Chaplygin gas in four dimensions, where it was shown that it always leads to the formation of a black hole.

In this paper, we investigate some gravitational collapse scenarios of shells with a variety of equations of state, including the GCG shell. To set the stage we first consider the collapse of a shell of pressureless dust. In dust collapse scenarios the evolution of the system is obtained by matching the inside and outside geometries using the junction conditions [20–22],

$$[g_{ij}] = 0, \quad [K_{ij}] = 0, \quad (1.1)$$

where $[h] \equiv h_+ - h_-$ and $(+)$ and $(-)$ represent exterior and interior spacetimes, respectively. However for shells with pressure the junction condition for the extrinsic curvature in Eq. (1.1) is no longer valid, since there is a nonvanishing surface stress-energy tensor on the boundary of the shell to take into account.

The main result of our investigation is that gravitational collapse in (2 + 1) dimensions does not necessarily lead to black hole formation for any of the fluid sources we study.

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The end points of collapse depend on the initial conditions, and can lead to either a black hole or the formation of a singularity and a Cauchy horizon.¹ This singularity is characterized by the onset of a divergent stress energy in the shell, whose intrinsic Ricci scalar also diverges in finite proper time for observers comoving with the shell. For pressureless dust the singularity develops when the shell collapses to zero size. However for shells with pressure the singularity develops at some nonzero size characterized by the equation of state. A similar scenario holds for the GCG shell. We also find that collapse is not the only possibility, but that shells can also expand out to infinity, possibly with a bounce depending on the initial conditions. Our results are consistent with earlier work on shell collapse in (2 + 1) dimensions [23,24], generalizing them to include a more detailed analysis of collapse to naked singularities, and to situations in which a more general relationship between density and pressure is assumed.

The outline of our paper is as follows. In Sec. II, we briefly present a formulation of the shell collapse and obtain the evolution equation for the dust shell radius. In Sec. III, the gravitational collapses of pressureless dust shell are studied and compared to the result of dust cloud collapse in [3]. In Sec. IV, we study a collapse of a shell with an arbitrary pressure with no loss of generality. In Sec. V, the collapse of GCG shell is studied and some possible collapse conditions are found. Finally, we shall summarize and discuss our results in Sec. VI. We consider the construction of some relevant Penrose diagrams and some basic properties of Jacobian elliptic functions in appendices.

II. SHELL COLLAPSE

We assume that the metrics in both regions, \mathcal{V}_+ (outside the shell) and \mathcal{V}_- (inside the shell) are given by

$$(ds)_{\mathcal{V}_{\pm}}^2 = -F_{\pm} dT^2 + \frac{dR^2}{F_{\pm}} + R^2 d\theta^2, \quad (2.1)$$

where F_+ and F_- are exterior and interior metrics, respectively. The surface stress-energy tensor for a fluid of energy density ρ and pressure p is

$$S^{ab} = \rho u^a u^b + p h^{ab}, \quad (2.2)$$

where $h_{ab} = g_{ab} + u_a u_b$ is an induced metric on Σ , and u^a is the shell's velocity. For dust $p = 0$, whereas $p = -A/\rho^\alpha$ for the generalized Chaplygin gas (GCG). We employ a coordinate system (τ, θ) on Σ ; at $R = \mathcal{R}(\tau)$ the induced metric is

$$\begin{aligned} (ds)_{\Sigma}^2 &= -F_{\pm} dT^2 + \frac{d\mathcal{R}^2}{F_{\pm}} + \mathcal{R}(\tau)^2 d\theta^2 \\ &= -d\tau^2 + r_0^2 a^2(\tau) d\theta^2, \end{aligned} \quad (2.3)$$

Continuity of the metric implies that $[g_{ij}] = 0$ or $F_{\pm}^2 (dT/d\tau)^2 = (d\mathcal{R}/d\tau)^2 + F_{\pm}$ and $\mathcal{R}^2 = r_0^2 a(\tau)^2$. However there exists a discontinuity in the extrinsic curvature of the shell, $[K_{ij}] \neq 0$, since nonvanishing surface stress-energy tensor exists. The extrinsic curvatures on \mathcal{V}_{\pm} are

$$\begin{aligned} K_{\tau\tau}^{\pm} &= -\frac{d}{d\mathcal{R}} \sqrt{\left(\frac{d\mathcal{R}}{d\tau}\right)^2 + F_{\pm}}, \\ K_{\theta\theta}^{\pm} &= \frac{1}{\mathcal{R}} \sqrt{\left(\frac{d\mathcal{R}}{d\tau}\right)^2 + F_{\pm}}. \end{aligned} \quad (2.4)$$

The surface stress-energy tensor is defined by

$$S_{ab} = -([K_{ab}] - [K]h_{ab}), \quad (2.5)$$

where h_{ab} is the induced metric on Σ . On the edge of the shell, the induced dyad and normal vector are

$$e_{(\tau)}^{\alpha} = \left(\frac{dT}{d\tau}, \frac{d\mathcal{R}}{d\tau}, 0\right), \quad e_{(\theta)}^{\alpha} = \left(0, 0, \frac{1}{\mathcal{R}}\right), \quad (2.6)$$

and

$$n_{\alpha} = \left(-\frac{d\mathcal{R}}{d\tau}, \frac{d\mathcal{R}}{d\tau}, 0\right), \quad (2.7)$$

where α is a coordinate system in the bulk, (T, R, θ) . The surface stress-energy tensor can be straightforwardly evaluated

$$S^{\tau}_{\tau} = \frac{1}{\mathcal{R}}(\beta_+ - \beta_-), \quad S^{\theta}_{\theta} = \frac{d}{d\mathcal{R}}(\beta_+ - \beta_-), \quad (2.8)$$

where $\beta_{\pm} \equiv \sqrt{(d\mathcal{R}/d\tau)^2 + F_{\pm}}$. Using Eqs. (2.2) and (2.8), we have two relations,

$$-\rho = \frac{1}{\mathcal{R}}(\beta_+ - \beta_-) \quad (2.9)$$

$$p = \frac{d}{d\mathcal{R}}(\beta_+ - \beta_-). \quad (2.10)$$

The preceding relations can be written in the form

$$\beta_+ - \beta_- + \rho\mathcal{R} = 0 \quad (2.11)$$

$$\frac{d}{d\mathcal{R}}(\rho\mathcal{R}) + p = 0 \quad (2.12)$$

and Eq. (2.11) implies for positive densities that $\beta_+ < \beta_-$, which in turn implies that

$$a_- \mathcal{R}^2/\ell^2 + k_- > a_+ \mathcal{R}^2/\ell^2 + k_+ \quad (2.13)$$

where the generic form of the metrics we study have $F_+ = (a_+ \mathcal{R}^2/\ell^2 + k_+)$ and $F_- = (a_- \mathcal{R}^2/\ell^2 + k_-)$. Here a_{\pm} and k_{\pm} are constants whose values, respectively, determine whether or not the spacetime is asymptotically AdS, dS, or flat, and whether or not the spacetime contains a point mass or a black hole. Differing magnitudes for a_{\pm} correspond to

¹This latter point has been overlooked in previous studies [4].

different values of the size of the cosmological constant inside and outside of the shell. Without loss of generality we can choose one of these to have unit magnitude, i.e. $|a_-| = 1$, though we shall not always exercise this option.

III. COLLAPSE OF A PRESSURELESS DUST SHELL

For the dust shell case, $p = 0$ and Eq. (2.10) becomes

$$0 = \frac{d}{d\mathcal{R}}(\beta_+ - \beta_-). \quad (3.1)$$

Equation (3.1) is easily solved; equating with Eq. (2.9) yields

$$(\beta_+ - \beta_-) = -\rho_0 = -\rho\mathcal{R}, \quad (3.2)$$

where ρ_0 is an integration constant. The density profile is therefore $\rho = \rho_0/\mathcal{R}$; if $\rho = 0$ then clearly $F_+ = F_-$.

Equation (3.2) yields the generic differential equation for the dust shell

$$\sqrt{\left(\frac{d\mathcal{R}}{d\tau}\right)^2 + \frac{a_+\mathcal{R}^2}{\ell^2} + k_+} - \sqrt{\left(\frac{d\mathcal{R}}{d\tau}\right)^2 + \frac{a_-\mathcal{R}^2}{\ell^2} + k_- + \rho_0} = 0. \quad (3.3)$$

Upon redefining $t \equiv \tau/\ell$, $x \equiv \mathcal{R}/\ell$, $\varrho \equiv \rho\ell$, and $\varrho_0 \equiv \rho_0\ell$, Eq. (3.3) can be written as

$$\sqrt{\dot{x}^2 + a_+x^2 + k_+} - \sqrt{\dot{x}^2 + a_-x^2 + k_- + \varrho_0} = 0, \quad (3.4)$$

which can be alternatively written in the form

$$\dot{x}^2 + V_{\text{eff}}(x) = 0, \quad (3.5)$$

where the effective potential is given by

$$V_{\text{eff}}(x) = -\frac{1}{4\varrho_0^2}(a_4x^4 - 2a_2x^2 + a_0), \quad (3.6)$$

with

$$\begin{aligned} a_4 &= (a_+ - a_-)^2 \\ a_2 &= (a_+ + a_-)\varrho_0^2 - (a_+ - a_-)(k_+ - k_-) \\ a_0 &= (\varrho_0^2 - (k_+ + k_-))^2 - 4k_+k_-. \end{aligned} \quad (3.7)$$

Equation (3.4) has the general solution

$$x(t) = -\frac{\gamma_-}{|a_+ - a_-|} \text{JacobiSN}\left[\frac{\gamma_+}{2\varrho_0}(t - t_0), \frac{\gamma_-}{\gamma_+}\right], \quad (3.8)$$

where

$$\gamma_{\pm}^2 = a_2 \pm \sqrt{a_2^2 - a_0a_4}, \quad (3.9)$$

and

$$\begin{aligned} t_0 &= \frac{2\varrho_0}{\gamma_+} \text{JacobiSN}^{-1}\left[\frac{|a_+ - a_-|}{\gamma_-}x_0, \frac{\gamma_-}{\gamma_+}\right] \\ &= \frac{2\varrho_0}{\gamma_+} \text{EllipticF}\left[\frac{|a_+ - a_-|}{\gamma_-}x_0, \frac{\gamma_-}{\gamma_+}\right]. \end{aligned} \quad (3.10)$$

The properties of the Jacobi elliptic functions are reviewed in an appendix.

In the special case that $a_+ = a_-$, $\gamma_- = 0$ and the solution becomes

$$x(t) = -\sqrt{\frac{a_0}{2a_2}} \sin\left[\frac{\sqrt{2a_2}}{2\varrho_0}(t - t_0)\right]. \quad (3.11)$$

Alternatively, if $a_0 = 0$, then the solution is

$$x(t) = -\sqrt{\frac{2a_2}{a_4}} \frac{1}{\sin\left[\frac{\sqrt{2a_2}}{2\varrho_0}(t - t_0)\right]}. \quad (3.12)$$

The qualitative behavior of the solutions will depend upon the relative signs of the four parameters a_{\pm} and k_{\pm} . In general there are 81 possibilities since each parameter can vanish, though of course not all of these are allowed. For example if $a_+ < 0$ then $k_+ > 0$ in order to preserve the metric signature. There are additional restrictions that arise from the reality of the collapse trajectory, which imply that the quantities γ_{\pm} must either be pure real or pure imaginary. This yields $a_2^2 - a_0a_4 > 0$, or

$$\begin{aligned} a_+a_-\varrho_0^2 + (a_+ - a_-)(a_+k_- - a_-k_+) \\ = a_+a_-(\varrho_0^2 - k_+ - k_-) + a_+^2k_- + a_-^2k_+ > 0. \end{aligned} \quad (3.13)$$

Much of the general behavior of the solution can be understood by noting that Eq. (3.5) describes the one-dimensional motion of a point particle of zero energy in the effective potential V_{eff} given in Eq. (3.6), which is sketched in Fig. 1. Note that only the $x > 0$ part of the potential is relevant; the behavior of the shell will depend on the number of roots of the effective potential in this region.

If there are no roots, then the shell will either collapse to zero size from some finite value, or it will expand indefinitely, depending upon the initial conditions. If there is one nondegenerate root then the shell will either expand indefinitely or contract to some finite size and then expand (for example Eq. (3.12) describes this situation). If there is one degenerate root, then the shell can either (a) sit in an unstable equilibrium at some fixed value $x = x_E = \sqrt{a_2/a_4}$ (provided $a_2^2 = a_0a_4$), (b) collapse to either a black hole or a naked singularity provided its initial size is such that $x_0 < x_E$, or (c) exhibit the behavior of the nondegenerate single root case, provided $x_0 > x_E$. If there are two roots, then there will either be collapse to a black hole or naked singularity, or else the behavior will be qualitatively similar to that of the nondegenerate single

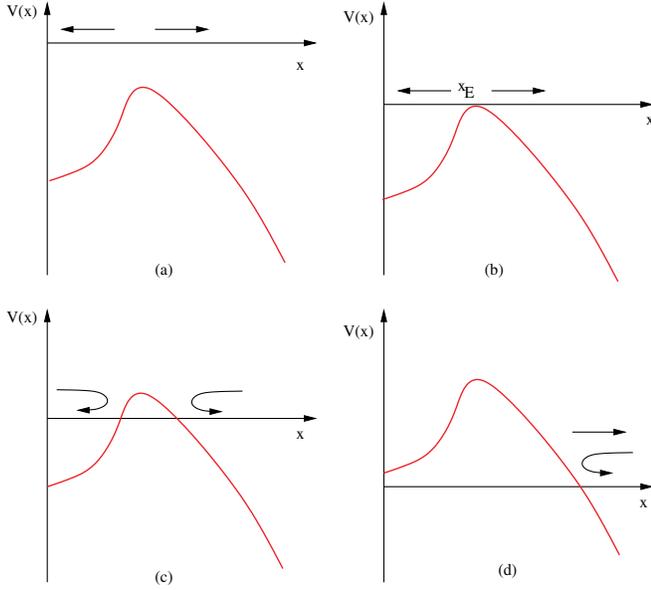


FIG. 1 (color online). Plots of the effective potential of the dust shell for existing roots.

root case. These various cases are illustrated in Fig. 1 by the arrows that indicate possible trajectories of the shell and resemble in part the higher-dimensional situation [25,26]. The key distinction here is the possible of collapse of the shell either to zero size or to a black hole, depending on the choice of parameters and the initial conditions. A discussion of the Penrose diagrams for a number of these scenarios appears in the appendix.

The preceding analysis assumed $a_4 \neq 0$. If $a_4 = 0$ (i.e. $a_+ = a_-$), then the effective potential is quadratic. If $a_+ > 0$ the shell will always collapse provided $a_0 > 0$, whereas if $a_+ < 0$ then the shell will either expand indefinitely or contract to some finite size and then expand, or—if $a_0 < 0$ —collapse to a naked singularity.

Note that collapse to a point mass is not a possible end state for the dust shell. One gets a hint of the underlying problem upon realizing that \dot{x} will not be zero at the end point of collapse. This suggests a bounce, but since the interior spacetime has shrunk to zero size, the future evolution of the spacetime after such a putative bounce is not

uniquely determined. Instead, as the shell collapses its induced curvature becomes singular as $t \rightarrow t_0$, as is clear from the expansion of the Ricci scalar associated with the induced metric (2.3)

$$R^\mu{}_\mu[\Sigma] = \frac{1}{x^2(t)} \left(x(t) \frac{d^2x}{dt^2} - \left(\frac{dx}{dt} \right)^2 \right) = -\frac{1}{2(t-t_0)^2} + \dots \quad (3.14)$$

which diverges quadratically regardless of the values of the parameters a_\pm and k_\pm .

If $k_+ < 0$ this singularity will be cloaked by an event horizon. However if $k_+ > 0$ the ostensible point mass final state suggested by the form of the exterior metric will actually be an incomplete spacetime, with a Cauchy horizon emerging from the singularity. There is nothing *a priori* to prevent this choice for k_+ , and so there will be a range of initial conditions (even for asymptotically flat space) in which the *final* state of collapse yields a naked singularity, in violation of cosmic censorship.

We shall now consider the evolution of the shell in more specific terms, categorizing our study by $a_+ > 0$ (exterior AdS space), $a_+ = 0$ (exterior flat space) and $a_+ < 0$ (exterior dS space).

A. External AdS space

If the spacetime is asymptotically anti-de Sitter, then $a_+ > 0$. The general solution is then given by (3.8)

$$x(t) = -\frac{\gamma_-}{|1-a_-|} \text{JacobiSN} \left[\frac{\gamma_+}{2\varrho_0} (t-t_0), \frac{\gamma_-}{\gamma_+} \right] \quad (3.15)$$

where γ_\pm is still given by Eq. (3.9), but with

$$\begin{aligned} a_4 &= (1-a_-)^2 \\ a_2 &= (1+a_-)\varrho_0^2 + (1-a_-)(M+k_-) \\ a_0 &= (\varrho_0^2 + (M-k_-))^2 + 4Mk_- \end{aligned} \quad (3.16)$$

where we have set $a_+ = 1$ without loss of generality and $k_+ = -M$ so that a naked singularity is avoided.

Collapse to an AdS₃ black hole with no angular momentum and with cosmological constant, $\Lambda = -1/\ell^2$ will take place provided

$$x(0) = x_0 < \frac{\sqrt{(1+a_-)\varrho_0^2 + (1-a_-)(M+k_-) - 2\varrho_0\sqrt{a_-(\varrho_0^2 + M - k_-)} + k_- - a_-^2 M}}{|1-a_-|} \quad (3.17)$$

and that the term under the second square root is positive, which is the condition (3.13). If (3.17) is satisfied, we have an additional condition for the collapse, $\dot{x}_0^2 + x_0^2 \geq M = x_H > -k$ (with \dot{x}_0 the initial velocity of the shell), which means that the initial radius cannot be smaller than the black hole horizon. If these conditions are satisfied and $\dot{x}_0 > 0$ then the shell will first expand to a maximal size given by the right-hand side of (3.17), and then collapse to a black hole; otherwise the shell will irreversibly collapse. If (3.17) is violated, then the shell collapses to a minimal radius

$$x_{\min} = \frac{\sqrt{(1 + a_-)\varrho_0^2 + (1 - a_-)(M + k_-) + 2\varrho_0\sqrt{a_-(\varrho_0^2 + M - k_-) + k_- - a_-^2 M}}}{|1 - a_-|} \quad (3.18)$$

and then expands to indefinitely large size (or else sits at a point of unstable equilibrium if $a_-(\varrho_0^2 + M - k_-) + k_- - a_-^2 M = 0$ and if the initial conditions are properly set).

If $k_- = 1$, the interior is pure AdS; if $1 > k_- > 0$, then the interior metric corresponds to a point mass in AdS spacetime; if $k_- < 0$ then the interior metric corresponds to a black hole. Since Eq. (2.13) implies that

$$\varrho_0 = \begin{cases} \text{positive,} & ((a_- - 1)x_0^2 + k_- + M > 0) \\ \text{negative,} & ((a_- - 1)x_0^2 + k_- + M < 0) \\ 0, & ((a_- - 1)x_0^2 + k_- + M = 0). \end{cases} \quad (3.19)$$

We see for positive energy density that in general the mass of the interior black hole must be smaller than M provided $0 < a_- < 1$, i.e. the magnitude of the interior cosmological constant is not as large as that of the exterior space. If $a_- \leq 0$ then $k_- > 0$ or else the metric signature of the interior is not properly preserved (alternatively condition (3.13) is not satisfied). If $a_- \geq 1$ then the interior black hole will have a larger mass than M , with the energy of the shell contributing negatively to the total energy of the spacetime.

The case $a_+ = a_- = 1$ merits special attention, since it corresponds to the previously analyzed collapse of a disk of dust [3]. We recover the solution (3.11), which can be written as

$$x(t) = -\frac{\sin(t - t_0)}{2\varrho_0} \sqrt{M^2 + 2(\varrho_0^2 + k_-)M + (\varrho_0^2 - k_-)^2}, \quad (3.20)$$

where

$$\begin{aligned} t_0 &= \arcsin \left[\frac{2\varrho_0 x_0}{\sqrt{M^2 + 2(\varrho_0^2 + k_-)M + (\varrho_0^2 - k_-)^2}} \right] \\ &= \arcsin \left[\frac{x_0}{\sqrt{\dot{x}_0^2 + x_0^2}} \right]. \end{aligned} \quad (3.21)$$

Collapse to a black hole takes place when $x(t_H) = x_H \equiv \mathcal{R}_H/\ell = \sqrt{M}$, where

$$t_H = \arcsin \left[\frac{x_0}{\sqrt{\dot{x}_0^2 + x_0^2}} \right] - \arcsin \left[\frac{\sqrt{M}}{\sqrt{\dot{x}_0^2 + x_0^2}} \right] \quad (3.22)$$

which is when (in comoving time) the radius of the shell is coincident with the event horizon.

Note that Eq. (3.20) yields the requirement that $x(t)$ be real, implying in turn that $a_0 > 0$ or

$$\begin{aligned} &M^2 + 2(\varrho_0^2 + k_-)M + (\varrho_0^2 - k_-)^2 \\ &= (\varrho_0^2 - (k_- + M))^2 + 4\varrho_0^2 M \geq 0 \end{aligned} \quad (3.23)$$

which is satisfied for all positive M . The effective potential is obtained by setting $a_+ = a_- = 1$, $k_+ = -M$ in Eq. (3.6),

$$V_{\text{eff}}(x) = \frac{1}{4\varrho_0^2} (4\varrho_0^2 x^2 - a_0) \quad (3.24)$$

and has a minimum at $x = 0$ with $a_0 > 0$, which implies that the shell will inevitably collapse to a black hole, regardless of the sign of its initial velocity, and it will shrink to the origin within finite time. Choosing initial conditions so that $\dot{x}_0 = 0$, Eq. (3.4) can be rewritten as

$$M = \varrho_0 [2(x_0^2 + k_-)^{1/2} - \varrho_0] - k_-, \quad (3.25)$$

which is analogous to the condition for dust ball collapse [3]. A black hole can form only if

$$\varrho_0 > (x_0^2 + k_-)^{1/2} - x_0 \quad (3.26)$$

otherwise $M < 0$ and the shell collapses to a naked singularity and a Cauchy horizon forms. Note that if $-M < k_- < 0$ then the condition (3.26) is always satisfied. It is curious that for sufficiently small initial shell density that cosmic censorship is violated.

The comoving time for the shell radius to become coincident with the event horizon is given by (3.22), and the time t_c for the shell to collapse from $x_i = x_0$ to $x_f = x(t_c) = 0$ is always finite for positive M and ϱ_0 . However, the coordinate time at which an observer outside the black hole observes the collapse is not finite since

$$\hat{T} = T_0 + \int_{x_0}^{\hat{x}} \frac{\ell dx}{(x^2 - M)} = T_0 - \frac{\ell}{\sqrt{M}} \operatorname{arctanh} \left[\frac{x}{\sqrt{M}} \right]_{x_0}^{\hat{x}}, \quad (3.27)$$

where \hat{T} is a coordinate time at which a signal emitted from the edge of the shell arrives at a certain point \hat{x} . This coordinate time is clearly divergent when $\hat{x} \rightarrow \sqrt{M} = x_H$, which implies that the collapse to the horizon takes infinite time, so that observers outside the black hole will not observe this collapse.

The redshift of light from the edge of the dust shell is

$$z = \frac{d\hat{T}}{dt} - 1 = \frac{\dot{x}}{(x - x_H)(x + x_H)} - 1, \quad (3.28)$$

which obviously diverges at $t = t_H(x = x_H)$. Thus the collapsing shell of dust will fade away from observer's sight as time goes by, as with the collapse of the dust ball [3].

Note that if $M = -m < 0$ then Eq. (3.23) is not necessarily satisfied. If $\rho_0 > 0$ then $1 > k > m$, and the conical deficit angle outside the shell is larger than that inside the shell. In this case the larger the shell density, the larger the exterior deficit angle relative to the interior one. The density is bounded by

$$\varrho_0 < (x_0^2 + k)^{1/2} - x_0 \quad (3.29)$$

which ensures that the exterior deficit angle is always less than 2π .

We close this subsection by noting that if $k_+ > 0$ then the shell can collapse to a naked singularity if the initial conditions are properly set. The alternative to this scenario is that the shell either expands indefinitely or collapses to a minimal size and then expands indefinitely. As noted previously, the specifics depend on the values of the parameters a_- and k_\pm .

B. External flat space

Now we turn to shell collapse where Minkowski and/or conical deficit space describing a point mass is inside and/or outside the shell, corresponding to the case $a_+ = 0$.

Consider first the case where $a_- = 0$ as well, yielding $F_\pm = k_\pm$, representing a flat space with conical deficit when $0 < k_\pm < 1$ (which vanish when $k_\pm = 1$), where $k_\pm > 0$ to preserve the sign of the metric. For positive energy density $k_+ < k_-$, ensuring that the exterior deficit angle is greater (corresponding to a larger mass) than the interior deficit angle as noted above. Then the equation of motion (3.4) has the solution

$$x(t) = x_0 + \dot{x}_0 t, \quad (3.30)$$

which is analogous to dust ball collapse in Minkowski space [3]. The coefficient of t corresponds to the initial velocity of the disk, which must be negative if collapse is to take place (if it is positive then the shell expands outward without resistance). The initial velocity is

$$\dot{x}_0 = \pm \frac{\sqrt{\varrho_0^4 - 2(k_+ + k_-)\varrho_0^2 + (k_+ - k_-)^2}}{2\varrho_0}, \quad (3.31)$$

which is real provided

$$\varrho_0 > \sqrt{k_-} - \sqrt{k_+}. \quad (3.32)$$

Note that there is no collapse unless the shell is given some initial inward velocity, i.e. $\dot{x}_0 < 0$, in which case the shell will collapse to a naked singularity with its associated Cauchy horizon.²

Next we consider the more general case of an interior spacetime with cosmological constant. Without loss of

²Note that even if the energy density of the shell is negative, collapse can still take place if $\dot{x}_0 < 0$. In this case the exterior deficit angle is smaller than the interior one.

generality we can set $a_- = \pm 1$. The effective potential is still a quartic given by Eq. (3.6) and the solution is given by (3.8), both with $a_+ = 0$ and $|a_-| = 1$. Positivity of the initial density $\varrho_0 > 0$ implies

$$\sqrt{a_- x_0^2 + k_-} - \sqrt{k_+} > 0 \rightarrow a_- x_0^2 > k_+ - k_- \quad (3.33)$$

and so either $k_+ > 0$ must be sufficiently small if the interior is AdS or else $k_- > 0$ must be sufficiently large if the interior is dS. In either case the shell will either expand indefinitely, undergo a bounce after which it expands indefinitely, or else collapse to a naked singularity.

C. External dS space

Finally, we assume that the exterior metric function is that for a dS spacetime, i.e. $a_+ = -1$, yielding $F_+ = (\mu - \mathcal{R}^2/\ell^2)$. The cosmological horizon is located at $\mathcal{R}_h = \sqrt{k_+}\ell$, where $k_+ = \mu > 0$ in order to have the metric signature correct. The exterior metric is that of dS spacetime with a conical deficit unless $\mu = 1$, in which case it is pure dS spacetime.

The general solution is again given by Eq. (3.8), where

$$x(t) = -\frac{\gamma_-}{|1 + a_-|} \text{JacobiSN}\left[\frac{\gamma_+}{2\varrho_0}(t - t_0), \frac{\gamma_-}{\gamma_+}\right], \quad (3.34)$$

with

$$\begin{aligned} a_4 &= (1 + a_-)^2 \\ a_2 &= -(1 - a_-)\varrho_0^2 + (a_- + 1)(\mu - k_-) \\ a_0 &= (\varrho_0^2 - (\mu + k_-))^2 - 4\mu k_-, \end{aligned} \quad (3.35)$$

and again γ_\pm is given by Eq. (3.9).

Positivity of energy now imposes the requirement

$$(a_- + 1)x_0^2 > \mu - k_- \quad (3.36)$$

which implies that μ must be sufficiently small relative to the other parameters. If the interior is AdS, then $a_- > 0$ and the initial shell size x_0 must be sufficiently large relative to the sum of the masses (if there is a black hole in the interior, with $k_- = -M < 0$) or their difference (if there is a point mass in the interior, with $0 < k_- < 1$). The same analysis holds true if the interior is a dS space with smaller cosmological constant (i.e. $-1 < a_- < 0$), though in this case $k_- > 0$. If $a_- < -1$ then k_- must be sufficiently large for the shell to have any allowed motion.

The possible trajectories of the shell have been covered at the beginning of this section. If the shell initially contracts, it will either collapse to a naked singularity or else it will bounce at some finite radius

$$x_{\min} = \frac{\sqrt{-(1 - a_-)\varrho_0^2 + (1 + a_-)(\mu - k_-) + 2\varrho_0\sqrt{-a_-(\varrho_0^2 - \mu - k_-) + k_- + a_-^2\mu}}}{|1 + a_-|}, \quad (3.37)$$

and then expand to indefinitely large size. If the shell initially expands, it will either expand indefinitely or it will bounce at

$$x_{\max} = \frac{\sqrt{-(1 - a_-)\varrho_0^2 + (1 + a_-)(\mu - k_-) - 2\varrho_0\sqrt{-a_-(\varrho_0^2 - \mu - k_-) + k_- + a_-^2\mu}}}{|1 + a_-|}, \quad (3.38)$$

and then collapse to a naked singularity. The remaining alternative is that of a shell in unstable equilibrium, which occurs if $-a_-(\varrho_0^2 - \mu - k_-) + k_- + a_-^2\mu$ and if the initial conditions are properly set. Collapse to a black hole is never possible.

IV. COLLAPSE OF A SHELL WITH A POLYTROPIC EQUATION OF STATE

We now consider the collapse of a shell with pressure, whose equation of state we take to be that of a polytrope

$$p = q\rho\left(\frac{\rho}{\rho_0}\right)^{1/n}, \quad (4.1)$$

where q is a constant, representing diverse choices for the matter content of the shell [27]. For example, $n = 0$, $n = 1$, and $n = 2$ respectively represent constant energy density, nonrelativistic degenerate fermions, and nonrelativistic matter or radiation pressure. Moreover, the equation of state for perfect fluids is achieved by setting $n \rightarrow \infty$.

The matching conditions (2.9) and (2.10) of the shell imply that

$$\frac{d}{d\mathcal{R}}(\rho\mathcal{R}) = -p = -q\rho^{(1+1)/n}, \quad (4.2)$$

which yields

$$\rho(\mathcal{R}) = \rho_0\left(-q + K\left(\frac{\mathcal{R}}{\ell}\right)^{1/n}\right)^{-n}, \quad (4.3)$$

where K is a constant of integration. For the perfect fluid, we obtain

$$\rho(\mathcal{R}) = \rho_0\left(\frac{\ell}{\mathcal{R}}\right)^{1+q}. \quad (4.4)$$

From these we respectively obtain the equations

$$\sqrt{\dot{x}^2 + a_+x^2 + k_+} - \sqrt{\dot{x}^2 + a_-x^2 + k_-} + \frac{\varrho_0x}{(-q + (1 + q)x^{1/n})^n} = 0, \quad (4.5)$$

for finite n , and

$$\sqrt{\dot{x}^2 + a_+x^2 + k_+} - \sqrt{\dot{x}^2 + a_-x^2 + k_-} + \frac{\varrho_0}{x^q} = 0, \quad (4.6)$$

for the perfect fluid, where again $x \equiv \mathcal{R}/\ell$, $\varrho_0 = \rho_0\ell$, and $K = (1 + q)$ so that $\rho = \rho_0$ when $x = 1$.

For all n , the equation of motion can be written as

$$\dot{x}^2 + V_{\text{eff}}(x) = 0, \quad (4.7)$$

where

$$V_{\text{eff}}(x) = -\frac{x^2}{4}\left(\rho^2 - 2(a_+ - a_-) + \frac{(a_+ - a_-)^2}{\rho^2}\right) + \frac{k_+ - k_-}{2} - \frac{(a_+ - a_-)(k_+ - k_-)}{\rho^2} + \frac{(k_+ - k_-)^2}{4\rho^2x^2} \quad (4.8)$$

is the effective potential. In general, it depends on many parameters, $(\varrho_0, q, a_{\pm}, k_{\pm})$ and so is somewhat unwieldy to analyze in full generality. Furthermore it is hard to obtain an exact solution of Eq. (4.5) since the equation of motion is highly nonlinear.

However the structure of the effective potential allows us to discern some basic features. First, for finite n and non-zero q , the shell will not collapse to a point, but rather to a ring of size $x = x_q \equiv q^n/(1 + q)^n$ in finite proper time. Physically we can think of the shell as developing an increasingly large internal pressure that diverges for some finite value of the shell radius. For the perfect fluid the shell can collapse to zero size in finite proper time. In either case the stress-energy tensor of the shell diverges. However there is no backreaction since the field equations force spacetime to have constant curvature in regions

where the stress-energy tensor vanishes, i.e. everywhere outside of the shell.

However the intrinsic Ricci scalar can be written in the form

$$R_{\mu}^{\mu}[\Sigma] = \frac{1}{x^2} [x\dot{x}^2 - \ddot{x}] = \frac{1}{2x^2} \left[\frac{d}{dx} V_{\text{eff}}(x) - 2xV_{\text{eff}}(x) \right] \quad (4.9)$$

by using Eq. (4.7). Since $V(x_q) \rightarrow -\infty$ and $dV(x_q)/dx \rightarrow \infty$, the intrinsic Ricci scalar of the shell diverges at $x = x_q$ as the shell approaches its minimal radius. Physically the internal pressure forbids the shell to be compressed without limit. As it shrinks in size, the pressure grows, eventually diverging (along with the density) at some finite shell radius.

The shell can exhibit several kinds of behavior, depending upon the values of the parameters, a_{\pm} , k_{\pm} , q , n , and ϱ_0 . We illustrate here the generic possibilities for the effective potential for specific values of these constants in Fig. 2 (for perfect fluids in Fig. 3). Generically the effective potential has 2 local maxima and one local minimum, and it diverges to minus infinity for large x and for some finite x . It is possible for the rightward local maximum to occur for positive values of V_{eff} , in which case the shell expands to

infinity, possibly after a bounce if it is given inward initial velocity. The event horizon for the black hole is always to the left of this maximum (and occurs where $V_{\text{eff}} < 0$) so for initial values of the shell radius between the event horizon and the smaller root of V_{eff} the shell will always collapse into a black hole, again possibly with a bounce if given outward initial velocity. If the rightward local maximum of V_{eff} occurs at $V_{\text{eff}} < 0$ then the shell will either expand outward to infinity or collapse to a black hole depending on whether the initial velocity is outward or inward. Variation of the parameters can cause the local minimum to disappear, leaving a single maximum for the effective potential, in which case the same qualitative behavior of the shell takes place as previously described. A numerical search indicates that there are no values of the parameters for which both local maxima occur for $V_{\text{eff}} > 0$, and so the shell can never undergo bouncing oscillations between maximum and minimum values.

However it is possible for the shell to collapse without forming an event horizon if $k_{+} > 0$. In this case the pressure and density diverge in finite proper time at some finite value of the shell radius (or zero value in the case of a perfect fluid) as discussed above. The stress-energy tensor and the intrinsic Ricci scalar of the shell both diverge, and it is not possible to evolve the shell beyond this point. In

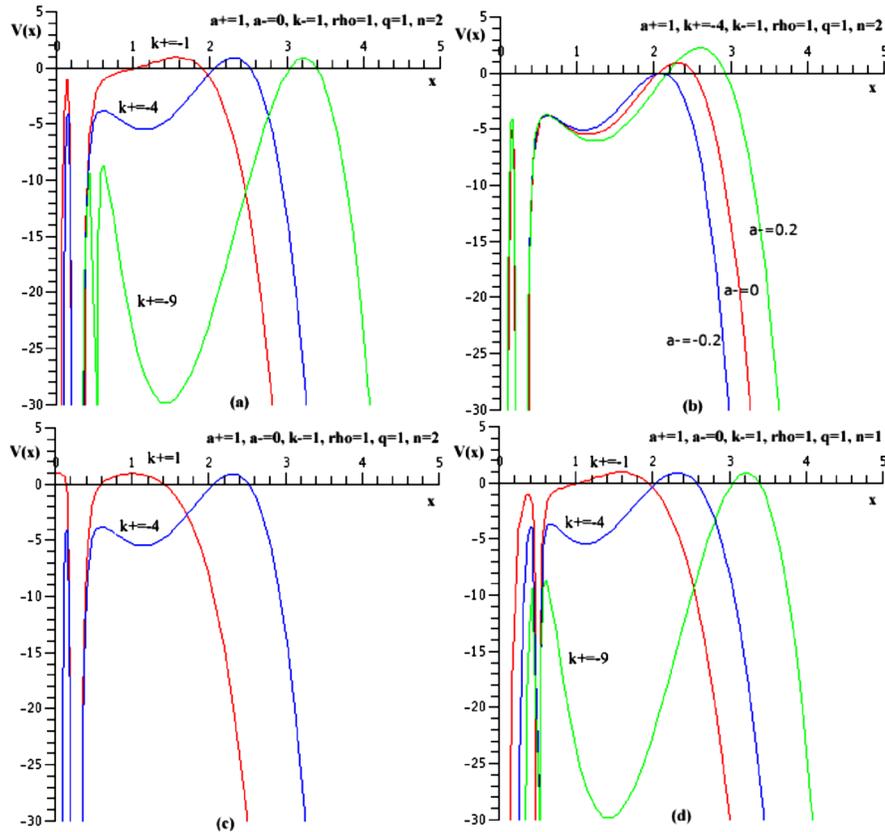


FIG. 2 (color online). Some plots of the effective potential of the shell with pressure. The parameters have been chosen so that the pressure and density diverge at $x = 1/2$; the effective potential is physically meaningful only for $x > 1/2$.

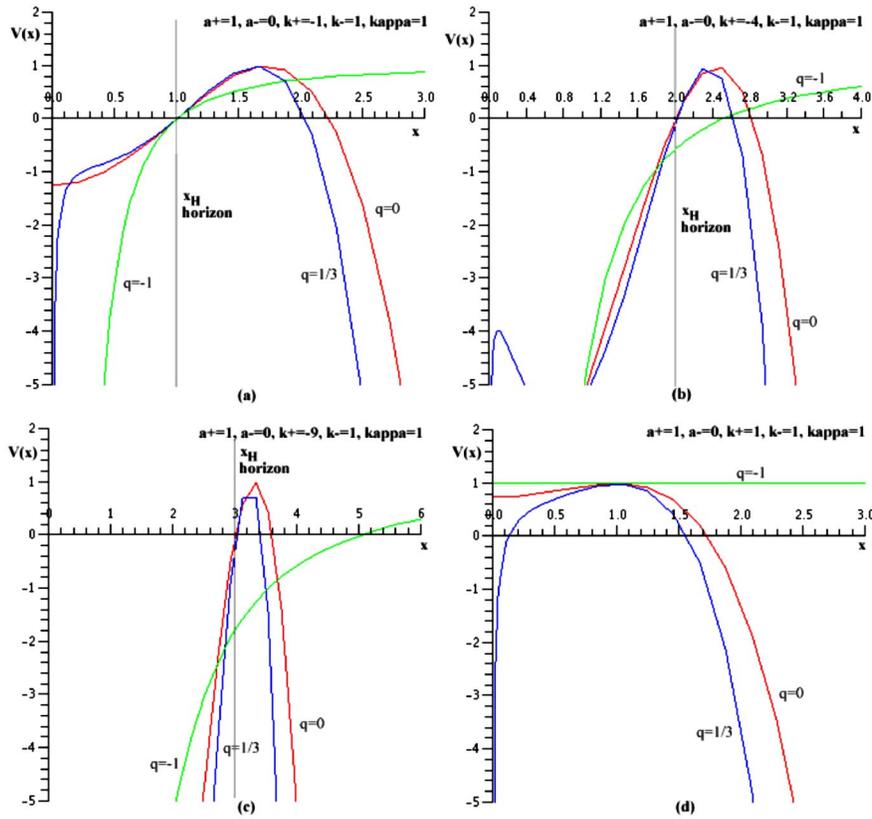


FIG. 3 (color online). Some plots of the effective potential of the shell of perfect fluids, for example, $q = 0$ (dust), $q = 1/3$ (radiating matter), and $q = -1$ (cosmological constant).

this sense we have a mild violation of cosmic censorship; although the curvature is finite everywhere outside the shell, it diverges on the shell, as with the case of pressureless dust. The pressure, rather than preventing a singularity, instead moves it out to finite shell radius.

V. COLLAPSE OF THE GENERALIZED CHAPLYGIN SHELL

We now turn to consideration of the gravitational collapse of a generalized Chaplygin gas (GCG) shell with an equation of state, $p = -\tilde{A}/\rho^\alpha = -A\rho_0(\rho_0/\rho)^\alpha$. We find that the density is

$$\rho = \frac{[A\rho_0^{\alpha+1}\mathcal{R}^{\alpha+1} + C]^{1/(\alpha+1)}}{\mathcal{R}} \quad (5.1)$$

where C is a constant of integration. Combining Eqs. (2.9) and (2.10) leads to a simple equation,

$$\sqrt{\left(\frac{d\mathcal{R}}{d\tau}\right)^2 + F_+} - \sqrt{\left(\frac{d\mathcal{R}}{d\tau}\right)^2 + F_-} + [A\rho_0^{\alpha+1}\mathcal{R}^{\alpha+1} + C]^{1/(\alpha+1)} = 0, \quad (5.2)$$

where C has been chosen so that $\rho = \rho_0$ when $x \equiv \mathcal{R}/\ell = 1$.

The setting is like that of a polytrope, but with $q < 0$ and $n = 1/(1 + \alpha)$. If we set $A = 0$ the situation reduces to that of the collapse of pressureless dust shell investigated in the previous section. If $A > 0$ Eq. (5.2) describes a GCG with a negative pressure. Provided $\alpha > -1$, the density will diverge at the origin for $C > 0$. If $C < 0$ it will converge to some finite value at some nonzero value of the shell radius \mathcal{R} , with the pressure diverging at that same radius. If $C = 0$ the density and pressure are constant for all values of \mathcal{R} .

We proceed as before by redefining parameters so that $x \equiv \mathcal{R}/\ell$, $\varrho_0 \equiv \rho_0\ell$, $\tau \equiv t/\ell$, and set $C = 1 - A\varrho_0^{(\alpha+1)}$, so that $\rho = \rho_0$ when $x = 1$. Then Eq. (5.2) becomes

$$\sqrt{\dot{x}^2 + a_+x^2 + k_+} - \sqrt{\dot{x}^2 + a_-x^2 + k_-} + \varrho_0[Ax^{\alpha+1} + 1 - A]^{1/(\alpha+1)} = 0. \quad (5.3)$$

The equation of motion can be rewritten as

$$\dot{x}(t)^2 + V_{\text{eff}}(x) = 0, \quad (5.4)$$

where the effective potential is

$$V_{\text{eff}}(x) = \frac{1}{4\mathcal{C}^2}[(\mathcal{C}^2 - ((a_+ + a_-)x^2 + k_+ + k_-))^2 - 4(a_+x^2 + k_+)(a_-x^2 + k_-)], \quad (5.5)$$

with $C = \varrho_0(Ax^{\alpha+1} + 1 - A)^{1/(\alpha+1)}$, whose shape depends upon several parameters, α , A , a_{\pm} , and k_{\pm} . Note that the effective potential will not diverge for any finite value of x if $A < 1$. Otherwise, for all $\alpha > -1$ it will diverge quadratically for large x to either positive or negative infinity, depending on the values of the parameters.

For example, $\alpha = 1$ describes a Chaplygin gas shell, whose effective potential is

$$V_{\text{eff}}(x) = -\frac{1}{4\varrho_0(\omega^2 x^2 + c)}(a_4 x^4 - 2a_2 x^2 + a_0), \quad (5.6)$$

where for convenience we have defined $\omega^2 = A\varrho_0^2$ and $c = (1 - A)\varrho_0^2$ so that

$$\begin{aligned} a_4 &= (\omega^2 - (a_+ + a_-))^2 - 4a_+ a_-, \\ a_2 &= (k_+ + k_- - c)\omega^2 + (a_+ + a_-)c \\ &\quad - (k_+ - k_-)(a_+ - a_-), \\ a_0 &= (c - (k_+ + k_-))^2 - 4k_+ k_-. \end{aligned} \quad (5.7)$$

The effective potential for arbitrary c is plotted in Fig. 4 for an exterior AdS black hole metric ($k_+ < 0$) and an exterior AdS point mass metric ($k_+ > 0$) outside the shell.

A glance at Fig. 4 indicates that the GCG shell will collapse to an AdS black hole within a finite time for $c \geq 0$ while this is always not the case if $c < 0$. If $x_H > x_s$, the shell will form a black hole while if $x_H < x_s$, it will collapse to a finite size of $x = x_s$, even if the exterior metric is an AdS black hole. The final state is a rather

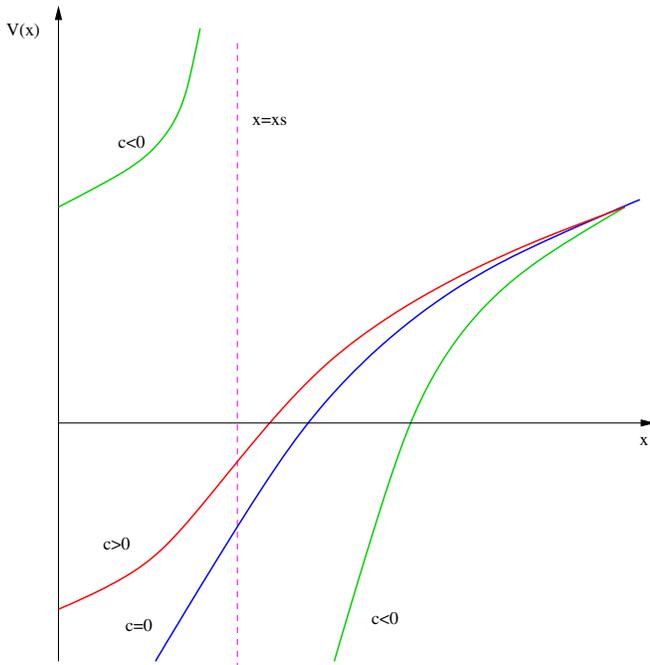


FIG. 4 (color online). Plot of the effective potential of the GCG shell with arbitrary c for $\alpha = 1$, where $x_s = \varrho_0\sqrt{1 - 1/A} = \sqrt{-c}/\omega$ only for $c < 0$.

unusual state in which the density is finite but the pressure diverges, again yielding a Cauchy horizon and a violation of cosmic censorship.

Consider the special case $C = 0$. This leads to the simpler form

$$\sqrt{\dot{x}^2 + a_+ x^2 + k_+} - \sqrt{\dot{x}^2 + a_- x^2 + k_-} + \omega x = 0, \quad (5.8)$$

or alternatively

$$\dot{x}^2 + V_{\text{eff}}(x) = 0, \quad (5.9)$$

where $V_{\text{eff}}(x)$ is an effective potential with

$$V_{\text{eff}}(x) = -\frac{1}{4\omega^2 x^2}[a_4 x^4 - 2a_2 x^2 + a_0], \quad (5.10)$$

where

$$\begin{aligned} a_4 &= (\omega^2 - a_+ - a_-)^2 - 4a_+ a_-, \\ a_2 &= (k_+ + k_-)\omega^2 - (k_+ - k_-)(a_+ - a_-) \\ a_0 &= (k_+ - k_-)^2. \end{aligned} \quad (5.11)$$

For this case the density and pressure are always constant. The specific shape of the potential will depend upon the values of a_4 , a_2 , and a_0 associated with a_{\pm} , k_{\pm} , and ω . For example, if $a_0 = 0$ (i.e. $k_+ = k_-$), it is simply described by the usual quadratic function. The shape of the effective potential for each case is plotted in Fig. 5.

For $a_0 > 0$, the behavior of $x \rightarrow 0$ is $V_{\text{eff}}(x \rightarrow 0) = -\infty$, regardless of the value of a_4 . For $a_4 > 0$, since the potential

has two roots at $x = \sqrt{a_2 \pm \sqrt{a_2^2 - a_4 a_0}}/\sqrt{a_4}$ and $V_{\text{eff}}(x \rightarrow \infty) = -\infty$, the curve of the potential is concave down. In this case the shell will expand to infinity for sufficiently large initial radius (with a bounce if the initial velocity is negative) or collapse to zero size (leaving behind a black hole or a naked singularity) for sufficiently small initial radius (with a bounce if the initial velocity is positive). For $a_4 < 0$ there is only one root and $V_{\text{eff}}(x \rightarrow \infty) = \infty$. In this case the shell always collapses to zero size, possibly preceded by a bounce if it is initially expanding outward. For $a_4 = 0$ a third possibility exists in which the shell can expand indefinitely but will continually decelerate. These scenarios are depicted on the left-hand-side of Fig. 5.

For $k_+ < 0$ (an exterior AdS black hole metric) and $a_4 \geq 0$ (middle and bottom lines in Fig. 5(a)), the shell can either collapse or expand, depending upon the initial velocity, \dot{x} . However for $a_4 < 0$, the shell will either collapse to a black hole or initially expand, bounce and then collapse to a black hole. For $k_+ > 0$ (an AdS point mass metric) and $a_4 \leq 0$ (top and middle lines in Fig. 5(b)), the shell will collapse to a point at $x = 0$ while if $a_4 > 0$, then the shell will either collapse to a point or expand indefinitely or collapse to a certain size and expand indefinitely again.

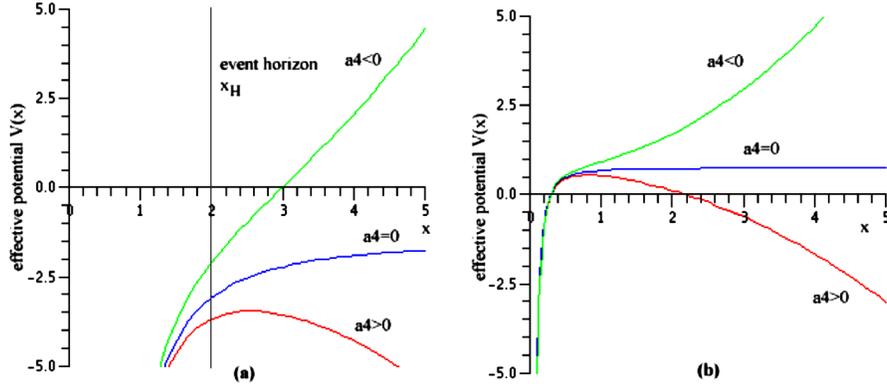


FIG. 5 (color online). Plots of the effective potential of the GCG shell for each a_4 , setting $\alpha = 1$, $k_- = 1$, and $k_+ = -4$ (left-hand side) and $k_+ = 0.5$ (right-hand side).

There are three classes of solutions, depending upon the sign of a_4 : $a_4 > 0$, $a_4 = 0$, and $a_4 < 0$. The general solutions of Eq. (5.8) are

$$x(t)^{a_4 > 0} = \frac{1}{\sqrt{a_4}} \left[a_2 \left(1 - \cosh \left[\frac{\sqrt{a_4}}{\omega} (t - t_0) \right] \right) + \sqrt{a_4 a_0} \sinh \left[\frac{\sqrt{a_4}}{\omega} (t - t_0) \right] \right]^{1/2} \quad (5.12)$$

$$x(t)^{a_4 < 0} = \frac{1}{\sqrt{\bar{a}_4}} \left[\sqrt{a_2^2 + a_0 \bar{a}_4} \sin \left[\arcsin \left(\frac{a_2}{\sqrt{a_2^2 + a_0 \bar{a}_4}} \right) - \frac{\sqrt{\bar{a}_4}}{\omega} (t - t_0) \right] - a_2 \right]^{1/2} \quad (5.13)$$

$$x(t)^{a_4 = 0} = \sqrt{\frac{a_0 \omega^2 - a_2^2 (t - t_0 + \frac{\sqrt{a_0} \omega}{a_2})^2}{2a_2 \omega^2}}, \quad (5.14)$$

where t_0 is the collapse time to $x = 0$, respectively, given by

$$t_0^{a_4 > 0} = \frac{\omega}{\sqrt{a_0}} \operatorname{arccosh} \left[\frac{a_2 (a_4 x_0^2 - a_2) + a_4 \sqrt{a_0 (a_4 x_0^4 - 2a_2 x_0^2 + a_0)}}{a_0 a_4 - a_2^2} \right] \quad (5.15)$$

$$t_0^{a_4 < 0} = \frac{\omega}{\sqrt{\bar{a}_4}} \left[\arcsin \left[\frac{a_2 + \bar{a}_4 x_0^2}{\sqrt{a_2^2 + a_0 \bar{a}_4}} \right] - \arcsin \left[\frac{a_2}{\sqrt{a_2^2 + a_0 \bar{a}_4}} \right] \right] \quad (5.16)$$

$$t_0^{a_4 = 0} = \frac{\omega}{a_2} (\sqrt{a_0} - \sqrt{a_0 - 2a_2 x_0^2}), \quad (5.17)$$

where $\bar{a}_4 = -a_4 > 0$.

In order to have a positive and real collapse time in Eq. (5.15), we should impose that the argument of the cosine hyperbolic function must be greater than 1, which leads to a condition,

$$a_0 a_4 - a_2^2 > 0, \quad (5.18)$$

while Eqs. (5.16) and (5.17) is always valid regardless of parameters.

The intrinsic scalar curvature of the shell can be evaluated by an expansion in terms of $t - t_0$ for each case,

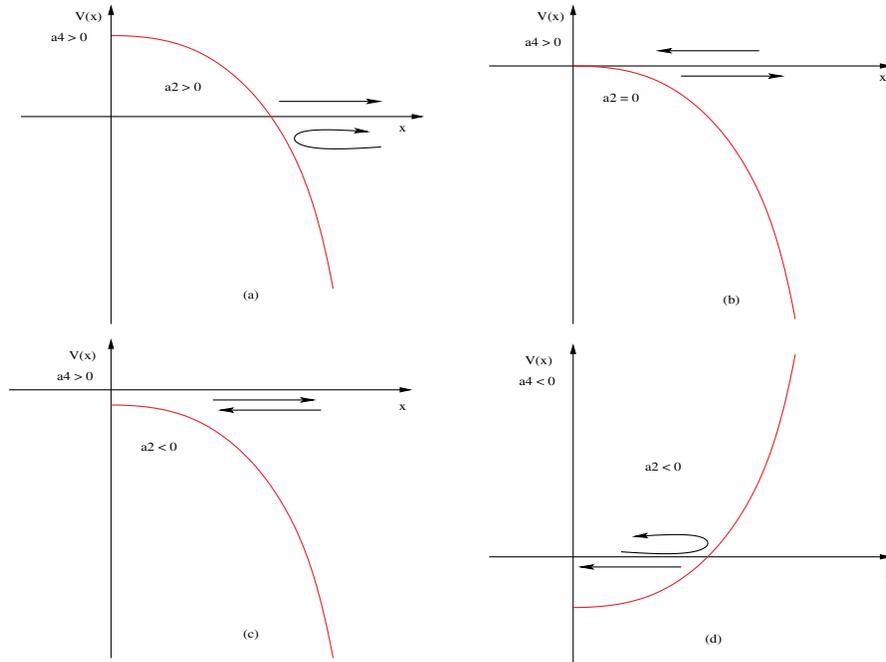
$$R_{\mu}^{\mu}[\Sigma] \sim -\frac{3}{8(t - t_0)^2} \dots \quad (5.19)$$

where we see that it is generically singular at the endpoint of collapse.

An interesting subcase is obtained by setting $k_+ = k_- = k$, which yields $a_4 = (\omega^2 - a_+ - a_-)^2 - 4a_+ a_-$, $a_2 = 2k\omega^2$, and $a_0 = 0$. The effective potential is now a quadratic function in the form of

$$V_{\text{eff}}(x) = -\frac{1}{4\omega^2} (a_4 x^2 - 2a_2), \quad (5.20)$$

and possible scenarios of collapse, depending upon a_4 and a_2 , are shown in Fig. 6. For $a_4 > 0$, there are three choices


 FIG. 6 (color online). Possible collapse scenarios of $k_+ = k_-$ case.

of collapse scenarios depending on the sign of a_2 . For an AdS space outside the shell ($a_+ = 1$), then $a_4 = (\omega^2 - 1 - a_-)^2 - 4a_- > 0$. When $a_2 > 0$, then $k > 0$, which implies that we have the metric of an AdS point mass outside the shell. The interior space is either (i) an AdS point mass if $a_- > 0$, (ii) a dS space if $a_- < 0$, or (iii) a point mass in a flat space if $a_- = 0$. Then the shell will either bounce at $x = \sqrt{2a_2/a_4}$ and expand again if it initially contracts or expand indefinitely, as shown in Fig. 6(a). If $a_2 = 0$, then $k = 0$. Thus $a_- > 0$ to preserve the metric signature, which describes an AdS vacuum in and outside of the shell. Then the shell will either expand indefinitely or contract to zero size, at which point its intrinsic Ricci scalar diverges, forming a naked singularity (see 6(b)).

Alternatively the situation $a_2 < 0$ describes two AdS black holes with $k = -M$ since $a_- > 0$ for preserving the metric signature. Note that these holes will have differing masses since $a_- > a_+$. In this case, the shell will either collapse to a black hole or expand, depending upon its initial motion. [Fig. 6(c)] Finally, for $a_4 < 0$, it is found that $a_+ a_- > 0$, i.e. both spaces should be AdS spaces. Since $a_2 = 2k\omega^2 < 0$, this also describes AdS black holes of differing mass, even if the shape of the effective potential is different from above case.

Note that the final state of collapse for all black hole scenarios is a singularity cloaked by an event horizon. Since $k_+ = k_-$, we obtain from Eqs. (5.12) and (5.13)

$$R_{\mu}^{\mu}[\Sigma]^{a_4 > 0} = -\frac{a_4}{8\omega^2} \left[1 - \frac{2}{\cosh\left[\frac{\sqrt{a_4}}{\omega}(t - t_0)\right] - 1} \right], \quad (5.21)$$

$$R_{\mu}^{\mu}[\Sigma]^{a_4 < 0} = -\frac{a_4}{8\omega^2} \left[1 - \frac{2}{\cos\left[\frac{\sqrt{a_4}}{\omega}(t - t_0)\right] - 1} \right], \quad (5.22)$$

for the intrinsic curvature scalars in each case, which are both singular as $t \rightarrow t_0$. Consequently [Figs. 6(c) and 6(d)] once the shell hits zero size a singularity is hit and then we lose predictability.

Now let us consider a point mass in flat space outside the shell. Then we have $a_+ = 0$ and $k > 0$, which implies $a_4 > 0$ and $a_2 > 0$, regardless of the sign of a_- . Then the shell will ultimately expand in this case.

If there is a dS space outside the shell (i.e. $a_+ < 0$ and $k > 0$), we also have $a_2 > 0$ while a_4 can have both signs, depending upon the choice of a_- . If $a_- \geq 0$ (a point mass in AdS or flat space), $a_4 > 0$ and this describes an expanding shell as shown in Fig. 6(a). However, if $a_- < 0$, it describes dS spaces in and outside the shell, which leads to an indefinitely expanding shell. [Fig. 6(a)]

VI. DISCUSSION

Perhaps the most intriguing result of this paper is that shell collapse in $(2 + 1)$ -dimensional gravity can violate—albeit somewhat mildly—cosmic censorship for a broad range of initial conditions, whether we have pressureless dust shells, shells with pressure, or GCG shells. The situation is markedly different from that of a scalar field [28], in which either a black hole is formed or the scalar field oscillates indefinitely without collapse, neither of which violates cosmic censorship. Here, as the shell collapses its density (and pressure, if any) diverge in finite proper time. Although the exterior spacetime develops no curvature singularities (since spacetime in $(2 + 1)$ dimensions has

TABLE I. Properties of collapse scenarios for each initial condition and background space-time.

initial velocity	initial position	exterior spacetime	properties
$\dot{x}_0 > 0$	$x_0 > x_{\min}$	all	expand
	$x_0 < x_{\max}$	AdS ($\Lambda < 0$)	bounce & collapse to BH/NS ⁴
		flat ($\Lambda = 0$)	bounce & collapse to NS
		dS ($\Lambda > 0$)	bounce & collapse to NS
$\dot{x}_0 < 0$	$x_0 > x_{\min}$	all	bounce & expand
	$x_0 < x_{\max}$	AdS ($\Lambda < 0$)	collapse to BH/NS
		flat ($\Lambda = 0$)	collapse to NS
		dS ($\Lambda > 0$)	collapse to NS

constant curvature outside of all matter sources), the stress-energy tensor of the shell diverges in finite proper time and so the Einstein equations (and the second junction condition in Eq. (1.1)) break down. Consequently the equation of motion describing the time-evolution of the shell is no longer valid. This yields a mild violation of cosmic censorship in that, strictly speaking, there is no definite manner in which to continue the spacetime beyond this event and its future light cone.³ A Cauchy horizon forms if this final state is not cloaked by an event horizon. A classification of collapse scenarios, depending on the initial conditions, is shown in the Table I.

We find that dust shells can collapse to zero size in an AdS background, displaying similar behavior to that of a pressureless disk of dust [3]. The final state of collapse in both cases is one in which the shell/disk has finite velocity when it achieves zero size, with a diverging intrinsic Ricci scalar. Unlike the situation in higher dimensions, this state need not be cloaked by an event horizon, in which case a Cauchy horizon is present. However one might take the viewpoint that it is natural to consider matching this spacetime to one in which the shell bounces repeatedly from zero size to a maximal value and back again; the exterior space will always be that corresponding to an AdS point mass, with the interior space being one of several possibilities as outlined in the discussion in section III. The collapse time is always finite. If a black hole is formed, the edge of the shell and the event horizon coincide in finite proper time, t_h .

For shells with pressure, the situation is more intriguing. In this case the final state of shell collapse has a finite radius, since the material of the shell is no longer infinitely compressible. The intrinsic Ricci scalar becomes singular in finite proper time and the effective potential diverges. In other words, the shell collapses to a singular ring with a finite size within finite proper time. If the external geometry is initially that of a black hole, that singular ring will be screened by an event horizon. However this need not be the case, and a naked singular ring with a finite size can be

formed if the exterior metric is that of a point mass. The role of pressure is that of shifting the singular point at $x = 0$ to some finite radial position, sustaining the shell with a finite size.

Despite the qualitatively different physics of the GCG shell, we find that it can exhibit similar behavior to the other two cases. It also presents a variety of scenarios (such as a collapse to a black hole or an indefinitely expanding shell) which depend upon the initial velocity and the shape of the effective potential. However there are some qualitative differences. It is possible for a GCG shell to collapse to a shell of finite radius in which the density is finite but the pressure diverges. There is still a curvature singularity on the surface of the shell at the radius $x = x_s$. Even if the initial external geometry is a black hole, the singular ring need not be cloaked by an event horizon, and scenarios similar to the previous cases ensue.

While collapsing shells in $(2 + 1)$ dimensions are not easily translatable into realistic scenarios in $(3 + 1)$ dimensions, our study is of more than passing interest. From a general relativistic viewpoint shell collapse highlights the importance of understanding what limits there may be to cosmic censorship. Indeed, since there are two possible final states for collapse (either a black hole or not) then there should be some kind of critical phenomenon associated with this scenario as with the scalar field [28]. In this context it would be interesting to extend our results to rotating black holes, where it has been shown that cosmic censorship holds in dust collapse for the addition of a small amount of angular momentum [23]. From a string-theoretic viewpoint it would be interesting to understand the implications of this work for the AdS/CFT correspondence.

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³For a discussion of possible violations of cosmic censorship in fluid collapse in higher dimensions, see [29].

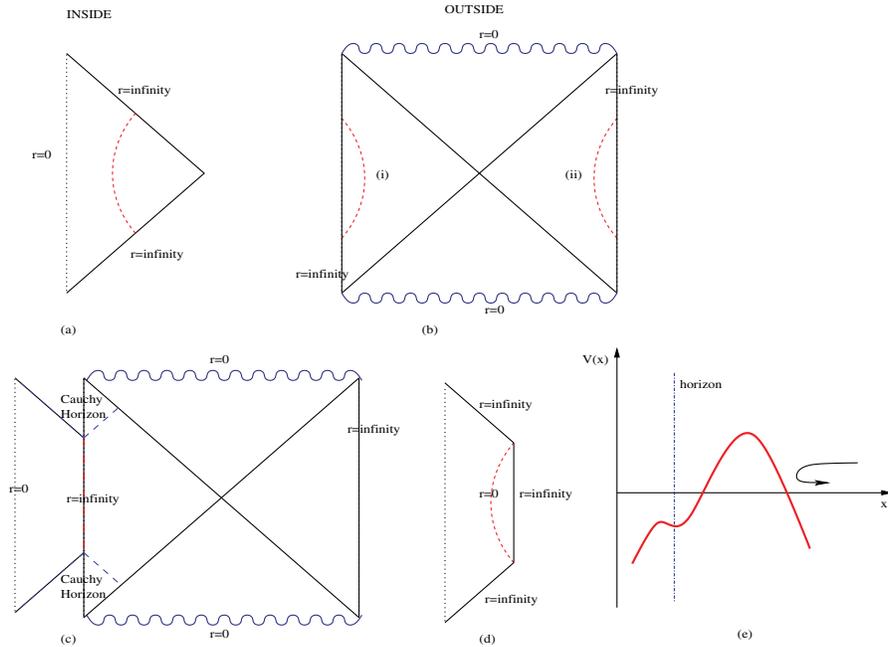


FIG. 7 (color online). Penrose diagrams with flat space inside and a black hole outside the expanding shell. Fig. (a) represents a flat geometry with a point mass and shell trajectory is shown in the red line for the effective potential (Fig. (e)). The external geometry has two possibilities of sign of β_+ ((i) and (ii)). Combined figures of both cases are shown in Figs. (c) (for (i)) and (d) (for (ii)). Note that for case (c) there is a Cauchy horizon.

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APPENDIX A: PENROSE DIAGRAMS

We consider here the construction of Penrose diagrams for the various collapse scenarios. With no loss of generality, the positivity of energy density gives rise to

$$\beta_- - \beta_+ > 0, \tag{A1}$$

where $|\beta_{\pm}| = \sqrt{\dot{x}^2 + F_{\pm}}$ can each be either positive or negative and $F_{\pm} = a_{\pm}x^2 + k_{\pm}$. A wide variety of collapse scenarios are possible under the constraint of Eq. (A1). For example, if $k_+ > 0$ and $k_- < 0$ (a black hole inside and a point mass outside the shell), taking $\beta_+ > 0$ for an expanding shell restricts $\beta_- > 0$ but otherwise provides no further constraints.

Cases with an AdS-exterior and dS-interior in the context of inflationary models have been treated before [26]. We shall therefore not consider this case, and concentrate only on a few of the remaining scenarios. While we have been primarily interested in shell collapse in this paper, we shall consider scenarios where the shell can expand as well.

Consider first a point mass inside and an AdS black hole outside the shell. Within this context we can have both expanding and collapsing shells, depending upon the direction of the initial velocity. For the expanding case, the positivity condition, Eq. (A1) leads to two possible final geometries upon matching to the exterior, as shown in

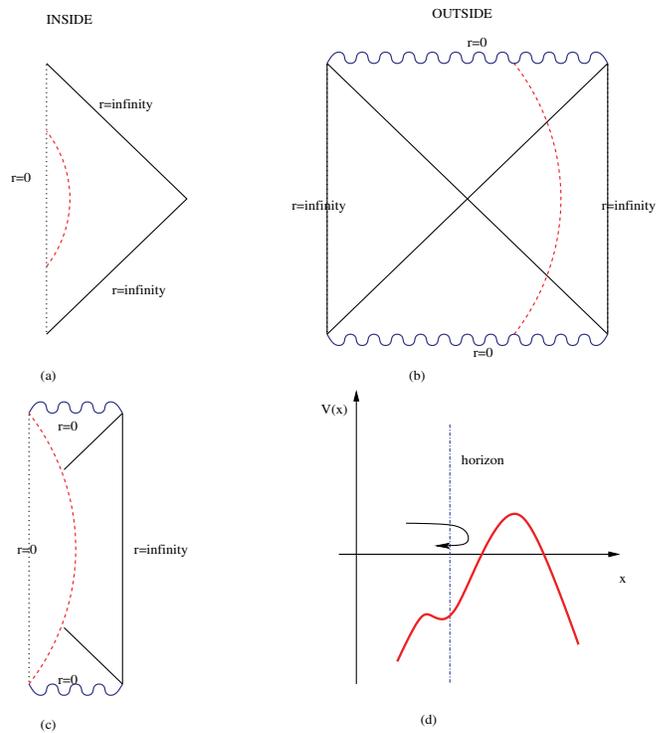


FIG. 8 (color online). Sketch of Penrose diagrams of a flat space inside and a black hole outside the collapsing shell (Figs. (a) and (b)) and a combined diagram (Fig. (c)).

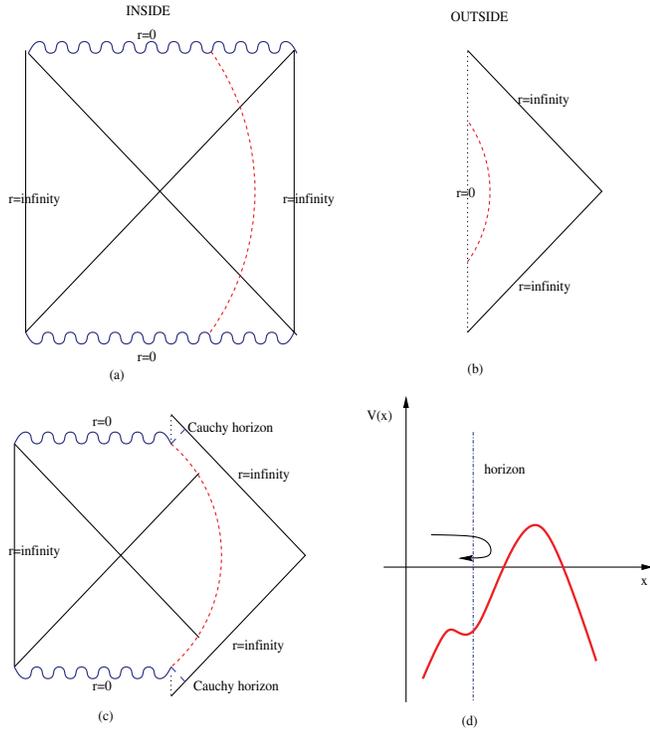


FIG. 9 (color online). Sketch of Penrose diagrams of a black hole inside and a flat space outside the collapsing shell (Figs. (a) and (b)) and a combined diagram (Fig. (c)); the latter shows that a Cauchy horizon appears.

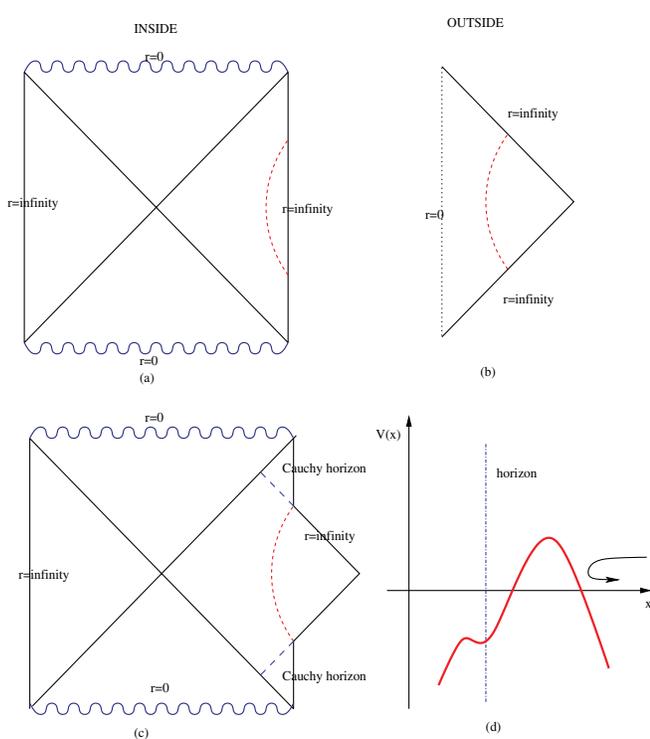


FIG. 10 (color online). Sketch of Penrose diagrams of a black hole inside and a flat space outside an expanding shell (Figs. (a) and (b)) and the combined diagram (Fig. (c)). Note the resultant Cauchy horizon in case (c).

Figs. 7(c) and 7(d). The former contains Cauchy horizons (though not from the viewpoint of observers on the right half of the diagram).

The diagrams for a collapsing shell can be obtained by considering the same procedure in Fig. 8. Here the shell forms an event horizon in finite proper time, collapsing into a black hole; the time-reversed version of this (in which the shell expands out of a white hole) is also shown in the diagram.

Inverting the exterior and interior, we obtain the situation depicted in Figs. 9 and 10 for contracting and expanding cases, respectively. An interesting feature can be found for the collapsing shell case (Fig. 9). In the combined figure [Fig. 9(c)], we see that an observer external to the shell will eventually have the singularity within his/her past light cone, signalling the appearance of a Cauchy horizon. This can be avoided for special trajectories of the shell, in which null infinity for the external observer ends at the endpoint of the shell trajectory in the Penrose diagram.

For a shell with pressure the stress-energy tensor diverges at $x = x_s$ (where the effective potential also diverges), forming a singular ring and a Cauchy horizon. The relevant diagrams are shown in Fig. 11 for a collapsing shell, which also yields a cosmic censorship violation.

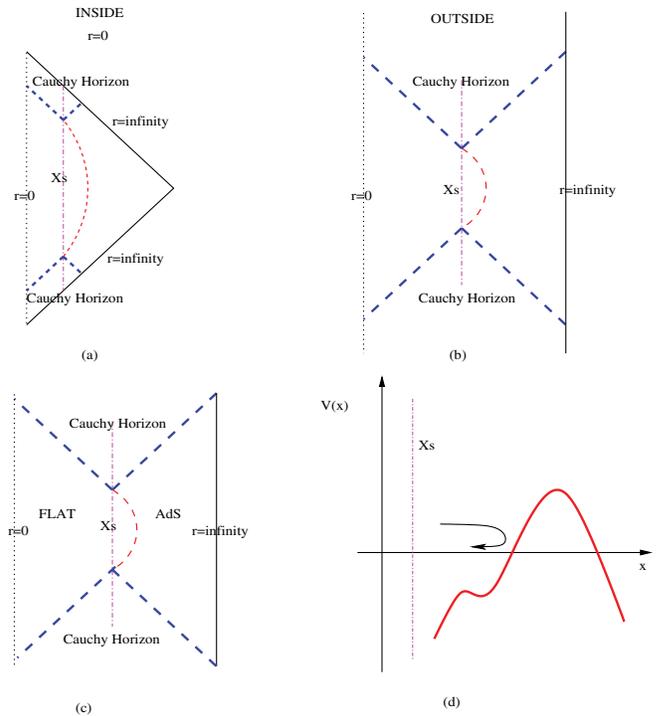


FIG. 11 (color online). Sketch of Penrose diagrams of a flat space inside and AdS spacetime outside a collapsing shell with pressure (Figs. (a) and (b)) and a combined diagram (Fig. (c)). In this case there is a Cauchy horizon that forms where stress-energy tensor diverges at $x = x_s$.

APPENDIX B: JACOBIAN ELLIPTIC FUNCTIONS

In this appendix, we shall briefly introduce Jacobian elliptic functions and its properties. We can define a doubly periodic elliptic function with real parameters, m and m_1 , where $m + m_1 = 1$, as

$$K(m) = K = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - m\sin^2\theta}},$$

$$iK'(m) = iK' = i \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - m_1\sin^2\theta}},$$
(B1)

where $0 \leq m \leq 1$. Note that K and K' are real numbers. Here we denote the points, $0, K, K + iK', iK'$ by s, c, d, n respectively, which are at the vertices of a rectangle and show a repeated pattern indefinitely. Now the Jacobian elliptic functions can be defined with respect to an integral,

$$u = \int_0^\varphi \frac{d\theta}{\sqrt{1 - m\sin^2\theta}},$$
(B2)

where the angle φ is called the amplitude. Then, we define

$$\text{JacobiSN}(u, k) = \sin\varphi,$$
(B3)

$$\text{JacobiCN}(u, k) = \cos\varphi,$$
(B4)

$$\text{JacobiDN}(u, k) = \sqrt{1 - m\sin^2\varphi}.$$
(B5)

Here we simply denote $\text{JacobiSN}(u, k)$ as $sn(u|m)$, hereafter, where $m = k^2$. There are some useful relations of the Jacobian functions to the copolar trio, sn, cn, dn , such that

$$cd(u|m) = \frac{cn(u|m)}{dn(u|m)}, \quad dc(u|m) = \frac{dn(u|m)}{cn(u|m)},$$

$$ns(u|m) = \frac{1}{sn(u|m)}, \quad sd(u|m) = \frac{sn(u|m)}{dn(u|m)},$$

$$nc(u|m) = \frac{1}{cn(u|m)}, \quad ds(u|m) = \frac{dn(u|m)}{sn(u|m)},$$

$$nd(u|m) = \frac{1}{dn(u|m)}, \quad sc(u|m) = \frac{sn(u|m)}{cn(u|m)},$$

$$cs(u|m) = \frac{cn(u|m)}{sn(u|m)}.$$

If the parameter m is a positive number, there are some useful relations for the negative parameter. Defining new parameters as $\mu = m/(1 + m)$, $\mu_1 = 1/(1 + m)$, and $v = u/\sqrt{\mu_1}$, where $0 < \mu < 1$, we have

$$sn(u|m) = \sqrt{\mu_1}sd(v|\mu),$$
(B6)

$$cn(u|m) = cd(v|\mu),$$
(B7)

$$dn(u|m) = nd(v|\mu).$$
(B8)

The Jacobian elliptic function is a real function for the real parameters and variables. If we consider the SN function, $SN(au, m)$, where a and m are constants, then the function has the following properties:

$$sn(au|m) \in \mathcal{R} \quad (a \in \mathcal{R}, m \in \mathcal{R} \quad \text{or} \quad a \in \mathcal{R}, m \in I)$$
(B9)

$$sn(au|m) \in I \quad (a \in I, m \in \mathcal{R} \quad \text{or} \quad a \in I, m \in I).$$
(B10)

There are Jacobi's imaginary transformations for the imaginary values of parameters,

$$sn(iu|m) = isc(u|m_1),$$

$$cn(iu|m) = nc(u|m_1),$$

$$dn(iu|m) = dc(u|m_1),$$
(B11)

which are useful to convert a function to a simple form. Moreover, there are some useful relations for real parameters, called Jacobi's real transformation. For $m > 0$, defining $\mu = 1/m$ and $v = \sqrt{m}u$, then we have

$$sn(u|m) = \sqrt{\mu}sn(v|\mu),$$
(B12)

$$cn(u|m) = dn(v|\mu),$$
(B13)

$$dn(u|m) = cn(v|\mu).$$
(B14)

If $m > 1$, then $m^{-1} = \mu < 1$, which implies that real parameters of elliptic functions always lies between 0 and 1. More details on the further properties on the Jacobian elliptic functions are shown in Ref. [30].

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